Where polynomial dynamics meets Fuchsian groups

(Based on joint works with Luo, Lyubich, Mj, Makarov, et al.)

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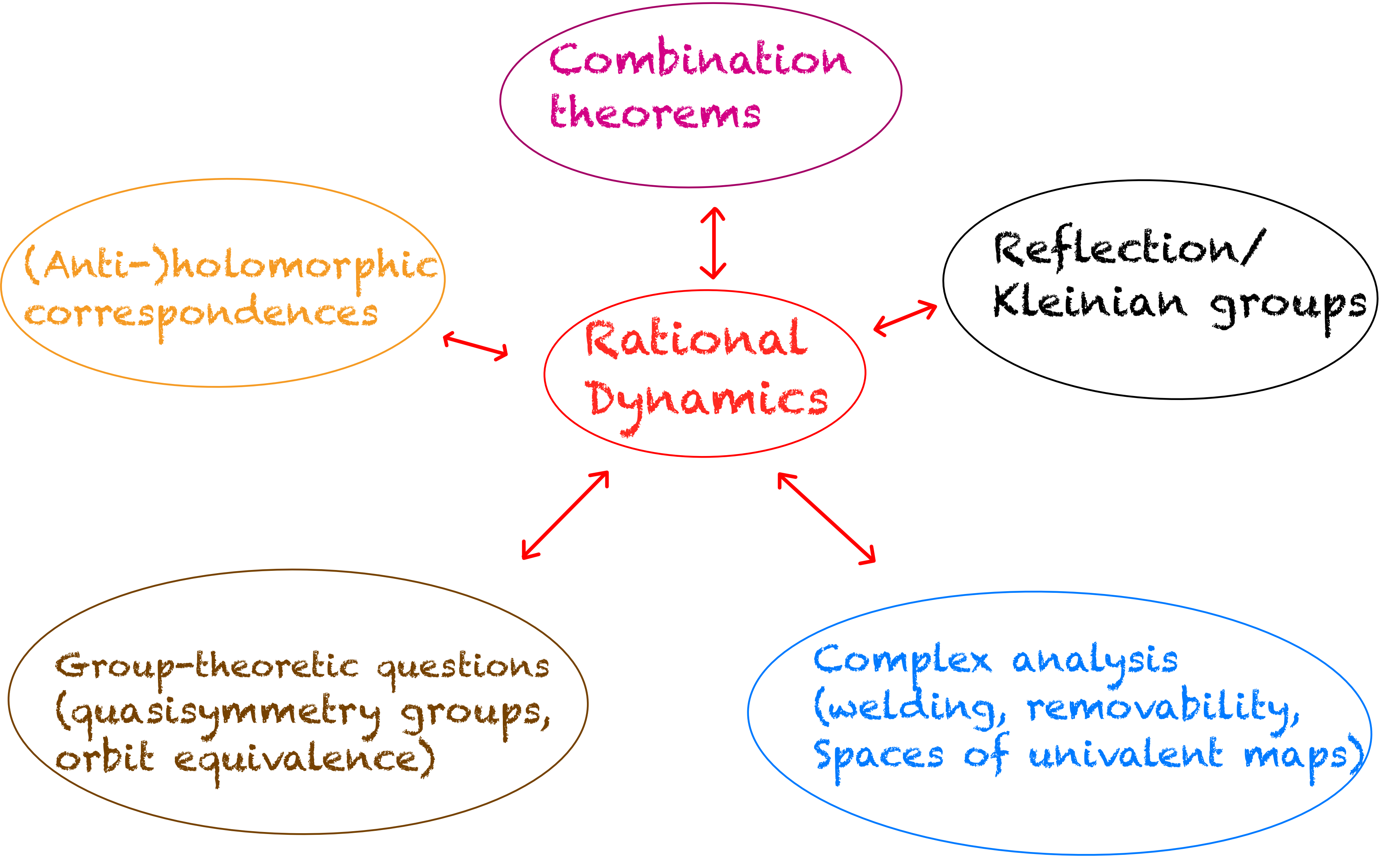

Combination theorems
(Anti-)holomorphic correspondences
Rational Dynamics
Reflection/ Kleinian groups
Group-theoretic questions (quasisymmetry groups, orbit equivalence)
Complex analysis (welding, removability, Spaces of univalent maps)

Conformal dynamics, and Fatou-Sullivan Dictionary

Parallel Universes

Rational maps
(Holo. dynamics on $\mathbb{CP}^1$)

Fatou set / Julia set
(Tame dynamics) (Chaotic dynamics)

Sullivan's no wandering domains theorem
(Complete classification of tame dynamics)

Blaschke products / Blaschke Space
(Holo. dynamics on unit disk $\mathbb{D}$)

