SOME COMMENTS ON THE STABLE BERNSTEIN CENTER

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Abstract. We prove several results concerning the stable Bernstein center of a connected reductive p-adic group G, which follow from a variant of a "μ-constancy result" of Shahidi in [Sha90]. One of these results says that the stable center conjecture holds for quasi-split classical groups. Towards this result, we introduce a notion of unitarily stable discrete series L-packets, and discuss criteria for detecting them, proving in particular that when $p \gg 0$, Kaletha's regular supercuspidal packets are unitarily stable. We also prove a weak but unconditional variant of Shahidi's constancy of the Plancherel μ-function in an L-packet, as well as of its transfer across inner forms. As a consequence, we deduce that the Plancherel μ-function is constant on unitarily stable discrete series packets on Levi subgroups of G (and thus, when $p \gg 0$, on regular supercuspidal packets on Levi subgroups of G). We also slightly refine a result of M. Oi on the depth preservation of the local Langlands correspondence for classical groups.

Contents

1. Introduction

In this introduction, unless otherwise stated, let G be a connected reductive group defined over a finite extension F of $\mathbb{Q}_p$, where p is a prime number. To keep this introduction simple, we will assume G to be quasi-split unless otherwise stated. Further, theorems in this introduction will typically be stated informally, but with a reference to their more precise versions in the body of the paper.

1.1. **Pre-introduction: Some of the main results proved in this paper.** This paper is a collection of a priori disparate seeming results, which are nevertheless related in their evolving from a variant of an argument from [Sha90], that we will informally refer to as 'Shahidi's μ-constancy argument'. To give the reader an idea of what to expect, let us state some consequences of these results:

(1) The stable center conjecture for quasi-split classical groups (See Corollary 5.1.4): If G is a quasi-split symplectic, special orthogonal, unitary, general symplectic or odd general spin group, then it satisfies a form of the stable center conjecture of Bezrukavnikov, Kazhdan, Varshavsky and Haines. If G is a general special orthogonal group, a weaker result involving an outer automorphism holds.
(2) Depth preservation for quasi-split classical groups (see Corollary 5.1.6): Combining this result with the stability of the depth r projector (proved in [BKV16]) and with the work of M. Oi in [Oi22] on depth preservation, we can slightly refine [Oi22, Theorem 1.2] into the following depth preservation result for local Langlands correspondence: if $p \gg 0$, and σ is an irreducible discrete series representation of a quasi-split symplectic, special orthogonal or unitary group over F, with Langlands parameter φ_σ, then
$$\operatorname{depth} \varphi_\sigma = \operatorname{depth} \sigma$$
(the unitary case of this was already known by [Oi22] and [Oi21], even without requiring G to be quasi-split).
(3) Comparing Kaletha's and Arthur's packets (see Remark 3.4.13): Assume that $p \gg 0$. Then for quasi-split symplectic, special orthogonal and unitary groups over F, regular supercuspidal packets constructed by Kaletha are also packets in the sense of Arthur's book, [Art13]. However, we are not able to compare their Langlands parameterizations.
(4) Regular supercuspidal representations and the constancy/transfer of the Plancherel measure (See Corollary 4.2.13): Assume that $p \gg 0$ (but G is arbitrary connected reductive over F). If Σ is a regular supercuspidal packet on $\mathrm{M}(F)$, where $\mathrm{M} \subset \mathrm{G}$ is a Levi subgroup, and μ stands for the Plancherel μ-function, then $\mu(\sigma_1) = \mu(\sigma_2)$ for all $\sigma_1, \sigma_2 \in \Sigma$. More generally, if M^* is a Levi subgroup of a quasi-split group G^* over F, such that there exists an inner twist from G^* to G that restricts to an inner twist from M^* to M, and if Σ^* is a regular supercuspidal packet on $\mathrm{M}^*(F)$ with the same Langlands parameter as Σ, then the values of the Plancherel μ-function on $\sigma \in \Sigma$ and $\sigma^* \in \Sigma^*$ are related by:
$$\gamma'''(\mathrm{G}^*|\mathrm{M}^*)\mu(\sigma^*) = \gamma'''(\mathrm{G}|\mathrm{M})\mu(\sigma).$$
Note that according to the formalism described in [Sha90, Section 9], extending Langlands-Shahidi L-functions outside the generic case, this tells us that even in many cases involving non-quasi-split groups, Langlands-Shahidi L-functions can be used to normalize intertwining operators.

These results are deduced from more general versions stated in terms of unitarily stable discrete series packets, a notion that we introduce in Definition 3.3.2. In Subsection 1.7 we try to state our case that this notion enjoys 'availability', utility and naturality, and is hence worth considering.

Shahidi had shown in [Sha90, Section 9], under some natural but strong assumptions concerning the existence of stable tempered L-packets, that for any Levi subgroup $\mathrm{M} \subset \mathrm{G}$ and any discrete series L-packet σ on $\mathrm{M}(F)$, the Plancherel μ-function is constant on Σ. It is to Shahidi's proof of this result that we refer, when we talk of 'Shahidi's μ-constancy argument'. A key input into the proofs of the main results of this paper is our variation on this μ-constancy result, as well as a generalization of it, and a corresponding generalization of the 'transfer to inner forms' variant of

Shahidi's result studied by Choiy and Heiermann (see [Cho14] and [Hei16]): see Proposition 1.6.1 later in this introduction for an informal statement, and Corollary 4.2.12 for a precise statement. In fact, we prove weaker and uglier but unconditional versions of these two results (see Proposition 1.3.1 in Subsection 1.3 for the former, and Proposition 1.4.1 in Subsection 1.4 for the latter).

In what follows, namely, the introduction proper to this paper, we will elaborate on the context and the objects that we concern ourselves with, and discuss our more general results that specialize to (1)-(4) above.

1.2. **The stable center conjecture.** Let $\mathcal{Z}(\mathrm{G})$ and $\Omega(\mathrm{G})$ respectively denote the Bernstein center and the Bernstein variety of G (see, e.g., [BDK86, Hai14, BKV15, BKV16]), so that $\mathcal{Z}(\mathrm{G})$ identifies with the ring $\mathbb{C}[\Omega(\mathrm{G})]$ of regular functions on $\Omega(\mathrm{G})$.

There are many conjecturally equivalent candidates for the definition of a ring $\mathcal{Z}_{\mathrm{st}}(\mathrm{G})$, called the stable Bernstein center of G, which is expected to map into $\mathcal{Z}(\mathrm{G})$ and inform the study of the local Langlands conjectures and related topics such as stability and endoscopy. In fact, according to [BKV15, the introduction], studying it can provide both a 'supporting evidence' and a 'step in the proof of the local Langlands conjecture'.

Since the work of Vogan in [Vog93], which is the earliest reference on this topic that the author is aware of, several conjectural descriptions of what should deserve to be called the stable Bernstein center have emerged, only some of which obviously map to $\mathcal{Z}(\mathrm{G})$. In this part, we will study the equality of the two harmonic analytically defined complex vector spaces $\mathcal{Z}_1(\mathrm{G})$ and $\mathcal{Z}_2(\mathrm{G}) \subset \mathcal{Z}(\mathrm{G})$ below.

Definition 1.2.1. (i) Let $\mathcal{Z}_1(\mathrm{G}) \subset \mathcal{Z}(\mathrm{G})$ be the *vector subspace* of elements z that, viewed as distributions on $\mathrm{G}(F)$, are stable.
(ii) Let $\mathcal{Z}_2(\mathrm{G}) \subset \mathcal{Z}(\mathrm{G})$ be the $\mathbb{C}$-*subalgebra* of elements z such that whenever $f \in C_c^\infty(\mathrm{G}(F))$ is unstable, so is $z * f$.

Let us first discuss the relation between $\mathcal{Z}_1(\mathrm{G})$ and $\mathcal{Z}_2(\mathrm{G})$. Recall that through out this introduction, we assume unless otherwise stated that G is quasi-split.

Remark 1.2.2. (i) While a priori $\mathcal{Z}_1(\mathrm{G})$ is only a $\mathbb{C}$-vector subspace of $\mathcal{Z}(\mathrm{G})$, $\mathcal{Z}_2(\mathrm{G})$ is a $\mathbb{C}$-subalgera of $\mathcal{Z}(\mathrm{G})$.
(ii) $\mathcal{Z}_2(\mathrm{G}) \subset \mathcal{Z}_1(\mathrm{G})$: since $z(f) = z * \check{f}(1)$, this follows from the fact that $f \mapsto f(1)$ is a stable distribution (see [Kot88, Proposition 1]).

A weak form of the stable center conjecture, namely [BKV15, Conjecture 3.1.4(a)], says:

Conjecture 1.2.3. $\mathcal{Z}_1(\mathrm{G}) \subset \mathcal{Z}(\mathrm{G})$ is a subalgebra.

By Remark 1.2.2, this conjecture follows from the following conjecture, which is thus a stronger form of the stable center conjecture:

Conjecture 1.2.4. $\mathcal{Z}_2(\mathrm{G}) = \mathcal{Z}_1(\mathrm{G})$.

As mentioned in the introduction of [BKV15], one expects $\mathcal{Z}_1(\mathrm{G})$ to be the set of elements in $\mathcal{Z}(\mathrm{G})$ with the property that, if π_1, π_2 are tempered representations of $\mathrm{G}(F)$ belonging to the same L-packet, then z acts as multiplication by the same scalar on π_1 and π_2. Note that this would transparently yield Conjecture 1.2.3 as well. It is easy to turn this comment into an easy proof of the stronger Conjecture 1.2.4, for those groups for which tempered L-packets have been defined and shown to satisfy the appropriate stability properties.

Namely, according to the formalism of Langlands and Arthur, one expects that the set $\mathrm{Irr}_{\mathrm{temp}}(\mathrm{G})$ of isomorphism classes of irreducible tempered representations of $\mathrm{G}(F)$ can be partitioned into finite subsets, called tempered L-packets, such that each such packet Σ supports a nonzero stable virtual character Θ_Σ, and such that the Θ_Σ form a basis for the space of stable tempered virtual characters on $\mathrm{G}(F)$. One makes a slightly more precise requirement: for each Levi subgroup M of G, one asks for a partition of the set $\mathrm{Irr}_2(\mathrm{M})$ of isomorphism classes of irreducible unitary square-integrable (modulo center) representations of $\mathrm{M}(F)$ into 'discrete series L-packets' Σ each supporting a nonzero stable virtual character Θ_Σ, and one asks for these Θ_Σ to constitute a basis for the space of stable elliptic virtual characters on $\mathrm{M}(F)$. If this condition is satisfied, we will

say that G satisfies the existence of tempered L-packets (thus implicitly assumed to have the appropriate stability properties). In the body of the present paper, this requirement is stated as Hypothesis 2.5.1, and referred to as the existence of tempered L-packets. Let us remark that in the body of the paper, including in Hypothesis 2.5.1, we work with a system $\{\mathcal{O}_{\mathrm{M}}\}_{\mathrm{M}}$ of automorphisms of Levi subgroups of G (essentially to deal with outer automorphisms of groups such as SO_{2n} or GSO_{2n}), but to keep this introduction simple, we will assume all these groups to be trivial.
Thus, one of the aims of this paper is to show the following fact, which is probably known to many experts but which the author cannot find in literature (see Theorem 4.4.2 for a precise version).

Theorem 1.2.5. *If* G *satisfies the existence of tempered L-packets, then Conjecture 1.2.4 is true, i.e.,* $\mathcal{Z}_2(\mathrm{G}) = \mathcal{Z}_1(\mathrm{G})$ *(and hence, so is Conjecture 1.2.3).*

As is well-known, in the case of quasi-split symplectic and special orthogonal groups, the monumental work of Arthur in [Art13] gives us such a description of tempered L-packets as well as a proof of character identities satisfied by them, while the work of Mok [Mok15] adapts the work of Arthur to quasi-split unitary groups. In [Xu18], Bin Xu proves analogous results for quasi-split general symplectic groups, and a weaker version for even general special orthogonal groups involving an outer automorphism. On the other hand, the work [Mg14] of Mœglin deals with quasi-split general spin groups in addition to the quasi-split classical groups considered by Arthur and Mok, and, proves a slightly weakened form of the character theoretic properties that these packets are expected to satisfy; her results for even special orthogonal and even general spin groups too involve an outer automorphism (and as far as the author understands, the results of [Mg14] do not depend on the twisted weighted fundamental lemma for non-split groups, or the articles referred to in [Art13] as [A25], [A26] or [A27]). What we know from these results is strictly stronger than the hypotheses necessary for Theorem 1.2.5, so that we can deduce Conjecture 1.2.4, and hence consequently also Conjecture 1.2.3, for quasi-split symplectic, special orthogonal, unitary, general symplectic and odd general spin groups, and a weaker result involving an outer automorphism for general special orthogonal groups; see Corollary 5.1.4, which gives (1) of the 'pre-introduction', Subsection 1.1. However, due to some technical reasons, we do not treat the case of even general spin groups (essentially because we do not yet know if certain transfer factors relevant to it are invariant under the appropriate outer automorphism).

1.3. **An unconditional variant of Shahidi's μ-constancy argument.** The proof of Theorem 1.2.5 follows Shahidi's proof of the constancy of the Plancherel μ-function on discrete series L-packets on Levi subgroups, under an assumption almost equivalent to (perhaps slightly weaker than) our assumption on the existence of tempered L-packets (see [Sha90, Proposition 9.3]). Indeed, Shahidi studied the Plancherel expansion of $f \mapsto f(1)$ to show that if Σ is a discrete series packet on a Levi subgroup $\mathrm{M} \subset \mathrm{G}$, then $\mu(\sigma_1) = \mu(\sigma_2)$ for all $\sigma_1, \sigma_2 \in \Sigma$, where μ is the Plancherel μ-function associated to the parabolic induction from M to G (see [Wal03, Section V]). Similarly, given $z \in \mathcal{Z}_1(\mathrm{G})$, the proof that $z \in \mathcal{Z}_2(\mathrm{G})$ (under the assumption that tempered L-packets exist) goes through first showing that if Σ is a discrete series packet on a Levi subgroup $\mathrm{M} \subset \mathrm{G}$, then $\hat{z}(\sigma_1) = \hat{z}(\sigma_2)$ for all $\sigma_1, \sigma_2 \in \Sigma$, where $\hat{z}(\sigma_i)$ refers to the scalar with which z acts on any irreducible subquotient of a representation of $\mathrm{G}(F)$ parabolically induced from σ_i. This uses, as well as combines with, the following deep result of Arthur: if $f \in C_c^\infty(\mathrm{G}(F))$ satisfies that $\Theta(f) = 0$ for all stable tempered virtual characters Θ on $\mathrm{G}(F)$, then f is unstable, i.e., all its stable orbital integrals vanish (this follows from [Art96, Theorems 6.1 and 6.2], but it may be more convenient to see this from the statement of the twisted version given in [MW16, Corollary XI.5.2]). The existence of tempered L-packets makes it convenient to check the condition that $\Theta(f) = 0$ for all stable tempered virtual characters Θ on $\mathrm{G}(F)$. While Shahidi considers the Plancherel expansion of $f \mapsto f(1)$, here one considers the expansion of $f \mapsto z(f^\vee) = z * f(1)$, where $f^\vee$ is given by $f^\vee(g) = f(g^{-1})$, to get the constancy of $\sigma \mapsto \hat{z}(\sigma)\mu(\sigma)$ on Σ, from which the constancy of $\sigma \mapsto \hat{z}(\sigma)$ on Σ follows (using Shahidi's result mentioned above, which can be recovered by taking z to be the Dirac measure at the identity element).

We find it convenient to use the above Plancherel expansion argument to prove the following unconditional result along the way, one that does not depend on a strong assumption like the existence of tempered L-packets (see Corollary 4.2.11(i) for more details):

Proposition 1.3.1. *Let $z \in \mathcal{Z}_1(\mathrm{G})$. Let $\zeta : \mathrm{A}_\mathrm{M}(F) \to \mathbb{C}^\times$ be a smooth unitary character, where A_M is the maximal split torus contained in the center of M. Write $\mathrm{Irr}_{\zeta,2}(\mathrm{M})$ for the subset of $\mathrm{Irr}_2(\mathrm{M})$ consisting of representations whose central character restricts to ζ on $\mathrm{A}_\mathrm{M}(F)$. Then*

$$\sum_{\sigma \in \mathrm{Irr}_{\zeta,2}(\mathrm{M})} d(\sigma)\hat{z}(\sigma)\mu(\sigma)\Theta_\sigma, \tag{1}$$

which makes sense as a distribution on $\mathrm{G}(F)$ by Remark 2.2.5, is stable.

1.4. **An unconditional 'inner form transfer' variant of Shahidi's μ-constancy argument.** Recall that already in [Sha90], Shahidi had proposed that his proof of the constancy of the μ-function on discrete series packets on Levi subgroups should generalize to transfer to inner forms, and should thus make local Langlands-Shahidi L-functions available for inner forms, modulo the generic packet conjecture. Such transfers have been known, at least in many cases, by the work of Choiy and Heiermann (see [Cho14] and [Hei16]).

Inspired by the transfer of μ-functions across inner forms by Choiy and Heiermann, one can ask if the stability of (1) given by Proposition 1.3.1 can be enhanced to a transfer between inner forms. This leads to the following proposition, which we state informally and refer to Corollary 4.2.11(ii) for more details:

Proposition 1.4.1. *Given an inner twist between G and its quasi-split inner form G^* that transfers a Levi subgroup $\mathrm{M} \subset \mathrm{G}$ to a Levi subgroup $\mathrm{M}^* \subset \mathrm{G}^*$, (1) above generalizes to:*

$$\sum_{\sigma^* \in \mathrm{Irr}_{\zeta,2}(\mathrm{M}^*)} d(\sigma^*)\hat{z}(\sigma^*)\mu(\sigma^*)\Theta_{\sigma^*} \text{ transfers to (scalar)} \cdot \sum_{\sigma \in \mathrm{Irr}_{\zeta,2}(\mathrm{M})} d(\sigma)\hat{z}(\sigma)\mu(\sigma)\Theta_\sigma. \tag{2}$$

The scalar in the above proposition is given sort of explicitly in Corollary 4.2.11(ii). Among other things, it involves the Kottwitz sign $e(\mathrm{G}) = e(\mathrm{M})$ of G.

While Proposition 1.3.1 used that $f \mapsto f(1)$ is stable (by [Kot88, Proposition 1]), Proposition (1.4.1) uses that the distribution $f^* \mapsto f^*(1)$ on $\mathrm{G}^*(F)$ transfers to the product of $e(\mathrm{G})$ and the distribution $f \mapsto f(1)$ on $\mathrm{G}(F)$ (by [Kot88, Proposition 2]). Instead of using the result of [Art96] that the instability of a function can be checked on stable characters, one uses that whether or not two functions have matching orbital integrals can be checked by seeing that various (non-explicit) character identities are satisfied; this follows from [Art96, Lemma 6.3], as explained in [LM20]: see the equivalence of the conditions (A) and (B) in page 587 of that reference.

1.5. **Unitarily stable discrete series packets.** This proposition suggests that, to conclude the equality $\hat{z}(\sigma_1)\mu(\sigma_1) = \hat{z}(\sigma_2)\mu(\sigma_2)$ for two given representations σ_1, σ_2 that belong to a candidate discrete series L-packet (such as a Kaletha packet), we might not need the full strength of the existence of tempered L-packets: it will suffice if we know that the function $d(\sigma_1)^{-1}f_{\sigma_1} - d(\sigma_2)^{-1}f_{\sigma_2}$ is unstable, where f_{σ_i} is a pseudocoefficient for σ_i among those representations of $\mathrm{M}(F)$ whose central character restricts to ζ on $\mathrm{A}_\mathrm{M}(F)$.

Thus, we consider what we call 'unitarily stable' discrete series L-packets, modifying terminology from [MY20]: a finite subset $\Sigma \subset \mathrm{Irr}_2(\mathrm{M})$ isunitarily stable if it supports a nonzero stable virtual character Θ_Σ with the property that every stable elliptic virtual character Θ on $\mathrm{M}(F)$ can be uniquely written as $c\Theta_\Sigma + \Theta'$, where c is a scalar and Θ' is supported outside Σ. This notion is a natural one and hence is almost certainly well-known to experts, but we could not find a reference in literature. Note that the notion of unitary stability is stronger than the notion of atomic stability found in [Kal22, Conjecture 2.2]. One then shows (see Corollary 4.2.12) that the Plancherel μ-function as well as $\hat{z}$, for any $z \in \mathcal{Z}_1(\mathrm{G})$, are constant on any unitarily stable discrete series packet on a Levi subgroup of G. While pursuing these considerations, it is not hard to see that for any unitarily stable discrete series packet Σ, Θ_Σ is a scalar multiple of $\sum d(\sigma)\Theta_\sigma$, with σ running over Σ and $d(\sigma)$ denoting the formal degree of σ.

If tempered L-packets are known to exist, then it is easy to see that every discrete series packet on a Levi subgroup is unitarily stable. It turns out that examples can be given even when tempered L-packets are not known to exist. We describe two ways to check that a given finite set of discrete series representations constitutes a unitarily stable discrete series packet. Again, this should be known to experts, since these two results are very simple consequences of [Art96], but we were unaware of them and could not find them in literature. It was a remark of Mœglin in [Mg14, Section 4.8] that suggested the first to us, and it was [LMW18, Section 4.6, Lemma 3] that suggested the second.

To describe the first, let M be a connected reductive group over F. Given a virtual discrete series character Θ on $\mathrm{M}(F)$, recalling that it is completely determined by the values it takes as a locally constant function on the set $\mathrm{M}(F)_{\mathrm{ell}}$ of strongly regular *elliptic* semisimple elements of $\mathrm{M}(F)$, let us denote by $\Theta^{\mathrm{st}} : \mathrm{M}(F)_{\mathrm{ell}} \to \mathbb{C}$ the function that takes γ to the average of the $\Theta(\gamma')$ as γ' varies over representatives for the conjugacy classes in the stable conjugacy class of γ. It is not obvious that Θ^{st} is the set of values taken by any virtual character on $\mathrm{M}(F)_{\mathrm{ell}}$, but one knows from a deep result of [Art96] that it is so. The first way to detect unitary stability is as follows (see Proposition 3.4.2):

Proposition 1.5.1. *A finite subset $\Sigma \subset \mathrm{Irr}_2(\mathrm{M})$ is a unitarily stable discrete series packet if and only if the following two conditions are satisfied:*

- *The $\Theta_\sigma^{\mathrm{st}}$, as σ varies over Σ, are all proportional to each other; and*
- *Some linear combination of the Θ_σ with all coefficients nonzero, as σ varies over Σ, is a stable distribution.*

As remarked earlier, the proof is not hard: if we assume for simplicity that M is semisimple, the result follows easily once one computes that $\Theta_\sigma^{\mathrm{st}}$ is the image of Θ_σ under the projection map $D_{\mathrm{ell}}(\mathrm{M}) \to SD_{\mathrm{ell}}(\mathrm{M})$, where $D_{\mathrm{ell}}(\mathrm{M})$ is the space of elliptic virtual characters on $\mathrm{M}(F)$, $SD_{\mathrm{ell}}(\mathrm{M}) \subset D_{\mathrm{ell}}(\mathrm{M})$ is the subspace of stable elliptic virtual characters on $\mathrm{M}(F)$, and the projection is with respect to the elliptic inner product.

The second way to detect unitary stability is only a sufficient condition, which we state slightly informally and imprecisely as follows; see Proposition 3.4.11 for the more precise statement:

Proposition 1.5.2. *If a finite subset $\Sigma \subset \mathrm{Irr}_2(\mathrm{M})$ has a crude 'endoscopic decomposition', in the sense that we can write*

$$\sum_{\sigma \in \Sigma} \mathbb{C}\Theta_\sigma = \sum_{\underline{\mathrm{H}}} \mathbb{C}\Theta_{\underline{\mathrm{H}}}^{\mathrm{M}},$$

where $\underline{\mathrm{H}}$ runs over a set of distinct relevant elliptic endoscopic data for M and $\Theta_{\underline{\mathrm{H}}}^{\mathrm{M}}$ is the transfer to $\mathrm{M}(F)$ of some stable elliptic virtual character on (a z-extension of) $\mathrm{H}(F)$ via $\underline{\mathrm{H}}$, then Σ is a unitarily stable discrete series packet.

This proposition follows easily from the result in [Art96] (though we follow the exposition in [LMW18]) that endoscopic transfer from stable elliptic virtual characters on relevant elliptic endoscopic groups gives us a decomposition of $D_{\mathrm{ell}}(\mathrm{M})$ that is orthogonal for the elliptic inner product. This way of detecting the property of being unitarily stable is harder to implement, but has the advantage that the necessary work has already been done by Kaletha in the case of regular supercuspidal packets when $p \gg 0$.

Thus , we conclude that when $p \gg 0$, Kaletha's regular supercuspidal packets are unitarily stable (see Corollary 3.4.12). This implies (see Remark 3.4.13 for a few more details) a weak compatibility result, comparing Kaletha's local Langlands correspondence with those of Arthur, Mœglin and Mok: when $p \gg 0$, Kaletha's regular supercuspidal packets on quasi-split special orthogonal and symplectic (resp., unitary) groups are also packets in the sense of [Art13] (resp., [Mok15]); an analogous comment applies with [Mg14] in place of [Art13] and [Mok15], provided one accounts for an outer automorphism in the case of even special orthogonal groups. However, we do not have any result on the compatibility between the relevant Langlands parametrizations. In any case, this justifies (3) of Subsection 1.1.

1.6. **μ-constancy for unitarily stable packets.** Given Propositions 1.3.1 and 1.4.1, the following informally stated proposition, whose first (resp., second) assertion generalizes Shahidi's constancy of the μ-function on discrete series L-packets (resp., the transfer of μ-functions as in the works of Choiy and Heiermann), is not hard to see; we refer to Corollary 4.2.12 for more details:

Proposition 1.6.1. *(i) If Σ is a unitarily stable discrete series packet on a Levi subgroup $\mathrm{M} \subset \mathrm{G}$, and $z \in \mathcal{Z}_1(\mathrm{G})$, then for all $\sigma_1, \sigma_2 \in \Sigma$ we have $\mu(\sigma_1) = \mu(\sigma_2)$ and $\hat{z}(\sigma_1) = \hat{z}(\sigma_2)$.*

(ii) In the setting of Proposition 1.4.1, if a unitarily stable discrete series packet Σ^* on M^* transfers to a unitarily stable discrete series packet Σ on M in a sense that is not hard to formulate, then for all $\sigma \in \Sigma$ and $\sigma^* \in \Sigma^*$, $\mu^*(\sigma^*)$ is the product of $\mu(\sigma)$ and an explicit constant that does not depend on Σ^* or Σ. If moreover $z \in \mathcal{Z}(\mathrm{G})$ is a transfer of $z^* \in \mathcal{Z}_1(\mathrm{G}^*)$ in the sense that as a distribution on $\mathrm{G}(F)$, z is the product of $e(\mathrm{G})$ and the endoscopic transfer of z^* viewed as a distribution on $\mathrm{G}^*(F)$, then for all $\sigma^* \in \Sigma^*$ and $\sigma \in \Sigma$, we have

$$\hat{z}^*(\sigma^*) = \hat{z}(\sigma). \tag{3}$$

In particular, when $p \gg 0$, since we have observed that the regular supercuspidal packets on Levi subgroups of G are unitary stable, the Plancherel μ-function associated to these packets transfers well across inner forms, justifying (4) in Subsection 1.1.

