Topology and geometry of quadrature domains via holomorphic dynamics

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Quadrature domains in statistical physics

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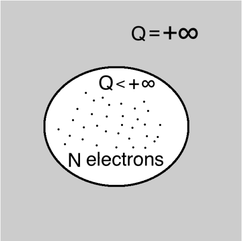

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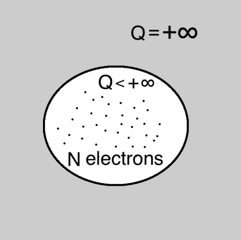

- The combined energy resulting from particle interaction and external potential is:

$$\mathcal{E}_Q(z_1, \cdots, z_N) = \sum_{i \neq j} \ln |z_i - z_j|^{-1} + N \sum_{j=1}^{N} Q(z_j).$$

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- In various physically interesting cases, the external potential Q satisfies some algebraic properties. The compact set T (on which electrons condensate) is then called an *algebraic droplet* of Q.
- In these situations, the complementary components of the droplet T admit global reflection maps. (Lee-Makarov)

From Algebraic Droplets to Quadrature Domains

Definition (Quadrature Domains – Aharonov, Shapiro)

A domain $\Omega \subsetneq \hat{\mathbb{C}}$ (with $\infty \notin \partial\Omega$ and $\mathrm{int}(\overline{\Omega}) = \Omega$) is called a *quadrature domain* if there exists a continuous function $\sigma : \overline{\Omega} \to \hat{\mathbb{C}}$ satisfying the following two properties:

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- Lee and Makarov used iteration of Schwarz reflection maps to study the topology of quadrature domains.

Quadrature Identities

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  There exists a rational map R_Ω with all poles inside Ω such that

  $$\int_\Omega \phi dA = \frac{1}{2i} \oint_{\partial\Omega} \phi(z) R_\Omega(z) dz \quad \left(= \sum c_k \phi^{(n_k)}(a_k)\right)$$

  for all $\phi \in H(\Omega) \cap C(\overline{\Omega})$ (if $\infty \in \Omega$, one also requires $\phi(\infty) = 0$).

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- By definition, $d_\Omega \equiv \mathrm{order}(\Omega) := \deg(R_\Omega)$.
- The rational map R_Ω and the Schwarz reflection map σ have the same (finite) set of poles.

Algebraic properties of quadrature domains

Simply Connected Quadrature Domains

- A simply connected domain $\Omega \subsetneq \hat{\mathbb{C}}$ with $\infty \notin \partial\Omega$ and $\mathrm{int}(\overline{\Omega}) = \Omega$ is a quadrature domain if and only if the Riemann map $\phi : \mathbb{D} \to \Omega$ extends as a rational map of $\widehat{\mathbb{C}}$.

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$$\begin{tikzcd}
\overline{\mathbb{D}} \arrow[r, "\varphi"] \arrow[d, "1/\overline{z}"] & \overline{\Omega} \arrow[d, "\sigma"] \\
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\overline{\mathbb{D}} & \xrightarrow{\varphi} & \overline{\Omega} \\
\Big\downarrow 1/\overline{z} & & \Big\downarrow \sigma \\
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\end{array}$$

- The rational map ϕ semi-conjugates the reflection map $1/\overline{z}$ of $\mathbb{D}$ to the Schwarz reflection map σ of Ω.

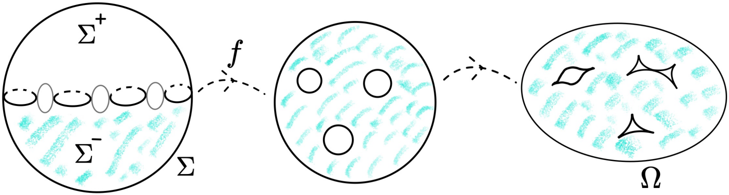
Quadrature domains and Schottky double

- $\Omega =$ Quadrature domain of connectivity k with Schwarz reflection map σ (i.e., $k = \#$ connected components of $\Omega^c = \widehat{\mathbb{C}} \setminus \Omega$).

Theorem (Gustafsson, Acta Appl. Math., 1983)

There exist a genus $k-1$ compact Riemann surface Σ, an anti-conformal involution ι of Σ, and a meromorphic map $f : \Sigma \to \widehat{\mathbb{C}}$ such that

1. $\mathrm{Fix}(\iota)$ *is a disjoint union of k circles;*
2. $\Sigma \setminus \mathrm{Fix}(\iota) = \Sigma^+ \sqcup \Sigma_-$, *where $\Sigma^\pm$ are connected;*
3. $f : \Sigma^- \to \Omega$ *is a conformal isomorphism; and*
4. $\sigma \equiv f \circ \iota \circ (f|_{\Sigma^-})^{-1}$.



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- σ is an algebraic function; and

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- The boundary $\partial\Omega$ is a real-algebraic curve whose singularities are cusps or double points.