Realizing branched covers as rational maps

Normal family arguments

Quasiconformal deformations

Iteration on Teich. spaces

$\mathbb{R}$-trees

Common techniques

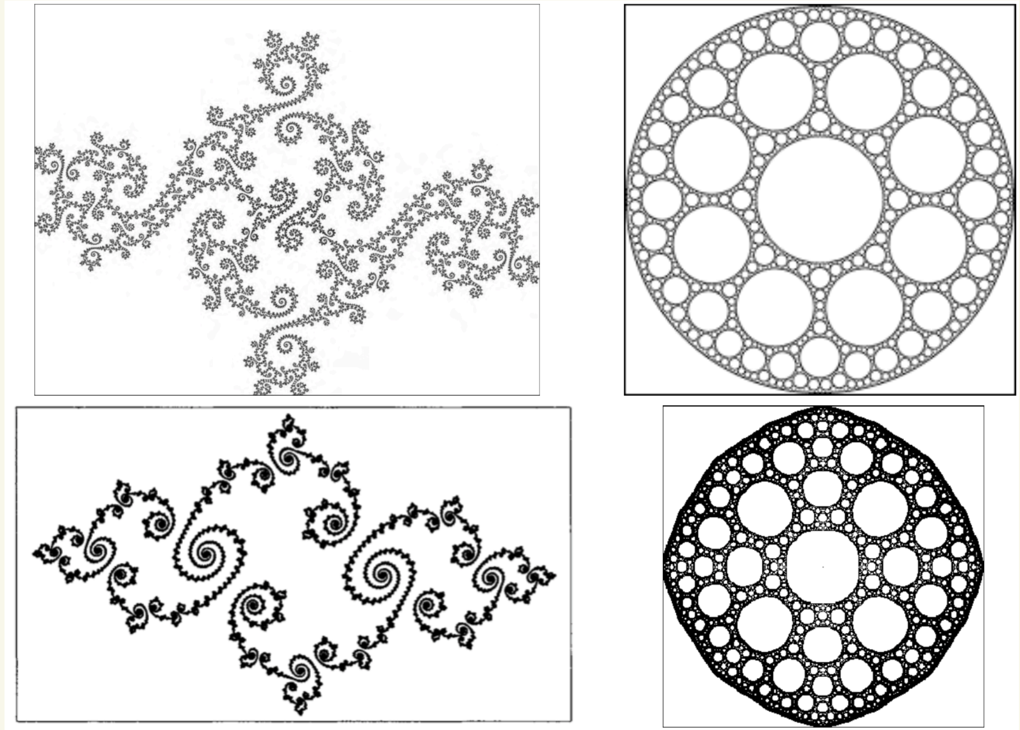
Kleinian groups
(Group action on $\mathbb{CP}^1$ via conf. automorphisms)

Discontinuity set / Limit set
(Tame dynamics) (Chaotic dynamics)

Ahlfors' finiteness theorem
(Complete classification of tame dynamics)

Fuchsian groups / Teichmüller space
(Group action on $\mathbb{D}$ via conf. automorphisms)

Hyperbolization of 3-manifolds



- $J(R)$, $\Lambda(G)$ = Entire sphere or nowhere dense.
- $J(R)$ = Closure of repelling periodic points,
  $\Lambda(G)$ = Closure of repelling fixed points.

Some features are lost in translation

Rational maps on $\mathbb{CP}^1$

Critical points,
Non-invertible,
Non-reversible

Positive area Julia sets

Connected, non locally conn. Julia sets

No genuine analogue

Kleinian groups

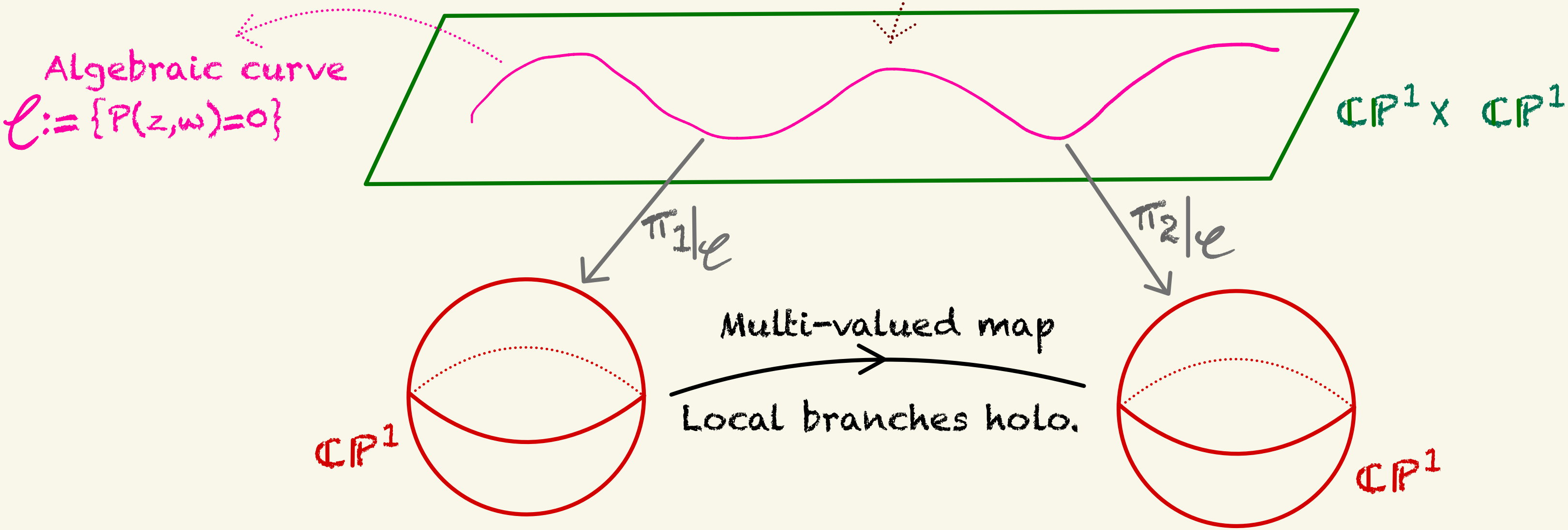
Invertible,
reversible,
many generators,

Area(limit set) = 0

Connected limit sets are locally conn.

Action on $\mathbb{H}^3$

Fatou (1920s): The similarities are probably not coincidental. These conformal dynamical systems live inside the galaxy of algebraic correspondences.



- $w - R(z) = 0$

  $R \to$ rational map

- $(w-\gamma_1(z))\dots(w-\gamma_2(z)) = 0$

  $\langle \gamma_1, \gamma_2, \dots, \gamma_k \rangle \to$ f.g. Kleinian group

A possible general theory?