1.7. **Unitarily stable packets, again.** We digress to summarize the virtues of unitarily stable discrete series packets:

- *There are provably many of them:* They include the regular supercuspidal packets of Kaletha when $p \gg 0$ (Corollary 3.4.12).
- *They have good properties:* Their defining property is precisely what it takes for Shahidi's μ-constancy argument to work (Corollary 4.2.12).
- *They are canonical:* If Σ is a unitarily stable discrete series packet on a Levi subgroup M of G, and if G satisfies the existence of tempered L-packets, then the resulting set of discrete series packets on M includes Σ. In particular, when $p \gg 0$, Kaletha's packets are also Arthur's (see Remark 3.4.13).

1.8. **Application to Langlands-Shahidi L-functions.** We discuss two applications for Proposition 1.6.1. First, the arguments of Shahidi following [Sha90, Conjecture 9.4] should now make available the normalization of intertwining operators using Langlands-Shahidi L-functions, for those unitarily stable discrete series packets on $\mathrm{G}(F)$ that transfer to unitarily stable discrete series packets on the quasi-split form $\mathrm{G}^*(F)$ that can be shown to be generic, and in particular for regular supercuspidal packets when $p \gg 0$. For some more explanation, see Subsubsection 4.3.1. Let us also remark that there is a much more delicate and subtle strengthening of the aforementioned transfer of Plancherel μ-functions, called the local intertwining relation (and which is due to Arthur), addressing which is beyond the scope of this paper. One can hope that forthcoming work of Kaletha will shed light on it. Let us also take this opportunity to mention that a 'relatively local' approach towards proving some special cases of the local intertwining relation when the induced representation is irreducible, is given by the Goldberg-Shahidi method of computing residues of intertwining operators: see [Sha92] and [Var]. We hope that, at least in some very special situations, and assuming $p \gg 0$, it could yield, by 'relatively local' methods, an answer to a question that the above considerations bring to the fore: whether the Langlands-Shahidi L-functions and ϵ-factors associated to regular supercuspidal packets on Levi subgroups agree with the corresponding Artin L-functions and ϵ-factors associated to the Langlands parameters assigned to them by Kaletha.

1.9. **Application to depth preservation.** To state the second application of Proposition 1.6.1, fix $r \geq 0$ and let $E_r \in \mathcal{Z}(\mathrm{G})$ be the depth r projector in the sense of [BKV16]; thus, for an irreducible admissible representation σ of $\mathrm{G}(F)$, $\hat{E}_r(\sigma)$ equals 1 or 0 depending on whether or not the depth of σ is at most r. Assuming $p \gg 0$, the second application of Proposition 1.6.1 gives the constancy of depth (see [MP96]) on unitarily stable discrete series L-packets, and the fact that transfer of unitarily stable discrete series L-packets across inner forms respects depth; we refer to Corollary 4.3.4 and Proposition 4.3.5 for more details.

Proposition 1.9.1. *(i) If p is very good for* G *in the sense of* [BKV16], *and* Σ *is a unitarily stable discrete series packet on a Levi subgroup* $\mathrm{M} \subset \mathrm{G}$, *then the elements of* Σ *have the same depth.*

(ii) Let G^* *be a quasi-split inner form of* G, *and let* E_r^* *be the depth r projector on it. Assume p to be very good for* G, *and that* $\mathfrak{g}$ *has a bilinear form that behaves well with respect to its Moy-Prasad filtrations (as in* [AR00, Proposition 4.1]*). Then:*

(a) E_r^* *belongs to* $\mathcal{Z}_1(\mathrm{G}^*)$ *and transfers as a stable Bernstein center element to* E_r, *in the sense that when viewed as distributions, and with* $\mathrm{G}(F)$ *and* $\mathrm{G}^*(F)$ *given compatible Haar measures,* E_r^* *transfers to* $e(\mathrm{G})E_r$.

(b) Moreover, if we are in the setting of Proposition 1.4.1, and if a unitarily stable discrete series packet Σ^* *on* M^* *transfers to a unitarily stable discrete series packet* Σ *on* M, *then for all* $\sigma \in \Sigma$ *and* $\sigma^* \in \Sigma^*$, *we have* $\mathrm{depth}(\sigma) = \mathrm{depth}(\sigma^*)$.

(i) of the above proposition is an immediate consequence of the stability of E_r (as given by [BKV16], since p is 'very good' for G) and Proposition 1.6.1(i). As for (ii) of the above proposition, the assertion (a) is Proposition 4.3.5, while the assertion (b) follows from the assertion (a) and Proposition 1.6.1(ii). In future joint work with Li and Oi alluded to above, we hope to generalize (ii)(a) of the above proposition to an assertion about the behavior of the depth r projector with respect to endoscopic transfer, when $p \gg 0$.

Using Proposition 1.9.1((i)) along with the existence of tempered L-packets for quasi-split classical groups (due to Arthur and Mok; see Proposition 5.1.2) and [Oi22, Theorem 1.2], it is easy to refine the latter theorem into the assertion that for $p \gg 0$, the local Langlands corresondence for quasi-split classical groups preserves depth, giving (2) of Subsection 1.1 (see Corollary 5.1.6).

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2. Some notation, preliminaries, preparation, and hypotheses

Throughout this paper, notation that we define for a group will be applied with obvious modification to other groups. For instance, once we define the object $D(\mathrm{M})$ or $\Omega(\mathrm{G})$ associated to the connected reductive group M or G, then for any connected reductive group G', we will use $D(\mathrm{G}')$ or $\Omega(\mathrm{G}')$ to denote the analogous object associated to G'.

2.1. Some notation.

2.1.1. *Miscellaneous notation.* For an abstract group $\mathscr{G}$ that acts on a mathematical object X, we will denote by $X^{\mathscr{G}}$ (resp., $X_{\mathscr{G}}$) the invariants (resp., the coinvariants) for the action of $\mathscr{G}$ on X, provided such a thing makes sense. For any mathematical object X, we will write $\mathrm{Aut}(X)$ for the group of automorphisms of X, when the meaning of 'automorphisms' is clear from the context. If $\mathscr{G}$ is a topological group, $\mathrm{Hom}_{\mathrm{cts}}(\mathscr{G}, \mathbb{C}^\times)$ will denote the group of (quasi-)characters of $\mathscr{G}$, i.e., of continuous homomorphisms $\mathscr{G} \to \mathbb{C}^\times$.

For a ring R, an R-algebra R', a module M over R and a scheme X over R, we will write $M_{R'}$ for $M \otimes_R R'$ and $X_{R'}$ or $X \times_R R'$ for the base-change of X from R to R'.

M^0 will denote the identity component of an algebraic group M defined over a field. The Lie algebra of an algebraic group denoted by a roman letter (e.g., G) will be denoted by the corresponding fraktur letter (e.g., $\mathfrak{g}$). If X is a variety (resp., algebraic group) over a valued field F, $\mathrm{X}(F)$ will be viewed as a topologial space (resp., topological group) with the "Hausdorff topology" associated to the valuation. Whenever X is a complex variety, we may abbreviate $\mathrm{X}(\mathbb{C})$ to X.

2.1.2. *Tori.* If T is a torus defined over a field F, we will denote by $X^*(\mathrm{T})$ (resp., $X_*(\mathrm{T})$) the character lattice (resp., the cocharacter lattice) of the base-change T_{F^s} of T to F^s, and view it together with the $\mathrm{Gal}(F^s/F)$-action on it, where F^s will be a separable closure of F that will be clear from the context. Moreover, given such a T, $\mathrm{A_T} \subset \mathrm{T}$ will denote the maximal split subtorus and $\mathrm{T} \to \mathrm{S_T}$ the maximal split quotient torus.

2.1.3. *Derived group, outer automorphisms etc.* If M is a connected reductive group over a field F, we will write $\mathrm{M_{der}}, \mathrm{M_{ad}}$ and $\mathrm{M_{sc}}$ respectively for the derived group of M, the adjoint group of M, and the simply connected cover of $\mathrm{M_{der}}$, respectively. Moreover, $\mathrm{Out(M)}$ will denote the group of outer automorphisms of the base-change $\mathrm{M}_{\bar{F}}$ of M to a suitable algebraic closure $\bar{F}$ of F (i.e., the group of all algebraic automorphisms of $\mathrm{M}_{\bar{F}}$ quotiented by the normal subgroup of the inner automorphisms $\mathrm{Int}\, m$, with m ranging over $\mathrm{M}(\bar{F})$).

2.1.4. *Twisted spaces, center and related notation.* For any algebraic group M over a field F, $\tilde{\mathrm{M}}$ will usually denote a twisted space for M — recall that this means that $\tilde{\mathrm{M}}$ is an algebraic variety over F that is given commuting left and right M-actions, which we will write as $(m, \delta) \mapsto m\delta$ and $(\delta, m) \mapsto \delta m$, that are both simply transitive, and satisfying $\tilde{\mathrm{M}}(F) \neq \emptyset$. Note that $m_1 \delta m_2$ has an unambiguous meaning for $m_1, m_2 \in \mathrm{M}(F)$ and $\delta \in \tilde{\mathrm{M}}(F)$, as either of the terms in the equality $(m_1\delta)m_2 = m_1(\delta m_2)$.
If δ is an element of a twisted space $\tilde{\mathrm{M}}$ over an algebraic group M over a field F, we will denote by $\mathrm{Int}\,\delta$ the unique automorphism of M such that $\delta \cdot m = \mathrm{Int}\,\delta(m) \cdot \delta$. $\mathrm{Ad}\,\delta$ will then denote the derivative of $\mathrm{Int}\,\delta$. $\mathrm{Z}_{\tilde{\mathrm{M}}}$ will denote the intersection of the kernels of the $\mathrm{Int}\,\delta$ as δ ranges over $\tilde{\mathrm{M}}$, and $\mathrm{A}_{\tilde{\mathrm{M}}}$ the maximal split torus contained in $\mathrm{Z}_{\tilde{\mathrm{M}}}$.
Often, an algebraic group M over a field F will be implicitly considered as a twisted space over itself using its left and right multiplication. The group $\mathrm{Z_M}$ thus defined coincides with the center of M, and, for each $\delta \in \mathrm{M}$, the automorphisms $\mathrm{Int}\,\delta$ and $\mathrm{Ad}\,\delta$ thus defined coincide with conjugation by δ and the adjoint action of δ, respectively.

2.1.5. *The p-adic field F and related notation.* Henceforth we fix a finite extension F of $\mathbb{Q}_p$ for some prime p, an algebraic closure $\bar{F}$ of F, and a uniformizer ϖ for the ring of integers of F. Let $\mathfrak{O} = \mathfrak{O}_F \subset F$ be the ring of integers of F, and q the cardinality of the residue field of F. Let $|\cdot| : \bar{F} \to \mathbb{R}$ denote the usual extension to $\bar{F}$ of the normalized absolute value on F. We will denote by $\Gamma := \mathrm{Gal}(\bar{F}/F)$ and by $W_F \subset \Gamma$ the absolute Galois group and the Weil group of F, by $I_F \subset W_F$ the inertia subgroup, and by $W'_F := W_F \times \mathrm{SL}_2(\mathbb{C})$ the Weil-Deligne group of F. Let $\mathrm{Fr} \in W_F/I_F$ stand for the element that induces the Frobenius automorphism of the residue field. We denote by $\|\cdot\| : W_F \to \mathbb{R}_{>0}$ the composite of the normalized absolute value on $F^\times$ and the abelianization homomorphism $W_F \to F^\times$ that is normalized to send (any representative for) Fr to a uniformizer in the ring of integers of F.

2.1.6. *Discrete series etc.* For a connected reductive group M over F, a 'discrete series representation of $\mathrm{M}(F)$' will refer to a unitary irreducible smooth representation of $\mathrm{M}(F)$ whose matrix coefficients are square-integrable modulo the center, while an 'essentially square-integrable representation of $\mathrm{M}(F)$' will refer to a twist of a discrete series representation of $\mathrm{M}(F)$ by a (not necessarily unitary) continuous (quasi-)character $\chi \in \mathrm{Hom_{cts}}(\mathrm{M}(F), \mathbb{C}^\times)$. If M is a connected reductive group over F, we denote by $\mathrm{Irr(M)}$(resp., $\mathrm{Irr_{temp}(M)}$; resp., $\mathrm{Irr_2(M)}$; resp., $\mathrm{Irr}_2^+(\mathrm{M})$) the set of isomorphism classes of irreducible representations of $\mathrm{M}(F)$ that are admissible (resp., tempered; resp., discrete series; resp., essentially square-integrable).
If $\mathcal{Z} \subset \mathrm{M}(F)$ is a central subgroup that is understood from the context, and $\zeta : \mathcal{Z} \to \mathbb{C}^\times$ is a smooth character, then we denote by $\mathrm{Irr_2(M)}_\zeta \subset \mathrm{Irr_2(M)}, \mathrm{Irr}_2^+(\mathrm{M})_\zeta \subset \mathrm{Irr}_2^+(\mathrm{M}), \mathrm{Irr_{temp}(M)}_\zeta \subset \mathrm{Irr_{temp}(M)}$

and $\mathrm{Irr}(\mathrm{M})_\zeta \subset \mathrm{Irr}(\mathrm{M})$ the subsets consisting of (isomorphism classes) of representations whose central character restricts to ζ on $\mathcal{Z}$.

2.1.7. *Levi subgroups and parabolic induction.* Let M be a connected reductive group over F. For each Levi subgroup $\mathrm{L} \subset \mathrm{M}$, we denote by $W_\mathrm{M}(\mathrm{L})$, and by $W(\mathrm{L})$ when M is understood from the context, the group of F-rational points of the quotient, of the normalizer of L in M, by L. Then every element of $W(\mathrm{L})$ can be represented by an element of $\mathrm{M}(F)$, letting us identify $W(\mathrm{L})$ with the quotient of the normalizer of $\mathrm{L}(F)$ in $\mathrm{M}(F)$, by $\mathrm{L}(F)$ (here is a quick argument: choosing a parabolic subgroup Q of M with L as a Levi subgroup and having unipotent radical N, $W(\mathrm{L})$ acts freely on the set of parabolic subgroups of M that are $\mathrm{M}(\bar{F})$-conjugate to Q and have L as a Levi subgroup; but the normalizer of $\mathrm{L}(F)$ in $\mathrm{M}(F)$ acts transitively on this set — use [Bor91, Proposition 20.5], which gives both the surjectivity of $\mathrm{M}(F) \to (\mathrm{Q}\backslash\mathrm{M})(F)$ and the fact that the set of Levi subgroups of Q is a torsor under $\mathrm{N}(F)$-conjugation).

Let $\mathrm{M}_1 \subset \mathrm{M}$ be a Levi subgroup (of a parabolic subgroup of M). Then for any parabolic subgroup $\mathrm{P}_1 \subset \mathrm{M}$ with Levi subgroup M_1, we will write $\mathrm{Ind}_{\mathrm{P}_1}^{\mathrm{M}}$ for the associated (normalized) parabolic induction functor, taking smooth representations of $\mathrm{M}_1(F)$ to smooth representations of $\mathrm{M}(F)$. The map induced by the functor $\mathrm{Ind}_{\mathrm{P}_1}^{\mathrm{M}}$ at the level of virtual characters is independent of the choice of P_1, and hence will be written $\mathrm{Ind}_{\mathrm{M}_1}^{\mathrm{M}}$. Sometimes we will refer to a subquotient of $\mathrm{Ind}_{\mathrm{M}_1}^{\mathrm{M}} \sigma$, by which we will mean a subquotient of $\mathrm{Ind}_{\mathrm{P}_1}^{\mathrm{P}} \sigma$ — this notion is independent of the choice of P_1, though $\mathrm{Ind}_{\mathrm{P}_1}^{\mathrm{M}} \sigma$ itself (and what its subrepresentations and quotient representations are) depends on P_1. We have not defined the twisted harmonic analytic version of the notation just introduced, as we will not need it.

2.1.8. *Twisted representations and virtual characters.* In this subsubsection, let M be an arbitrary reductive group over F and $\tilde{\mathrm{M}}$ a twisted space associated to M, with the property that for some $\delta \in \tilde{\mathrm{M}}(F)$, $\mathrm{Int}\,\delta$ is a semisimple automorphism of M whose restriction to $\mathrm{A}_{\tilde{\mathrm{M}}}$ is of finite order. Let $\omega : \mathrm{M}(F) \to \mathbb{C}^\times$ be a continuous character attached to a cocycle $\mathbf{a} \in H^1(W_F, \mathrm{Z}_{\hat{\mathrm{M}}})$. Assume that ω is unitary.

An ω-invariant distribution on $\tilde{\mathrm{M}}(F)$ refers to a $\mathbb{C}$-linear map $C_c^\infty(\tilde{\mathrm{M}}(F)) \to \mathbb{C}$ such that $D(\tilde{f} \circ \mathrm{Int}\, m) = \omega(m) D(\tilde{f})$ for all $m \in \mathrm{M}(F)$.

Recall that a representation of $(\tilde{\mathrm{M}}(F), \omega)$, or an ω-representation of $\tilde{\mathrm{M}}(F)$, is a representation (σ, V) of $\mathrm{M}(F)$ together with a map $\tilde{\sigma} : \tilde{\mathrm{M}}(F) \to \mathrm{Aut}_\mathbb{C}(V)$, such that $\tilde{\sigma}(m_1 \delta m_2) = \omega(m_2)\sigma(m_1)\tilde{\sigma}(\delta)\sigma(m_2)$ for all $m_1, m_2 \in \mathrm{M}(F)$ and $\delta \in \tilde{\mathrm{M}}(F)$. We will refer to σ as the representation of $\mathrm{M}(F)$ underlying $\tilde{\sigma}$. We will say that $\tilde{\sigma}$ is smooth or admissible or $\mathrm{M}(F)$-irreducible or of finite length if the representation σ of $\mathrm{M}(F)$ that underlies it, is (see [MgW18, Section 2.5]). Note that $\mathrm{M}(F)$-irreducibility is stronger than the 'obvious' notion of irreducibility. We will refer to $\tilde{\sigma}$ as tempered if $\tilde{\sigma}$ is unitary and σ is tempered.

Given an admissible representation $\tilde{\sigma}$ of $\tilde{\mathrm{M}}(F)$ (the underlying representation σ of $\mathrm{M}(F)$ being suppressed from the notation), we will denote by $\Theta_{\tilde{\sigma}}$ the (easily checked to be ω-invariant) distribution $C_c^\infty(\tilde{\mathrm{M}}(F)) \to \mathbb{C}$ that takes $\tilde{f} \in C_c^\infty(\tilde{\mathrm{M}}(F))$ to $\mathrm{tr}\,\tilde{\sigma}(\tilde{f})$, where:

$$\tilde{\sigma}(\tilde{f}) = \left(v \mapsto \int_{\tilde{\mathrm{M}}(F)} \tilde{f}(\delta) \cdot \tilde{\sigma}(\delta) v \, d\delta \right),$$

for some fixed choice of a measure on $\tilde{\mathrm{M}}(F)$ obtained by transferring a Haar measure on $\mathrm{M}(F)$ via any isomorphism $\mathrm{M}(F) \to \tilde{\mathrm{M}}(F)$ obtained as $m \mapsto \delta \cdot m$ or $m \mapsto m \cdot \delta$ for some $\delta \in \tilde{\mathrm{M}}(F)$. All these isomorphisms indeed yield the same measure on $\tilde{\mathrm{M}}(F)$ that is independent of δ. One knows that any such $\Theta_{\tilde{\sigma}}$ can be realized by integration against a locally integrable function on $\tilde{\mathrm{M}}(F)$ that is locally constant on the set of regular semisimple elements of $\tilde{\mathrm{M}}(F)$ (see [LH17, Corollary 5.8.3] and use, as discussed in [LMW18, Section 3.1], that F has characteristic zero). By abuse of notation, we will use $\Theta_{\tilde{\sigma}}$ to also denote this function, called the Harish-Chandra character of $\tilde{\sigma}$.

We can talk of formal complex linear combinations $\sum c_i \tilde{\sigma}_i$ of finite-length admissible ω-representations $\tilde{\sigma}_i$ of $\tilde{\mathrm{M}}(F)$, and thus make sense of character distributions or Harish-Chandra characters associated to such formal linear combinations as well: $\Theta_{\sum c_i \tilde{\sigma}_i} = \sum c_i \Theta_{\tilde{\sigma}_i}$. Such distributions and functions will be referred to as virtual characters associated to the ω-representation theory of $\tilde{\mathrm{M}}(F)$. Let Θ be such a virtual character. Θ is said to be supported on a set Σ of isomorphism classes of M-irreducible admissible ω-representations of $\tilde{\mathrm{M}}(F)$, if we can write $\Theta = \sum_i c_i \Theta_{\tilde{\sigma}_i}$ with $\tilde{\sigma}_i \in \Sigma$ for each i. Θ is said to be supported outside another such set Σ', if Σ can be chosen so that no $\mathrm{M}(F)$-representation underlying an element of Σ underlies an element of Σ'.

2.1.9. *Some spaces of distributions.*

Notation 2.1.1. Let $\mathrm{M}, \tilde{\mathrm{M}}, \omega$ be as in Subsubsection 2.1.8.

(i) Following [LMW18, Section 3.1], or the definition of "$D_{temp}(\tilde{G}(F), \omega)$" in [MgW18, Section 2.9], let $D(\tilde{\mathrm{M}}, \omega)$ denote the complex vector space of ω-invariant distributions on $\mathrm{M}(F)$ spanned by the characters of tempered $\mathrm{M}(F)$-irreducible ω-representations of $\tilde{\mathrm{M}}(F)$; it is spanned by characters of representations $\tilde{\sigma}_{\boldsymbol{\tau}}$ associated to certain triplets τ as in [MgW18, Section 2.9].

(ii) Following [MgW18, Section 2.12], we consider the subspace $D_{\mathrm{ell}}(\tilde{\mathrm{M}}, \omega) \subset D(\tilde{\mathrm{M}}, \omega)$ spanned by the characters of those $\tilde{\sigma}_{\boldsymbol{\tau}}$ such that the triplet τ is elliptic as defined in [MgW18, Section 2.11]; it is the twisted version of the analogous notion considered by Arthur.

(iii) We refer to [MW16] for the notion of orbital integrals $O(\gamma, \omega, \cdot)$, and their special cases $O(\gamma, \cdot) = O(\gamma, \mathbb{1}, \cdot)$, defined on appropriate function spaces (like suitable $C^\infty_\mu(\tilde{\mathrm{M}}(F))$ as below).

(iv) Let $\mathscr{Z} \subset \mathrm{M}(F)$ be a central subgroup, and $\mu : \mathscr{Z} \to \mathbb{C}^\times$ a continuous character. In such a situation we will use the following notation, often suppressing from the notation the dependence on $\mathscr{Z}$ when it is understood in the context.
- We will let $C^\infty_\mu(\tilde{\mathrm{M}}(F))$ be the space of smooth functions $f_1 : \tilde{\mathrm{M}}(F) \to \mathbb{C}$, compactly supported modulo $\mathscr{Z}$, such that $f_1(z_1\gamma_1) = \mu(z_1)^{-1} f_1(\gamma_1)$ for all $z_1 \in \mathscr{Z}$ and $\gamma_1 \in \tilde{\mathrm{M}}(F)$. If $\mathscr{Z}$ is not clear from the context, or if $\mathscr{Z} = \mathrm{C}_1(F)$ with $\mathrm{C}_1 \subset \mathrm{M}$ a central subgroup the dependence on which we do not wish to suppress, we will write $C^\infty_{\mathscr{Z},\mu}(\tilde{\mathrm{M}}(F))$ or $C^\infty_{\mathrm{C}_1,\mu}(\tilde{\mathrm{M}}(F))$ or $C^\infty_{\mathrm{C}_1(F),\mu}(\tilde{\mathrm{M}}(F))$ in place of $C^\infty_\mu(\tilde{\mathrm{M}}(F))$.
- We will let $D_{\mathscr{Z},\mu}(\tilde{\mathrm{M}}, \omega) = D_\mu(\tilde{\mathrm{M}}, \omega)$ (resp., $D_{\mathcal{Z},\mu,\mathrm{ell}}(\tilde{\mathrm{M}}, \omega) = D_{\mu,\mathrm{ell}}(\tilde{\mathrm{M}}, \omega)$) denote the subspace of $D(\tilde{\mathrm{M}}, \omega)$ (resp., $D_{\mathrm{ell}}(\tilde{\mathrm{M}}, \omega)$) generated by characters of ω-representations $(\pi, \tilde{\pi})$ of $\tilde{\mathrm{M}}(F)$ with the property that π has a central character that restricts to μ on $\mathscr{Z}$; this agrees with the notation in [LMW18, Sections 4.3 and 4.4].

(v) The above notation will be adapted, without further comment, to deal with usual invariant harmonic analysis — $D_{\mathrm{ell}}(\tilde{\mathrm{M}}), D_{\mathscr{Z},\mu}(\tilde{\mathrm{M}}), D_{\mu,\mathrm{ell}}(\tilde{\mathrm{M}})$ etc. will denote $D_{\mathrm{ell}}(\tilde{\mathrm{M}}, \mathbb{1})$, $D_{\mathscr{Z},\mu}(\tilde{\mathrm{M}}, \mathbb{1})$, $D_{\mu,\mathrm{ell}}(\tilde{\mathrm{M}}, \mathbb{1})$ etc., where $\mathbb{1}$ denotes the trivial character of $\mathrm{M}(F)$. Further, $D_{\mathrm{ell}}(\mathrm{M}), D_{\mathscr{Z},\mu}(\mathrm{M}), D_{\mu,\mathrm{ell}}(\mathrm{M})$ etc. will denote $D_{\mathrm{ell}}(\tilde{\mathrm{M}}'), D_{\mathscr{Z},\mu}(\tilde{\mathrm{M}}'), D_{\mu,\mathrm{ell}}(\tilde{\mathrm{M}}')$ etc., where $\tilde{\mathrm{M}}'$ equals M thought of as a twisted space over itself using left and right multiplication.

