Dynamical systems - Problem Set 1

(1) *Middle-thirds Cantor Set as a Non-escaping Set.*

Prove that the set

$$C = \left\{ x \in [0,1] : E_3^{\circ k}(x) \notin \left(\frac{1}{3}, \frac{2}{3} \right) \forall\, k \in \mathbb{N} \cup \{0\} \right\}$$

is the standard middle-thirds Cantor set, where $E_3(x) = 3x \pmod 1$. Conclude that $\frac{1}{4} \in C$.

(2) *ω-Limit Set.*

Let (X, d) be a non empty compact set and $T : X \to X$ continuous. Let $x \in X$. Show that the set $\omega(x)$ consisting of the accumulation points of the orbit of x, i.e.

$$\omega(x) := \{ y \in X |\ \exists n_i \uparrow +\infty,\ y = \lim_{i\to\infty} T^{\circ n_i}(x) \},$$

is compact, non-empty, and invariant.

(3) *Conjugacy between Linear Maps.*

a) Prove that the set of all invertible linear contractions is open in $M_n(\mathbb{R})$.

b) Let $A\colon \mathbb{R}^n \to \mathbb{R}^n$ be an invertible linear contraction. Prove that any linear map $B\colon \mathbb{R}^n \to \mathbb{R}^n$, with $||B - A||$ sufficiently small, is topologically conjugate to A.

c) Deduce that any two orientation-preserving (respectively reversing) invertible linear contractions are topologically conjugate.

d) Conclude that the maps $x \mapsto \frac{x}{2}$ and $x \mapsto \frac{x}{3}$ of the real line are topologically conjugate, but not smoothly conjugate.

(4) *Periodic Points of Subshifts of Finite Type.*

Let A be an $m \times m$ matrix of zeroes and ones, and Σ_A be the one-sided subshift of finite type determined by A (see Section 1.4 of Brin and Stuck). Prove that:

(a) the number of fixed points in Σ_A is the trace of A;
(b) the number of permissible words of length $n+1$ beginning with the symbol i and ending with j is the (i,j)-th entry of A^n; and
(c) the number of periodic points of period n in Σ_A is the trace of A^n.

(5) *Contraction Mappings.*

Let (X, d) be a non-empty compact set and $T : X \to X$ continuous.

(a) Suppose that $T^{\circ p}$ is strictly contracting for some integer $p \geq 1$. Show that T has a unique fixed point x_0 and $\{T^{\circ n}(x)\}$ converges to x_0 as $n \to +\infty$ for all $x \in X$.
(b) Suppose that $d(Tx, Ty) < d(x, y)$ for all $x \neq y \in X$. Show that T has a fixed point. Can the assumption be relaxed to $d(Tx, Ty) \leq d(x, y)$?

(6) *Real Quadratic Maps.*

For $\mu \geq 1$, consider the real quadratic map

$$q_\mu : \mathbb{R} \to \mathbb{R}, \quad x \mapsto \mu x(1 - x).$$

Prove the following statements.

(a) The forward orbit of each point outside $[0,1]$ tends to $-\infty$.
(b) For $1 \leq \mu \leq 4$, the interval $[0,1]$ is invariant under q_μ.
(c) For $1 \leq \mu < 3$, the forward orbit of each point in $(0,1)$ converges to the fixed point $(1 - \frac{1}{\mu})$. What happens for $\mu = 3$?
(d) For $\mu = 4$, the dynamical system is *chaotic*; i.e., there is a dense forward orbit, and the periodic points of q_μ are dense on $[0,1]$.
(e) For $\mu = 5$, the *non-escaping set* $\{x \in \mathbb{R} : (q_\mu^{\circ n}(x))_n \text{ is bounded}\}$ is a Cantor set.