Topology/geometry of quadrature domains via dynamics

Controlling topology/geometry of quadrature domains

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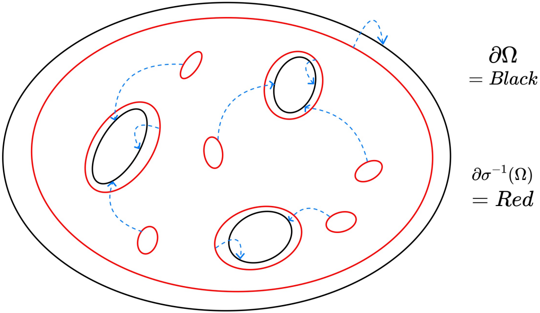
  2. Obtain the desired bound by studying antiholomorphic dynamics of Schwarz reflection maps.

Key step of Lee–Makarov proof

- Look at the degree $d_f - 1$ branched covering $\sigma : \sigma^{-1}(\Omega) \to \Omega$; i.e., forget the action of σ over Ω^c.



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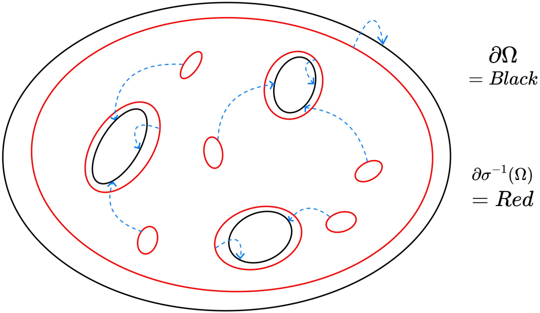

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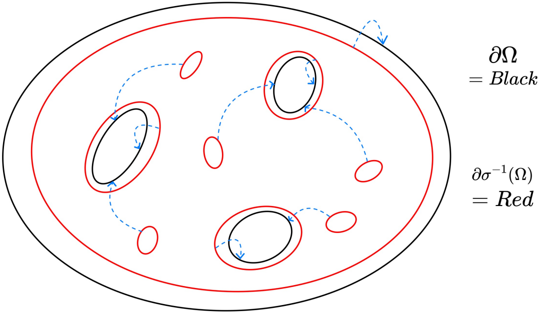

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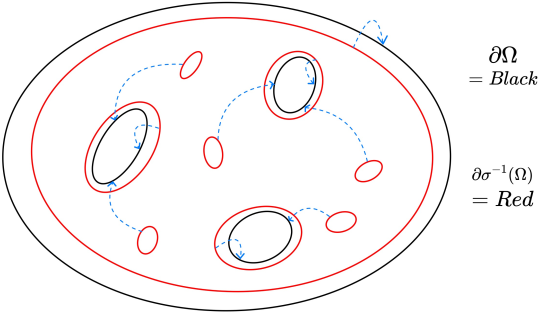

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- Use the fact that such a map has at most $2(d_f - 1) - 2 = 2d_f - 4$ attracting fixed points.

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- The harder part: *Use the dynamics of the Schwarz reflection σ to assign a critical point of f to each component of Ω^c and each double point on $\partial\Omega$.*

The key lemma

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critical points.
- Hence, $3k + \#\,\mathrm{Double\ points} \leq 2d_f + 2k - 4$; i.e.,
$$k + \#\,\mathrm{Double\ points} \leq 2d_f - 4.$$

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- **Sample case III: m=1 and ∂T has 2 cusps.** Dynamics of σ on $\bigcup_{j\geq 0} \sigma^{-j}(T)$ + Local parabolic dynamics at double points + Denjoy-Wolff theorem.

Mating phenomena

The deltoid reflection map

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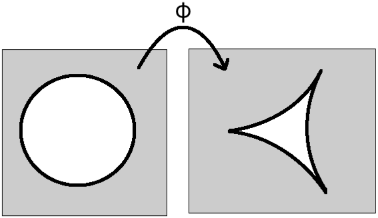
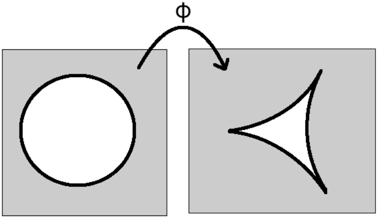
- The complement of the deltoid has a Riemann map $\phi(z) = z + \frac{1}{2z^2}$, so it is a quadrature domain.



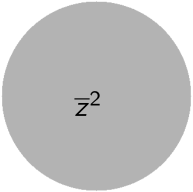
The deltoid reflection map

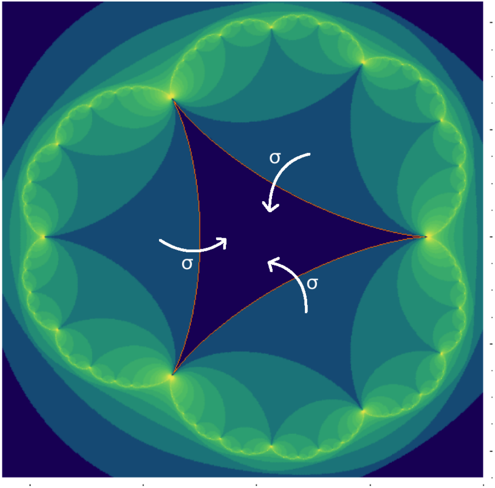
- For the cubic potential $Q(z) = |z|^2 - \mathrm{Re}(z^3)$, the deltoid appears as a "maximal" algebraic droplet.
- The complement of the deltoid has a Riemann map $\phi(z) = z + \frac{1}{2z^2}$, so it is a quadrature domain.