(vi) Now suppose further that we are in the case where $\tilde{\mathrm{M}}$ has the property that, for all $\delta \in \tilde{\mathrm{M}}(\bar{F})$, the automorphism $\mathrm{Int}\,\delta$ of M is inner (i.e., equal to $\mathrm{Int}\,m$ for some $m \in \mathrm{M}_{\mathrm{ad}}(\bar{F})$). The latter property is what [MW16] refers to as $\tilde{\mathrm{M}}$ being 'á torsion intérieure'. Further, assume that ω is trivial, and that either M is quasi-split, or that we are in the case where the twisted space $\tilde{\mathrm{M}}$ is isomorphic to M acting on itself by left and right multiplication. [1] In this case:
- We refer to [KS99] or [MW16] for the notion of the stable orbital integrals $SO(\gamma, \cdot)$.
- In the setting of twisted endoscopy, a function belonging to $C^\infty_c(\tilde{\mathrm{M}}(F))$ or some suitable $C^\infty_\mu(\tilde{\mathrm{M}}(F))$, whose stable orbital integrals all vanish, will be called unstable. The condition that the stable orbitals vanish only needs to be checked at semisimple elements that are strongly regular in the sense of having an abelian centralizer.
- A stable distribution is one that vanishes on unstable functions (in the context of an appropriate space of distributions).

[1] This combination of assumptions may not be very natural, but we stick to it for simplicity.

- Therefore, various spaces of distributions defined above have their stable variants, which are their subspaces consisting of those distributions that are stable: $SD(\tilde{\mathrm{M}}) \subset D(\tilde{\mathrm{M}}), SD_{\mathrm{ell}}(\tilde{\mathrm{M}}) \subset D_{\mathrm{ell}}(\tilde{\mathrm{M}}), SD_{\mathscr{Z},\mu,\mathrm{ell}}(\tilde{\mathrm{M}}) \subset D_{\mathscr{Z},\mu,\mathrm{ell}}(\tilde{\mathrm{M}})$ etc. Again, this makes sense of $SD_{\mathrm{ell}}(\mathrm{M}), SD(\mathrm{M})$ etc., thinking of M as a twisted space over itself under left and right multiplication.

(vii) Let $\mathscr{Z} \subset \mathrm{M}(F)$ be a central subgroup. Any choice of a Haar measure on $\mathscr{Z}$ gives us an obvious map $C_c^\infty(\tilde{\mathrm{M}}(F)) \to C_\mu^\infty(\tilde{\mathrm{M}}(F))$, through which the elements of $D_\mu(\tilde{\mathrm{M}}, \omega)$ and $D_{\mu,\mathrm{ell}}(\tilde{\mathrm{M}}, \omega)$ factor, letting us view $D_\mu(\tilde{\mathrm{M}}, \omega)$ and $D_{\mu,\mathrm{ell}}(\tilde{\mathrm{M}}, \omega)$ as linear forms on $C_\mu^\infty(\tilde{\mathrm{M}}(F))$. We will similarly make sense of $SD_\mu(\tilde{\mathrm{M}})$ and $SD_{\mu,\mathrm{ell}}(\tilde{\mathrm{M}})$ as linear forms on $C_\mu^\infty(\tilde{\mathrm{M}}(F))$, in those contexts in which we have defined SD (see (vi) above).

We will use the above notation only when μ is unitary.

2.2. **Review of facts about Langlands classification, and about stable virtual characters.** In this subsection, let M be a connected reductive group over F.

Definition 2.2.1. Let $\mathrm{Q} \subset \mathrm{M}$ be a parabolic subgroup, and $\chi : \mathrm{Q}(F) \to \mathbb{C}^\times$ an unramified character (this notion is recalled in Notation 2.3.1 below). Then χ is said to be Q-dominant if for some (or equivalently, any) maximal split torus A_0 of M contained in Q, and any coroot $\lambda : \mathbb{G}_m \to \mathrm{A}_0$ associated to a root of A_0 in the unipotent radical of Q (one knows that the coroots λ belong to $X_*(\mathrm{A}_0)$ and not just to $X_*(\mathrm{A}_0) \otimes \mathbb{Q}$), the character $\chi \circ \lambda : F^\times \to \mathbb{C}^\times$ is of the form $|\cdot|^s$, where the complex number s, well-defined modulo $2\pi i(\log q)^{-1}\mathbb{Z} \subset \mathbb{C}$, has a nonnegative real part.

The following notation will be used only in this section.

Notation 2.2.2. Let L be a Levi subgroup of a parabolic subgroup Q of M, and let $\upsilon \in \mathrm{Irr}_2^+(\mathrm{L})$. One knows that one can write $\upsilon = \upsilon' \otimes \chi'$, where $\upsilon' \in \mathrm{Irr}_2(\mathrm{L})$ and $\chi' : \mathrm{L}(F) \to \mathbb{C}^\times$ is an unramified character. We say that υ is Q-dominant if χ', viewed as a character $\mathrm{Q}(F) \to \mathbb{C}^\times$ by inflation, is. This notion is independent of the decomposition $\upsilon = \upsilon' \otimes \chi'$, since given two such decompositions $\upsilon' \otimes \chi'$ and $\upsilon'' \otimes \chi''$ of υ, $\chi'(\chi'')^{-1}$ is unitary.

We now recall the version of the Langlands classification involving essentially square-integrable representations:

Proposition 2.2.3. *(i) Given $\sigma \in \mathrm{Irr}(\mathrm{M})$, there exists a pair (L, υ) consisting of a Levi subgroup* L *of* M *and a representation $\upsilon \in \mathrm{Irr}_2^+(\mathrm{L})$, uniquely determined up to $\mathrm{M}(F)$-conjugacy, such that σ is an irreducible quotient (not just subquotient) of $\mathrm{Ind}_{\mathrm{Q}}^{\mathrm{M}} \upsilon$, where* Q *is a choice of a parabolic subgroup of* M *such that* Q *has* L *as a Levi subgroup and υ is* Q*-dominant (it is standard that such a* Q *exists).*

(ii) Sending σ to the $\mathrm{M}(F)$-conjugacy class of (L, υ) as in (i) gives a finite-to-one map from $\mathrm{Irr}(\mathrm{M})$ *to the set of $\mathrm{M}(F)$-conjugacy classes of pairs (L, υ) with $\mathrm{L} \subset \mathrm{M}$ a Levi subgroup and $\upsilon \in \mathrm{Irr}_2^+(\mathrm{L})$. Thus, we get a finite-to-one surjective map*

$$\mathrm{Irr}(\mathrm{M}) \to \bigsqcup_{\mathrm{L}} \mathrm{Irr}_2^+(\mathrm{L})/W_{\mathrm{M}}(\mathrm{L}), \tag{4}$$

where L *runs over any set of representatives for the $\mathrm{M}(F)$-conjugacy classes of Levi subgroups of* M *(and $W_{\mathrm{M}}(\mathrm{L})$ is as in Subsubsection 2.1.7).*

(iii) For a pair (L, υ), with L *occurring in* (4) *and $\upsilon \in \mathrm{Irr}_2^+(\mathrm{L})$, the fiber of* (4) *over the image of υ in $\mathrm{Irr}_2^+(\mathrm{L})/W_{\mathrm{L}}$ consists of all the irreducible* quotients *of $\mathrm{Ind}_{\mathrm{Q}}^{\mathrm{M}} \upsilon$, where* Q *is any choice of a parabolic subgroup of* M *such that* L *is a Levi subgroup of* Q *and υ is* Q*-dominant.*

Proof. We omit the proof, since it is well-known, and can be found in [ABPS14, Theorem 1.2]. Let us remark that the proof combines two ingredients, the first being [Wal03, Proposition III.4.1], which asserts the existence of a finite-to-one surjective map defined similarly as in (4):

$$\mathrm{Irr}_{\mathrm{temp}}(\mathrm{M}) \to \bigsqcup_{\mathrm{L}} \mathrm{Irr}_2(\mathrm{L})/W_{\mathrm{M}}(\mathrm{L}). \tag{5}$$

(5) is the restriction of (4) to $\mathrm{Irr}_{\mathrm{temp}}(\mathrm{M})$, and its fibers have a description similar to the one given for (4) in Proposition 2.2.3(iii). The second ingredient is the usual statement of Langlands classification (e.g., [SZ18, Theorem 1.4]). □

Remark 2.2.4. We now recall some easy facts about stable virtual characters that we will use. Let $\sum_{\sigma\in\Sigma} c_\sigma\Theta_\sigma$ be a stable virtual character on $\mathrm{M}(F)$, for some $\Sigma\subset\mathrm{Irr}(\mathrm{M})$.

(i) For any central subgroup $Z\subset\mathrm{M}(F)$ and any smooth character $\chi: Z\to\mathbb{C}^\times$, $\sum_{\sigma\in\Sigma_\chi} c_\sigma\Theta_\sigma$ is also a stable virtual character, where $\Sigma_\chi\subset\Sigma$ is the subset consisting of representations whose central character restricts to χ on Z.

(ii) For any isomorphism $\mathrm{M}\to\mathrm{M}'$ of reductive groups over F, and any smooth character χ' : $\mathrm{M}'(F)\to\mathbb{C}^\times$ that is trivial on $\mathrm{M}'_{\mathrm{der}}(F)$, $\sum_{\sigma\in\Sigma} c_\sigma\Theta_{(\sigma\circ\beta^{-1})\otimes\chi'} = ((\sum_{\sigma\in\Sigma} c_\sigma\Theta_\sigma)\circ\beta^{-1})\chi'$ is a stable virtual character on $\mathrm{M}'(F)$.

Remark 2.2.5. Later, we will have use for distributions on $\mathrm{M}(F)$ of the form:

$$\sum_{\sigma\in\mathrm{Irr}_2(\mathrm{M})_\zeta} c_\sigma\Theta_\sigma,$$

where $\zeta:\mathcal{Z}\to\mathbb{C}^\times$ is a smooth character of a central subgroup $\mathscr{Z}$ of $\mathrm{M}(F)$ containing $\mathrm{A}_{\mathrm{M}}(F)$, and $c_\sigma\in\mathbb{C}$ for each $\sigma\in\mathrm{Irr}_2(\mathrm{M})_\zeta$. We claim that such infinite sums makes sense, i.e., for each $f\in C_c^\infty(\mathrm{M}(F))$, or equivalently for each $f\in C_\zeta^\infty(\mathrm{M}(F))$, $\Theta_\sigma(f)=0$ for all but finitely many $\sigma\in\mathrm{Irr}_2(\mathrm{M})_\zeta$. This is easy to deduce from [Wal03, Theorem VIII.1.2] using standard facts, as observed in [MW16, Corollary XI.4.1].

2.3. Unramified characters.

Notation 2.3.1. Suppose P is a linear algebraic group over F, which is not necessarily reductive.

(i) We denote by S_{P} the maximal split torus quotient of P.

(ii) Recall that a character $\chi:\mathrm{P}(F)\to\mathbb{C}^\times$ is said to be unramified if $\chi(x)=1$ for all $x\in\mathrm{P}(F)$ such that $|\mu(x)|=1$ for all algebraic characters $\mu:\mathrm{P}\to\mathbb{G}_m$. We denote by $X^{\mathrm{unr}}(\mathrm{P})$ the group of unramified characters $\mathrm{P}(F)\to\mathbb{C}^\times$. Then $X^{\mathrm{unr}}(\mathrm{P})\subset\mathrm{Hom}_{\mathrm{cts}}(\mathrm{P}(F),\mathbb{C}^\times)$.

(iii) Let $X^{\mathrm{unr-uni}}(\mathrm{P})\subset X^{\mathrm{unr}}(\mathrm{P})$ be the subgroup of unitary characters, and $X^{\mathrm{unr}}(\mathrm{P})_{>0}\subset X^{\mathrm{unr}}(\mathrm{P})$ the subgroup consisting of characters taking values in the multiplicative group $\mathbb{R}_{>0}$ of positive real numbers.

(iv) $X^{\mathrm{unr}}(\mathrm{P})$ will be viewed as a complex torus in the usual way (see Remark 2.3.2(ii) below). The product map

$$X^{\mathrm{unr}}(\mathrm{P})_{>0}\times X^{\mathrm{unr-uni}}(\mathrm{P})\to X^{\mathrm{unr}}(\mathrm{P})$$

is easily seen to be an isomorphism, identifying $X^{\mathrm{unr}}(\mathrm{P})_{>0}$ with the set of hyperbolic elements of $X^{\mathrm{unr}}(\mathrm{P})$ in the sense of [SZ18, Section 5.1], and $X^{\mathrm{unr-uni}}(\mathrm{P})$ with the maximal compact subgroup of $X^{\mathrm{unr}}(\mathrm{P})$.

Remark 2.3.2. (i) Pull-back gives an isomorphism $X^{\mathrm{unr}}(\mathrm{M})\cong X^{\mathrm{unr}}(\mathrm{P})$, where M is the Levi quotient of P.

(ii) Recall that $X^{\mathrm{unr}}(\mathrm{S}_{\mathrm{P}})\to X^{\mathrm{unr}}(\mathrm{P})$ is surjective with finite kernel. Indeed, for surjectivity, combine the injectivity of the abelian group $\mathbb{C}^\times$ with the fact that for any $\chi\in X^{\mathrm{unr}}(\mathrm{P})$ and $p\in\mathrm{P}(F)$, $\chi(p)$ depends only on the image of p under $\mathrm{P}(F)\to\mathrm{S}_{\mathrm{P}}(F)\to\mathrm{S}_{\mathrm{P}}(F)/\mathrm{S}_{\mathrm{P}}(\mathfrak{O})$ (use that any algebraic character $\mu:\mathrm{P}\to\mathbb{G}_m$ factors through $\mathrm{P}\to\mathrm{S}_{\mathrm{P}}$). For the finiteness of the kernel, one immediately reduces to the case where $\mathrm{P}=\mathrm{M}$ is reductive, and notes that we have a chain of restriction maps $X^{\mathrm{unr}}(\mathrm{S}_{\mathrm{M}})\to X^{\mathrm{unr}}(\mathrm{M})\to X^{\mathrm{unr}}(\mathrm{A}_{\mathrm{M}})$, whose composite has finite kernel since $\mathrm{A}_{\mathrm{M}}\to\mathrm{S}_{\mathrm{M}}$ is an isogeny.

Since $X^{\mathrm{unr}}(\mathrm{S}_{\mathrm{P}})\cong\mathrm{Hom}(X_*(\mathrm{S}_{\mathrm{P}}),\mathbb{C}^\times)$ is a complex torus, it follows that so is $X^{\mathrm{unr}}(\mathrm{P})$. Clearly, $X^{\mathrm{unr}}(\mathrm{S}_{\mathrm{P}})\to X^{\mathrm{unr}}(\mathrm{P})$ restricts to an isomorphism $X^{\mathrm{unr}}(\mathrm{S}_{\mathrm{P}})_{>0}\to X^{\mathrm{unr}}(\mathrm{P})_{>0}$.

(iii) If $\chi\in\mathrm{Hom}_{\mathrm{cts}}(\mathrm{P}(F),\mathbb{C}^\times)$ is valued in $\mathbb{R}_{>0}$, then $\chi\in X^{\mathrm{unr}}(\mathrm{P})$ — assuming without loss of generality that P is reductive, and using that $\mathbb{R}_{>0}$ is torsion free, this follows from the well-known fact that the subgroup of $p\in\mathrm{P}(F)$ such that $|\mu(p)|=1$ for all algebraic characters $\mu:\mathrm{P}\to\mathbb{G}_m$, is generated by the compact subgroups of $\mathrm{P}(F)$ (e.g., [FP21, Lemma 4.8]).

2.4. **Groups of automorphisms 'up to which' we will work.** We will now fix our connected reductive group G over F, as well as a collection $\{\mathcal{O}_{\mathrm{M}}\}_{\mathrm{M}}$ indexed by Levi subgroups $\mathrm{M} \subset \mathrm{G}$ (subject the to some conditions), where each $\mathcal{O}_{\mathrm{M}}$ is a subgroup of $\mathrm{Aut}(\mathrm{M})$.

Notation 2.4.1. (i) For the rest of this paper, let G be a fixed connected reductive group over F.

(ii) We fix a collection $\{\mathcal{O}_{\mathrm{M}}\}_{\mathrm{M}}$ indexed by Levi subgroups $\mathrm{M} \subset \mathrm{G}$, where $\mathcal{O}_{\mathrm{M}} \subset \mathrm{Aut}(\mathrm{M})$ is a group of (F-rational algebraic) automorphisms of M, subject to the conditions of (iv) below; we will abbreviate $\mathcal{O}_{\mathrm{G}}$ to $\mathcal{O}$.

(iii) Given Levi subgroups $\mathrm{L}, \mathrm{L}_1, \mathrm{L}_2 \subset \mathrm{M} \subset \mathrm{G}$, we define groups $\mathcal{O}^+_{\mathrm{M}}(\mathrm{L}_2, \mathrm{L}_1), \mathcal{O}^+_{\mathrm{M},\mathrm{L}}$ and $\mathcal{O}^+_{\mathrm{M}}$ as follows:
  - We let $\mathcal{O}^+_{\mathrm{M}}(\mathrm{L}_2, \mathrm{L}_1)$ be the set of all $\beta|_{\mathrm{L}_1} : \mathrm{L}_1 \to \mathrm{L}_2$, as β runs over the elements of $\mathcal{O}_{\mathrm{M}} \circ \mathrm{Int}\,\mathrm{M}(F)$ (which we will soon abbreviate to $\mathcal{O}^+_{\mathrm{M}}$) such that $\beta(\mathrm{L}_1) = \mathrm{L}_2$. We abbreviate $\mathcal{O}^+_{\mathrm{M}}(\mathrm{L}, \mathrm{L})$ to $\mathcal{O}^+_{\mathrm{M},\mathrm{L}}$ and $\mathcal{O}^+_{\mathrm{M},\mathrm{M}} = \mathcal{O}_{\mathrm{M}} \circ \mathrm{Int}\,\mathrm{M}(F)$ to $\mathcal{O}^+_{\mathrm{M}}$;

(iv) We subject the collection $\{\mathcal{O}_{\mathrm{M}}\}_{\mathrm{M}}$ to the following conditions:
  (a) For each M, each element of $\mathcal{O}_{\mathrm{M}}$ acts as the identity on A_{M};
  (b) If $\mathrm{L}, \mathrm{M} \subset \mathrm{G}$ are Levi subgroups, and $\beta \in \mathcal{O}^+_{\mathrm{G}} = \mathcal{O}_{\mathrm{G}} \circ \mathrm{Int}\,\mathrm{G}(F)$ satisfies that $\beta(\mathrm{L}) \subset \mathrm{M}$, then under the map $\mathrm{Aut}(\mathrm{L}) \to \mathrm{Aut}(\beta(\mathrm{L}))$ given by transport by β, the image of $\mathcal{O}^+_{\mathrm{L}} = \mathcal{O}_{\mathrm{L}} \circ \mathrm{Int}\,\mathrm{L}(F)$ is contained in $\mathcal{O}^+_{\mathrm{M},\beta(\mathrm{L})}$ (so, as an important special case, $\mathcal{O}_{\mathrm{L}} \subset \mathcal{O}^+_{\mathrm{L}} \subset \mathcal{O}^+_{\mathrm{M},\mathrm{L}}$);
  (c) The image of $\mathcal{O}_{\mathrm{G}}$ in $\mathrm{Out}(\mathrm{G})$ is finite.

Remark 2.4.2. (i) An important example of a collection $\{\mathcal{O}_{\mathrm{M}}\}_{\mathrm{M}}$ as above is the one where each $\mathcal{O}_{\mathrm{M}}$ is trivial. We get another example by fixing any group $\mathcal{O}$ of automorphisms of G with finite image in $\mathrm{Out}(\mathrm{G})$, and taking $\mathcal{O}_{\mathrm{M}}$ to be the group of automorphisms of M induced by those elements of $\mathcal{O}^+_{\mathrm{G}} = \mathcal{O} \circ \mathrm{Int}\,\mathrm{G}(F)$ that act as the identity on Z_{M} (and hence preserve the centralizer M of Z_{M}).

(ii) The typical example we have in mind for a nontrivial collection $\{\mathcal{O}_{\mathrm{M}}\}$ is in the case where G is a quasi-split form of SO_{2n}, GSO_{2n} or GSpin_{2n}, with $\mathcal{O} = \mathcal{O}_{\mathrm{G}}$ a two element group of automorphisms of G, one of which is outer, and the $\mathcal{O}_{\mathrm{M}}$ as in the latter example of (i) above (i.e., consisting of those automorphisms of M induced by elements of $\mathcal{O}^+_{\mathrm{G}}$ that act as the identity on Z_{M}).

(iii) As (ii) suggests, the reason for introducing the collection $\{\mathcal{O}_{\mathrm{M}}\}_{\mathrm{M}}$ is to be able to make a weaker statement in cases where we don't have a 'canonical' collection of stable packets for the M as such, but only one up to the action of the $\mathcal{O}_{\mathrm{M}}$; this applies to the study of quasi-split forms of SO_{2n} in [Mg14], of GSO_{2n} in [Xu16] and [Xu18], and of GSpin_{2n} in [Mg14]. A reader who is not particular about cases of this sort may assume each $\mathcal{O}_{\mathrm{M}}$ to be trivial or to be simply the group of all inner conjugations $\mathrm{Int}\, m$ with $m \in \mathrm{M}(F)$, in which case a lot of the definitions and results below simplify.

(iv) For each Levi subgroup of $\mathrm{M} \subset \mathrm{G}$, $\mathcal{O}^+_{\mathrm{G},\mathrm{M}}$ contains the group of conjugations of M by the elements of the normalizer $\mathrm{N}_{\mathrm{G}}(\mathrm{M})(F)$ of $\mathrm{M}(F)$ in $\mathrm{G}(F)$. If $\mathcal{O}_{\mathrm{G}}$ is trivial, then this containment is easily checked to be an equality.

(v) We will only consider the action of $\mathcal{O}_{\mathrm{M}}$ on objects related to invariant harmonic analysis on M, so replacing $\mathcal{O}_{\mathrm{M}}$ by $\mathcal{O}^+_{\mathrm{M}}$ will not change any of the analysis that follows. The only reason we write $\mathcal{O}_{\mathrm{M}}$ instead of $\mathcal{O}^+_{\mathrm{M}}$ in what follows, is that it can be convenient to think of a finite group of automorphisms (which $\mathcal{O}^+_{\mathrm{M}}$ almost never is, while $\mathcal{O}_{\mathrm{M}}$ is allowed, though not required, to be trivial).

(vi) A lot of the time we will consider only the action of $\mathcal{O}_{\mathrm{M}}$ on objects such as a set of 'L-packets' or a set of (non-enhanced) Langlands parameters associated to M, so our dependence on $\mathcal{O}_{\mathrm{M}}$ will often, though not always, be only through its image in $\mathrm{Out}(\mathrm{M})$.

Lemma 2.4.3. *Let $\mathrm{L} \subset \mathrm{M} \subset \mathrm{G}$ be Levi subgroups, and let $\beta \in \mathcal{O}^+_{\mathrm{G}} = \mathcal{O} \circ \mathrm{Int}\,\mathrm{G}(F)$.*

(i) β transports $\mathcal{O}^+_{\mathrm{M},\mathrm{L}}$ isomorphically onto $\mathcal{O}^+_{\beta(\mathrm{M}),\beta(\mathrm{L})}$.

(ii) $\mathrm{Int}\,\mathrm{M}(F) \subset \mathcal{O}^+_{\mathrm{G},\mathrm{M}}$ is of finite index. Equivalently, the image of $\mathcal{O}^+_{\mathrm{G},\mathrm{M}}$ in $\mathrm{Out}(\mathrm{M})$ is finite.

(iii) *For any Levi subgroup* $\mathrm{M}' \subset \mathrm{G}$*, the collection* $\{\mathcal{O}_{\mathrm{L}'}\}_{\mathrm{L}'}$*, as* L' *varies over the Levi subgroups of* M'*, satisfies the analogues, for* M' *in place of* G*, of the hypotheses imposed on the collection* $\{\mathcal{O}_{\mathrm{M}}\}_{\mathrm{M}}$ *in Notation 2.4.1(iv).*

Proof. By (iv)b of Notation 2.4.1, β transports $\mathcal{O}^+_{\mathrm{M},\mathrm{M}} = \mathcal{O}^+_{\mathrm{M}}$ into $\mathcal{O}^+_{\beta(\mathrm{M}),\beta(\mathrm{M})} = \mathcal{O}^+_{\beta(\mathrm{M})}$. In doing so, it clearly transports the set $\mathcal{O}^+_{\mathrm{M},\mathrm{L}}$ of automorphisms in $\mathcal{O}^+_{\mathrm{M}}$ that preserve L to the set $\mathcal{O}^+_{\beta(\mathrm{M}),\beta(\mathrm{L})}$ of automorphisms in $\mathcal{O}^+_{\beta(\mathrm{M})}$ that preserve $\beta(\mathrm{L})$. By making a similar argument with β^{-1}, (i) follows. Since $\mathcal{O}^+_{\mathrm{G}}$ has finite image in $\mathrm{Out}(\mathrm{G})$, and $\mathrm{Int}\,\mathrm{G}(F)$ has finite index in $\mathrm{Int}\,\mathrm{G}_{\mathrm{ad}}(F)$, $\mathrm{Int}\,\mathrm{G}(F)$ has finite index in $\mathcal{O}^+_{\mathrm{G}}$. Thus, some finite-index subgroup of $\mathcal{O}^+_{\mathrm{G},\mathrm{M}}$ acts on M by restrictions of elements of $\mathrm{Int}\,\mathrm{G}(F)$. Since the normalizer of $\mathrm{M}(F)$ in $\mathrm{G}(F)$ has finite image in $\mathrm{Out}(\mathrm{M})$, some smaller finite-index subgroup of $\mathcal{O}^+_{\mathrm{G},\mathrm{M}}$ acts on M by elements of $\mathrm{Int}\,\mathrm{M}(F)$. From this, (ii) follows. Now (iii) is easy to verify (using (ii)). □

Lemma 2.4.4. *Let* M *be a Levi subgroup of* G*. Suppose* $\beta \in \mathrm{Aut}(\mathrm{G})$ *preserves* M *and restricts to an element of* $\mathcal{O}^+_{\mathrm{M}}$ *on it. Then for each parabolic subgroup* $\mathrm{P} \subset \mathrm{G}$ *with* M *as a Levi subgroup, we have* $\beta(\mathrm{P}) = \mathrm{P}$*.*

Proof. If $\mathrm{P} \subset \mathrm{G}$ is a parabolic subgroup with M as a Levi subgroup, there exists a homomorphism $\mu : \mathbb{G}_m \to \mathrm{A}_{\mathrm{M}}$ such that $\mathrm{Lie}\,\mathrm{P}$ is the subspace of $\mathrm{Lie}\,\mathrm{G}$ on which $\mathrm{Ad} \circ \mu$ acts by nonnegative weights. Hence the lemma follows from the fact that $\mathcal{O}^+_{\mathrm{M}}$ acts trivially on A_{M}. □

Lemma 2.4.5. *Suppose* M *is a Levi subgroup of* G*. Then the obvious actions of* $\mathcal{O}^+_{\mathrm{M}}$ *on* $X^{\mathrm{unr}}(\mathrm{M})$ *and* $X^{\mathrm{unr}}(\mathrm{S}_{\mathrm{M}})$ *are trivial.*

Proof. The assertion for $X^{\mathrm{unr}}(\mathrm{S}_{\mathrm{M}})$ follows from the fact that $\mathrm{A}_{\mathrm{M}} \to \mathrm{S}_{\mathrm{M}}$ is an isogeny, so that the elements of $\mathcal{O}^+_{\mathrm{M}}$ induce the identity automorphism of S_{M}. The assertion for $X^{\mathrm{unr}}(\mathrm{M})$ follows from the assertion for $X^{\mathrm{unr}}(\mathrm{S}_{\mathrm{M}})$, since the restriction map $X^{\mathrm{unr}}(\mathrm{S}_{\mathrm{M}}) \to X^{\mathrm{unr}}(\mathrm{M})$ is surjective and respects the action of $\mathcal{O}^+_{\mathrm{M}}$. □

2.5. *L*-packets from the point of view of stability of distributions.

