Correspondences on hyperelliptic surfaces, combination theorems, and Hurwitz spaces

Sabya Mukherjee

Tata Institute of Fundamental Research

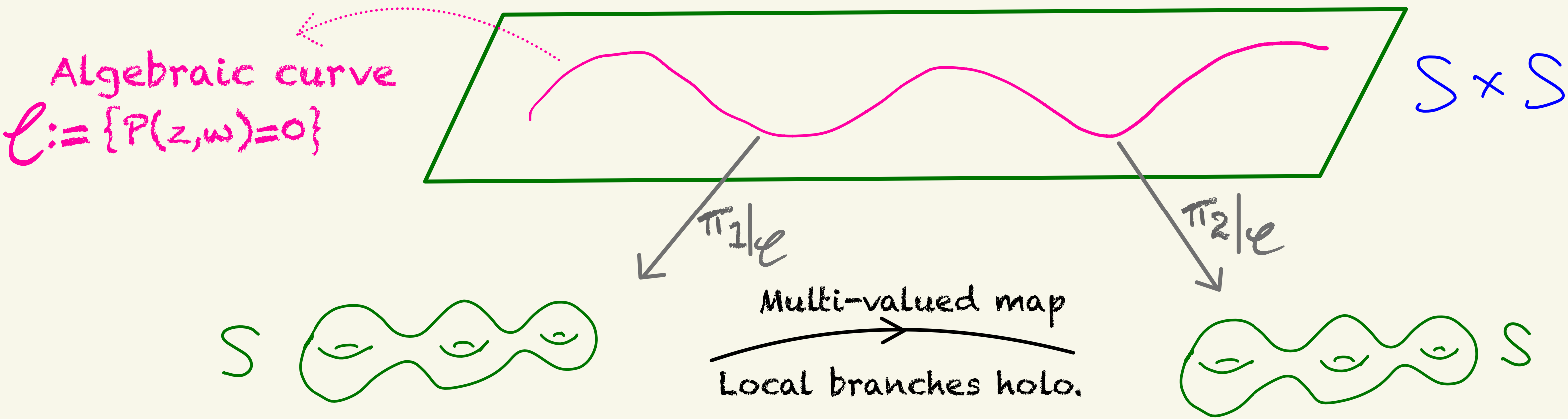
Pacific Quasiworld

November 2025

Holomorphic correspondences

- Holomorphic/algebraic correspondence: a finite-to-finite map on a compact Riemann surface S with holomorphic local branches.

- Equivalently,



Some correspondences on $\hat{\mathbb{C}}$

- $w - R(z) = 0$

  $R \to$ rational map

- $(w-\gamma_1(z))\ldots(w-\gamma_k(z)) = 0$

  $\langle \gamma_1, \gamma_2, \ldots, \gamma_k \rangle \to$ f.g. Kleinian group

- Holomorphic correspondences provide a general framework for studying rational dynamics and Kleinian groups.

  (Sullivan's dictionary)

- However, a general dynamical theory of correspondences is out of reach at present.

A Program

Explore compatibility between these systems to combine them as corr., and put their parameter spaces in a unified Setting

Understand connections between various notions of mating

Use this framework to get a window into the dynamics of corr.

Piecing together

polynomials and Fuchsian groups

to produce correspondences

Main difficulty in interbreeding:

Invertible group vs Non-invertible map

Strategy:

1. Encode the group in a map

2. Mate this map with a rational map

3. Describe the conformal mating 'explicitly'

4. Promote the conformal mating to a correspondence

Theorem (Mj-M, Luo-Lyubich-M)

- A large class of genus zero orbifolds (including punctured spheres and Hecke orbifolds) can be mated with 'tame' polynomials of appropriate degree.

- The matings are realized as correspondences on nodal spheres.

- Products of parameter spaces of polynomials and Teichmüller spaces of the above surfaces embed into parameter spaces of algebraic correspondences.

(Tame = At most finitely renormalizable, no Siegel/Cremer)

Theorem (M-Viswanathan)

- An arbitrary collection of expanding Blaschke products and Fuchsian punctured sphere/Hecke groups can be combined into a correspondence on a hyperelliptic Riemann surface.

- This yields a dynamically natural injection of products of Teichmüller and Blaschke spaces into a Hurwitz space (moduli space of ramified covers of the Riemann sphere).

- This is a hybrid analog of the following facts:

Multiple Blaschke products can be combined in a rational map.