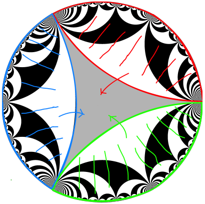


- The corresponding Schwarz reflection map is unicritical, and has a super-attracting fixed point at ∞.

Deltoid Reflection as a mating







Theorem (Deltoid Reflection as a mating)

1) The dynamical plane of the Schwarz reflection σ of the deltoid can be partitioned as

$$\hat{\mathbb{C}} = T^\infty \sqcup \Gamma \sqcup A(\infty),$$

where T^∞ is the tiling set, $A(\infty)$ is the basin of infinity, and Γ is their common boundary (which we call the limit set). Moreover, Γ is a locally connected Jordan curve.

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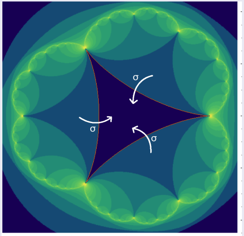

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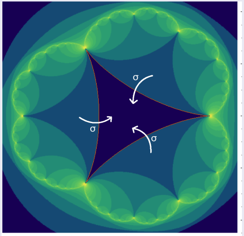

2) σ is the unique conformal mating of the reflection map ρ and the anti-polynomial $z \mapsto \overline{z}^2 : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$.

Generality: dynamical partition of Schwarz dynamical plane

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- The *tiling set* $T^\infty(\sigma)$ of σ is defined as the set of points in $\overline{\Omega}$ that eventually escape to T^0; i.e. $T^\infty(\sigma) = \bigcup_{k=0}^{\infty} \sigma^{-k}(T^0)$.

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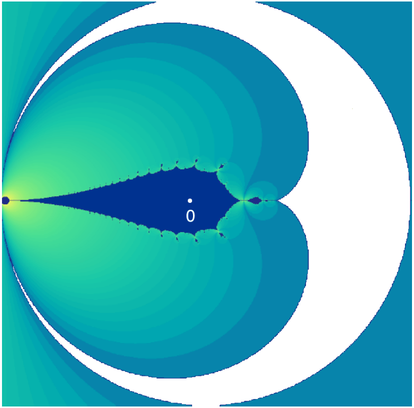
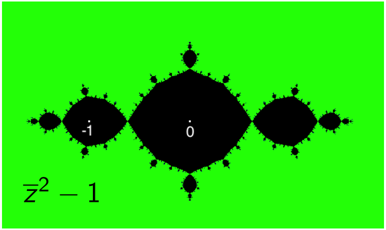
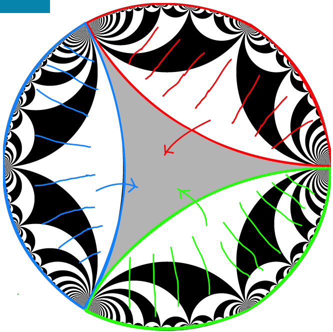
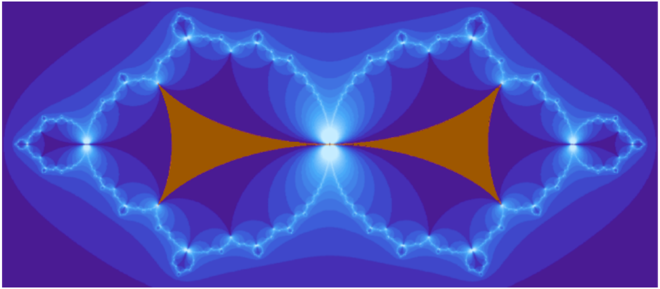
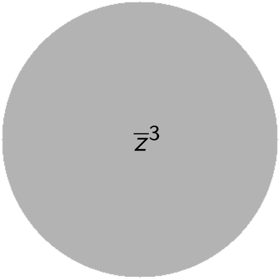
- The *non-escaping set* $K(\sigma)$ of σ is the complement $\hat{\mathbb{C}} \setminus T^\infty(\sigma)$.

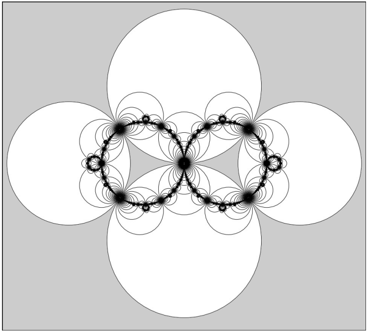

0


-1
0
$\bar{z}^2 - 1$






$\bar{z}^3$



A general combination theorem

Theorem (Luo–Lyubich–M)

Let f be a generic degree d anti-polynomial with connected Julia set. Then, there exists a Schwarz reflection map, unique up to Möbius conjugacy, that is a conformal mating between f and the Nielsen map of an ideal $(d+1)-$gon reflection group.

Thank you!