2.5.1. *The main hypothesis for L-packets to be defined from the perspective of stability.* In much of what follows, Remark 2.4.2(v) can be helpful to keep in mind. Informally, the following hypothesis says that '$\mathcal{O}$-coarsened' tempered L-packets can be defined based on the notion of stability of distributions (see also [Sha90, Section 9], and the notion of being unitarily stable in [MY20, Section 4]).

Hypothesis 2.5.1 (Existence of tempered L-packets)**.** For each Levi subgroup M of G, there exists a collection $\Phi_2(\mathrm{M})$ of finite subsets of $\mathrm{Irr}_2(\mathrm{M})$ partitioning it, and a virtual character Θ_Σ for each $\Sigma \in \Phi_2(\mathrm{M})$, such that the following two properties are satisfied:

(i) For each $\Sigma \in \Phi_2(\mathrm{M})$, Θ_Σ is a nonzero stable $\mathcal{O}_{\mathrm{M}}$-invariant (or equivalently, $\mathcal{O}^+_{\mathrm{M}}$-invariant) virtual character on $\mathrm{M}(F)$ of the form $\sum_{\sigma\in\Sigma} c_\sigma \Theta_\sigma$ (thus, Σ is $\mathcal{O}^+_{\mathrm{M}}$-invariant as well).

(ii) $\{\Theta_\Sigma \mid \Sigma \in \Phi_2(\mathrm{M})\}$ is a complex vector space basis for the subspace $SD_{\mathrm{ell}}(\mathrm{M})^{\mathcal{O}_{\mathrm{M}}} = SD_{\mathrm{ell}}(\mathrm{M})^{\mathcal{O}^+_{\mathrm{M}}}$ of $SD_{\mathrm{ell}}(\mathrm{M})$ fixed by $\mathcal{O}_{\mathrm{M}}$ or equivalently by $\mathcal{O}^+_{\mathrm{M}}$.

Proposition 2.5.2. *Suppose Hypothesis 2.5.1 is satisfied. Then* Θ_Σ *is a multiple of* $\sum_{\sigma\in\Sigma} d(\sigma)\Theta_\sigma$*, where* $d(\sigma)$ *stands for the formal degree of* σ*. In particular, if* $\Theta_\Sigma = \sum_{\sigma\in\Sigma} c_\sigma\Theta_\sigma$*, then* $c_\sigma \neq 0$ *for each* $\sigma \in \Sigma$*.*

Proof. This is proved exactly as in Proposition 3.3.6(ii) below, and in fact follows from it. Note that, while we do not prove Proposition 3.3.6(ii) either, its proof is an easier variant of the proof of Proposition 3.3.7(ii) below it. □

Lemma 2.5.3. *Assume Hypothesis 2.5.1. Let* $\mathrm{M} \subset \mathrm{G}$ *be a Levi subgroup.*

(i) $\Phi_2(\mathrm{M})$ *can be described as the set of* $\Sigma \subset \mathrm{Irr}_2(\mathrm{M})$ *satisfying the following condition: there exists a nonzero stable* $\mathcal{O}_{\mathrm{M}}$*-invariant virtual character* Θ'_Σ *supported on* Σ*, with the property that every stable* $\mathcal{O}_{\mathrm{M}}$*-invariant virtual character* $\Theta \in SD_{\mathrm{ell}}(\mathrm{M})^{\mathcal{O}_{\mathrm{M}}}$ *can be uniquely* *written in the form* $c_1\Theta'_\Sigma + c_2\Theta'$ *for a (automatically stable and* $\mathcal{O}_{\mathrm{M}}$*-invariant) virtual character* Θ' *supported outside* Σ*, and complex numbers* c_1, c_2*.*

(ii) *If* $\beta \in \mathcal{O}_{\mathrm{G}}^+$ *and* $\mathrm{M}' = \beta(\mathrm{M})$*, and* $\chi' : \mathrm{M}'(F) \to \mathbb{C}^\times$ *is a smooth unitary character on which* $\mathcal{O}_{\mathrm{M}'}$ *acts trivially (this is automatic if* χ *is unramified, by Lemma 2.4.5), then we have a bijection* $\Phi_2(\mathrm{M}) \to \Phi_2(\mathrm{M}')$ *sending each* $\Sigma \in \Phi_2(\mathrm{M})$ *to* $\Sigma' := \{(\sigma \circ \beta^{-1}) \otimes \chi' \mid \sigma \in \Sigma\}$*. Moreover,* $\Theta_{\Sigma'}$ *is a scalar multiple of* $(\Theta_\Sigma \circ \beta^{-1})\chi'$*.*

Remark 2.5.4. (i) of the lemma implies that, when Hypothesis 2.5.1 is satisfied, $\Phi_2(\mathrm{M})$ is uniquely determined, and not a choice made along with assuming the hypothesis. On the other hand, each Θ_Σ is uniquely determined up to a nonzero scalar.

Proof of Lemma 2.5.3. (i) is immediate, but it needs that $c_\sigma \neq 0$ whenever $\Sigma \in \Phi_2(\mathrm{M})$ and $\sigma \in \Sigma$ (to prevent proper subsets of such a Σ from satisfying the condition of (i)), a consequence of Proposition 2.5.2. (ii) follows from (i) and the fact that β transports $\mathcal{O}_{\mathrm{M}}^+$ to $(\mathcal{O}_{\mathrm{M}'})^+$ (see Lemma 2.4.3(i)) and $SD_{\mathrm{ell}}(\mathrm{M})$ to $SD_{\mathrm{ell}}(\mathrm{M}')$, etc. □

We will now define analogous sets $\Phi(\mathrm{M}), \Phi_{\mathrm{temp}}(\mathrm{M})$ and $\Phi_2^+(\mathrm{M})$, using the Langlands classification of Proposition 2.2.3, or rather its corollary in the form of the following $\mathcal{O}_{\mathrm{M}}$-invariant version:

Corollary 2.5.5. *Let* $\mathrm{M} \subset \mathrm{G}$ *be a Levi subgroup.*

(i) *The map* (4) *induces a finite-to-one surjective map*

$$\mathrm{Irr}(\mathrm{M})/\mathcal{O}_{\mathrm{M}} = \mathrm{Irr}(\mathrm{M})/\mathcal{O}_{\mathrm{M}}^+ \to \bigsqcup_{\mathrm{L}} \mathrm{Irr}_2^+(\mathrm{L})/\mathcal{O}_{\mathrm{M,L}}^+, \tag{6}$$

where the L *runs over a set of representatives for the* $\mathcal{O}_{\mathrm{M}}^+$*-orbits of Levi subgroups of* M*. It restricts to an analogously defined map:*

$$\mathrm{Irr}_{\mathrm{temp}}(\mathrm{M})/\mathcal{O}_{\mathrm{M}} = \mathrm{Irr}_{\mathrm{temp}}(\mathrm{M})/\mathcal{O}_{\mathrm{M}}^+ \to \bigsqcup_{\mathrm{L}} \mathrm{Irr}_2(\mathrm{L})/\mathcal{O}_{\mathrm{M,L}}^+. \tag{7}$$

(ii) *For a pair* (L, υ)*, with* L *occurring in* (6) *and* $\upsilon \in \mathrm{Irr}_2^+(\mathrm{L})$*, the fiber of* (6) *over the image of* υ *in* $\mathrm{Irr}_2^+(\mathrm{L})/\mathcal{O}_{\mathrm{M,L}}^+$ *is the union of the* $\mathcal{O}_{\mathrm{M}}^+$*-orbits (or equivalently the* $\mathcal{O}_{\mathrm{M}}$*-orbits) of the irreducible* quotients *of* $\mathrm{Ind}_{\mathrm{Q}}^{\mathrm{M}} \upsilon$*, where* Q *is a choice as in Proposition 2.2.3(iii). An analogous description applies to the fibers of* (7)*.*

Proof. One gets (i) from Proposition 2.2.3(ii) simply by quotienting with $\mathcal{O}_{\mathrm{M}}^+ \supset \mathrm{Int}\,\mathrm{M}(F)$. The finite-to-one-ness follows from the fact that $\mathrm{Int}\,\mathrm{M}(F) \subset \mathcal{O}_{\mathrm{M}}^+$ is of finite index (see Lemma 2.4.3(ii)). (ii) is an immediate consequence of Proposition 2.2.3(iii). □

Notation 2.5.6. Henceforth, whenever Hypothesis 2.5.1 is satisfied, in addition to fixing the Θ_Σ as in it (the $\Phi_2(\mathrm{M})$ being automatically fixed — see Remark 2.5.4), we also define the following objects:

(i) For each Levi subgroup $\mathrm{M} \subset \mathrm{G}$, we define $\Phi_2^+(\mathrm{M}) = \{\Sigma \otimes \chi \mid \Sigma \in \Phi_2(\mathrm{M}), \chi \in X^{\mathrm{unr}}(\mathrm{M})\}$.

(ii) If $\mathrm{L} \subset \mathrm{M} \subset \mathrm{G}$ are Levi subgroups and $\Upsilon \in \Phi_2^+(\mathrm{L})$ (as defined in (i)), then we let Υ^{M} be the preimage, under (6), of the image of Υ in $\mathrm{Irr}_2^+(\mathrm{L})/\mathcal{O}_{\mathrm{M,L}}^+$: here we assume without loss of generality that L occurs on the right-hand side of (6). In other words, Υ^{M} is the collection of the $\mathcal{O}_{\mathrm{M}}$-conjugates of the irreducible quotients of $\mathrm{Ind}_{\mathrm{Q}}^{\mathrm{M}} \upsilon$, where υ runs over Υ and Q is as in Proposition 2.2.3(iii): note that the same Q works for all $\upsilon \in \Upsilon$. If $\Upsilon \in \Phi_2(\mathrm{L})$, then by unitarity, we can replace the word 'quotients' by 'subquotients' in the previous sentence.

(iii) For $\mathrm{M} \subset \mathrm{G}$ a Levi subgroup, we let $\Phi(\mathrm{M})$ (resp., $\Phi_{\mathrm{temp}}(\mathrm{M})$) be the set of all Υ^{M} as (L, Υ) ranges over pairs consisting of a Levi subgroup $\mathrm{L} \subset \mathrm{M}$ and $\Upsilon \in \Phi_2^+(\mathrm{L})$ (resp., $\Upsilon \in \Phi_2(\mathrm{L})$).

(iv) If $\Sigma \in \Phi_{\mathrm{temp}}(\mathrm{M})$, choosing (L, Υ) such that $\Sigma = \Upsilon^{\mathrm{M}}$, we let $\Theta_\Sigma = \mathrm{Avg}_{\mathcal{O}_{\mathrm{M}}}(\mathrm{Ind}_{\mathrm{L}}^{\mathrm{M}} \Theta_\Upsilon)$, where $\mathrm{Avg}_{\mathcal{O}_{\mathrm{M}}}$ refers to averaging with respect to the action of $\mathcal{O}_{\mathrm{M}}$ (which acts through the finite quotient $\mathcal{O}_{\mathrm{M}}^+/\mathrm{Int}\,\mathrm{M}(F)$). Note that Θ_Σ, which is a virtual character supported on Σ, is well-defined, since (L, Υ) is well-defined up to $\mathcal{O}_{\mathrm{M}}^+$-conjugation by (6).

Lemma 2.5.7. *Let* $\mathrm{M} \subset \mathrm{G}$ *be a Levi subgroup.*

(i) *$\Phi(\mathrm{M}), \Phi_2^+(\mathrm{M})$ and $\Phi_{\mathrm{temp}}(\mathrm{M})$ consist of $\mathcal{O}_{\mathrm{M}}^+$-invariant sets, and they are partitions of* $\mathrm{Irr}(\mathrm{M}), \mathrm{Irr}_2^+(\mathrm{M})$ *and* $\mathrm{Irr}_{\mathrm{temp}}(\mathrm{M})$*, respectively.*

(ii) Θ_Σ *is well-defined for each* $\Sigma \in \Phi_{\mathrm{temp}}(\mathrm{M})$*, and the collection of the* Θ_Σ *forms a basis for* $SD(\mathrm{M})^{\mathcal{O}_{\mathrm{M}}}$.

Proof. Since every element of $\mathrm{Irr}_2^+(\mathrm{M})$ can be written as $\sigma\otimes\chi$ with $\sigma \in \mathrm{Irr}_2(\mathrm{M})$ and $\chi \in X^{\mathrm{unr}}(\mathrm{M})$, it is immediate that the union of the $\Phi_2^+(\mathrm{M})$ equals $\mathrm{Irr}_2^+(\mathrm{M})$. By Lemma 2.4.5, each element of $\Phi_2^+(\mathrm{M})$ is also $\mathcal{O}_{\mathrm{M}}^+$-invariant. If $\Sigma_1, \Sigma_2 \in \Phi_2^+(\mathrm{M})$, $\chi_1, \chi_2 \in X^{\mathrm{unr}}(\mathrm{M})$ and $\Sigma_1\otimes\chi_1$ intersects $\Sigma_2\otimes\chi_2$, then $\chi_1\chi_2^{-1} \in X^{\mathrm{unr}}(\mathrm{M})$ restricts to a unitary character on $\mathrm{Z}_{\mathrm{M}}(F)$ and is hence unitary, so that $\Sigma_1 \otimes \chi_1 = \Sigma_2 \otimes \chi_2$ by Lemma 2.5.3(ii).
Thus, we have proved the assertion of (i) for $\Phi_2^+(\mathrm{M})$. Applying this with M replaced by various Levi subgroups $\mathrm{L} \subset \mathrm{M}$, the assertion of (i) for $\Phi(\mathrm{M})$ (resp., $\Phi_{\mathrm{temp}}(\mathrm{M})$) then follows from (6) (resp., (7)) and the fact that the elements of $\mathcal{O}_{\mathrm{M,L}}^+$ permute $\Phi_2^+(\mathrm{L})$ (resp., $\Phi_2(\mathrm{L})$), by Lemma 2.5.3(ii).
(ii) follows from Proposition 3.2.8 later below, applied with M in place of G, and the corresponding restriction of the collection $\{\mathcal{O}_{\mathrm{M}}\}_{\mathrm{M}}$ (as justified by Lemma 2.4.3(iii)). □

Remark 2.5.8. Thus, Hypothesis 2.5.1 also has the consequence that each $SD(\mathrm{M})^{\mathcal{O}_{\mathrm{M}}}$ and in particular $SD(\mathrm{G})^{\mathcal{O}_{\mathrm{G}}} = SD(\mathrm{G})^{\mathcal{O}}$, has a basis consisting of virtual characters whose supports are pairwise disjoint and together exhaustive. By the same argument as in Lemma 2.5.3(i), the elements of such a basis are uniquely determined up to scaling, and hence $\Phi_{\mathrm{temp}}(\mathrm{M})$ has an alternate characterization as in Lemma 2.5.3(i).

Notation 2.5.9. Henceforth, for any Levi subgroup $\mathrm{M} \subset \mathrm{G}$, the elements of $\Phi_2(\mathrm{M})$ (resp., $\Phi_{\mathrm{temp}}(\mathrm{M})$) will be referred to as the discrete series L-packets (resp., tempered L-packets) on $\mathrm{M}(F)$ up to the action of $\mathcal{O}_{\mathrm{M}}$ in the sense of Hypothesis 2.5.1.

Remark 2.5.10. Suppose that Hypothesis 2.5.1 is satisfied with the collection $\{\mathcal{O}_{\mathrm{M}}\}_{\mathrm{M}}$ replaced by a collection $\{\mathcal{O}'_{\mathrm{M}}\}_{\mathrm{M}}$ satisfying analogous conditions, where $\mathcal{O}'_{\mathrm{M}}$ is a normal subgroup of $\mathcal{O}_{\mathrm{M}}$ for each Levi subgroup $\mathrm{M} \subset \mathrm{G}$. Since we may assume that $c_\sigma > 0$ for each $\sigma \in \Sigma$ by Proposition 2.5.2, it is easy to see by averaging and using the idea of the proof of Lemma 2.5.3, that Hypothesis 2.5.1 is satisfied (without replacing $\{\mathcal{O}_{\mathrm{M}}\}_{\mathrm{M}}$ by $\{\mathcal{O}'_{\mathrm{M}}\}_{\mathrm{M}}$).

3. Some results on stable virtual characters and unitarily stable packets

3.1. **Elliptic characters and endoscopic transfer.**

We will typically assume the three hypotheses stated in [MW16, Section I.1.5]:

Notation 3.1.1. Let $(\mathrm{M}, \tilde{\mathrm{M}}, \mathbf{a})$ be a triple where $(\mathrm{M}, \tilde{\mathrm{M}})$ is a twisted space (see Subsubsection 2.1.1), and $\mathbf{a}$ is a cocycle representing an element of $H^1(W_F, \mathrm{Z}_{\hat{\mathrm{M}}})$. Let $\omega : \mathrm{M}(F) \to \mathbb{C}^\times$ be the quasi-character associated to $\mathbf{a}$, which we assume to be unitary. The purpose of this notation is to record the following hypothesis (to be imposed later):

(i) $\tilde{\mathrm{M}}(F) \neq \emptyset$;
(ii) θ^* has finite order, where θ^* is the object constructed towards the end of [MW16, Section I.1.2], as an automorphism of 'the pinned Borel pair' attached to M.
(iii) ω is trivial on $\mathrm{Z}_{\tilde{\mathrm{M}}}(F)$ (else the theory is empty).

Notation 3.1.2. For a triple $(\mathrm{M}, \tilde{\mathrm{M}}, \mathbf{a})$ and the associated character $\omega : \mathrm{M}(F) \to \mathbb{C}^\times$ as in Notation 3.1.1, satisfying the hypotheses of that notation, we will often use the following notation:

(i) As in [MW16, I.4.11], $\mathcal{E}(\tilde{\mathrm{M}}, \mathbf{a})$ will denote the set of isomorphism classes of relevant elliptic endoscopic data for $(\tilde{\mathrm{M}}, \mathbf{a})$. If we simply write $\mathcal{E}(\mathrm{M})$, it will stand for the set $\mathcal{E}(\mathrm{M}, \mathbb{1})$, where M is thought of as a twisted space over itself with respect to left and right multiplication, and $\mathbb{1}$ stands for the zero element of $H^1(W_F, \mathrm{Z}_{\hat{\mathrm{M}}})$ (thus, $\mathcal{E}(\mathrm{M})$ consists of endoscopic data for standard, untwisted, endoscopy).
(ii) We will write a typical element of $\mathcal{E}(\tilde{\mathrm{M}}, \mathbf{a})$ or $\mathcal{E}(\mathrm{M})$ as $\underline{\mathrm{H}}$, and given such an $\underline{\mathrm{H}}$, write H for its underlying endoscopic group. This is an abuse of notation, since H does not determine $\underline{\mathrm{H}}$.

(iii) For each endoscopic datum $\underline{\mathrm{H}} = (\mathrm{H}, \mathcal{H}, \tilde{s}) \in \mathcal{E}(\tilde{\mathrm{M}}, \mathbf{a})$ (the notation is chosen as in [MW16, Section I.1.5] — we will recall more of it in a later section when it becomes necessary), we will denote by $(\mathrm{H}, \tilde{\mathrm{H}})$ the associated twisted space as in [MW16, Section I.1.7]; it has the property that for each $\gamma \in \tilde{\mathrm{H}}(\bar{F})$, $\mathrm{Int}\,\gamma$ is of the form $\mathrm{Int}\,h$ for some $h \in \mathrm{H}_{\mathrm{ad}}(\bar{F})$ (this is the meaning of 'est á torsion intérieure' in (3) of [MW16, Section I.1.7]). For each such $\underline{\mathrm{H}}$, we will also often choose some auxiliary data as in [MW16, Section I.2.1], but also satisfying the extra condition of [MW16, Section I.7.1, (3)] (which may be imposed as ω is unitary); these yield for us a 5-tuple $(\mathrm{H}_1 \to \mathrm{H}, \hat{\xi}_1, \tilde{\mathrm{H}}_1 \to \tilde{\mathrm{H}}, \mathrm{C}_1, \mu)$, where:
- $\mathrm{H}_1 \to \mathrm{H}$ is a z-extension, i.e., its kernel is an induced torus and the derived group of H_1 is simply connected;
- $\hat{\xi}_1$ will be recalled later when it becomes necessary;
- C_1 is the kernel of $\mathrm{H}_1 \to \mathrm{H}$ (and is hence an induced torus);
- $\mu : \mathrm{C}_1(F) \to \mathbb{C}^\times$ is a character (this is the λ_1 of [MW16, Section I.2.1]), which is unitary since we have imposed [MW16, Section I.7.1, (3)] (see towards the end of [MW16, Section I.7.1]);
- $\tilde{\mathrm{H}}_1$ is a twisted space with underlying group H_1, satisfying $\tilde{\mathrm{H}}_1(F) \neq \emptyset$, and the map $\tilde{\mathrm{H}}_1 \to \tilde{\mathrm{H}}$ is compatible in the obvious way with the homomorphism $\mathrm{H}_1 \to \mathrm{H}$.

Typically, when we make these choices, we will suppress the dependence of these objects on $\underline{\mathrm{H}}$ for lightness of notation.

(iv) There is a notion of endoscopic transfer of functions, which is a linear map from $C_c^\infty(\tilde{\mathrm{M}}(F))$ to the quotient of $C_\mu^\infty(\tilde{\mathrm{H}}_1(F))$ by the subspace consisting of the unstable functions in it, i.e., functions whose stable orbital integrals all vanish (see, e.g., [MW16, Section 1.2.4]). By [MW16, Corollary XI.5.1] (keeping in mind the convention from [MW16, Section XI.1] of calling an ω-representation just a representation), dual to this map is a map $\mathbf{T}_{\underline{\mathrm{H}}} : SD_\mu(\tilde{\mathrm{H}}_1) \to D(\tilde{\mathrm{M}}, \omega)$, restricting to a map $\mathbf{T}_{\underline{\mathrm{H}},\mathrm{ell}} : SD_{\mu,\mathrm{ell}}(\tilde{\mathrm{H}}_1) \to D_{\mathrm{ell}}(\tilde{\mathrm{M}}, \omega)$ (thus, one can show that pulling back under endoscopic transfer of functions takes $SD_\mu(\tilde{\mathrm{H}}_1)$ to $D(\tilde{\mathrm{M}}, \omega)$ and $SD_{\mu,\mathrm{ell}}(\tilde{\mathrm{H}}_1)$ to $D_{\mathrm{ell}}(\tilde{\mathrm{M}}, \omega)$). As explained around [LMW18, Section 4.4, (4)], the latter factors through the projection $SD_{\mu,\mathrm{ell}}(\tilde{\mathrm{H}}_1) \to SD_{\mu,\mathrm{ell}}(\tilde{\mathrm{H}}_1)_{\mathrm{Aut}(\underline{\mathrm{H}})}$ from $SD_{\mu,\mathrm{ell}}(\tilde{\mathrm{H}}_1)$ to its space of coinvariants for an action of a certain outer automorphism group $\mathrm{Aut}(\underline{\mathrm{H}})$ of H determined by the endoscopic datum $\underline{\mathrm{H}} \in \mathcal{E}(\tilde{\mathrm{M}}, \omega)$ (this group $\mathrm{Aut}(\underline{\mathrm{H}})$ is recalled in [MW16, I.1.5]), and these add together to give us an isomorphism of complex vector spaces :

$$\bigoplus_{\underline{\mathrm{H}} \in \mathcal{E}(\tilde{\mathrm{M}}, \mathbf{a})} \mathbf{T}_{\underline{\mathrm{H}}} = \bigoplus_{\underline{\mathrm{H}} \in \mathcal{E}(\tilde{\mathrm{M}}, \mathbf{a})} \mathbf{T}_{\underline{\mathrm{H}},\mathrm{ell}} : \bigoplus_{\underline{\mathrm{H}} \in \mathcal{E}(\tilde{\mathrm{M}}, \mathbf{a})} SD_{\mu,\mathrm{ell}}(\tilde{\mathrm{H}}_1)_{\mathrm{Aut}(\underline{\mathrm{H}})} \to D_{\mathrm{ell}}(\tilde{\mathrm{M}}, \omega). \tag{8}$$

For $\Theta \in D_{\mathrm{ell}}(\tilde{\mathrm{M}}, \omega)$ and $\underline{\mathrm{H}} \in \mathcal{E}(\tilde{\mathrm{M}}, \mathbf{a})$, we will let $\Theta^{\underline{\mathrm{H}}}$ denote the component of Θ along the subspace of $D_{\mathrm{ell}}(\tilde{\mathrm{M}}, \omega)$ obtained from the contribution of $\underline{\mathrm{H}}$ in the above decomposition of $D_{\mathrm{ell}}(\tilde{\mathrm{M}}, \omega)$.

Remark 3.1.3. We emphasize that, in (8), each $\underline{\mathrm{H}}$ contributes a different 'μ', i.e., the 'μ' of $SD_{\mu,\mathrm{ell}}(\tilde{\mathrm{H}}_1)_{\mathrm{Aut}(\underline{\mathrm{H}})}$ depends on $\underline{\mathrm{H}}$ as well. This dependence is suppressed from notation for lightness.

Remark 3.1.4. Fix $(\mathrm{M}, \tilde{\mathrm{M}}, \mathbf{a}), \omega, \underline{\mathrm{H}}$ and $(\mathrm{H}_1 \to \mathrm{H}, \hat{\xi}_1, \tilde{\mathrm{H}}_1 \to \tilde{\mathrm{H}}, \mathrm{C}_1, \mu)$ as in Notation 3.1.2, except that we do not yet assume that $\underline{\mathrm{H}}$ is elliptic.

