Dynamical systems - Problem Set 2

(1) *Iteration of Möbius Maps.*

(a) Let

$$M(z) = \frac{3z-2}{2z-1}.$$

Compute an explicit formula for $M^{\circ n}$, and deduce that for each $z \in \widehat{\mathbb{C}}$, the orbit $R^{\circ n}(z) \to 1$ as $n \to +\infty$. Is the convergence uniform on a neighborhood of 1?

(b) More generally, show that if a Möbius map M has a unique fixed point in $\widehat{\mathbb{C}}$, then all orbits under M converge to this fixed point. (Such maps are called *parabolic* Möbius transformations.)

(c) On the other hand, argue that if M has exactly two distinct fixed points in $\widehat{\mathbb{C}}$, then either M is conjugate to a rotation (via a Möbius map), or all orbits under M converge to one of the fixed points of M.

(2) *Attracting Dynamics on $\mathbb{D}$.*

If a holomorphic map $f : \mathbb{D} \to \mathbb{D}$ fixes the origin and is not a rotation, prove that the orbit $\{f^{\circ n}(z)\}_{n \geq 0}$ converges to the origin for all $z \in \mathbb{D}$. Prove that this convergence is uniform on compact subsets of $\mathbb{D}$.

(Hint: Schwarz lemma.)

(3) *Conjugacy to a Polynomial.*

Let $R(z) := (2z^2 - 2z + 1)/(3z^2 - 4z + 2)$. Does there exist a Möbius map M such that $M \circ R \circ M^{-1}$ is a polynomial?

(4) *Dynamics of Blaschke Products.*

For any $a \in \mathbb{D}$, the map

$$\varphi_a(z) = (z - a)/(1 - \bar{a}z)$$

carries the unit disk $\mathbb{D}$ conformally onto itself. A finite product of the form

$$f(z) = e^{i\vartheta}\varphi_{a_1}(z)\varphi_{a_2}(z)\cdots\varphi_{a_n}(z)$$

with $a_i \in \mathbb{D}$, $\vartheta \in \mathbb{R}$, is called a Blaschke product of degree n.

(a) Show that every such f is a rational map which carries $\mathbb{D}$ onto $\mathbb{D}$ and $\widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}$ onto $\widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}$. Conclude that the Julia set of f is contained in the unit circle.

(b) If $n \geq 2$ and if some $a_i = 0$, show that f has attracting fixed points at 0 and ∞, and that the Julia set of f is the entire unit circle.

(c) Study the local dynamics of the Blaschke product $f(z) = \frac{3z^2+1}{3+z^2}$ near the fixed point 1 (hint: look at its power series expansion at $z = 1$). Use this to show that all orbits in $\mathbb{D}$ and $\widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}$ converge to 1. What is the Julia set of f?