Topology and geometry of quadrature domains via holomorphic dynamics

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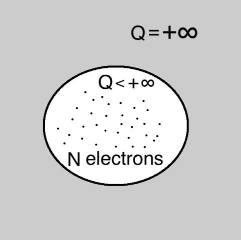
Quadrature domains in statistical physics

Motivation from physics: 2D Coulomb gas ensembles

- Consider N electrons placed in the complex plane at points $\{z_j\}_{j=1}^N$, influenced by a strong external (magnetic/electrostatic) field with uniform density.
  Let the potential of the external field be $NQ : \mathbb{C} \to \mathbb{R} \cup \{+\infty\}$.



- The combined energy resulting from particle interaction and external potential is:

$$\mathcal{E}_Q(z_1, \cdots, z_N) = \sum_{i \neq j} \ln |z_i - z_j|^{-1} + N \sum_{j=1}^{N} Q(z_j).$$

Large N behavior of 2D Coulomb gas ensembles

- It is more likely to find configurations of electrons with low energy. This leads to the following joint density of states

$$\frac{\exp(-\mathcal{E}_Q(z_1, \cdots, z_N))}{Z_N} dVol_{2N} \in \mathrm{Prob}(\mathbb{C}^N).$$

- We are interested in the limiting behavior of the point process as the number of electrons grows to infinity.
- In the limit, the electrons condensate on a compact set T, and they are distributed according to the normalized area measure of T. (Wiegmann, Zabrodin, Elbau, Felder, Hedenmalm, Makarov, et al.)

Random normal matrix ensemble

- For a potential Q, consider the probability measure

$$e^{-N \text{ trace } Q(M)} dM/Z_N$$

  on the set of $N \times N$ complex normal matrices with spectrum in some compact set.

- (Elbau–Felder, Hedenmalm–Makarov) The corresponding eigenvalues $\{\lambda_j\} \in \mathbb{C}^N$ are distributed according to the probability measure

$$e^{-\mathcal{E}_Q(\lambda_1, \cdots, \lambda_N)} dA(\lambda_1) \cdots dA(\lambda_N)/Z_N$$

- In the large N limit, the eigenvalues are distributed according to the normalized area measure of a compact set T.

- In many physically interesting cases, the potential Q satisfies some algebraic properties. The set T (on which eigenvalues/electrons condensate) is then called an *algebraic droplet* of Q.

From algebraic droplets to quadrature domains

- Algebraic potential: $Q(z) = |z|^2 - \Re(H(z))$, where ∂H is a rational function.
- In this situation, the complementary components of the droplet T admit global reflection maps. (Lee-Makarov)

Definition (Quadrature Domains – Aharonov, Shapiro)

A domain $\Omega \subsetneq \hat{\mathbb{C}}$ (with $\infty \notin \partial\Omega$ and $\mathrm{int}(\overline{\Omega}) = \Omega$) is called a *quadrature domain* if there exists a continuous function $\sigma : \overline{\Omega} \to \hat{\mathbb{C}}$ satisfying the following two properties:

1. $\sigma = \mathrm{id}$ on $\partial\Omega$.
2. σ is anti-meromorphic on Ω.

- The map σ is called the *Schwarz reflection map* of Ω.

Quadrature domains

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- The map σ is called the *Schwarz reflection map* of Ω.
- Thus, the study of algebraic droplets is related to the study of quadrature domains.
- Lee and Makarov used iteration of Schwarz reflection maps to study the topology of quadrature domains.

Classical perspective: quadrature identity

- A domain $\Omega \subsetneq \widehat{\mathbb{C}}$ (with $\infty \notin \partial\Omega$ and $\mathrm{int}(\overline{\Omega}) = \Omega$) is a quadrature domain $\iff$
  There exists a rational map R_Ω with all poles inside Ω such that
  $$\int_\Omega \phi dA = \frac{1}{2i} \oint_{\partial\Omega} \phi(z) R_\Omega(z) dz \quad \left(= \sum c_k \phi^{(n_k)}(a_k)\right)$$
  for all $\phi \in H(\Omega) \cap C(\overline{\Omega})$ (if $\infty \in \Omega$, one also requires $\phi(\infty) = 0$).