(i) Suppose $\underline{\mathrm{H}}$ is elliptic and relevant. Let $(\mathrm{L}_{\mathrm{H}}, \tilde{\mathrm{L}}_{\mathrm{H}})$ be a Levi subspace of $(\mathrm{H}, \tilde{\mathrm{H}})$, and $(\mathrm{L}_1, \tilde{\mathrm{L}}_1)$ its inverse image in $(\mathrm{H}_1, \tilde{\mathrm{H}}_1)$. We now state the compatibility between parabolic induction and endoscopic transfer as follows.
- If $\mathrm{L}_{\mathrm{H}} \subset \mathrm{H}$ is not relevant in the sense described in [MW16, Section I.3.4], then under the endoscopic transfer map $SD_\mu(\tilde{\mathrm{H}}_1) \to D(\tilde{\mathrm{M}}, \omega)$, the image of any virtual character parabolically induced from $\tilde{\mathrm{L}}_1$ is zero.
- Suppose L_{H} is relevant in the sense described in [MW16, Section I.3.4]. Thus, [MW16, Section I.3.4] constructs a Levi subspace $(\mathrm{L}, \tilde{\mathrm{L}}) \subset (\mathrm{M}, \tilde{\mathrm{M}})$ and an elliptic relevant endoscopic datum $\underline{\mathrm{L}}_{\mathrm{H}}$ for $(\tilde{\mathrm{L}}, \mathbf{a}_{\tilde{\mathrm{L}}})$ with underlying group L_{H}, where $\mathbf{a}_{\tilde{\mathrm{L}}}$ is a cocycle representing the image of $\mathbf{a}$ in $H^1(W_F, \mathrm{Z}_{\hat{\mathrm{L}}})$. Then $(\mathrm{L}_1, \tilde{\mathrm{L}}_1, \mu)$ is part of a choice

of auxiliary data for $\underline{\mathrm{L}}_{\mathrm{H}}$ obtained from those for $\underline{\mathrm{H}}$, as discussed in [MW16, Section I.3.3 or Section I.3.4]. For any virtual character $\Theta_1 \in SD_\mu(\tilde{\mathrm{L}}_1)$ parabolically inducing to $\mathrm{Ind}_{\tilde{\mathrm{L}}_1}^{\tilde{\mathrm{H}}_1} \Theta_1 =: \Theta_1^{\tilde{\mathrm{H}}_1} \in D_\mu(\tilde{\mathrm{H}}_1)$ and endoscopically transferring via $\underline{\mathrm{L}}_{\mathrm{H}}$ to $\Theta \in D(\tilde{\mathrm{L}}, \omega|_{\mathrm{L}(F)})$, $\Theta_1^{\tilde{\mathrm{H}}_1}$ belongs to $SD_\mu(\tilde{\mathrm{H}}_1)$, and its endoscopic transfer to $D(\tilde{\mathrm{M}}, \omega)$ under $\underline{\mathrm{H}}$ equals the parabolically induced character $\mathrm{Ind}_{\tilde{\mathrm{L}}}^{\tilde{\mathrm{M}}} \Theta =: \Theta^{\tilde{\mathrm{M}}}$.

These assertions are present in [MW16, Section I.4.11]. Slightly more precisely, recalling that parabolic induction is dual to the 'constant term' map, and using the assertion from the discussion below [MW16, Proposition I.4.11] that the image of the map [MW16, I.4.11(4)] is contained in the space denoted $I_+^{\mathcal{E}}(\tilde{G}(F), \omega)$ there, the former (resp., the latter) assertion follows from the condition (3) (resp., the condition (2)) in the definition of $I_+^{\mathcal{E}}(\tilde{G}(F), \omega)$ given at the beginning of [MW16, Section I.4.11].

(ii) For simplicity, we now assume that we are in the situation of standard endoscopy, and suppose that $\underline{\mathrm{H}}$ is not elliptic, i.e., the obvious injection $\mathrm{Z}_{\hat{\mathrm{M}}}^{\Gamma,0} \to \mathrm{Z}_{\hat{\mathrm{H}}}^{\Gamma,0}$ is not bijective. Thus, $\dim \mathrm{A}_{\mathrm{H}} > \dim \mathrm{A}_{\mathrm{M}}$, and it is easy to see that no elliptic strongly regular semisimple element of $\mathrm{M}(F)$ matches any semisimple element of $\mathrm{H}(F)$. Therefore, the image of the endoscopic transfer map $SD_\mu(\mathrm{H}_1) \to D(\mathrm{M})$ consists of virtual characters that vanish on the set $\mathrm{M}(F)_{\mathrm{ell}}$ of elliptic strongly regular elements of $\mathrm{M}(F)$. This implies (using a standard fact, (11) below) that this image is contained in the span of virtual characters that are fully induced from proper Levi subgroups of M.

3.2. **Unstable functions and stable characters on a non-quasi-split group.** Many of the descriptions concerning $SD(\mathrm{G})$ in [Art96] are only given explicitly in the case where G is quasi-split. The purpose of this subsection is to explain that similar descriptions for non-quasi-split G are implicitly present in [Art96], subsumed into Arthur's formalism concerning the stabilization of the trace formula.

Notation 3.2.1. (i) In this subsection, given a connected reductive group M over F, we will denote by M^* its quasi-split inner form, and implicitly fix an inner twist $\psi_{\mathrm{M}^*} : \mathrm{M}^*_{\bar{F}} \to \mathrm{M}_{\bar{F}}$ from M^* to M unless otherwise specified. Note that ψ_{M^*} fixes an identification ${}^L\mathrm{M}^* = {}^L\mathrm{M}$, helping realize M^* as the endoscopic group underlying some $\underline{\mathrm{M}}^* \in \mathcal{E}(\mathrm{M})$, which is uniquely determined up to isomorphism. For $\underline{\mathrm{H}} = \underline{\mathrm{M}}^*$, we may and shall assume that the associated auxiliary data as in Notation 3.1.2(iii) satisfy $\mu = 1, \tilde{\mathrm{H}} = \mathrm{H}$ and $\tilde{\mathrm{H}}_1 = \mathrm{H}_1$, and identify $C_\mu^\infty(\mathrm{H}_1(F))$ with $C_c^\infty(\mathrm{M}^*(F))$, $SD_{\mu,\mathrm{ell}}(\tilde{\mathrm{H}}_1)$ with $SD_{\mathrm{ell}}(\mathrm{M}^*)$ etc. When we talk of endoscopic transfer between M and M^* (i.e., between $C_c^\infty(\mathrm{M}(F))$ and $C_c^\infty(\mathrm{M}^*(F))$ or the pull-back from $SD(\mathrm{M}^*)$ to $D(\mathrm{M})$), the reference will be to such a fixed endoscopic datum.

(ii) It is easy to see that the inner twist ψ_{M^*} fixed in (i) above identifies $\mathrm{A}_{\mathrm{M}^*}$, $\mathrm{Z}_{\mathrm{M}^*}$, $\mathrm{S}_{\mathrm{M}^*}$, $X^{\mathrm{unr}}(\mathrm{M}^*)$, $X^{\mathrm{unr-uni}}(\mathrm{M}^*)$ etc. with $\mathrm{A}_{\mathrm{M}}, \mathrm{Z}_{\mathrm{M}}, \mathrm{S}_{\mathrm{M}}, X^{\mathrm{unr}}(\mathrm{M}), X^{\mathrm{unr-uni}}(\mathrm{M})$ etc. We will use this to transfer central characters, unramified characters etc. between $\mathrm{M}^*(F)$ and $\mathrm{M}(F)$.

(iii) Sometimes we will consider a 'variant with central character' of these notions: if $\mathrm{Z} \subset \mathrm{M}$ is a central subgroup and $\zeta : \mathrm{Z}(F) \to \mathbb{C}^\times$ is a unitary character, then endoscopic transfer also defines a map from $C_{\mathrm{Z},\zeta}^\infty(\mathrm{M}(F))$ to the quotient of $C_{\mathrm{Z},\zeta}^\infty(\mathrm{M}^*(F))$ by its subspace of unstable functions, where these function spaces are as in Notation 2.1.1(iv), and where Z is also viewed as a central subgroup of M^* as described in (ii) above. The map $SD_{\mathrm{Z},\zeta}(\mathrm{M}^*) \to SD_{\mathrm{Z},\zeta}(\mathrm{M})$ dual to this transfer (between $C_{\mathrm{Z},\zeta}^\infty(\mathrm{M}(F))$ and $C_{\mathrm{Z},\zeta}^\infty(\mathrm{M}^*(F))$) is also obtained by restricting the dual map $SD(\mathrm{M}^*) \to SD(\mathrm{M})$ for the transfer between $C_c^\infty(\mathrm{M}(F))$ and $C_c^\infty(\mathrm{M}^*(F))$.

(iv) Sometimes, we will give $\mathrm{M}(F)$ and $\mathrm{M}^*(F)$ measures that are compatible in the sense explained in [Kot88, page 631]: this means that, for some scalar $c > 0$ and some top-degree differential form ω on M defined over F, these measures are $c|\omega|$ and $c|(\psi_{\mathrm{M}^*})^*(\omega)|$.

(v) There is an injection from the set of $\mathrm{M}(F)$-conjugacy classes of Levi subgroups of M to the set of $\mathrm{M}^*(F)$-conjugacy classes of Levi subgroups of M^*, under which the conjugacy class of $\mathrm{M}_1 \subset \mathrm{M}$ maps to that of $\mathrm{M}_1^* \subset \mathrm{M}^*$ if and only if $\psi_{\mathrm{M}^*}((\mathrm{M}_1^*)_{\bar{F}})$ is $\mathrm{M}(\bar{F})$-conjugate to $(\mathrm{M}_1)_{\bar{F}}$, or equivalently, M_1^* and M_1 correspond to the same conjugacy class of Levi subgroups of

${}^L\mathrm{M}^* = {}^L\mathrm{M}$ (the identification ${}^L\mathrm{M}^* = {}^L\mathrm{M}$ is obtained from ψ_{M^*} or equivalently from $\underline{\mathrm{M}}^*$). Here, to make sense of the former description of this injection, we use Solleveld's result that conjugacy of Levi subgroups may be checked after base-changing to $\bar{F}$ (see [Sol20, Theorem A]). A Levi subgroup $\mathrm{M}_1^* \subset \mathrm{M}^*$ is said to be $\underline{\mathrm{M}}^*$-relevant if its conjugacy class lies in the image of this map; this agrees with the notion of relevance from [MW16, Section I.3.4], that we used earlier.

(vi) Now let $\mathrm{M}_1 \subset \mathrm{M}$ be a Levi subgroup, and consider inner twists ψ^* in $\psi_{\mathrm{M}^*} \circ \operatorname{Int} \mathrm{M}^*(\bar{F}) = \operatorname{Int} \mathrm{M}(\bar{F}) \circ \psi_{\mathrm{M}^*}$ such that $(\psi^*)^{-1}$ takes, for some or equivalently any parabolic subgroup $\mathrm{Q} \subset \mathrm{M}$ with Levi subgroup M_1, $(\mathrm{Q}_{\bar{F}}, (\mathrm{M}_1)_{\bar{F}})$ to $(\mathrm{Q}^*_{\bar{F}}, (\mathrm{M}^*_1)_{\bar{F}})$ for some parabolic-Levi pair $(\mathrm{Q}^*, \mathrm{M}^*_1)$ in M^*: to see that this condition is independent of Q, note that these are precisely the inner twists $\psi^* \in \psi_{\mathrm{M}^*} \circ \operatorname{Int} \mathrm{M}^*(\bar{F})$, that satisfy the property that ${}^\sigma\psi^* \circ (\psi^*)^{-1} \in \operatorname{Int} \mathrm{M}_1(\bar{F})$ for all $\sigma \in \operatorname{Gal}(\bar{F}/F)$, and hence satisfy the same property with Q replaced by any other parabolic subgroup $\mathrm{Q}' \subset \mathrm{M}$ with M_1 as a Levi subgroup. Given any such inner twist ψ^*, $(\psi^*)^{-1}((\mathrm{M}_1)_{\bar{F}})$ is of the form $(\mathrm{M}^*_1)_{\bar{F}}$ for some Levi subgroup $\mathrm{M}^*_1 \subset \mathrm{M}^*$. Thus, any such ψ^* restricts to an inner twist $\psi_{\mathrm{M}^*_1}$ from such an M^*_1 to M_1, realizing an endoscopic datum $\underline{\mathrm{M}}^*_1$ for M_1 with M^*_1 as the unerlying group. Here is a second way to describe the resulting identification ${}^L\mathrm{M}_1 = {}^L\mathrm{M}^*_1$ up to $\operatorname{Int} \hat{\mathrm{M}}_1$-conjugacy. We can choose parabolic subgroups $\mathrm{Q} \subset \mathrm{M}$ and $\mathrm{Q}^* \subset \mathrm{M}^*$ with $\psi^*(\mathrm{Q}^*_{\bar{F}}) = \mathrm{Q}_{\bar{F}}$, so that Q and Q^* correspond to the conjugacy class of a common parabolic subgroup $\mathcal{Q} \subset {}^L\mathrm{M} = {}^L\mathrm{M}^*$. Choosing a Levi subgroup $\mathcal{L} \subset \mathcal{Q}$, we get using the pairs $(\mathrm{Q}, \mathrm{M}_1), (\mathrm{Q}^*, \mathrm{M}^*_1)$ and $(\mathcal{Q}, \mathcal{L})$ embeddings $\iota_{\mathrm{M},\mathrm{M}_1} : {}^L\mathrm{M}_1 \to {}^L\mathrm{M}$ and $\iota_{\mathrm{M}^*,\mathrm{M}^*_1} : {}^L\mathrm{M}^*_1 \to {}^L\mathrm{M}^*$ with the same image $\mathcal{L}$, and using these embeddings, a realization of M^*_1 as an elliptic endoscopic group of M_1, that can also be seen to agree with $\underline{\mathrm{M}}^*_1$. Henceforth, given a Levi subgroup $\mathrm{M}_1 \subset \mathrm{M}$ as above, we will often choose 'Levi subgroup matching data' consisting of a Levi subgroup $\mathrm{M}^*_1 \subset \mathrm{M}^*$ together with an inner twist $\psi^* = \psi_{\mathrm{M}^*} \circ \operatorname{Int} m^*$ restricting to $\psi_{\mathrm{M}^*_1}$ as above, and the resulting realization $\underline{\mathrm{M}}^*_1$ of M^*_1 as an elliptic endoscopic group of M_1. Sometimes, we will also fix auxiliary choices $\mathrm{Q}, \mathrm{Q}^*, \mathcal{Q}, \psi_{\mathrm{M}^*_1}$ etc. as above. This endoscopic datum and the resulting map $SD(\mathrm{M}^*_1) \to D(\mathrm{M}_1)$, as well as various isomorphisms such as the map $W(\mathrm{M}^*_1) \to W(\mathrm{M}_1)$ considered in (vii) below, depend on these auxiliary choices, but in a harmless way. In what follows, this dependence will be suppressed for lightness of notation.

(vii) Suppose $\mathrm{M}_1 \subset \mathrm{M}$ is a Levi subgroup, and a pair $(\mathrm{M}^*_1, \psi_{\mathrm{M}^*_1})$ is assigned to M_1 as in (vi) above. Let us study the impact of changing the choice of $(\mathrm{M}^*_1, \psi_{\mathrm{M}^*_1})$ to a different one, $((\mathrm{M}^*_1)', \psi_{(\mathrm{M}^*_1)'})$. Choose a parabolic subgroup $\mathrm{Q} \subset \mathrm{M}$ with M_1 as a Levi subgroup, and note that $(\mathrm{M}^*_1, \psi^{-1}_{\mathrm{M}^*_1}(\mathrm{Q}_{\bar{F}}))$ and $((\mathrm{M}^*_1)', \psi^{-1}_{(\mathrm{M}^*_1)'}(\mathrm{Q}_{\bar{F}}))$ are obtained by base-change from conjugate parabolic pairs in M^*. It follows that $\psi_{(\mathrm{M}^*_1)'} = \psi_{\mathrm{M}^*_1} \circ \operatorname{Int}(m^*_1 w)$ for some $w \in \mathrm{M}^*(F)$ transporting $(\mathrm{M}^*_1)'$ to M^*_1 and some $m^*_1 \in \mathrm{M}^*_1(\bar{F})$. It is then easy to see that the identifications ${}^L\mathrm{M}_1 = {}^L\mathrm{M}^*_1$ and ${}^L\mathrm{M}_1 = {}^L(\mathrm{M}'_1)^*$ as in (vi) differ from each other by the isomorphism ${}^L\mathrm{M}^*_1 = {}^L(\mathrm{M}'_1)^*$ that is dual to $\operatorname{Int} w : (\mathrm{M}'_1)^* \to \mathrm{M}^*_1$.

(viii) It is easy to see that $\psi_{\mathrm{M}^*_1}$, though not defined over F, induces an (F-)isomorphism $W(\mathrm{M}_1) \cong W(\mathrm{M}^*_1)$ between the Weyl groups of M_1 in M and M^*_1 in M^*, where $W(\mathrm{M}^*_1)$ and $W(\mathrm{M}_1)$ are described in terms of $\bar{F}$-points using the discussion of Subsubsection 2.1.7.

Remark 3.2.2. Let M be a connected reductive group over F. We collect a few useful facts concerning the endoscopic transfer between M and M^* (see Notation 3.2.1).

(i) The transfer factors between M^* and M can be normalized such that, if the stable conjugacy classes of $\gamma^* \in \mathrm{M}^*(F)$ and $\gamma \in \mathrm{M}(F)$ correspond to each other, then $\Delta(\gamma^*, \gamma) = 1$ (while this surely exists somewhere in the literature, since we have managed to not be able to locate a reference, here is a summary: as per [LS87], the transfer factor Δ_I and the relative transfer factor $\Delta_1 = \Delta_{III,1}$ are trivial because the element 's' in the endoscopic datum is the identity, the transfer factors Δ_{II} and Δ_{IV} are trivial because all roots of M come from M^*, and the transfer factor $\Delta_2 = \Delta_{III,2}$ is trivial because, in the notation of [LS87, (3.5)], we have $\xi \circ \xi_{T_H} = \xi_T$).

(ii) (i), together with the fact that the set of stable conjugacy classes of strongly regular semisimple elements of $\mathrm{M}(F)$ injects into the analogous set for $\mathrm{M}^*(F)$ under the matching of semisimple elements in the theory of endoscopy (see [Kot82, Section 6]), implies that:
- If $\mathrm{Z} \subset \mathrm{M}$ is a central torus and $\zeta : \mathrm{Z}(F) \to \mathbb{C}^\times$ is a unitary character, then a function $f \in C_c^\infty(\mathrm{M}(F))$ (resp., $C^\infty_{\mathrm{Z},\zeta}(\mathrm{M}(F))$) is unstable if and only if some or equivalently any endoscopic transfer $f^* \in C_c^\infty(\mathrm{M}^*(F))$ (resp., $f^* \in C^\infty_{\mathrm{Z},\zeta}(\mathrm{M}^*(F))$) of f to $\mathrm{M}^*(F)$ is unstable.
- At the level of distributions, it follows that endoscopic transfer takes stable distributions on $\mathrm{M}^*(F)$ to stable distributions on $\mathrm{M}(F)$, and $SD(\mathrm{M}^*)$ to $SD(\mathrm{M}) \subset D(\mathrm{M})$. Restricting to $SD_{\mathrm{ell}}(\mathrm{M})$ and using [MW16, Theorem XI.4], it also induces a map $SD_{\mathrm{ell}}(\mathrm{M}) \to SD_{\mathrm{ell}}(\mathrm{M}^*)$.

(iii) The compatibility between parabolic induction and endoscopic transfer (see Remark 3.1.4) simplifies in this situation. Let $\mathrm{M}_1^* \subset \mathrm{M}^*$ be a Levi subgroup, and $\Theta^* \in SD(\mathrm{M}_1^*)$ a stable tempered character on $\mathrm{M}_1^*(F)$. First, $\mathrm{Ind}_{\mathrm{M}_1^*}^{\mathrm{M}^*} \Theta^*$ is then a stable tempered character on $\mathrm{M}^*(F)$, so it transfers to a distribution on $\mathrm{M}(F)$ under $SD(\mathrm{M}^*) \to SD(\mathrm{M})$. There turn out to be two cases, depending on whether or not $\mathrm{M}_1^* \subset \mathrm{M}^*$ is $\underline{\mathrm{M}}^*$-relevant.
- If M_1^* is not $\underline{\mathrm{M}}^*$-relevant, the assertion is that the image of $\mathrm{Ind}_{\mathrm{M}_1^*}^{\mathrm{M}^*} \Theta^*$ under $SD(\mathrm{M}^*) \to SD(\mathrm{M})$ is 0.
- Suppose M_1^* is relevant, and let the Levi subgroup $\mathrm{M}_1 \subset \mathrm{M}$ and various auxiliary choices be as in Notation 3.2.1(vi). The assertion in this case is that Θ^* transfers to some tempered character Θ on $\mathrm{M}_1(F)$ under the resulting transfer $SD(\mathrm{M}_1^*) \to SD(\mathrm{M}_1)$, and moreover, the stable character $\mathrm{Ind}_{\mathrm{M}_1^*}^{\mathrm{M}^*} \Theta^*$ transfers to $\mathrm{Ind}_{\mathrm{M}_1}^{\mathrm{M}} \Theta$ under the transfer $SD(\mathrm{M}^*) \to SD(\mathrm{M})$, independently of the auxiliary choices of parabolic subgroups involved in Notation 3.2.1(vi).

(iv) Let Levi subgroups $\mathrm{M}_1^* \subset \mathrm{M}^*$ and $\mathrm{M}_1 \subset \mathrm{M}$ and Levi subgroup matching data be chosen as in Notation 3.2.1(vi). It is now easy from the definitions in [LS87] that the transfer of stable conjugacy classes from M_1^* to M_1, and hence by (i) also the endoscopic transfer map $SD(\mathrm{M}_1^*) \to SD(\mathrm{M}_1)$, respects conjugacy under $W(\mathrm{M}_1) = W(\mathrm{M}_1^*)$.

Lemma 3.2.3. *Let* M *be a connected reductive group over* F*. Then:*

(i) The map $SD(\mathrm{M}^*) \to SD(\mathrm{M})$ *respects 'central characters', i.e., the eigendecomposition with respect to* $\mathrm{Z}_{\mathrm{M}}(F) = \mathrm{Z}_{\mathrm{M}^*}(F) \supset \mathrm{A}_{\mathrm{M}^*}(F) = \mathrm{A}_{\mathrm{M}}(F)$*, as well as twisting by* $X^{\mathrm{unr-uni}}(\mathrm{M}^*) = X^{\mathrm{unr-uni}}(\mathrm{M})$ *(see Notation 3.2.1(ii) for these identifications).*

(ii) Let $\mathrm{M}_1^* \subset \mathrm{M}^*, \mathrm{M}_1 \subset \mathrm{M}$ *be as in Notation 3.2.1(vi). Let* $\mathcal{O}'_{\mathrm{M}_1} \subset \mathrm{Aut}(\mathrm{M}_1), \mathcal{O}'_{\mathrm{M}_1^*} \subset \mathrm{Aut}(\mathrm{M}_1^*)$ *be subgroups with the same image* $\bar{\mathcal{O}}'_{\mathrm{M}_1} = \bar{\mathcal{O}}'_{\mathrm{M}_1^*}$ *in* $\mathrm{Out}(\mathrm{M}_1) = \mathrm{Out}(\mathrm{M}_1^*)$ *(e.g., we could have* $\bar{\mathcal{O}}'_{\mathrm{M}_1} = W(\mathrm{M}_1)$ *and* $\bar{\mathcal{O}}'_{\mathrm{M}_1^*} = W(\mathrm{M}_1^*)$*, by the discussion in Notation 3.2.1(viii)). Then the transfer of stable conjugacy classes from* $\mathrm{M}_1^*(F)$ *to* $\mathrm{M}_1(F)$*, as well as the endoscopic transfer map* $SD(\mathrm{M}_1^*) \to SD(\mathrm{M}_1)$*, are equivariant under* $\bar{\mathcal{O}}'_{\mathrm{M}_1} = \bar{\mathcal{O}}'_{\mathrm{M}_1^*}$ *(through which the actions of* $\mathcal{O}'_{\mathrm{M}_1}$ *and* $\mathcal{O}'_{\mathrm{M}_1^*}$ *clearly factor).*

Remark 3.2.4. Of course, one can prove a more general version of (i) of the above lemma, involving twisting by a group of characters that is larger than $X^{\mathrm{unr-uni}}(\mathrm{M})$, but we will not need it.

Proof of Lemma 3.2.3. These assertions being well-known (part of (i) was used in Notation 3.2.1(iii)), we will only sketch the proof. In what follows, we will use that the 'correspondence' of semisimple conjugacy classes between $\mathrm{M}(F)$ and $\mathrm{M}^*(F)$ has the following easy description: the conjugacy classes of semisimple elements $m \in \mathrm{M}(F)$ and $m^* \in \mathrm{M}^*(F)$ correspond if and only if m is $\mathrm{M}(\bar{F})$-conjugate to $\psi(m^*)$.

The assertion in (i) concerning central characters follows from Remark 3.2.2(i) together with the easy observation that the left-regular action of $\mathrm{Z}_{\mathrm{M}}(F) = \mathrm{Z}_{\mathrm{M}^*}(F)$ respects the transfer of stable conjugacy classes from $\mathrm{M}^*(F)$ to $\mathrm{M}(F)$. For the assertion in (i) concerning twisting by $X^{\mathrm{unr-uni}}(\mathrm{M}^*) = X^{\mathrm{unr-uni}}(\mathrm{M})$, combine Remark 3.2.2(i) with the easy observation that the inner twist ψ_{M^*} gives us an identification of the map $\mathrm{M}^* \to \mathrm{S}_{\mathrm{M}^*}$ with the map $\mathrm{M} \to \mathrm{S}_{\mathrm{M}}$ that is manifestly compatible with the transfer of stable conjugacy classes. For (ii), note that if $\beta \in \mathcal{O}'_{\mathrm{M}_1}$ and $\beta^* \in \mathcal{O}'_{\mathrm{M}_1^*}$ have the same image in $\bar{\mathcal{O}}'_{\mathrm{M}_1} = \bar{\mathcal{O}}'_{\mathrm{M}_1^*}$, then strongly regular semisimple elements $m \in \mathrm{M}_1(F)$ and $m^* \in \mathrm{M}_1^*(F)$ match if and only if $\beta(m)$ and $\beta^*(m^*)$ do; now use Remark 3.2.2(i). □

Remark 3.2.5. Let $\mathrm{M}_1^*, (\mathrm{M}_1')^*$ etc. and $w \in \mathrm{M}(F)$ be as in the setting of Notation 3.2.1(vii), from where we recall that the identifications ${}^L\mathrm{M}_1 = {}^L\mathrm{M}_1^*$ and ${}^L\mathrm{M}_1 = {}^L(\mathrm{M}_1')^*$ differ by an isomorphism ${}^L\mathrm{M}_1^* \to {}^L(\mathrm{M}_1')^*$ dual to $\mathrm{Int}\, w$. On the other hand, it is easy to see from Remark 3.2.2(i), as in the proof of the assertions of Lemma 3.2.3, that the maps $SD(\mathrm{M}_1^*) \to SD(\mathrm{M}_1)$ and $SD((\mathrm{M}_1')^*) \to SD(\mathrm{M}_1)$, and hence their restrictions $SD_{\mathrm{ell}}(\mathrm{M}_1^*) \to SD_{\mathrm{ell}}(\mathrm{M}_1)$ and $SD_{\mathrm{ell}}((\mathrm{M}_1')^*) \to SD_{\mathrm{ell}}(\mathrm{M}_1)$, differ by $\mathrm{Int}\, w$.