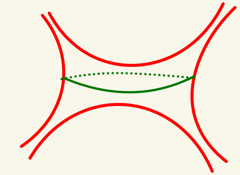
Multiple Fuchsian groups can be combined in a Kleinian group.

Strategy:

1. Replace a group with a map that remembers the dynamics of the group, and is compatible with rational dynamics.
2. ....
3. ....
4. ....

Step I: Circle coverings from groups

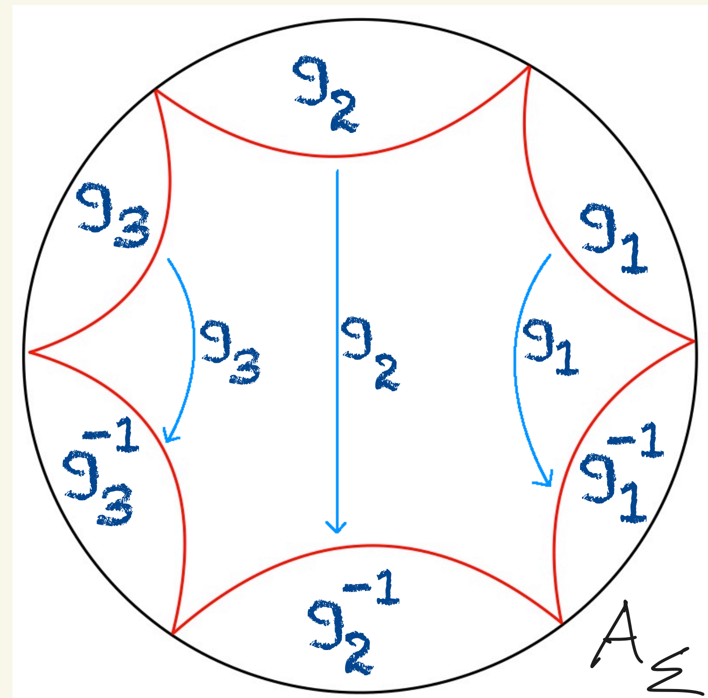
$\Sigma = S_{0,4}$



A. Punctured spheres

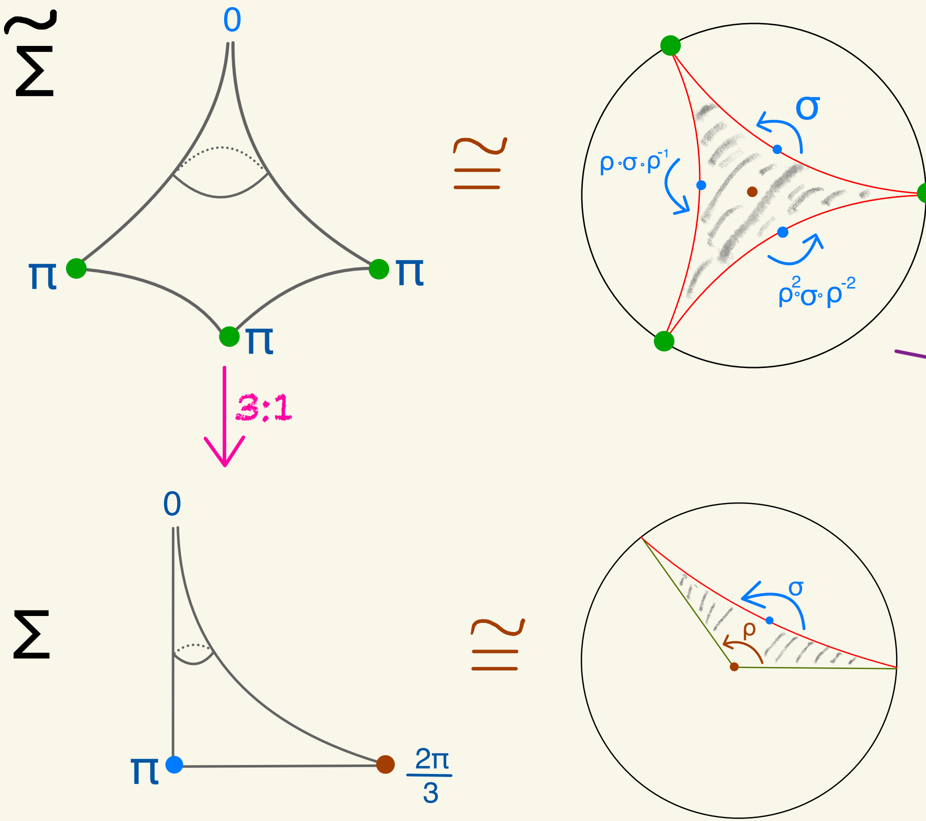
$S_{0,d+1} \longrightarrow$ Bowen-Series map,
degree $2d-1$
circle covering

i) piecewise analytic,
ii) expansive circle covering,
iii) orbit equivalent to the group.