Probably too ambitious

<u>Desire:</u> Want to devise a mechanism for producing correspondences that display both features simultaneously.

To fulfill such a dream, need to

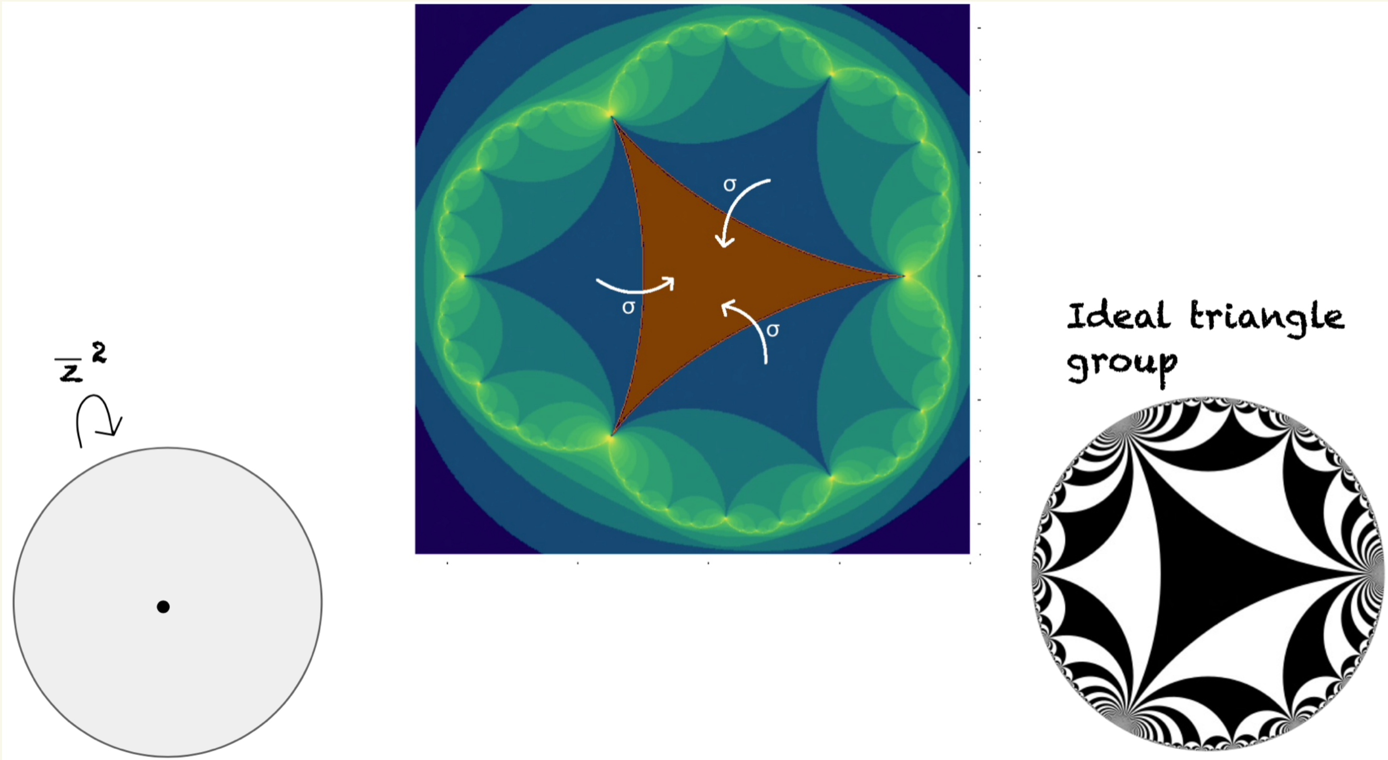
1) establish combination/mating theorems for rational maps and Kleinian groups,

2) study parameter spaces of such matings, revealing how moduli spaces of groups and maps co-exist in spaces of correspondences.

☆ Bullett–Penrose gave us some hope in the 1990s by constructing examples of matings of quadratic maps with PSL(2,Z).

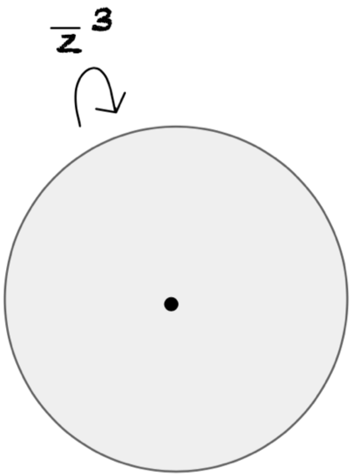
Schwarz reflection maps, and a combination phenomena

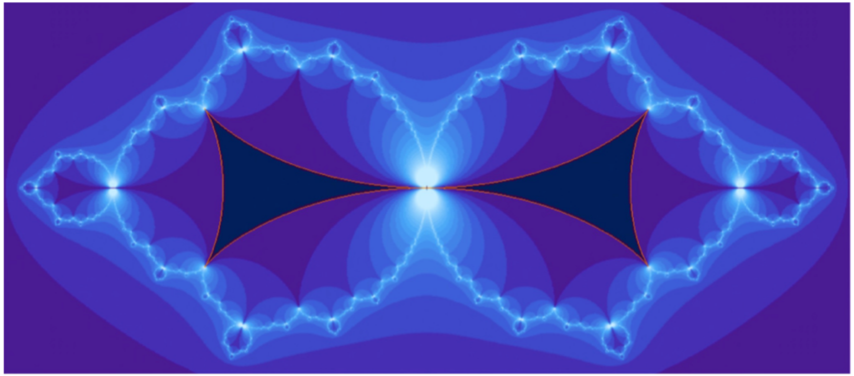
"Schwarz reflection maps": surprising connections

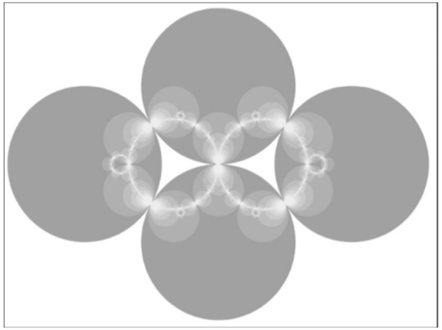


☆ Schwarz reflection in a deltoid combines $\bar{z}^2$ with the ideal triangle reflection group.