- By definition, $\mathrm{order}(\Omega) := \deg(R_\Omega)$.

- The rational map R_Ω and the Schwarz reflection map σ have the same (finite) set of poles.

Algebraic properties, and complexity

Simply connected quadrature domains

- A simply connected domain $\Omega \subsetneq \hat{\mathbb{C}}$ with $\infty \notin \partial\Omega$ and $\mathrm{int}(\overline{\Omega}) = \Omega$ is a quadrature domain if and only if the Riemann map $\phi : \mathbb{D} \to \Omega$ extends as a rational map of $\widehat{\mathbb{C}}$.

$$\begin{tikzcd} \overline{\mathbb{D}} \arrow[r, "\varphi"] \arrow[d, "1/\overline{z}"] & \overline{\Omega} \arrow[d, "\sigma"] \\ \hat{\mathbb{C}} \setminus \mathbb{D} \arrow[r, "\varphi"] & \hat{\mathbb{C}} \end{tikzcd}$$

- The rational map ϕ semi-conjugates the reflection map $1/\overline{z}$ of $\mathbb{D}$ to the Schwarz reflection map σ of Ω.

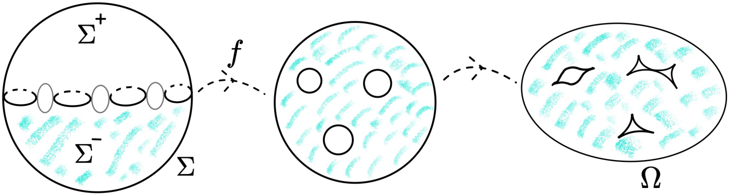
Quadrature domains and Schottky double

- $\Omega =$ Quadrature domain of connectivity k with Schwarz reflection map σ (i.e., $k = \#$ connected components of $\Omega^c = \widehat{\mathbb{C}} \setminus \Omega$).

Theorem (Gustafsson, Acta Appl. Math., 1983)

There exist a genus $k-1$ compact Riemann surface Σ, an anti-conformal involution ι of Σ, and a meromorphic map $f : \Sigma \to \widehat{\mathbb{C}}$ such that

1. $\mathrm{Fix}(\iota)$ *is a disjoint union of k circles;*
2. $\Sigma \setminus \mathrm{Fix}(\iota) = \Sigma^+ \sqcup \Sigma_-$, *where $\Sigma^\pm$ are connected;*
3. $f : \Sigma^- \to \Omega$ *is a conformal isomorphism; and*
4. $\sigma \equiv f \circ \iota \circ (f|_{\Sigma^-})^{-1}$.



Complexity of quadrature domains

- σ is an algebraic function; and

$$\deg(\sigma) \quad \text{(complexity of Schwarz reflection)}$$

$$\approx \quad \mathrm{order}(\Omega) := \mathbf{deg}(R_\Omega) \quad \text{(complexity of quadrature identity)}$$

$$\approx \quad \mathsf{deg}(\partial H) \quad \text{(complexity of external field)}$$

$$\approx \quad d_f := \deg\left(f : \Sigma \to \widehat{\mathbb{C}}\right) \quad \text{(algebraic complexity).}$$

- The boundary $\partial\Omega$ is a real-algebraic curve whose singularities are cusps or double points.
- Can the topology + geometry of a quadrature domain Ω be controlled by d_f?

Topology/geometry of quadrature domains via dynamics

Controlling topology/geometry of quadrature domains

- Goal: Get upper bounds on the connectivity and the number of singular points of Ω in terms of d_f.

Theorem (Gustafsson, J. Analyse Math., 1988)

$$\mathrm{Conn}(\Omega) + \#\ \mathrm{Cusps} + 2 \cdot \#\ \mathrm{Double\ points} \leq d_f^2.$$

- Proof idea: genus-degree formula for algebraic curves + Bézout's theorem.