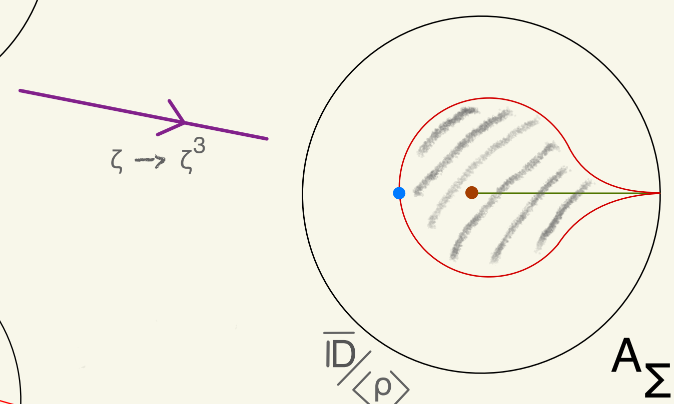


B. Hecke surface (genus zero; (2,n,∞))

$A^{BS}_{\tilde\Sigma}$ (Mildly discontinuous, commutes with ρ)



Modular surface



i) piecewise analytic, expansive circle covering,

ii) virtually orbit equivalent to the group.

Strategy:

1. Replace a group with a map that remembers enough about the group, but is also compatible with rational dynamics.

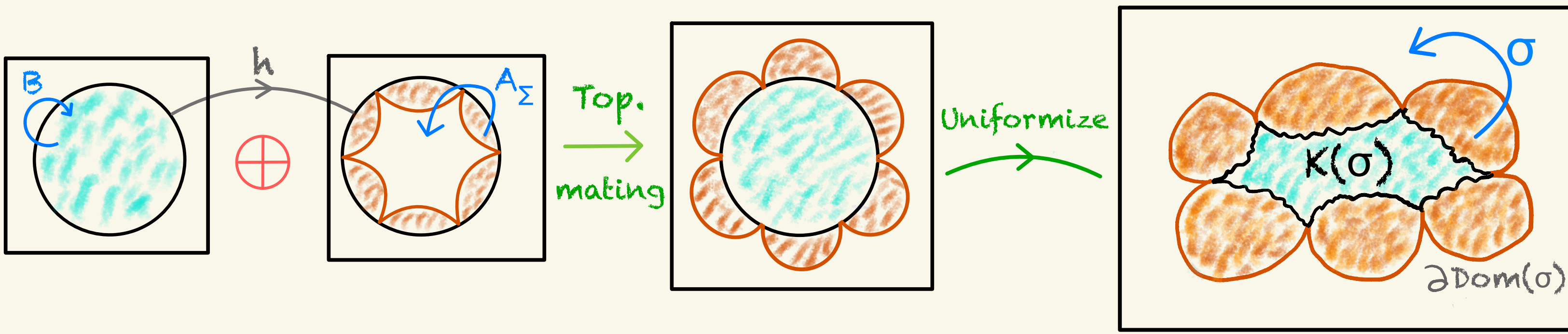
2. Combine this map with a rational map producing a "conformal mating" (surgery).

3. ....

4. ....