Proposition 3.2.6. *Let* M *be a connected reductive group over* F*, and let* $D \in D_{\mathrm{ell}}(\mathrm{M})$ *have the property that its restriction to the set* $\mathrm{M}(F)_{\mathrm{ell,srss}} \subset \mathrm{M}(F)$ *consisting of elliptic strongly regular semisimple elements is stable — in other words, recalling that* D *can be viewed as a function* $\mathrm{M}(F) \to \mathbb{C}$ *that is locally constant on the set* $\mathrm{M}(F)_{\mathrm{srss}}$ *of strongly regular semisimple elements of* $\mathrm{M}(F)$ *and locally integrable on* $\mathrm{M}(F)$*,* $D(\gamma) = D(\gamma')$ *whenever* $\gamma, \gamma' \in \mathrm{M}(F)_{\mathrm{ell,srss}}$ *are such that* γ' *is* $\mathrm{M}(\bar{F})$*-conjugate to* γ*. Then:*

(i) D *is stable, i.e.,* $D \in SD_{\mathrm{ell}}(\mathrm{M})$*.*

(ii) D *is the transfer of a stable distribution* $D^* \in SD_{\mathrm{ell}}(\mathrm{M}^*)$ *(in the sense of Notation 3.2.1).*

Proof. In the case where M is quasi-split, (i) (and hence trivially also (ii)) is a well-known result of Arthur; see [MW16, Theorem XI.3]. The general case can be easily deduced from this and some standard facts, as we will see now.

By the discussion in Remark 3.2.2(ii), (ii) implies (i), so it suffices to prove (ii).

As before, we will write $\underline{\mathrm{H}}$ for a typical element of $\mathcal{E}(\mathrm{M})$ and, given $\underline{\mathrm{H}}$, H for the corresponding endoscopic group. (8) specializes to an isomorphism:

$$\bigoplus_{\underline{\mathrm{H}} \in \mathcal{E}(\mathrm{M})} \mathbf{T}_{\underline{\mathrm{H}}} : \bigoplus_{\underline{\mathrm{H}} \in \mathcal{E}(\mathrm{M})} SD_{\mu,\mathrm{ell}}(\mathrm{H}_1)_{\mathrm{Aut}\underline{\mathrm{H}}} \to D_{\mathrm{ell}}(\mathrm{M}). \tag{9}$$

Recall that, when $\underline{\mathrm{H}}$ equals the endoscopic datum $\underline{\mathrm{M}}^* \in \mathcal{E}(\mathrm{M})$ as in Notation 3.2.1, we identify the factor $SD_{\mu,\mathrm{ell}}(\mathrm{H}_1)_{\mathrm{Aut}\underline{\mathrm{H}}}$ with $SD(\mathrm{H})_{\mathrm{Aut}\underline{\mathrm{H}}} = SD_{\mathrm{ell}}(\mathrm{M}^*)$. It suffices to show that D belongs to $\mathbf{T}_{\underline{\mathrm{M}}^*}(SD_{\mathrm{ell}}(\underline{\mathrm{M}}^*))$ under (9).

For this, we recall more specific details on the realization of (9). In this proof, write $C^\infty_{c,\mathrm{cusp}}(\mathrm{M}(F)) \subset C^\infty_c(\mathrm{M}(F))$ for the subspace consisting of cuspidal functions in the sense of [MgW18, Section 7.1], i.e., whose nonelliptic strongly regular semisimple orbital integrals all vanish. Let $\mathcal{I}_{\mathrm{cusp}}(\mathrm{M})$ be the quotient of $C^\infty_{c,\mathrm{cusp}}(\mathrm{M}(F))$ by the subspace consisting of those functions all of whose strongly regular semisimple orbital integrals vanish; this agrees with the notation in [MW16, towards the end of I.3.1]. Similarly, by [MW16, towards the end of Section I.3.1 and towards the end of Section I.2.5], for each $\underline{\mathrm{H}} \in \mathcal{E}(\mathrm{M})$, we have a space $\mathcal{SI}_{\mathrm{cusp}}(\underline{\mathrm{H}})$, a space of stable orbital integrals for functions, not on $\mathrm{H}(F)$, but lying in a space $C^\infty_\mu(\mathrm{H}_1(F))$ associated to a fixed choice of auxiliary data as in Notation 3.1.2(iii), which we now make.

By [Art96, Proposition 3.5], or by [MW16, Proposition I.4.11], as invoked in [LMW18, Section 4.4, (3)], endoscopic transfer from M along the $\underline{\mathrm{H}}$, as $\underline{\mathrm{H}}$ varies over $\mathcal{E}(\mathrm{M})$, descends to an isomorphism of vector spaces:

$$\mathcal{I}_{\mathrm{cusp}}(\mathrm{M}) \overset{\cong}{\to} \bigoplus_{\underline{\mathrm{H}} \in \mathcal{E}(\mathrm{M})} \mathcal{SI}_{\mathrm{cusp}}(\underline{\mathrm{H}})^{\mathrm{Aut}(\underline{\mathrm{H}})}, \tag{10}$$

where $\mathrm{Aut}(\underline{\mathrm{H}})$ is as in (8) (implicit in this isomorphism is the assertion that, if the orbital integrals of $f \in C^\infty_c(\mathrm{M}(F))$ at strongly regular nonelliptic semisimple elements of $\mathrm{M}(F)$ all vanish, then for any $\underline{\mathrm{H}} \in \mathcal{E}(\mathrm{M})$, the stable orbital integrals of any transfer $f^{\underline{\mathrm{H}}} \in C^\infty_\mu(\mathrm{H}_1(F))$ of f satisfy a similar property).

We have a map $D_{\mathrm{ell}}(\mathrm{M}) \to \mathrm{Hom}_{\mathbb{C}}(\mathcal{I}_{\mathrm{cusp}}(\mathrm{M}), \mathbb{C})$, obtained by restricting an element of $D_{\mathrm{ell}}(\mathrm{M})$ to the space $C^\infty_{c,\mathrm{cusp}}(\mathrm{M}(F)) \subset C^\infty_c(\mathrm{M}(F))$. As recalled in [LMW18, Section 4.3, a bit below (5)], this map lets us identify $D_{\mathrm{ell}}(\mathrm{M})$ as a linear subspace of $\mathrm{Hom}_{\mathbb{C}}(\mathcal{I}_{\mathrm{cusp}}(\mathrm{M}), \mathbb{C})$. A similar prescription identifies $SD_{\mu,\mathrm{ell}}(\mathrm{H}_1)$ with a linear subspace of $\mathrm{Hom}_{\mathbb{C}}(\mathcal{SI}_{\mathrm{cusp}}(\underline{\mathrm{H}}), \mathbb{C})$, for each $\underline{\mathrm{H}} \in \mathcal{E}(\mathrm{M})$. Moreover, with these identifications, as explained in [LMW18, Section 4.4, (4)], (9) is obtained by restricting the isomorphism

$$\mathrm{Hom}_{\mathbb{C}}(\mathcal{I}_{\mathrm{cusp}}(\mathrm{M}), \mathbb{C}) \overset{\cong}{\to} \bigoplus_{\underline{\mathrm{H}} \in \mathcal{E}(\mathrm{M})} \mathrm{Hom}_{\mathbb{C}}(\mathcal{SI}_{\mathrm{cusp}}(\mathrm{H})^{\mathrm{Aut}(\underline{\mathrm{H}})}, \mathbb{C})$$

obtained by applying $\mathrm{Hom}_{\mathbb{C}}(-, \mathbb{C})$ to (10) (because $\mathbf{T}_{\underline{\mathrm{H}}}$ is dual to endoscopic transfer).
Using this and the fact that (10) is an isomorphism, it now suffices to show that D, viewed inside $\mathrm{Hom}_{\mathbb{C}}(\mathcal{I}_{\mathrm{cusp}}(\mathrm{M}), \mathbb{C})$, factors as the composite of the projection $\mathcal{I}_{\mathrm{cusp}}(\mathrm{M}) \to \mathcal{SI}_{\mathrm{cusp}}(\underline{\mathrm{M}}^*)^{\mathrm{Aut}(\underline{\mathrm{M}}^*)}$ and some element of $\mathrm{Hom}_{\mathbb{C}}(\mathcal{SI}_{\mathrm{cusp}}(\underline{\mathrm{M}}^*)^{\mathrm{Aut}(\underline{\mathrm{M}}^*)}, \mathbb{C})$ (a priori not necessarily the one obtained from D using (9)). Thus, by Remark 3.2.2(i), it suffices to show that if $f \in C^\infty_{c,\mathrm{cusp}}(\mathrm{M}(F))$, then $D(f)$ depends only on the set of stable orbital integrals of f at strongly regular semisimple elements of $\mathrm{M}(F)$. This follows from the hypothesis on D (that its restriction to the set of elliptic strongly regular semisimple elements is stable), the fact that f belongs to $C^\infty_{c,\mathrm{cusp}}(\mathrm{M}(F))$, and the Weyl integration formula. □

Corollary 3.2.7. *Let* M *be a connected reductive group over* F*. The transfer* $SD(\mathrm{M}^*) \to D(\mathrm{M})$*, in the sense of Notation 3.2.1, takes* $SD_{\mathrm{ell}}(\mathrm{M}^*)$ *isomorphically onto* $SD_{\mathrm{ell}}(\mathrm{M})$*.*

Proof. The map $SD_{\mathrm{ell}}(\mathrm{M}^*) \to SD_{\mathrm{ell}}(\mathrm{M})$ is injective, since (9) isi an isomorphism, and since $\mathrm{Aut}(\underline{\mathrm{M}}^*)$ is trivial (use the identification ${}^L\mathrm{M}^* = {}^L\mathrm{M}$). The surjectivity of this map $SD_{\mathrm{ell}}(\mathrm{M}^*) \to SD_{\mathrm{ell}}(\mathrm{M})$ follows from Proposition 3.2.6. □

The above proof has the following corollary, in which $W(\mathrm{M})$ is as in Subsubsection 2.1.7:

Proposition 3.2.8. *Let* $\tilde{\mathcal{L}}$ *denote the set of all Levi subgroups of* G*. Then, inside the space:*

$$D(\mathrm{G}) = \bigoplus_{\mathrm{M} \in \tilde{\mathcal{L}}/\mathrm{G}(F)} \mathrm{Ind}_{\mathrm{M}}^{\mathrm{G}} D_{\mathrm{ell}}(\mathrm{M})^{W(\mathrm{M})} \tag{11}$$

(this identification is defined by parabolic induction; for a proof, see [MgW18, Proposition 2.12]*), we have compatibly an equality*

$$SD(\mathrm{G}) = \bigoplus_{\mathrm{M} \in \tilde{\mathcal{L}}/\mathrm{G}(F)} \mathrm{Ind}_{\mathrm{M}}^{\mathrm{G}} SD_{\mathrm{ell}}(\mathrm{M})^{W(\mathrm{M})}. \tag{12}$$

Moreover we also have the following compatible equality, where we recall our abbreviation $\mathcal{O} = \mathcal{O}_{\mathrm{G}}$*:*

$$SD(\mathrm{G})^{\mathcal{O}} = \bigoplus_{\mathrm{M} \in \tilde{\mathcal{L}}/\mathcal{O}_{\mathrm{G}}^+} \mathrm{Avg}_{\mathcal{O}} \left(\mathrm{Ind}_{\mathrm{M}}^{\mathrm{G}} SD_{\mathrm{ell}}(\mathrm{M})^{\mathcal{O}_{\mathrm{M}}} \right),$$

where $\mathrm{Avg}_{\mathcal{O}}$ *refers to averaging with respect to the action of* $\mathcal{O}$ *(which makes sense as* $\mathcal{O}$ *acts via a finite quotient; see the proof of Lemma 2.4.3(ii)).*

Proof. It is easy to deduce the latter assertion from the former, noting that

$$o \cdot \mathrm{Ind}_{\mathrm{M}}^{\mathrm{G}} SD_{\mathrm{ell}}(\mathrm{M})^{W(\mathrm{M})} = o \cdot \mathrm{Ind}_{\mathrm{M}}^{\mathrm{G}} SD_{\mathrm{ell}}(\mathrm{M}) = \mathrm{Ind}_{o\cdot\mathrm{M}}^{\mathrm{G}} SD_{\mathrm{ell}}(o \cdot \mathrm{M}) = \mathrm{Ind}_{o\cdot\mathrm{M}}^{\mathrm{G}} SD_{\mathrm{ell}}(o \cdot \mathrm{M})^{W(o\cdot\mathrm{M})},$$

for each $o \in \mathcal{O}_{\mathrm{G}}^+$ and each $\mathrm{M} \in \tilde{\mathcal{L}}$. Therefore let us prove the former.
As observed in [MW16, VIII.2.4] and [LMW18, Remark 3.4, around (2)], when M is quasi-split, this assertion (and even a twisted version of it) follows from [MW16, Corollary XI.3.1]. What we describe will be essentially the proof in [MgW18, Corollary XI.3.1], with only a slight variance, so we will be brief.
The inclusion '⊃' is immediate, since parabolic induction preserves the stability of virtual characters (a convenient reference for which is [KV16, Corollary 6.13]). To prove the inclusion '⊂', fix $\Theta \in SD(G)$, and, using fixed representatives for $\tilde{\mathcal{L}}/\mathrm{G}(F)$, chosen so as to contain a common minimal Levi subgroup, write $\Theta = \sum_{\mathrm{M}} \mathrm{Ind}_{\mathrm{M}}^{\mathrm{G}} \Theta_{\mathrm{M}}$ according to the decomposition in (11). It is enough to show that the element Θ_{M} of $D_{\mathrm{ell}}(\mathrm{M})$ is stable for each M. By an easy induction argument involving $\tilde{\mathcal{L}}$, partially ordered under reverse inclusion up to conjugacy (see [MgW18, the proof of Corollary XI.3.1]), we may assume that for some fixed $L \in \tilde{\mathcal{L}}$, $\Theta_{\mathrm{M}} = 0$ if M contains a

conjugate of L properly, and then prove that Θ_{L} is stable. If $\mathrm{M} \in \tilde{\mathcal{L}}$ is such that $g\gamma g^{-1} \in \mathrm{M}(F)$ for some $g \in \mathrm{G}(F)$ and some γ in the set $\mathrm{L}(F)_{\mathrm{ell,srss}}$ of elliptic strongly regular semisimple elements of $\mathrm{L}(F)$, then $g^{-1}\mathrm{M}g \supset \mathrm{L}$ by hypothesis (because $\mathrm{A_L}$ equals the maximal split torus in the centralizer of γ by ellipticity, and hence contains $g^{-1}\mathrm{A_M}g$), and hence $\Theta_{\mathrm{M}} = 0$ unless $\mathrm{M} = \mathrm{L}$. Using this, the fact that Θ_{L} was chosen to be fixed under $W(\mathrm{L})$, and van Dijk's formula for induced characters ([vD72, Theorem 3], which takes a particularly simple form at elliptic elements of the Levi subgroup under consideration), it follows that up to a ratio of discriminant factors, which is invariant under $W(\mathrm{L})$ and under stable conjugacy, Θ equals a multiple of Θ_{L} on $\mathrm{L}(F)_{\mathrm{ell,srss}}$. Thus, Θ_{L} is stable when restricted to $\mathrm{L}(F)_{\mathrm{ell,srss}}$, in the sense explained in Proposition 3.2.6. Hence Proposition 3.2.6 implies that Θ_{L} is stable, as desired. □

Corollary 3.2.9. *Let* M *be a connected reductive group over* F*. The transfer* $SD(\mathrm{M}^*) \to D(\mathrm{M})$*, in the sense of Notation 3.2.1, takes* $SD(\mathrm{M}^*)$ *surjectively onto* $SD(\mathrm{M})$*.*

Proof. In view of the compatibility between endoscopic transfer and parabolic induction (see Remark 3.2.2(iii)), this follows from Proposition 3.2.8 and Corollary 3.2.7. □

Proposition 3.2.10. *Let* M *be a connected reductive group over* F*. Let* $\mathrm{Z} \subset \mathrm{M}$ *be a central induced torus, and* $\zeta : \mathrm{Z}(F) \to \mathbb{C}^\times$ *a smooth unitary character. Recall the space* $C^\infty_{\mathrm{Z},\zeta}(\mathrm{M}(F))$*, and for each Levi subgroup* $\mathrm{L} \subset \mathrm{M}$*, the space* $SD_{\mathrm{Z},\zeta,\mathrm{ell}}(\mathrm{L}) := SD_{\mathrm{Z}(F),\zeta,\mathrm{ell}}(\mathrm{L})$ *(see Notation 2.1.1). Suppose* $f \in C^\infty_{\mathrm{Z},\zeta}(\mathrm{M}(F))$ *has the property that* $\left(\mathrm{Ind}_{\mathrm{L}}^{\mathrm{M}}\Theta\right)(f) = 0$ *for every Levi subgroup* $\mathrm{L} \subset \mathrm{M}$ *and every stable elliptic virtual character* $\Theta \in SD_{\mathrm{Z},\zeta,\mathrm{ell}}(\mathrm{L})$*. Then* f *is unstable.*

Remark 3.2.11. If M is quasi-split, this is immediate from [Art96, Lemma 6.3], as explained in [LM20, page 587]. When Z is trivial, an alternative reference that is more convenient to cross-check (in this quasi-split case) is the combination of [MW16, Corollary XI.5.2(i)] and the description in [MW16, Corollary XI.3.1], invoked earlier, of the space $SD(\mathrm{M})$.

Proof of Proposition 3.2.10. We choose a quasi-split form M^* of M, and fix an endoscopic datum $\underline{\mathrm{M}}^*$ and an inner twist from M^* to M as in Notation 3.2.1(i). By Remark 3.2.2(ii), it is enough to show that any transfer $f^* \in C^\infty_{\mathrm{Z},\zeta}(\mathrm{M}^*(F))$ of f to the quasi-split form M^* of M is unstable (recall from Notation 3.2.1(iii) that Z is viewed, using the fixed inner twist, as a subgroup of M^* as well). The proposition being already known with M replaced by the quasi-split group M^* ([Art96, Lemma 6.3] — here we use that Z is an induced torus), it suffices to show that for any Levi subgroup $\mathrm{L}^* \subset \mathrm{M}^*$ and any $\Theta \in SD_{\mathrm{Z},\zeta,\mathrm{ell}}(\mathrm{L}^*)$, we have $(\mathrm{Ind}_{\mathrm{L}^*}^{\mathrm{M}^*}\Theta)(f^*) = 0$. This follows from the hypothesis of the proposition together with the fact that endoscopic transfer between M and M^* is compatible with parabolic induction (see Remark 3.2.2(iii)), as well as with the central character condition involving (Z, ζ) (see the discussion in Notation 3.2.1(iii)), and takes stable virtual characters to stable virtual characters. □

The following corollary will be useful later:

Corollary 3.2.12. *Assume Hypothesis 2.5.1. Suppose* $f \in C_c^\infty(\mathrm{G}(F))$ *has* $\mathcal{O}$*-invariant image in the space* $\mathcal{I}(\mathrm{G})$ *of coinvariants for the* $\mathrm{G}(F)$*-conjugation action on* $C_c^\infty(\mathrm{G}(F))$*, and suppose that* $D(f) = 0$ *whenever* D *is a virtual character on* $\mathrm{G}(F)$ *obtained by* $\mathcal{O}$*-averaging* $\mathrm{Ind}_{\mathrm{M}}^{\mathrm{G}}\Theta$ *for some Levi subgroup* $\mathrm{M} \subset \mathrm{G}$ *and some* $\Theta \in SD_{\mathrm{ell}}(\mathrm{M})^{\mathcal{O}_{\mathrm{M}}}$*. Then* f *is unstable.*

Proof. Let $\mathrm{M} \subset \mathrm{G}$ be a Levi subgroup, and let $\Theta' \in SD_{\mathrm{ell}}(\mathrm{M})$. By Proposition 3.2.10, it suffices to show that $D'(f) = 0$, where $D' = \mathrm{Ind}_{\mathrm{M}}^{\mathrm{G}}\Theta'$. Let $\Theta \in SD_{\mathrm{ell}}(\mathrm{M})^{\mathcal{O}_{\mathrm{M}}}$ be the $\mathcal{O}_{\mathrm{M}}$-average of Θ', and let $D_0 = \mathrm{Ind}_{\mathrm{M}}^{\mathrm{G}}\Theta$.
Let D be the $\mathcal{O}$-average of D'. Since elements of $\mathcal{O}_{\mathrm{M}}$ are obtained by restricting from $\mathcal{O}_{\mathrm{G}}^+$ (by (iv)b of Notation 2.4.1), and since $\mathcal{O}$ and $\mathcal{O}_{\mathrm{G}}^+$ have the same orbit on the space of invariant distributions on $\mathrm{G}(F)$, D is also the $\mathcal{O}$-average of $D_0 = \mathrm{Ind}_{\mathrm{M}}^{\mathrm{G}}\Theta$, so that $D(f) = 0$. Therefore, using that f has $\mathcal{O}$-invariant image in $\mathcal{I}(\mathrm{G})$, we have $D'(f) = D(f) = 0$, as desired. □

We record the following variant of Proposition 3.2.10 for use elsewhere.

Proposition 3.2.13. *Suppose* H_1 *is a quasi-split reductive group over* F*, and* $\tilde{\mathrm{H}}_1$ *is a twisted space over* H_1 *with the property that for all* $\gamma_1 \in \tilde{\mathrm{H}}_1(F)$*, the automorphism* $\mathrm{Int}\,\gamma_1$ *of* H_1 *is inner in the sense of being given by conjugation under an element of* $\mathrm{H}_{1,\mathrm{ad}}(F)$*. Assume further that* $\tilde{\mathrm{H}}_1(F) \neq \emptyset$*. Let* $\mathrm{C}_1 \subset \mathrm{H}_1$ *be a central induced torus, and* $\mu : \mathrm{C}_1(F) \to \mathbb{C}^\times$ *a unitary character. Suppose* $f_1 \in C^\infty_\mu(\tilde{\mathrm{H}}_1(F))$ *has the property that* $\Theta_1(f_1) = 0$ *for all* $\Theta_1 \in SD_\mu(\tilde{\mathrm{H}}_1)$*. Then* f_1 *is unstable.*

Proof. Since $\tilde{\mathrm{H}}_1(F) \neq \emptyset$, by [MW16, Proposition III.2.3], one has an embedding $\mathrm{H}_1 \hookrightarrow \mathrm{H}_2$ of H_1 into a quasi-split reductive group H_2 with the same derived group as H_1, and with the property that, as an H_1-bitorsor, $\tilde{\mathrm{H}}_1$ can be identified with a coset of H_1 in H_2 (and hence with a fiber of $\mathrm{H}_2 \to \mathrm{H}_2/\mathrm{H}_1$). We can find a compact (usually not open) subgroup Z_2 of $\mathrm{Z}_{\mathrm{H}_2}(F)$, containing the identity, with the property that the multiplication map $\tilde{\mathrm{H}}_1(F) \times Z_2 \to \mathrm{H}_2(F)$ identifies the product $\tilde{\mathrm{H}}_1(F) \times Z_2$, as a topological space, with an open subset of $\mathrm{H}_2(F)$. Let f_2 be the pushforward of $f_1 \otimes \mathbb{1}_{Z_2} \in C^\infty_\mu(\tilde{\mathrm{H}}_1(F) \times Z_2)$ to an element of $C^\infty_\mu(\mathrm{H}_2(F))$, where $C^\infty_\mu(\mathrm{H}_2(F))$ is defined just like $C^\infty_\mu(\tilde{\mathrm{H}}_1(F))$, using the same subgroup $\mathrm{C}_1(F)$.
For appropriate choices of measures, it is easy to see that the stable orbital integral of f_2 at any strongly regular semisimple element of $\mathrm{H}_2(F)$ is either zero or equal to the stable orbital integral of f_1 at some strongly regular semisimple element of $\mathrm{H}_1(F)$. Therefore, it suffices to show that $f_2 \in C^\infty_\mu(\mathrm{H}_2(F))$ is unstable.
Since H_2 is quasi-split and C_1 is an induced torus, this in turn follows from [Art96, Lemma 6.3] if we show that $\Theta_2(f_2) = 0$ for all stable tempered virtual characters $\Theta_2 \in SD_\mu(\mathrm{H}_2)$.
Viewing Θ_2 as a locally integrable function on $\mathrm{H}_2(F)$, it is easy to see that its restriction to $\tilde{\mathrm{H}}_1(F)$, call it Θ_1, belongs to $SD_\mu(\tilde{\mathrm{H}}_1)$: if (π_2, V_2) is an irreducible smooth representation of $\mathrm{H}_2(F)$, the restriction of Θ_{π_2} to $\tilde{\mathrm{H}}_1(F)$ is the character of $\tilde{\mathrm{H}}_1(F)$ acting on the subspace of V_2 that is spanned by those irreducible $\mathrm{H}_1(F)$-subrepresentations that are preserved by $\tilde{\mathrm{H}}_1(F)$. Therefore, $\Theta_1(f_1) = 0$. On the other hand, it is easy to see that $\Theta_2(f_2)$ is some scalar multiple of $\Theta_1(f_1)$, so that $\Theta_2(f_2) = 0$ as well, as desired. □

3.3. **Unitarily stable discrete series L-packets.** In many situations where Hypothesis 2.5.1 hasn't been proved, we do have finite sets of representations that deserve to be called discrete series L-packets, in the sense that they satisfy a property of being 'unitarily stable' related to atomic stability as in [MY20, Section 4], and hence are necessarily automatically elements of $\Phi_2(\mathrm{M})$ the moment Hypothesis 2.5.1 is true. We will see that this is the case with the notion of L-packets given in Definition 3.3.2 below.

Notation 3.3.1.
(i) Henceforth, for a connected reductive group M over F, $e(\mathrm{M})$ denotes its Kottwitz sign (see [Kot83]).
(ii) For this subsection alone, we fix a pair $(\mathrm{M}, \mathcal{O}'_{\mathrm{M}})$ consisting of reductive group M over F, and a group $\mathcal{O}'_{\mathrm{M}}$ of automorphisms of M with finite image in $\mathrm{Out}(\mathrm{M})$. For example, M could be a Levi subgroup of G and $\mathcal{O}'_{\mathrm{M}}$ could equal $\mathcal{O}_{\mathrm{M}}$.