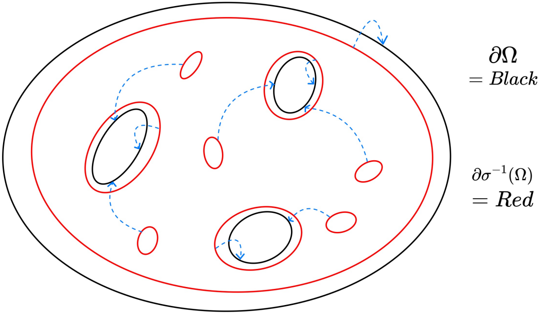
Theorem (Lee–Makarov, J. Amer. Math. Soc., 2016)

$$\mathrm{Conn}(\Omega) \leq 2d_f - 4.$$

- Proof idea:
  1. Use PDE (Hele-Shaw flow) to reduce to the non-singular case.
  2. Obtain the desired bound by studying antiholomorphic dynamics of Schwarz reflection maps.

Key step of Lee–Makarov proof

- Look at the degree $\approx d_f$ branched covering $\sigma : \sigma^{-1}(\Omega) \to \Omega$; i.e., forget the action of σ over Ω^c.



- Extend it to a topological branched covering $\widetilde{\sigma}$ of $\widehat{\mathbb{C}}$ with an attracting fixed point in each component of Ω^c.
- Promote $\widetilde{\sigma}$ to a holomorphic (rational) map using quasiconformal methods; smoothness of $\partial\Omega$ is essential for this step.
- A degree $\approx d_f$ rational map has at most $O(d_f)$ attracting fixed points.

Improving previously known bounds

- Revised goal: Get linear upper bounds (in d_f) on the connectivity and the number of singular points of Ω.

Theorem (Rashmita–M)

1. $\mathrm{Conn}(\Omega) + \#\ \text{Double points} \leq 2d_f - 4.$
2. $\#\ \text{Cusps} + 2 \cdot \#\ \text{Double points}\ \leq 6d_f - 12.$

- The easy part: Cusps come from (simple) critical points of f. Apply Riemann-Hurwitz formula to the branched covering $f : \Sigma \to \widehat{\mathbb{C}}$ to deduce control cusps.
- The harder part: Use the dynamics of the Schwarz reflection σ to assign a critical point of f to each component of Ω^c and each double point on $\partial\Omega$.

Key lemma

Lemma

Let T be a component of Ω^c with m double points. Then, at least $(m+3)$ critical values of f lie in $\bigcup_{j\geq 0} \sigma^{-j}(T)$.

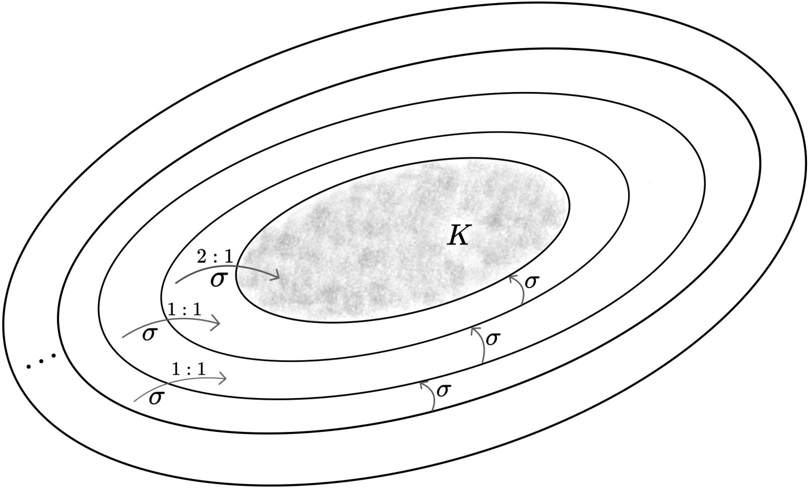
Proof of main theorem assuming the lemma:

- Let $k = \mathrm{Conn}(\Omega)$, and $T_1, \cdots, T_k$ be the components of Ω^c with $m_1, \cdots, m_k$ double points.
- By the previous lemma, this requires
$$\sum_{i=1}^{k}(3+m_i) = 3k + \#\ \text{Double points}$$
critical points of f.
- By Riemann-Hurwitz, f has $(2d_f + 2k - 4)$ critical points.
- Hence, $3k + \#\ \text{Double points} \leq 2d_f + 2k - 4$,
$$\implies k + \#\ \text{Double points} \leq 2d_f - 4.$$