Step II: Putting Blaschke and Fuchsian actions into a single map

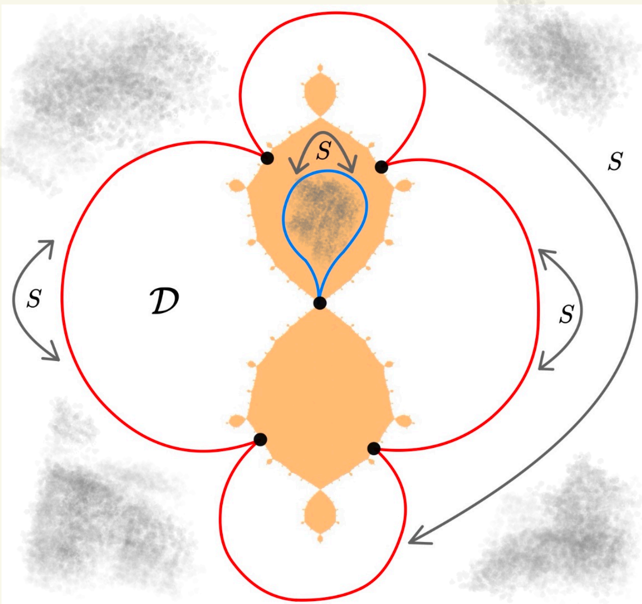
Mating a pair:

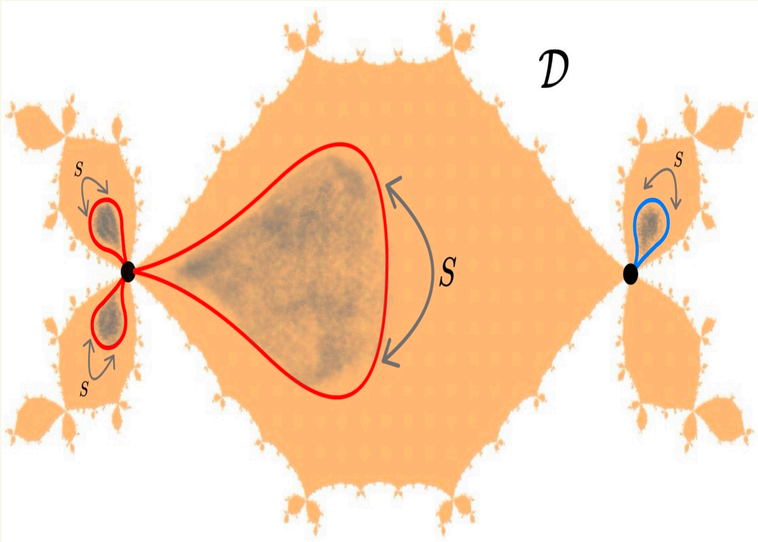


Conformal mating

- h conjugates B to A_Σ, and extends to the disk as a David homeomorphism.

Mating multiple Blaschke products and Fuchsian groups





- The constituents can be housed in a critically fixed polynomial.

Strategy:

1. Replace a group with a map that remembers enough about the group, but is also compatible with rational dynamics.

2. Combine this map with a rational map producing a "conformal mating" (surgery).

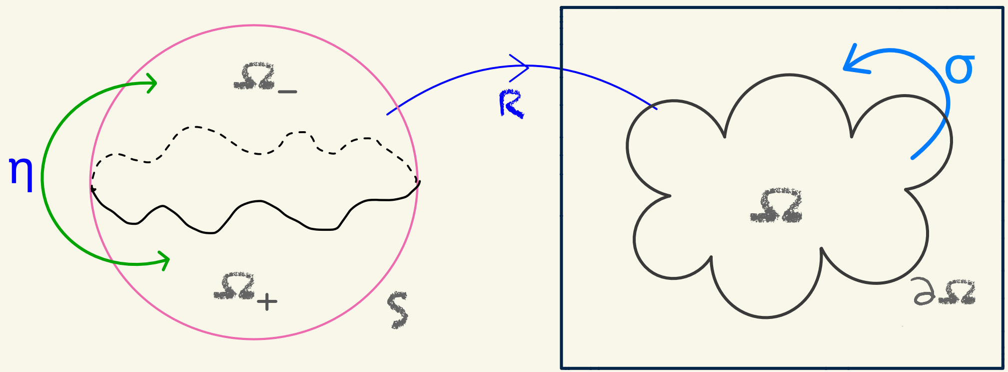
3. Prove algebraicity of the conformal mating.

4. ....

Step III: Algebraic description of conformal matings

- **Boundary involution:** "meromorphic map σ inducing an orientation-reversing involution on the boundary"

→ Weld two copies of Ω along the boundary using σ.