Definition 3.3.2. Let $\Sigma \subset \mathrm{Irr}_2(\mathrm{M})$ be finite. We say that Σ is an $\mathcal{O}'_{\mathrm{M}}$-unitarily stable L-packet of discrete series representations of $\mathrm{M}(F)$, if there exists a nonzero stable virtual character $\Theta_\Sigma = \sum_{\sigma \in \Sigma} c_\sigma \Theta_\sigma$ on $\mathrm{M}(F)$ supported on Σ, such that Σ and Θ_Σ are preserved under the action of $\mathcal{O}'_{\mathrm{M}}$, and such that every $\mathcal{O}'_{\mathrm{M}}$-invariant stable elliptic virtual character $\Theta \in SD_{\mathrm{ell}}(\mathrm{M})^{\mathcal{O}'_{\mathrm{M}}}$ on $\mathrm{M}(F)$ can be uniquely written in the form $c_1\Theta_\Sigma + c_2\Theta'$ for a (automatically stable and $\mathcal{O}'_{\mathrm{M}}$-invariant) virtual character Θ' supported outside Σ and complex numbers c_1, c_2. By a unitarily stable discrete series L-packet, we refer to an $\mathcal{O}''_{\mathrm{M}}$-unitarily stable discrete series L-packet in the sense just defined, but with $\mathcal{O}''_{\mathrm{M}} \subset \mathrm{Aut}(\mathrm{M})$ taken to be the trivial group.

Remark 3.3.3. The property of being 'unitarily stable' is a priori stronger than the property of 'atomic stability' as in [Kal22, Conjecture 2.2], though one hopes these two properties to be ultimately equivalent.

Remark 3.3.4. Later, we will see in Proposition 3.4.11, that any finite set Σ of discrete series representations of $\mathrm{M}(F)$ for which one can establish an 'endoscopic decomposition' (in the sense of Definition 3.4.9) satisfies the above property. Thanks to the fact that Kaletha and others have established endoscopic decompositions for various supercuspidal L-packets they have constructed (e.g., see [Kal15]), the scope of the above definition is not subordinate to that of Hypothesis 2.5.1.

Remark 3.3.5. Assume that $\mathcal{O}'_{\mathrm{M}}$ fixes A_{M} pointwise, Then the following lemma says that, for any Σ as in Definition 3.3.2, one can take $\Theta_\Sigma = \sum_{\sigma\in\Sigma} d(\sigma)\Theta_\sigma$, where $d(\sigma) \in \mathbb{R}_{>0}$ is the formal degree of σ with respect to any choice of Haar measure on $\mathrm{M}(F)/\mathrm{A}_{\mathrm{M}}(F)$.

Proposition 3.3.6. *Suppose Σ and Θ_Σ are as in Definition 3.3.2, and assume that $\mathcal{O}'_{\mathrm{M}}$ fixes A_{M} pointwise. Fix any Haar measure on $\mathrm{M}(F)/\mathrm{A}_{\mathrm{M}}(F)$. Then:*

(i) The central characters of the elements of Σ agree on $\mathrm{Z}_{\mathrm{M}}(F)^{\mathcal{O}'_{\mathrm{M}}} \supset \mathrm{A}_{\mathrm{M}}$. In particular, there exists a smooth character $\zeta : \mathrm{A}_{\mathrm{M}}(F) \to \mathbb{C}^\times$ such that the central character of each $\sigma \in \Sigma$ restricts to ζ on $\mathrm{A}_{\mathrm{M}}(F)$.

(ii) For some $c \in \mathbb{C}\setminus\{0\}$, $\Theta_\Sigma = c\sum_{\sigma\in\Sigma} d(\sigma)\Theta_\sigma$, where $d(\sigma)$ denotes the formal degree of σ with respect to the chosen Haar measure.

(iii) Suppose we are in the situation of (ii), and let $\zeta : \mathrm{A}_{\mathrm{M}}(F) \to \mathbb{C}^\times$ be as in (i). Suppose that Θ is an $\mathcal{O}'_{\mathrm{M}}$-invariant distribution on $\mathrm{M}(F)$ defined by a possibly infinite sum:

$$\Theta = \sum_{\sigma\in\mathrm{Irr}_2(\mathrm{M})_\zeta} c(\sigma)\Theta_\sigma,$$

where $c(\sigma) \in \mathbb{C}$ for each σ (see Remark 2.2.5 for why this infinite sum is well-defined). If further Θ is stable, then for all $\sigma_1, \sigma_2 \in \Sigma$ we have $c(\sigma_1)d(\sigma_1)^{-1} = c(\sigma_2)d(\sigma_2)^{-1}$.

Proof. (i) is an easy consequence of the definitions together with Remark 2.2.4(i).
The proofs of (ii) and (iii) are easier versions of the proofs of (ii) and (iii) of Proposition 3.3.7 that we will prove below, so we will be brief, referring the reader to the proof of Proposition 3.3.7 for more details including of some of the notation. Let us first prove (ii). Write $\Theta_\Sigma = \sum_{\sigma\in\Sigma} c(\sigma)\Theta_\sigma$. Suppose $\sigma_1, \sigma_2 \in \Sigma$; (ii) follows if we show that $c(\sigma_2)d(\sigma_1) = c(\sigma_1)d(\sigma_2)$. For $i = 1, 2$, we let $f_{\sigma_i} \in C^\infty_{\mathrm{A}_{\mathrm{M}}(F),\zeta}(\mathrm{M}(F))$ be a pseudocoefficient for σ_i from among those representations of $\mathrm{M}(F)$ whose central character restricts to ζ on $\mathrm{A}_{\mathrm{M}}(F)$, and let $f_i \in C^\infty_{\mathrm{A}_{\mathrm{M}}(F),\zeta}(\mathrm{M}(F))$ be the average of the pseudocoefficients $f_{\sigma_i} \circ \beta^{-1}$ of the representations $\sigma_i \circ \beta^{-1} \in \Sigma$, as β runs over a set of representatives in $\mathcal{O}'_{\mathrm{M}}$ for the finite group $\mathcal{O}'_{\mathrm{M}} \cdot \mathrm{Int}\,\mathrm{M}(F)/\mathrm{Int}\,\mathrm{M}(F)$.
We claim that $c(\sigma_2)f_1 - c(\sigma_1)f_2 \in C^\infty_{\mathrm{A}_{\mathrm{M}}(F),\zeta}(\mathrm{M}(F))$ is unstable; this is the analogue of Claim 1 in the proof of Proposition 3.3.7 below. By Proposition 3.2.10, this claim follows if we show that $\Theta(c(\sigma_2)f_1 - c(\sigma_1)f_2) = 0$ for all $\Theta \in SD(\mathrm{M})$. More precisely, the same proposition, together with the fact that $\sigma_1, \sigma_2 \in \mathrm{Irr}_2(\mathrm{M})$, in fact implies that this needs to be checked only for $\Theta \in SD_{\mathrm{ell}}(\mathrm{M})$. Moreover, by the "$\mathcal{O}'_{\mathrm{M}}$-averaging" process used to define the f_i, we may assume that $\Theta \in SD_{\mathrm{ell}}(\mathrm{M})^{\mathcal{O}'_{\mathrm{M}}}$, and then using Definition 3.3.2, that either $\Theta = \Theta_\Sigma$, or Θ is supported outside Σ. If $\Theta = \Theta_\Sigma$, then we have $\Theta(c(\sigma_2)f_1 - c(\sigma_1)f_2) = \Theta(c(\sigma_2)f_{\sigma_1} - c(\sigma_1)f_{\sigma_2}) = c(\sigma_2)c(\sigma_1) - c(\sigma_1)c(\sigma_2) = 0$, while if Θ is supported outside Σ we have $\Theta(c(\sigma_1)f_1 - c(\sigma_2)f_2) = 0 - 0 = 0$; in both cases, we used the definition of pseudocoefficients and the $\mathcal{O}'_{\mathrm{M}}$-invariance of Θ.
This proves that $c(\sigma_2)f_1 - c(\sigma_1)f_2$ is unstable. By [Kot88, Section 3, Proposition 1], we get $c(\sigma_2)f_1(1) - c(\sigma_1)f_2(1) = 0$, and since $\mathcal{O}'_{\mathrm{M}}$-averaging of functions preserves evaluation of functions at the identity, we get $c(\sigma_2)f_{\sigma_1}(1) - c(\sigma_1)f_{\sigma_2}(1) = 0$. But it is easy to see that $f_{\sigma_i}(1) = d(\sigma_i) \neq 0$ for $i = 1, 2$ (see [DKV84, Proposition A.3.g]), so we get $c(\sigma_2)d(\sigma_1) = c(\sigma_1)d(\sigma_2)$, as desired.
This proves (ii). Coming to (iii), (ii) and its proof now give us that $d(\sigma_2)f_1 - d(\sigma_1)f_2$ is unstable, applying which, along with the $\mathcal{O}'_{\mathrm{M}}$-invariance of Θ, we get:

$$c(\sigma_1)d(\sigma_2) = \Theta(d(\sigma_2)f_{\sigma_1}) = \Theta(d(\sigma_2)f_1) = \Theta(d(\sigma_1)f_2) = \Theta(d(\sigma_1)f_{\sigma_2}) = c(\sigma_2)d(\sigma_1),$$

which yields $c(\sigma_1)d(\sigma_1)^{-1} = c(\sigma_2)d(\sigma_2)^{-1}$, as desired. □

Proposition 3.3.7. *Let the quasi-split form M^* of F and various auxiliary data such as the inner twist ψ_{M} and the endoscopic datum $\underline{\mathrm{M}}^*$ be as in Notation 3.2.1(i). Let $\mathcal{O}'_{\mathrm{M}^*} \subset \mathrm{Aut}(\mathrm{M}^*)$ be a subgroup with finite image in $\mathrm{Out}(\mathrm{M}^*)$. Let Σ be an $\mathcal{O}'_{\mathrm{M}}$-unitarily stable discrete series L-packet on $\mathrm{M}(F)$, and Σ^* an $\mathcal{O}'_{\mathrm{M}^*}$-unitarily stable discrete series L-packet on $\mathrm{M}^*(F)$. Assume that $\mathcal{O}'_{\mathrm{M}}$ and*

$\mathcal{O}'_{\mathrm{M}^*}$ *fix* $\mathrm{A_M}$ *and* $\mathrm{A_{M^*}}$ *pointwise, and that their finite images* $\bar{\mathcal{O}}'_{\mathrm{M}}$ *and* $\bar{\mathcal{O}}'_{\mathrm{M}^*}$ *in* $\mathrm{Out(M)} = \mathrm{Out(M^*)}$ *are equal. Let* $\Theta_\Sigma \in SD_{\mathrm{ell}}(\mathrm{M})^{\mathcal{O}'_{\mathrm{M}}}$ *and* $\Theta_{\Sigma^*} \in SD_{\mathrm{ell}}(\mathrm{M}^*)^{\mathcal{O}'_{\mathrm{M}^*}}$ *be as in Definition 3.3.2. Assume that the image of* Θ_{Σ^*} *under the isomorphism* $SD_{\mathrm{ell}}(\mathrm{M}^*) \to SD_{\mathrm{ell}}(\mathrm{M})$ *(see Corollary 3.2.7) is supported in* Σ.

(i) There exists a smooth character $\zeta : \mathrm{A_M}(F) = \mathrm{A_{M^*}}(F) \to \mathbb{C}^\times$ *(the identification* $\mathrm{A_M} = \mathrm{A_{M^*}}$ *made using the inner twist* ψ_{M}*), such that each* $\sigma \in \Sigma$ *and* $\sigma^* \in \Sigma^*$ *has a central character restricting to* ζ *on* $\mathrm{A_M}(F) = \mathrm{A_{M^*}}(F)$.

(ii) We normalize the transfer factors as in Remark 3.2.2(i), and give $\mathrm{M}^*(F)$ *and* $\mathrm{M}(F)$ *compatible Haar measures (see Notation 3.2.1(iv)), and similarly with* $\mathrm{A_{M^*}}(F)$ *and* $\mathrm{A_M}(F)$*, so that we get compatible quotient measures on* $(\mathrm{M}^*/\mathrm{A_{M^*}})(F)$ *and* $(\mathrm{M}/\mathrm{A_M})(F)$*. If we use Proposition 3.3.6(ii) to choose* $\Theta_\Sigma = \sum_{\sigma\in\Sigma} d(\sigma)\Theta_\sigma$ *and* $\Theta_{\Sigma^*} = \sum_{\sigma^*\in\Sigma^*} d(\sigma^*)\Theta_{\sigma^*}$*, the image of* Θ_{Σ^*} *under the isomorphism* $SD_{\mathrm{ell}}(\mathrm{M}^*) \to SD_{\mathrm{ell}}(\mathrm{M})$ *equals* $e(\mathrm{M})\Theta_\Sigma$.

(iii) Suppose we are in the situation of (ii). Suppose Θ *is an* $\mathcal{O}'_{\mathrm{M}}$*-invariant stable distribution on* $\mathrm{M}(F)$*, and* Θ^* *an* $\mathcal{O}'_{\mathrm{M}^*}$*-invariant stable distribution on* $\mathrm{M}^*(F)$*, defined by infinite but well-defined (by* [Wal03, Theorem VIII.1.2]*, as explained in Remark 2.2.5) sums:*

$$\Theta = \sum_{\sigma\in\mathrm{Irr}_2(\mathrm{M})_\zeta} c(\sigma)\Theta_\sigma, \qquad \text{and} \qquad \Theta^* = \sum_{\sigma\in\mathrm{Irr}_2(\mathrm{M}^*)_\zeta} c(\sigma^*)\Theta_{\sigma^*},$$

where each $c(\sigma), c(\sigma^*) \in \mathbb{C}$*. If further* Θ *is the image of* Θ^* *under the endoscopic transfer of distributions between* M *and* M**, then we have* $c(\sigma) = e(\mathrm{M}) \cdot d(\sigma)c(\sigma^*) \cdot d(\sigma^*)^{-1}$ *for each* $\sigma \in \Sigma$ *and* $\sigma^* \in \Sigma^*$.

Proof of Proposition 3.3.7. (i) immediately follows from Lemma 3.2.3(i).
Now let us prove (ii), for which we write $\mathscr{Z} = \mathrm{A_M}(F) = \mathrm{A_{M^*}}(F)$, this identification made using ψ_{M}; it will not create confusion, though $\mathscr{Z}$ is being viewed as a subgroup of two different groups. Let '$\mathscr{Z}$-central character' stand for 'the restriction of the central character to $\mathscr{Z}$', so that ζ is the common $\mathscr{Z}$-central character of the elements of Σ as well as of σ^*. For $\sigma \in \Sigma$, let $f_\sigma \in C^\infty_{\mathscr{Z},\zeta}(\mathrm{M}(F))$ be a pseudocoefficient for σ from among those representations of $\mathrm{M}(F)$ with $\mathscr{Z}$-central character ζ, i.e., for every $\sigma' \in \mathrm{Irr_{temp}}(F)$ with $\mathscr{Z}$-central character ζ, we have that $\mathrm{tr}\,\sigma'(f_\sigma)$ equals 0 if $\sigma' \not\cong \sigma$, and that it equals 1 otherwise (here $\sigma'(f_\sigma)$ is defined using an integral over $\mathrm{M}(F)/\mathscr{Z} = (\mathrm{M}/\mathrm{A_M})(F)$). Similarly, we can talk of pseudocoefficients $f_{\sigma^*} \in C^\infty_{\mathscr{Z},\zeta}(\mathrm{M}^*(F))$ for each $\sigma^* \in \Sigma^*$. Let $f \in C^\infty_{\mathscr{Z},\zeta}(\mathrm{M}(F))$ be the average of the pseudocoefficients $f_\sigma \circ \beta^{-1}$ of the representations $\sigma \circ \beta^{-1} \in \Sigma$, as β runs over a set of representatives in $\mathcal{O}'_{\mathrm{M}}$ for the finite group $\mathcal{O}'_{\mathrm{M}} \cdot \mathrm{Int}\,\mathrm{M}(F)/\,\mathrm{Int}\,\mathrm{M}(F)$. Similarly, define f^* by averaging the $f_{\sigma^*} \circ \beta^{-1}$, as β runs over a set of representatives for $\mathcal{O}'_{\mathrm{M}^*} \cdot \mathrm{Int}\,\mathrm{M}^*(F)/\,\mathrm{Int}\,\mathrm{M}^*(F)$.
We can talk of elements in $C^\infty_{\mathscr{Z},\zeta}(\mathrm{M}(F))$ and $C^\infty_{\mathscr{Z},\zeta}(\mathrm{M}^*(F))$ having matching orbital integrals. Further, by Lemma 3.2.3(i), the map $SD(\mathrm{M}^*) \to SD(\mathrm{M})$ takes $SD_\zeta(\mathrm{M}^*)$ to $SD_\zeta(\mathrm{M})$. Note that the actions of $\mathcal{O}'_{\mathrm{M}}$ and $\mathcal{O}'_{\mathrm{M}^*}$ on $SD_\zeta(\mathrm{M})$ and $SD_\zeta(\mathrm{M}^*)$ (which are well-defined as $\mathcal{O}'_{\mathrm{M}}$ and $\mathcal{O}'_{\mathrm{M}^*}$ fix $\mathscr{Z}$ pointwise) each factor through $\bar{\mathcal{O}}'_{\mathrm{M}} = \bar{\mathcal{O}}'_{\mathrm{M}^*}$, and the map $SD_\zeta(\mathrm{M}) \to SD_\zeta(\mathrm{M}^*)$ is equivariant for $\bar{\mathcal{O}}'_{\mathrm{M}} = \bar{\mathcal{O}}'_{\mathrm{M}^*}$ by Lemma 3.2.3(ii). Thus, the image of Θ_{Σ^*} under $SD_{\mathrm{ell}}(\mathrm{M}^*) \to SD_{\mathrm{ell}}(\mathrm{M})$, which is supported in Σ by hypothesis, is also $\mathcal{O}'_{\mathrm{M}}$-invariant, and nonzero (by Corollary 3.2.7), and can hence be written as $a\Theta_\Sigma$ for some nonzero $a \in \mathbb{C}$.

Claim 1. $a^{-1} \cdot d(\sigma)^{-1} f \in C^\infty_{\mathscr{Z},\zeta}(\mathrm{M}(F))$ and $d(\sigma^*)^{-1} f^* \in C^\infty_{\mathscr{Z},\zeta}(\mathrm{M}^*(F))$ have matching orbital integrals.
Since $\mathrm{A_M} = \mathrm{A_{M^*}}$ is split and in particular induced, it is an easy consequence of [Art96, Lemma 6.3], as explained in [LM20, page 587] (see the equivalence of the conditions (A) and (B) there), that Claim 1 follows if we show that for every $\Theta^* \in SD_\zeta(\mathrm{M}^*)$ with image $\Theta \in SD_\zeta(\mathrm{M})$, we have

$$\Theta^*(d(\sigma^*)^{-1} f^*) = \Theta(a^{-1} d(\sigma)^{-1} f). \tag{13}$$

f and f^* are linear combinations of pseudocoefficients of discrete series representations. Therefore, using Proposition 3.2.8 and the compatibility between endoscopic transfer and parabolic induction (see Remark 3.2.2(iii)), we may assume without loss of generality that $\Theta^* \in SD_{\zeta,\mathrm{ell}}(\mathrm{M}^*)$ and $\Theta \in SD_{\zeta,\mathrm{ell}}(\mathrm{M})$. The image of f^* in the space $\mathcal{I}_{\mathscr{Z},\zeta}(\mathrm{M}^*)$ of $\mathrm{Int}\,\mathrm{M}^*(F)$-coinvariants for $C^\infty_{\mathscr{Z},\zeta}(\mathrm{M}^*(F))$ is $\mathcal{O}'_{\mathrm{M}^*}$-invariant. Combining this the analogous observation for f, the hypothesis $\bar{\mathcal{O}}'_{\mathrm{M}} = \bar{\mathcal{O}}_{\mathrm{M}^*}$, and Lemma 3.2.3(ii), we may and do replace Θ^* by its well-defined $\bar{\mathcal{O}}'_{\mathrm{M}^*}$-average and Θ by its $\bar{\mathcal{O}}'_{\mathrm{M}}$-average, to assume that $\Theta^* \in SD_{\zeta,\mathrm{ell}}(\mathrm{M}^*)^{\mathcal{O}'_{\mathrm{M}^*}}$ and $\Theta \in SD_{\zeta,\mathrm{ell}}(\mathrm{M})^{\mathcal{O}'_{\mathrm{M}}}$. Using Definition 3.3.2, we can write $\Theta^* = b\Theta_{\Sigma^*} + \Theta_1^*$, where $b \in \mathbb{C}$, and $\Theta_1^* \in SD_{\zeta,\mathrm{ell}}(\mathrm{M}^*)$ is supported outside Σ^*. Accordingly, we can write $\Theta = ab\Theta_\Sigma + \Theta_1$, where Θ_1 is the image of Θ_1^* under $SD_{\zeta,\mathrm{ell}}(\mathrm{M}^*) \to SD_{\zeta,\mathrm{ell}}(\mathrm{M})$. On the other hand, we can instead apply Definition 3.3.2 to $\Theta \in SD_{\zeta,\mathrm{ell}}(\mathrm{M})^{\mathcal{O}'_{\mathrm{M}}}$, to write $\Theta = b'\Theta_\Sigma + \Theta'_1$, where $b' \in \mathbb{C}$, and $\Theta'_1 \in SD_{\zeta,\mathrm{ell}}(\mathrm{M})$ is supported outside Σ. Using the $\mathcal{O}'_{\mathrm{M}^*}$-invariance of Θ^*, the $\mathcal{O}'_{\mathrm{M}}$-invariance of Θ and the definition of pseudocoefficients, we get:

$$b\Theta_{\Sigma^*}(d(\sigma^*)^{-1}f^*) = b\Theta_{\Sigma^*}(d(\sigma^*)^{-1}f_{\sigma^*}) = b = (ab\Theta_\Sigma)(a^{-1}d(\sigma)^{-1}f_\sigma) = (ab\Theta_\Sigma)(a^{-1}d(\sigma)^{-1}f).$$

From this, and recalling that $\Theta^* = b\Theta_{\Sigma^*} + \Theta_1^*$ and $\Theta = ab\Theta_\Sigma + \Theta_1$, (13), and hence also Claim 1, follows if we show that $\Theta_1^*(d(\sigma^*)^{-1}f^*) = 0 = \Theta_1(a^{-1}d(\sigma)^{-1}f)$. The definition of pseudocoefficients gives us $\Theta_1^*(d(\sigma^*)^{-1}f^*) = 0 = \Theta'_1(a^{-1}d(\sigma)^{-1}f)$ instead, so Claim 1 follows if we prove Claim 2 below.

Claim 2 . We have $\Theta_1 = \Theta'_1$ (and consequently we have $ab = b'$ as well).

Let us give a proof of Claim 2; it will involve some basic facts about the elliptic inner products on $SD_{\zeta,\mathrm{ell}}(\mathrm{M}^*)$ and $D_{\zeta,\mathrm{ell}}(\mathrm{M}) \supset SD_{\zeta,\mathrm{ell}}(\mathrm{M})$, about which more references and explanation are given in the proof of Proposition 3.4.11 below, which uses the same idea in a slightly more general setting. Claim 2 follows if we show that Θ_1 is orthogonal to Θ_Σ under the elliptic inner product on $D_{\zeta,\mathrm{ell}}(\mathrm{M})$, a property that Θ'_1 clearly satisfies (because Θ_Σ and Θ'_1 have disjoint supports). But since Θ_1^* is orthogonal to Θ_{Σ^*} for the elliptic inner product on $SD_{\zeta,\mathrm{ell}}(\mathrm{M}^*)$ (as it is a multiple of the restriction of the elliptic inner product on $D_{\zeta,\mathrm{ell}}(\mathrm{M}^*)$, by [LMW18, Section 4.6, Lemma 3]), Claim 2 follows from the fact that the map $SD_{\zeta,\mathrm{ell}}(\mathrm{M}^*) \to D_{\zeta,\mathrm{ell}}(\mathrm{M})$ is known to take the elliptic inner product on the former space to a multiple of the elliptic inner product on the latter (again by [LMW18, Section 4.6, Lemma 3]).

Thus, we have proved Claim 2, and hence also Claim 1. By [Kot88, Section 3, Proposition 2], given that our choice of measures is compatible with that in [Kot88], we conclude that $e(\mathrm{M}) \cdot a^{-1} \cdot d(\sigma)^{-1}f(1) = d(\sigma^*)^{-1}f^*(1)$ (here, the Kottwitz sign $e(\mathrm{M})$ comes from the definition of singular stable orbital integrals in [Kot88, page 638]; the Kottwitz sign of M^* equals 1 since M^* is quasi-split). Since $\mathrm{Aut}(\mathrm{M})$ and $\mathrm{Aut}(\mathrm{M}^*)$ preserve evaluation at the identity element, we get $e(\mathrm{M}) \cdot a^{-1} \cdot d(\sigma)^{-1}f_\sigma(1) = d(\sigma^*)^{-1}f_{\sigma^*}(1)$. But it is easy to see that $f_\sigma(1) = d(\sigma) \neq 0$ and $f_{\sigma^*}(1) = d(\sigma^*) \neq 0$ (see [DKV84, Proposition A.3.g]), so we get $a = e(\mathrm{M})$, giving (ii).

To see (iii), apply Claim 1 to Θ and Θ^*; we then get:

$$c(\sigma)e(\mathrm{M})^{-1}d(\sigma)^{-1} = \Theta(e(\mathrm{M})^{-1}d(\sigma)^{-1}f_\sigma) = \Theta(e(\mathrm{M})^{-1}d(\sigma)^{-1}f) = \Theta^*(d(\sigma^*)^{-1}f^*) = \Theta^*(d(\sigma^*)^{-1}f_{\sigma^*}) = c(\sigma^*)d(\sigma^*)^{-1},$$

giving (iii). □

Lemma 3.3.8. *Suppose Σ, Σ' are $\mathcal{O}'_{\mathrm{M}}$-unitarily stable discrete series L-packets on $\mathrm{M}(F)$. Assume that $\mathcal{O}'_{\mathrm{M}}$ fixes A_{M} pointwise. Then:*

- *(i) The space of nonzero stable $\mathcal{O}'_{\mathrm{M}}$-invariant virtual characters on $\mathrm{M}(F)$ supported on Σ is one-dimensional. Thus, Σ determines Θ_Σ up to a nonzero complex multiple.*
- *(ii) Σ and Σ' are either equal or disjoint.*
- *(iii) If M is a Levi subgroup of G, $\mathcal{O}'_{\mathrm{M}} = \mathcal{O}_{\mathrm{M}}$, and Hypothesis 2.5.1 is satisfied, then $\Sigma \in \Phi_2(\mathrm{M})$.*
- *(iv) Let θ be an F-rational automorphism of M normalizing $\mathcal{O}'_{\mathrm{M}}$, and suppose χ belongs to the group $\mathrm{Hom}_{\mathrm{cts}}(\mathrm{M}(F), \mathbb{C}^\times)^{\mathcal{O}'_{\mathrm{M}}}$ of (quasi-)characters of $\mathrm{M}(F)$ fixed by $\mathcal{O}'_{\mathrm{M}}$. Assume that χ is unitary. Then*
$$(\Sigma \circ \theta) \otimes \chi := \{(\sigma \circ \theta) \otimes \chi \mid \sigma \in \Sigma\}$$
is an $\mathcal{O}'_{\mathrm{M}}$-unitarily stable discrete series L-packet on $\mathrm{M}(F)$, supporting the stable $\mathcal{O}'_{\mathrm{M}}$-invariant virtual character $\Theta_{(\Sigma\circ\theta)\otimes\chi} := (\Theta_\Sigma \circ \theta)\chi$.