Comments on the proof of the key lemma

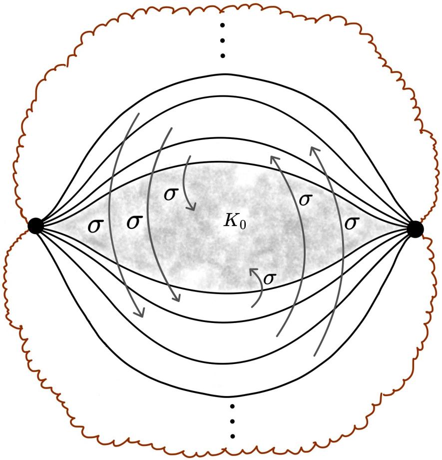
- While the Lee–Makarov proof 'forgets' the action of σ over Ω^c, we analyze the set of points escaping to Ω^c to prove the key lemma.
- Our proof relies only on classical holomorphic dynamics techniques; no quasiconformal surgery or Hele-Shaw flow involved.
- The bound on connectivity and double points is sharp.

The non-singular case



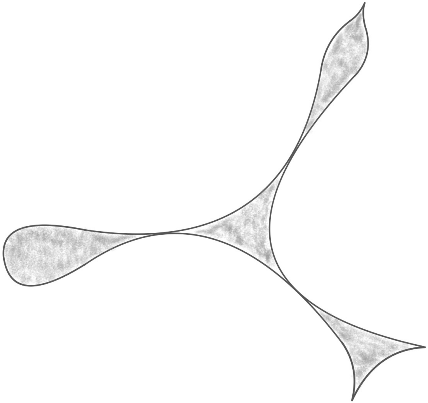
- Dynamics of σ + moduli of annuli argument.

The case of two cusps



- Dynamics of σ + contraction of hyperbolic metric + parabolic dynamics at cusps.

The case of double points

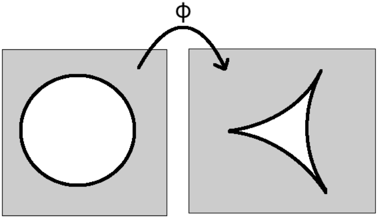


- Dynamics of σ + hyperbolic contraction + parabolic dynamics at cusps + (easy) graph theory.

Mating phenomena

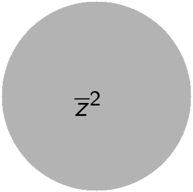
The deltoid reflection map

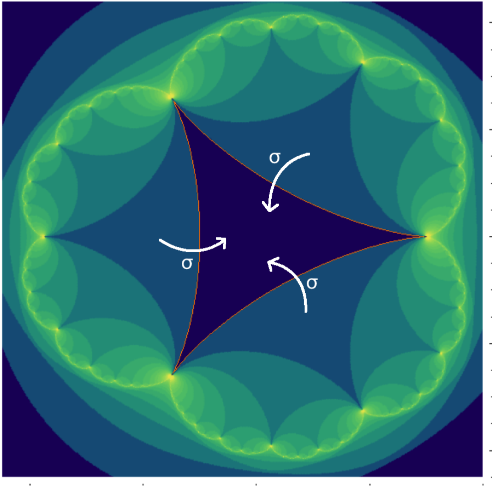
- For the cubic potential $Q(z) = |z|^2 - \mathrm{Re}(z^3)$, the deltoid appears as a "maximal" algebraic droplet.
- The complement of the deltoid has a Riemann map $\phi(z) = z + \frac{1}{2z^2}$, so it is a quadrature domain.

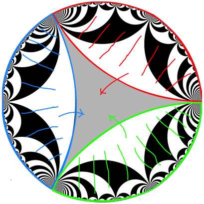


- The corresponding Schwarz reflection map is unicritical, and has a super-attracting fixed point at ∞.