"Schwarz reflection maps": surprising connections (continued)

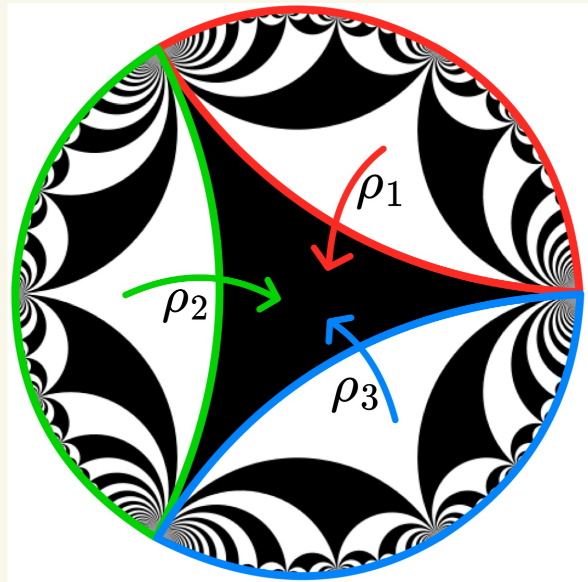






☆ Schwarz reflection in a curve with a tacnode/double point combines $\bar{z}^3$ with a more interesting reflection group.

Key takeaway: Nielsen map



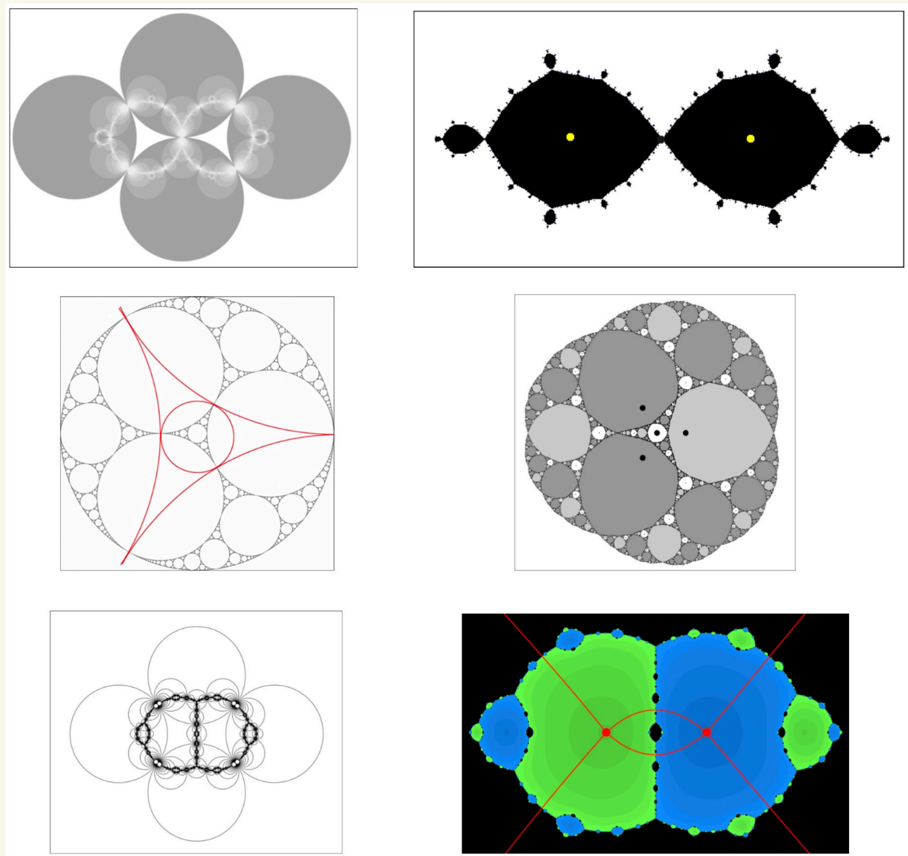
☆ The Nielsen map of an ideal (d+1)-gon reflection group

i) remembers all the generators of the gruop, and

ii) is conjugate to $\bar{z}^d$ on the circle.

