Dynamical systems - Problem Set 3

(1) *Totally Invariant Fatou Components.*

Let $R : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a rational map of degree at least 2, and U be a totally invariant component of the Fatou set $\mathcal{F}(R)$ (i.e., $R(U) = R^{-1}(U) = U$).

(a) Prove that $\partial U = \mathcal{J}(R)$.

(b) Conclude that the other components of $\mathcal{F}(R)$ are Jordan domains.

(Hint: iterated pre-images of a Julia point are dense on the Julia set.)

(2) *Finitely many Fatou components.*

Let $R : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a rational map of degree at least 2.

(a) Prove that if the Fatou set $\mathcal{F}(R)$ has finitely many components, then the number of Fatou components is $0, 1$, or 2.

(b) Prove that if $\mathcal{F}(R)$ has two totally invariant Fatou components, then these are the only components of $\mathcal{F}(R)$.

(Hint: Riemann-Hurwitz formula + classification of Fatou components, and their relations with critical orbits.)

Remark. Precise analogues of the results of Problems (1) and (2) hold true for finitely generated Kleinian groups. These are instances of the so-called *Sullivan dictionary* between rational dynamics and Kleinian groups.

(3) *Siegel Disks.*

Let R be a rational function of degree 2 or more with an irrationally neutral fixed point z_0 (i.e., $R(z_0) = z_0,\ |R'(z_0)| = 1$). Prove that the following three conditions are equivalent.

(a) R is locally linearizable around z_0; i.e., there exists a neighborhood $B(0, \varepsilon)$ and an injective holomorphic map $\varphi : B(0, \varepsilon) \to \mathbb{C}$ such that

$$\varphi(0) = 0, \text{ and } \varphi(R(z)) = R'(z_0) \cdot \varphi(z),\ \forall\, z \in B(0, \varepsilon).$$

(b) z_0 belongs to the Fatou set $\mathcal{F}(R)$.

(c) The connected component U of $\mathcal{F}(R)$ containing z_0 is conformally isomorphic to the unit disk $\mathbb{D}$ under an isomorphism which conjugates $R|_U$ to multiplication by $R'(z_0)$ on $\mathbb{D}$.