Deltoid Reflection as a mating





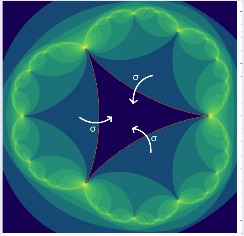


Theorem (Deltoid Reflection as a mating)

1) The dynamical plane of the Schwarz reflection σ of the deltoid can be partitioned as

$$\hat{\mathbb{C}} = T^{\infty} \sqcup \Gamma \sqcup A(\infty),$$

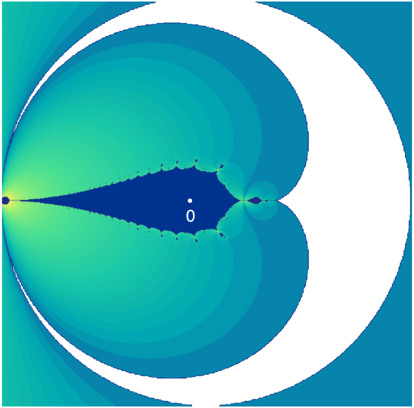
where T^{∞} is the tiling set, $A(\infty)$ is the basin of infinity, and Γ is their common boundary (which we call the limit set). Moreover, Γ is a locally connected Jordan curve.

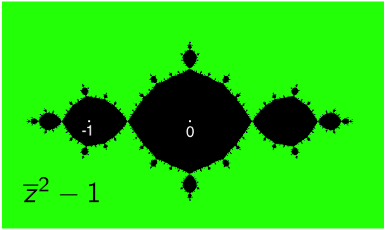


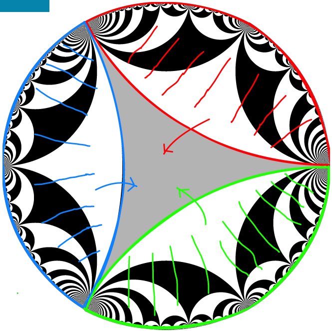
2) σ is the unique conformal mating of the reflection map ρ and the anti-polynomial $z \mapsto \overline{z}^2 : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$.

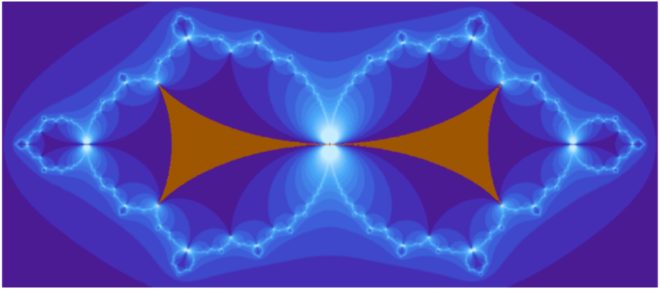
Generality: dynamical partition of Schwarz dynamical plane

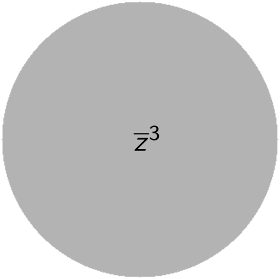
- $T^0 := \Omega^c \setminus$ Singular points.
- The *tiling set* $T^\infty(\sigma)$ of σ is defined as the set of points in $\overline{\Omega}$ that eventually escape to T^0; i.e. $T^\infty(\sigma) = \bigcup_{k=0}^{\infty} \sigma^{-k}(T^0)$.
- The *non-escaping set* $K(\sigma)$ of σ is the complement $\hat{\mathbb{C}} \setminus T^\infty(\sigma)$.

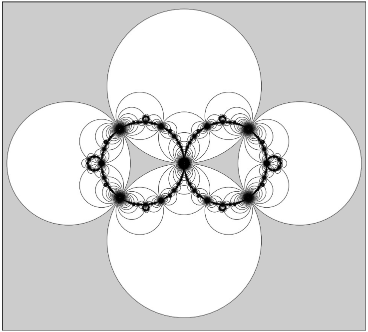

0


-1
0
$\bar{z}^2 - 1$






$\bar{z}^3$



A general combination theorem

Theorem (Luo–Lyubich–M)

Let f be a 'generic' degree d anti-polynomial with connected Julia set. Then, there exists a Schwarz reflection map, unique up to Möbius conjugacy, that is a conformal mating between f and the Nielsen map of an ideal $(d+1)-$gon reflection group.

- Requires new surgery/uniformization techniques.
- The same surgery machinery yields sharpness of our upper bounds.

Thank you!