Equivariant homeomorphism between Julia and limit sets

(Nielsen maps + promoting topological maps to analytic ones)

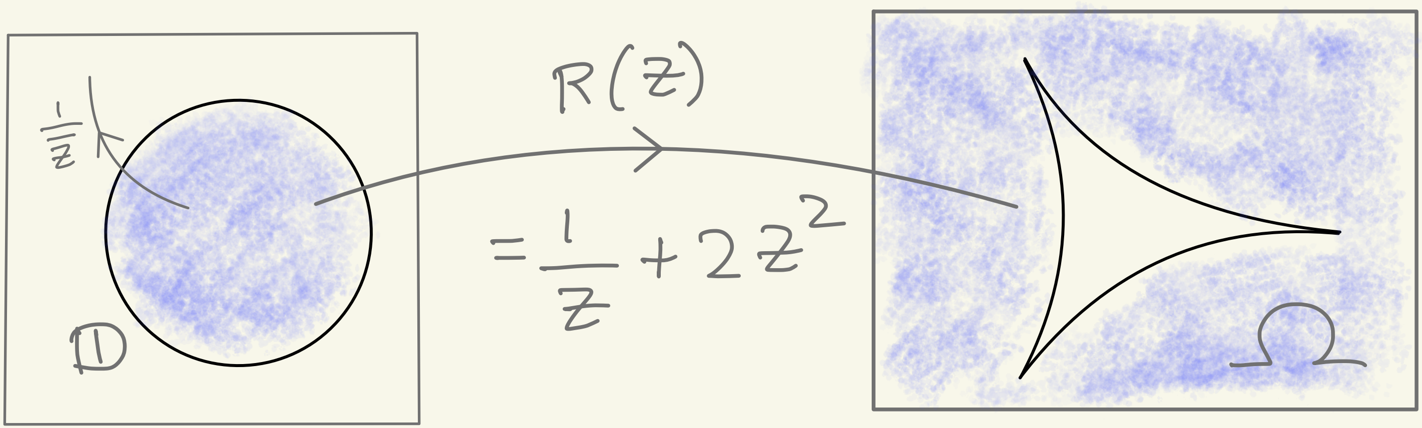


- Schwarz reflections in the above curves are special:

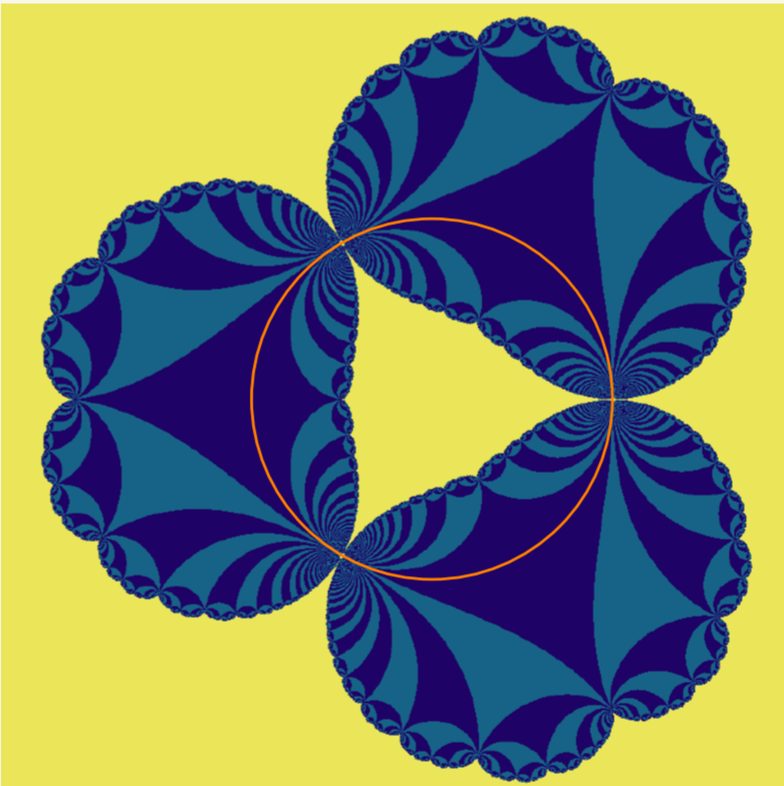
- The curves are images of S^1 under rational functions R that are injective on $\mathbb{D}$.

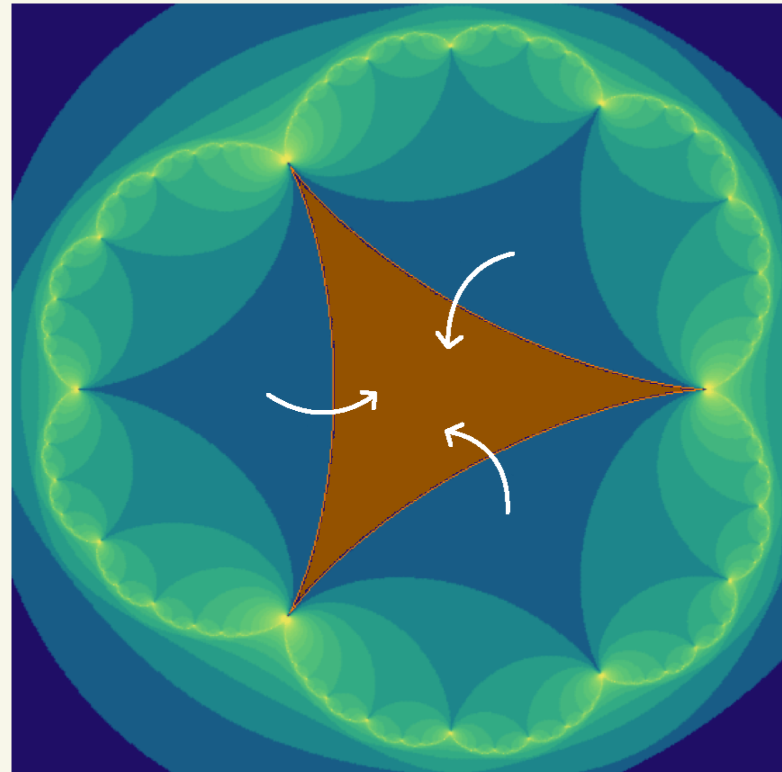
- The Schwarz reflections are algebraic functions of the form