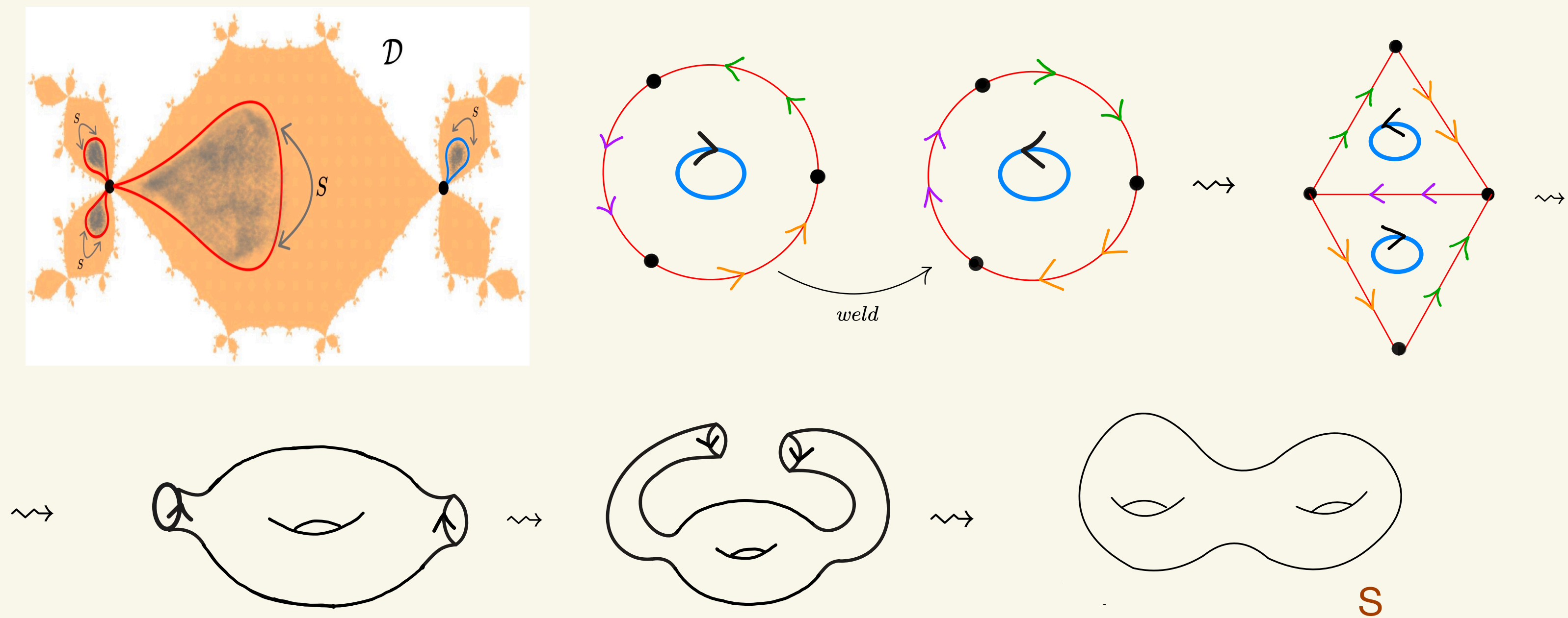
$$\sigma = R \circ \eta \circ \left(R|_{\Omega_+}\right)^{-1}$$

$$\eta : \Omega_\pm \to \Omega_\mp$$
$$\eta^2 = \mathrm{Id}$$

$$R : S \to \hat{\mathbb{C}} := \begin{cases} \mathrm{Id}, & \text{on } cl(\Omega_+) \\ \sigma, & \text{on } \Omega_- \end{cases}$$

Branched covering

Welding creates topology



- S carries an involution η with $S/\langle\eta\rangle \cong \hat{\mathbb{C}}$ ($\Rightarrow$ S is hyperelliptic)

- $\exists$ $R: S \to \hat{\mathbb{C}}$ meromorphic, injective on one-half of S, with

$$\sigma = R \circ \eta \circ \left(R|_{\overline{\Omega_+}}\right)^{-1}$$

Strategy:

1. Replace a group with a map that remembers enough about the group, but is also compatible with rational dynamics.

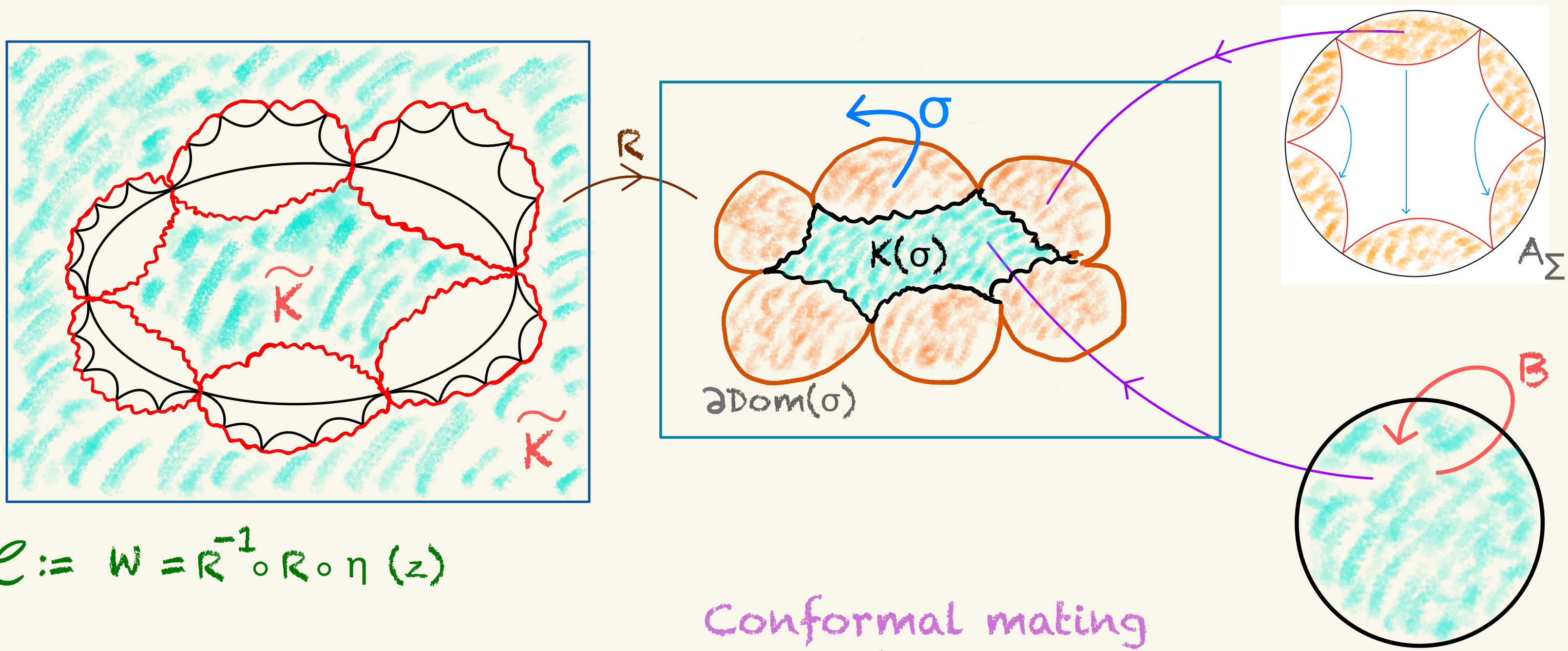
2. Combine this map with a rational map producing a "conformal mating" (surgery).

3. Prove algebraicity of the conformal mating.

4. Use the algebraic description to construct a correspondence.

Step IV: From boundary involutions to correspondences

- Lift the σ-dynamics by the meromorphic map R to define a correspondence $(z,w) \in \mathcal{C} \iff R(w) = R(\eta(z))$.



$\mathcal{C} := W = R^{-1} \circ R \circ \eta\,(z)$

Conformal mating plane

Correspondence plane

- In general, the triple (S,η,R) defines a correspondence on S:

$$\mathcal{C} := \quad w = R^{-1} \circ R \circ \eta\,(z)$$

- The Correspondence $\mathcal{C}$ combines the constituent Blaschke products and Fuchsian groups.

- Once the degrees of the Blaschke products and the topology of the surfaces are fixed, the ambient parameter space of the correspondences is given by the:

'Hurwitz space' =

{The space of pairs (S,R); i.e., the collection of all ramified covers of the sphere by Riemann surfaces of a fixed genus g.} $/\sim$

- This yields a natural injection of products of Teichmüller and Blaschke spaces into the above Hurwitz space.

Tasks:

- Study boundedness/degeneration in the Hurwitz space, as the Blaschke products/ Fuchsian groups degenerate.
- Study the dynamics of boundary correspondences.
- Using tuning, combinatorial continuity and rigidity arguments, construct embeddings of Mandelbrot sets in 'Hurwitz spaces'.

Thank you!