Exercice 1. (The non-wandering set.) Let (X,T) be a TDS. A point x is called *wandering* for T if there exists an open neighborhood U of x such that the sets $T^{-n}U$, $n \geq 0$ are pairwise disjoint. The *non-wandering* set for T, $\Omega(T)$, consist of all the points of X that are not wandering for T:

$$\Omega(T) = \{x \in T : \text{for every neighborhood } U \text{ of } x \ \exists n \geq 1 \text{ with } T^{-n}U \cap U \neq \emptyset\}.$$

(1) Prove that

(a) $\Omega(T)$ is closed.

(b) All periodic points belong to $\Omega(T)$.

(c) $\bigcup_{x\in X} \omega(x) \subset \Omega(T)$.

(d) $T(\Omega(T)) \subset \Omega(T)$.

(e)

$$\Omega(T) = \{x \in T : \text{for every neighborhood } U \text{ of } x \text{ and every } N \geq 1 \\ \text{there exists } n \geq N \text{ with } T^{-n}U \cap U \neq \emptyset\}.$$

(Hint: distinguish the cases x periodic and x non periodic).

(2) Define $\Omega_1(T) = \Omega(T)$ and by induction $\Omega_n(T) = \Omega(T_{|\Omega_{n-1}(T)})$.

(a) The set $\Omega_\infty = \bigcap_{n\geq 1} \Omega_n$ is called the center of T. Let μ be a T-invariant Borel probability measure. Prove by induction that $\mu(\Omega_n(T)) = 1$ for all $n \geq 1$, hence $\mu(\Omega_\infty) = 1$.

(b) Prove that if there exists a T-invariant Borel probability measure μ which gives positive mass to any non-empty open subset, then $X = \Omega(T)$.

(c) If X is the closed unique disc and $T(re^{2i\pi\theta}) = \sqrt{r}e^{2i\pi(\theta^2+1-r \mod 1)}$ $(0 \leq r \leq 1,\ \theta \in [0,1))$, show that $\Omega_1(T) = \{(0,0)\} \cup \partial X$ and $\Omega_2(T) = \{(0,0)\} \cup \{1,0\}$. Determine all the T-invariant measures in this case.

Exercice 2. (Unique ergodicity.) Let (X,T) be a TDS. (X,T) is said *uniquely ergodic* if there exists only one T-invariant Borel probability measure on X: $M(X,T) = \{\mu\}$. The measure is then necessarily ergodic.

(1) (a) Suppose that (X,T) is uniquely ergodic. Show that (X,T) is minimal if and only if $\mu(U) > 0$ for every non empty open set U.

(b) Suppose that X is the unit circle and $T : K \to K$ is the mapping $T(e^{2\pi\theta}) = e^{2\pi\theta^2}$, $\theta \in [0,1]$. Show that (X,T) is uniquely ergodic but not minimal.

(2) Show that the following properties are equivalent:

(a) (X,T) is uniquely ergodic;

(b) for every $f \in C(X)$, $\frac{1}{n}\sum_{k=0}^{n-1} f(T^k x)$ converges uniformy to a constant;

(c) for every $f \in C(X)$, $\frac{1}{n}\sum_{k=0}^{n-1} f(T^k x)$ converges pointwise to a constant;