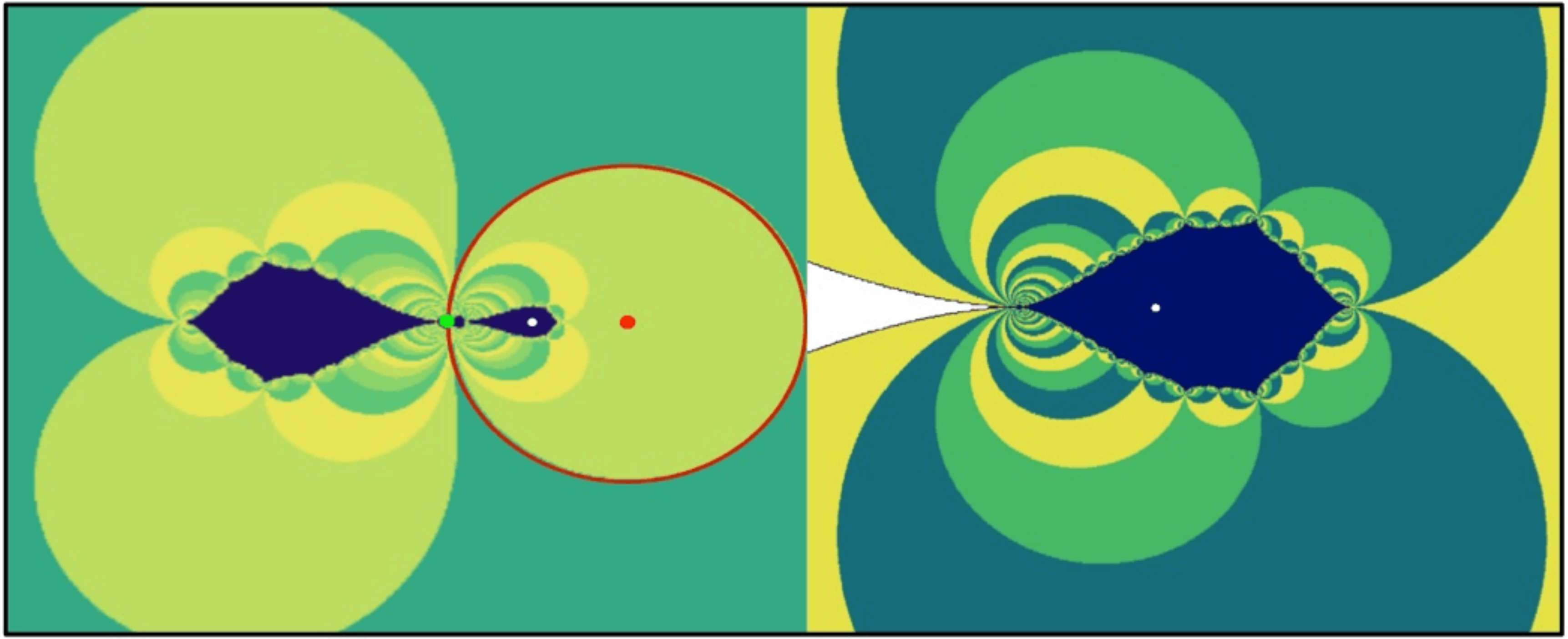
$$\sigma\big|_{\Omega} = R \circ \left(\frac{1}{\bar{z}}\right) \circ \left(R\big|_{\mathbb{D}}\right)^{-1}$$



☆ Thus, the Schwarz reflections can be lifted by R to get algebraic correspondences that combine groups with maps.







From special examples to a general program

1. Replace a group with a map that remembers enough about the group, but is also compatible with rational dynamics.

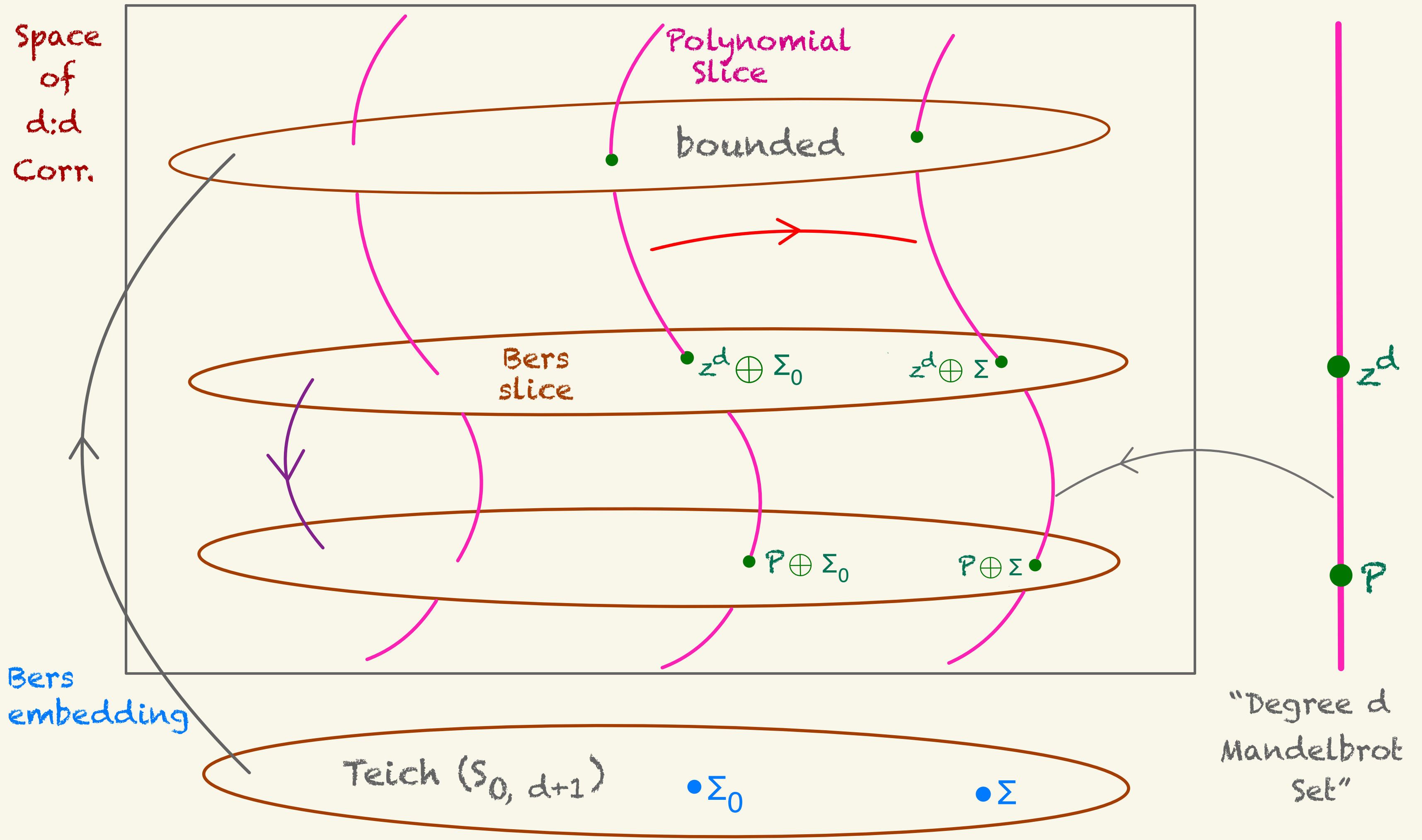
2. Combine this map with a rational map producing a "conformal mating".

3. Justify algebraicity of the conformal mating.

4. Use the algebraic description to construct a correspondence that combines the rational map with the group.

Theorem (Mj-M, Luo-Lyubich-M)

- Large classes of genus zero orbifolds (including punctured spheres and Hecke orbifolds) can be mated with generic polynomials of appropriate degree.
- The matings are realized as correspondences on nodal spheres.
- Products of parameter spaces of polynomials and Teichmüller spaces of the above surfaces embed into parameter spaces of algebraic correspondences.



Space
of
d:d
Corr.
Polynomial
Slice
bounded
Bers
slice
$z^d \oplus \Sigma_0$
$z^d \oplus \Sigma$
$P \oplus \Sigma_0$
$P \oplus \Sigma$
z^d
P
Bers
embedding
Teich $(S_{0,\, d+1})$
Σ_0
Σ
"Degree d
Mandelbrot
Set"

New techniques:

1. Construction of continuous analogs of "Bowen-Series maps".
2. Development of (David) surgery tools, and combinatorial continuity/rigidity arguments.
3. Identification of meromorphic maps with certain boundary behavior as algebraic functions (welding to produce compact Riemann surfaces).
4. Complex-analytic study of correspondences on nodal surfaces, and their degeneration behavior (rescaling limits of correspondences).

Some tasks:

1) Study the dynamics of Bers boundary correspondences.

2) Do the limit sets of Bers boundary correspondences have zero area? Hausdorff dimension 2?

3) Classify Bers boundary correspondences (combinatorial rigidity = "ending lamination").

4) Study geometric limit vs. algebraic limit.

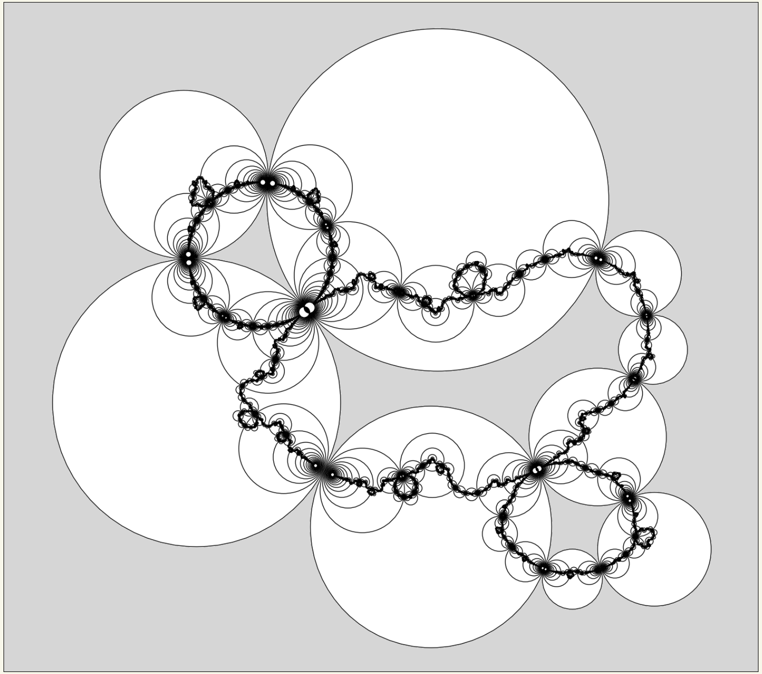
5) Study the topology of Bers boundaries (non-local conn., self-bumping).

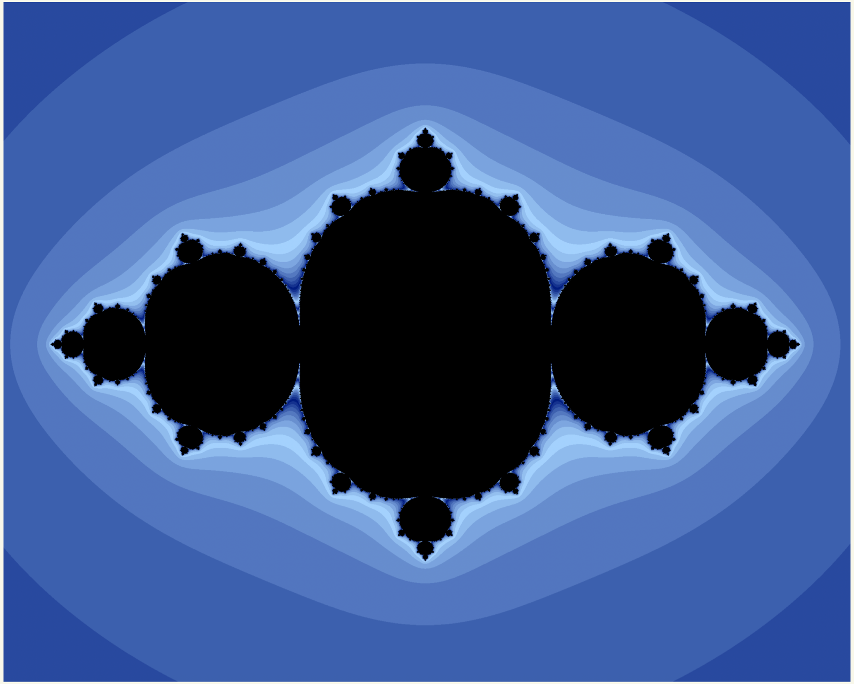
6) Combine co-compact Fuchsian groups with polynomials.

7) Study correspondences outside the mating locus (discrete, wild?).

Applications

Conformal removability of Julia and limit sets

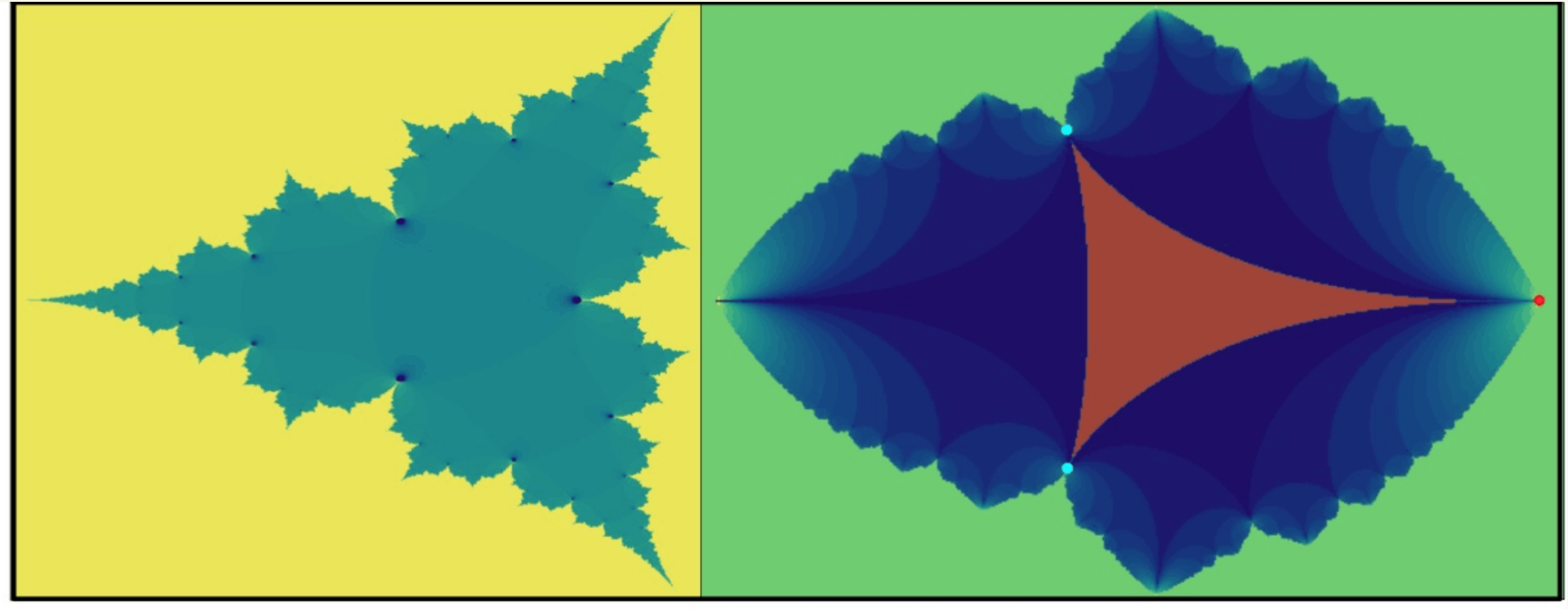




☆ These are images of $W^{1,1}$-removable sets under global David homeos.

(Lyubich-Merenkov-M-Ntalampekos)

Non-quasisymmetric welding homeomorphisms



☆ Non-type-preserving conjugacies between 'nice' expansive circle coverings are welding homeos.

(Lyubich-Merenkov-M-Ntalampekos)

- Failure of topological orbit equivalence rigidity for Fuchsian groups.

  (Mj-M)

- Extension of Bers' Simultaneous Uniformization Theorem for non-homeomorphic orbifolds.

  (Mj-M)

- Correspondences on hyperelliptic Riemann surfaces as matings.

  Embedding products of "Mandelbrot sets" and Teichmüller spaces in Hurwitz spaces.

  (M-Viswanathan)

- Sharp control of topology and singularities of certain real-algebraic curves.

  (M-Rashmita)

Thank you!