Proof. All assertions are easy. (i) is immediate from the definitions, and (iii) follows from Lemma 2.5.3(i). For (ii), if $\Sigma \cap \Sigma' \neq \emptyset$, then expanding $\Theta_{\Sigma'}$ as $c_1\Theta_\Sigma + c_2\Theta'$ as in Definition 3.3.2 gives the inclusion $\Sigma \subset \Sigma'$, where we use the consequence of Proposition 3.3.6(ii) that the coefficients of $\Theta_{\sigma'}$ in $\Theta_{\Sigma'}$ and Θ_σ in Θ_Σ are nonzero for all $\sigma \in \Sigma$ and $\sigma' \in \Sigma'$; similarly $\Sigma' \subset \Sigma$, so $\Sigma = \Sigma'$. For (iv), one uses that $\Sigma \circ \theta$ and χ are $\mathcal{O}'_{\mathrm{M}}$-invariant, the former since θ normalizes $\mathcal{O}'_{\mathrm{M}}$. □

Thanks to (ii) and (iv) of the above lemma, we get the following easy corollary:

Corollary 3.3.9. *Let $\mathcal{O}''_{\mathrm{M}}$ be some group of F-rational automorphisms of* M *normalizing $\mathcal{O}'_{\mathrm{M}}$. Let $\mathcal{F}_0$ be a set of $\mathcal{O}'_{\mathrm{M}}$-unitarily stable discrete series L-packets on* $\mathrm{M}(F)$ *in the sense of Definition 3.3.2, and let*

$$\begin{aligned}\mathcal{F} &= \{(\Sigma \circ \theta) \otimes \chi \mid \Sigma \in \mathcal{F}_0, \theta \in \mathcal{O}''_{\mathrm{M}}, \chi \in \mathrm{Hom}_{\mathrm{cts}}(\mathrm{M}(F), \mathbb{C}^\times)^{\mathcal{O}'_{\mathrm{M}}} \textit{ is unitary}\} \\ &= \{(\Sigma \otimes \chi) \circ \theta \mid \Sigma \in \mathcal{F}_0, \theta \in \mathcal{O}''_{\mathrm{M}}, \chi \in \mathrm{Hom}_{\mathrm{cts}}(\mathrm{M}(F), \mathbb{C}^\times)^{\mathcal{O}'_{\mathrm{M}}} \textit{ is unitary}\}.\end{aligned}$$

Extend the definition of Θ_Σ to $\mathcal{F}$ as follows: for $\Sigma \in \mathcal{F}$, make a choice of $\Sigma_0 \in \mathcal{F}_0, \theta \in \mathcal{O}''_{\mathrm{M}}$ and $\chi \in \mathrm{Hom}_{\mathrm{cts}}(\mathrm{M}(F), \mathbb{C}^\times)^{\mathcal{O}'_{\mathrm{M}}}$ such that $\Sigma = (\Sigma_0 \circ \theta) \otimes \chi$, and set $\Theta_\Sigma = (\Theta_{\Sigma_0} \circ \theta)\chi$. Then:

(i) *The (distinct) elements of $\mathcal{F}$ are all disjoint.*

(ii) *Given $\Theta \in SD_{\mathrm{ell}}(\mathrm{M})^{\mathcal{O}'_{\mathrm{M}}}$, write $\Theta = \Theta_1 + \Theta_2$, where Θ_1 (resp., Θ_2) is supported outside (resp., inside) the union of the members of $\mathcal{F}$. Then $\Theta_1, \Theta_2 \in SD_{\mathrm{ell}}(\mathrm{M})^{\mathcal{O}'_{\mathrm{M}}}$, and Θ_2 is uniquely a linear combination of the Θ_Σ, as Σ runs over $\mathcal{F}$.*

Proof. (i) follows from (ii) and (iv) of Lemma 3.3.8. Given (i), and using that each element of $\mathcal{F}$ is also an $\mathcal{O}'_{\mathrm{M}}$-unitarily stable discrete series L-packet (by (iv) of Lemma 3.3.8), (ii) then follows by induction. □

3.4. **Some criteria to prove unitary stability.** The main results of this subsection are Propositions 3.4.2 and 3.4.11, each of which gives a 'character theoretic' criterion intended to help verify whether a given 'candidate packet', in the form of a finite set of discrete series representations, forms a unitarily stable discrete series L-packet. Proposition 3.4.2 is inspired by, and at least aspires to be a commentary on, a remark in [Mg14, Section 4.8]. We feel that it should be possible to check the criterion in this proposition whenever one can verify stability for the given candidate packet.

The criterion of Proposition 3.4.11 is more involved, since it almost amounts to verifying the endoscopic character identities for the candidate packet, but has the advantage that it has already been verified by Kaletha for his regular supercuspidal packets when $p \gg 0$, allowing us to conclude that these packets are unitarily stable. Thus, there are indeed many provably unitarily stable packets, partially justifying the point of Definition 3.3.2.

These two propositions should be well-known to experts, and the proof of Proposition 3.4.11 seems to have some similarities with that of [MY20, Proposition 4.2].

Notation 3.4.1. (i) In this subsection, given a twisted space $(\mathrm{M}, \tilde{\mathrm{M}})$ over F, with M reductive, $\tilde{\mathrm{M}}(F)_{\mathrm{ell}}$ will denote the set of strongly regular semisimple elliptic elements of $\tilde{\mathrm{M}}(F)$; it is an open subset of $\tilde{\mathrm{M}}(F)$. In this subsection, given a triple $(\mathrm{M}, \tilde{\mathrm{M}}, \mathbf{a})$ and the associated unitary character $\omega : \mathrm{M}(F) \to \mathbb{C}^\times$ as in Notation 3.1.1, an element $\Theta \in D_{\mathrm{ell}}(\tilde{\mathrm{M}}, \omega)$ will be viewed by restriction as a locally constant function $\tilde{\mathrm{M}}(F)_{\mathrm{ell}} \to \mathbb{C}$ (this restriction determines Θ, as follows from [MgW18, Theorem 7.3]). Thus, $D_{\mathrm{ell}}(\tilde{\mathrm{M}}, \omega)$ can be viewed as a collection of locally constant functions $\tilde{\mathrm{M}}(F)_{\mathrm{ell}} \to \mathbb{C}$.

(ii) If $(\mathrm{M}, \tilde{\mathrm{M}})$ is a twisted space over F, with M quasi-split reductive and $\tilde{\mathrm{M}}$ of inner torsion (i.e., $\mathrm{Int}\, \tilde{m}$ is an inner automorphism of M for each $\tilde{m} \in \tilde{\mathrm{M}}(F)$), then for each $\gamma \in \tilde{\mathrm{M}}(F)_{\mathrm{ell}}$, we let $\kappa(\gamma)$ be the number of conjugacy classes in the stable conjugacy class of γ. Moreover, given any virtual character $\Theta \in D_{\mathrm{ell}}(\tilde{\mathrm{M}}(F))$, we define Θ^{st} to be the function $\tilde{\mathrm{M}}(F)_{\mathrm{ell}} \to \mathbb{C}$, given by:

$$\Theta^{\mathrm{st}}(\gamma) = \kappa(\gamma)^{-1} \sum_{\gamma'} \Theta(\gamma'),$$

where γ' runs over a set of representatives for the $\mathrm{M}(F)$-conjugacy classes in the stable conjugacy class of γ.

(iii) For the rest of this subsection, let M be a connected reductive group over F; we will put ourselves in the situation of Notation 3.3.1, but with $\mathcal{O}'_{\mathrm{M}}$ trivial.

Now we can state the first main result of this subsection.

Proposition 3.4.2. *Let $\Sigma \subset \mathrm{Irr}_2(\mathrm{M})$ be a finite subset. Then Σ is a unitarily stable discrete series L-packet (see Definition 3.3.2) if and only if the following conditions are satisfied:*

(a) The $\Theta_\sigma^{\mathrm{st}}$, as σ varies over Σ, are all proportional to each other; and
(b) There exist nonzero complex numbers c_σ for each $\sigma \in \Sigma$, such that $\Theta_\Sigma := \sum_{\sigma\in\Sigma} c_\sigma \Theta_\sigma$ is stable, i.e., belongs to $SD_{\mathrm{ell}}(\mathrm{M}) \subset D_{\mathrm{ell}}(\mathrm{M})$.

Moreover, when these conditions are satisfied, we have that $d(\sigma)^{-1}\Theta_\sigma^{\mathrm{st}} = d(\sigma')^{-1}\Theta_{\sigma'}^{\mathrm{st}}$ for any $\sigma, \sigma' \in \Sigma$, and that for any $\sigma_0 \in \Sigma$:

$$\left(\sum_{\sigma\in\Sigma} d(\sigma)^2\right)^{-1} \cdot \sum_{\sigma\in\Sigma} d(\sigma)\Theta_\sigma = d(\sigma_0)^{-1} \cdot \Theta_{\sigma_0}^{\mathrm{st}}.$$

Remark 3.4.3. It would be satisfying if Proposition 3.4.2 could be interpreted as giving a 'stable' version of the classical result that orbital integrals of pseudocoefficients at elliptic strongly regular elements yield character values (e.g., the much simpler elliptic untwisted case of [MgW18, Theorem 7.2]). However, we do not know if such an interpretation is appropriate.

We now proceed to do some preparations for the proof of Proposition 3.4.2.

Notation 3.4.4. Let a triple $(\mathrm{M}, \tilde{\mathrm{M}}, \mathbf{a})$ and the associated unitary character $\omega : \mathrm{M}(F) \to \mathbb{C}^\times$ be as in Notation 3.1.1, and assume notation from Notation 3.1.2. Note that for any closed subgroup $\mathscr{Z} \subset \mathrm{Z}_{\mathrm{M}}(F)$ we have a decomposition

$$D_{\mathrm{ell}}(\tilde{\mathrm{M}}, \omega) = \bigoplus_\zeta D_{\mathscr{Z},\zeta,\mathrm{ell}}(\tilde{\mathrm{M}}, \omega), \tag{14}$$

where ζ varies over unitary characters of $\mathscr{Z}$. We define an inner product $\langle\cdot,\cdot\rangle$ on $D_{\mathrm{ell}}(\tilde{\mathrm{M}}, \omega)$, using (14) with $\mathscr{Z}$ taken to be by $\mathrm{A}_{\tilde{\mathrm{M}}}(F)$: we require $\langle\cdot,\cdot\rangle$ to restrict to the elliptic inner product (see [MgW18, Section 7.3]) on each $D_{\mathrm{A}_{\tilde{\mathrm{M}}}(F),\zeta,\mathrm{ell}}(\tilde{\mathrm{M}}, \omega)$, and the $D_{\mathrm{A}_{\tilde{\mathrm{M}}}(F),\zeta,\mathrm{ell}}(\tilde{\mathrm{M}}, \omega)$ for distinct ζ to be orthogonal to each other. If either $\tilde{\mathrm{M}} = \mathrm{M}$ or if M is quasi-split and $\tilde{\mathrm{M}}$ is of inner torsion, and if ω is trivial, we get by restriction an inner product on $SD_{\mathrm{ell}}(\tilde{\mathrm{M}}) \subset D_{\mathrm{ell}}(\tilde{\mathrm{M}})$. We will use the following well-known property of this inner product (see [MgW18, Theorem 7.3(i)]): if $\tilde\sigma_1, \tilde\sigma_2$ are ω-representations of $\tilde{\mathrm{M}}(F)$, whose underlying $\mathrm{M}(F)$-representations σ_1, σ_2 belong to $\mathrm{Irr}_2(\mathrm{M})$, then $\langle\Theta_{\sigma_1}, \Theta_{\sigma_2}\rangle = 0$ unless $\sigma_1 \cong \sigma_2$.

The inner product on $D_{\mathrm{ell}}(\mathrm{M})$ having been defined, we can now state the following lemma, to be proved later, and modulo which Proposition 3.4.2 is almost formal.

Lemma 3.4.5. *If $\Theta \in D_{\mathrm{ell}}(\mathrm{M})$, then Θ^{st} is the image of Θ under the orthogonal projection $D_{\mathrm{ell}}(\mathrm{M}) \to SD_{\mathrm{ell}}(\mathrm{M})$. In particular, $\Theta^{\mathrm{st}} \in SD_{\mathrm{ell}}(\mathrm{M})$ (when viewed as in Notation 3.4.1(i)).*

Proof of Proposition 3.4.2, assuming Lemma 3.4.5. Since the condition (b) concerning the stable virtual character Θ_Σ is clearly necessary, we may and do assume it.
It is easy to see from the definition of the elliptic inner product (see Notation 3.4.4) that Σ is a unitarily stable discrete series L-packet if and only if each $\Theta \in SD_{\mathrm{ell}}(\mathrm{M})$ that is orthogonal to Θ_Σ in $SD_{\mathrm{ell}}(\mathrm{M}) \subset D_{\mathrm{ell}}(\mathrm{M})$ is also orthogonal in $D_{\mathrm{ell}}(\mathrm{M})$ to Θ_σ for each $\sigma \in \Sigma$. For each $\Theta \in SD_{\mathrm{ell}}(\mathrm{M})$ and each $\Theta' \in D_{\mathrm{ell}}(\mathrm{M})$ with projection $\bar\Theta'$ in $SD_{\mathrm{ell}}(\mathrm{M})$, the elliptic inner product $\langle\Theta, \Theta'\rangle$ inside $D_{\mathrm{ell}}(\mathrm{M})$ equals $\langle\Theta, \bar\Theta'\rangle$ taken inside $SD_{\mathrm{ell}}(\mathrm{M})$. Therefore, using Lemma 3.4.5, we conclude that Σ is a unitarily stable discrete series L-packet if and only if each $\Theta \in SD_{\mathrm{ell}}(\mathrm{M})$ that is orthogonal to Θ_Σ in $SD_{\mathrm{ell}}(\mathrm{M})$ is also orthogonal in $SD_{\mathrm{ell}}(\mathrm{M})$ to $\Theta_\sigma^{\mathrm{st}}$ for each $\sigma \in \Sigma$. This is clearly equivalent to Θ_Σ being proportional to $\Theta_\sigma^{\mathrm{st}}$ for each $\sigma \in \Sigma$, which is easily seen to be equivalent to the condition (a). Here, we note that each $\Theta_\sigma^{\mathrm{st}}$ is nonzero: this follows from Lemma 3.4.5 and [Mg14, Proposition 2.1].
For the last assertion, note from Proposition 3.3.6(ii) that $\sum d(\sigma)\Theta_\sigma$ is stable, where σ runs over Σ. Further, it is easy to see that $d(\sigma)^{-1}\Theta_\sigma^{\mathrm{st}}$ is independent of $\sigma \in \Sigma$: either use the argument of [Mg14, Proposition 2.1], or note that for $\sigma_1, \sigma_2 \in \Sigma$, the constant of proportionality between the $\Theta_{\sigma_i}^{\mathrm{st}}$ equals that between the $\langle\Theta_{\sigma_i}^{\mathrm{st}}, \sum d(\sigma)\Theta_\sigma\rangle = \langle\Theta_{\sigma_i}, \sum d(\sigma)\Theta_\sigma\rangle = d(\sigma_i)$. Using these two

observations:

$$\sum d(\sigma)\Theta_\sigma = (\sum d(\sigma)\Theta_\sigma)^{\mathrm{st}} = \sum d(\sigma)^2 \cdot (d(\sigma)^{-1}\Theta_\sigma)^{\mathrm{st}} = \Big(\sum d(\sigma)^2\Big) \cdot d(\sigma_0)^{-1}\Theta^{\mathrm{st}}_{\sigma_0},$$

where each sum is over $\sigma \in \Sigma$. This gives the last assertion of the lemma. □

We still need to prove Lemma 3.4.5 to complete the proof of Proposition 3.4.2, for which we now make some further preparations.

Remark 3.4.6. In what follows, we will use a lot of observations from [LMW18]. In each case it will be implicitly left to the reader to verify that, though the setting of [LMW18] involves an unramified quasi-split group in place of our M, the observations that we will use do not depend on these assumptions.

The following remark will also introduce some notation.

Remark 3.4.7. Let a triple $(\mathrm{M}, \tilde{\mathrm{M}}, \mathbf{a})$ and the associated unitary character $\omega : \mathrm{M}(F) \to \mathbb{C}^\times$ be as in Notation 3.1.1, and assume notation from Notation 3.1.2. Suppose $\mathscr{Z}$ is a closed subgroup of $\mathrm{Z}_{\tilde{\mathrm{M}}}(F)$, such that $\mathscr{Z} \cap \mathrm{A}_{\tilde{\mathrm{M}}}(F)$ is of finite index in $\mathrm{A}_{\tilde{\mathrm{M}}}(F)$. Let $\zeta : \mathscr{Z} \to \mathbb{C}^\times$ be a unitary character, and fix Haar measures on $\mathscr{Z}$ and $\mathrm{M}(F)$, the latter also giving a measure on $\tilde{\mathrm{M}}(F)$.

(i) Let $C^\infty_{\zeta,\mathrm{cusp}}(\tilde{\mathrm{M}}(F)) \subset C^\infty_\zeta(\tilde{\mathrm{M}}(F))$ be the subspace consisting of functions that are cuspidal in the sense of [MgW18, Sections 7.1 and 7.2] (i.e., whose nonelliptic strongly regular semisimple orbital integrals vanish), and let $\mathcal{I}_{\mathscr{Z},\zeta,\mathrm{cusp}}(\tilde{\mathrm{M}}, \omega)$ be the quotient of $C^\infty_{\zeta,\mathrm{cusp}}(\tilde{\mathrm{M}}(F))$ by the subspace consisting of functions whose strongly regular semisimple ω-twisted orbital integrals vanish. When M is quasi-split and $\tilde{\mathrm{M}}$ has inner torsion, and ω is trivial, we define $\mathcal{SI}_{\mathscr{Z},\zeta,\mathrm{cusp}}(\tilde{\mathrm{M}})$ as the quotient of $C^\infty_{\zeta,\mathrm{cusp}}(\tilde{\mathrm{M}}(F))$ by the subspace consisting of functions whose strongly regular semisimple stable orbital integrals vanish (this is a priori slightly different from the 'variant with central character' of the definition in [MW16, page 57], but equivalent to it, thanks to the surjectivity of the obvious map $\mathcal{I}_{\mathscr{Z},\zeta,\mathrm{cusp}}(\tilde{\mathrm{M}}) \to \mathcal{SI}_{\mathscr{Z},\zeta,\mathrm{cusp}}(\tilde{\mathrm{M}})$; see [MW16, Proposition I.4.11], or rather its variant with central character, discussed below).

(ii) For $\gamma \in \tilde{\mathrm{M}}(F)_{\mathrm{ell}}$, we will normalize, in this subsection alone, the ω-twisted orbital integral of $f \in C^\infty_{\zeta,\mathrm{cusp}}(\tilde{\mathrm{M}}(F))$ at γ as:

$$O(\gamma, \omega, f) := \int_{\mathscr{Z}\backslash \mathrm{M}(F)} \omega(m) f(m^{-1}\gamma m)\, dm, \tag{15}$$

where we use the measures on $\mathrm{M}(F)$ and $\mathscr{Z}$ fixed in the present collection of notation (and recall that ω is trivial on $\mathrm{Z}_{\tilde{\mathrm{M}}}(F) \supset \mathscr{Z}$, since we are imposing the conditions of Notation 3.1.1). The integral defining $O(\gamma, \omega, f)$ is absolutely convergent by ellipticity. Recall that if ω is trivial, we write $O(\gamma, f) = O(\gamma, \omega, f)$. Recall that the stable orbital integrals $SO(\gamma, f)$, when M is quasi-split and $\tilde{\mathrm{M}}$ has inner torsion and ω is trivial, are defined by summing the $O(\gamma', f)$ as γ' runs over a set of representatives for the $\mathrm{M}(F)$-conjugacy classes in the stable conjugacy class of γ.

(iii) If $a \in \mathcal{I}_{\zeta,\mathrm{cusp}}(\tilde{\mathrm{M}}, \omega)$ and $\gamma \in \tilde{\mathrm{M}}(F)_{\mathrm{ell}}$, $a(\gamma)$ will denote $O(\gamma, \omega, f)$, for any $f \in C^\infty_{\zeta,\mathrm{cusp}}(\tilde{\mathrm{M}}(F))$ with image a; it is independent of a by the density of strongly regular semisimple orbital integrals in the space of invariant distributions.

(iv) For each $\underline{\mathrm{H}} \in \mathcal{E}(\tilde{\mathrm{M}}, \mathbf{a})$, we fix associated auxiliary data such as the z-extension H_1 of H (see Notation 3.1.2(iii)). Let us now recall from [LMW18, Section 4.5] a subset $\mathcal{E}(\tilde{\mathrm{M}}, \mathbf{a})_\zeta \subset \mathcal{E}(\tilde{\mathrm{M}}, \mathbf{a})$, and for each $\underline{\mathrm{H}} \in \mathcal{E}(\tilde{\mathrm{M}}, \mathbf{a})$ a pair $(\mathscr{Z}_1, \zeta_1)$ consisting of a closed subgroup $\mathscr{Z}_1 \subset \mathrm{Z}_{\mathrm{H}_1}(F)$ and a unitary character $\zeta_1 : \mathscr{Z}_1 \to \mathbb{C}^\times$ ($\mathcal{E}(\tilde{\mathrm{M}}, \mathbf{a})_\zeta$ is the $\mathfrak{E}_{\mathcal{Z},\mu}$ defined in [LMW18, Section 4.5, between (4) and (5)], while $(\mathscr{Z}_1, \zeta_1)$ is denoted $(\mathcal{Z}'_1, \mu'_1)$ in [LMW18]; we suppress the dependence of these on $\underline{\mathrm{H}}$ and the auxiliary choices for lightness of notation). $\mathscr{Z}_1$ is the inverse image in $\mathrm{Z}_{\mathrm{H}_1}(F)$ of the image $\mathscr{Z}_{\mathrm{H}}$ of $\mathscr{Z} \hookrightarrow \mathrm{Z}_{\mathrm{M}}(F) \to \mathrm{Z}_{\mathrm{H}}(F)$. $\mathcal{E}(\tilde{\mathrm{M}}, \mathbf{a})_\zeta \subset \mathcal{E}(\tilde{\mathrm{M}}, \mathbf{a})$ is defined to be the subset consisting of $\underline{\mathrm{H}}$ such that there exists a (necessarily unitary) character $\zeta_1 : \mathscr{Z}_1 \to \mathbb{C}^\times$ with the property that whenever strongly regular semisimple elements $\delta_1 \in \tilde{\mathrm{H}}_1(F)$ and $\tilde{\gamma} \in \tilde{\mathrm{M}}(F)$ match (such δ_1 and $\tilde{\gamma}$ exist as $\underline{\mathrm{H}}$ is relevant), we have an equality of transfer factors:

$$\Delta(z_1\delta_1, z\tilde{\gamma}) = \zeta_1(z_1)^{-1}\zeta(z)\Delta(\delta_1, \tilde{\gamma}),$$

for all $z_1 \in \mathscr{Z}_1$ and $z \in \mathscr{Z}$ with the same image in $\mathscr{Z}_{\mathrm{H}}$. In view of Remark 3.4.6, we remark that the '$\lambda_{\mathfrak{z}}$' of [LMW18, Section 4.5] is the 'λ_C' of [KS99, page 53]. Note that it follows from the ellipticity of $\underline{\mathrm{H}}$ that $\mathscr{Z}_1 \cap \mathrm{A}_{\mathrm{H}_1}(F)$ is of finite index in $\mathrm{A}_{\mathrm{H}_1}(F)$.

(v) As in [LMW18, Section 4.5, (5)], endoscopic transfer 'with $\mathscr{Z}$-central character ζ' defines an isomorphism of vector spaces:

$$\mathcal{I}_{\mathscr{Z},\zeta,\mathrm{cusp}}(\tilde{\mathrm{M}}, \omega) \to \bigoplus_{\underline{\mathrm{H}} \in \mathcal{E}(\tilde{\mathrm{M}},\mathbf{a})_\zeta} \mathcal{SI}_{\mathscr{Z}_1,\zeta_1,\mathrm{cusp}}(\tilde{\mathrm{H}}_1)^{\mathrm{Aut}(\underline{\mathrm{H}})}. \tag{16}$$

Dually, as in [LMW18, Section 4.5, (6)], we get an isomorphism:

$$\bigoplus_{\underline{\mathrm{H}} \in \mathcal{E}(\mathrm{M},\mathbf{a})_\zeta} \mathbf{T}_{\underline{\mathrm{H}}} = \bigoplus_{\underline{\mathrm{H}} \in \mathcal{E}(\mathrm{M},\mathbf{a})_\zeta} \mathbf{T}_{\underline{\mathrm{H}},\mathrm{ell}} : SD_{\mathscr{Z}_1,\zeta_1,\mathrm{ell}}(\tilde{\mathrm{H}}_1)_{\mathrm{Aut}(\underline{\mathrm{H}})} \to D_{\mathscr{Z},\zeta,\mathrm{ell}}(\tilde{\mathrm{M}}, \omega), \tag{17}$$

which is a restriction of (8) (any dependence on measures is at the level of the transfer of functions, and not at the level of the transfer of Harish-Chandra characters).

(vi) As in [LMW18, Section 4.4, just below (4)], we average with respect to $\mathrm{Aut}(\underline{\mathrm{H}})$ to identify each $SD_{\mathscr{Z}_1,\zeta_1,\mathrm{ell}}(\tilde{\mathrm{H}}_1)_{\mathrm{Aut}(\underline{\mathrm{H}})}$ with the space $SD_{\mathscr{Z}_1,\zeta_1,\mathrm{ell}}(\tilde{\mathrm{H}}_1)^{\mathrm{Aut}(\underline{\mathrm{H}})}$ of $\mathrm{Aut}(\underline{\mathrm{H}})$-invariants. This space gets an inner product from its inclusion in $D_{\mathscr{Z}_1,\zeta_1,\mathrm{ell}}(\tilde{\mathrm{H}}_1)$ (which has an inner product as in Notation 3.4.4), also denoted $\langle \cdot, \cdot \rangle$. For $\Theta_1^{\underline{\mathrm{H}}}, \Theta_2^{\underline{\mathrm{H}}} \in SD_{\mathscr{Z}_1,\zeta_1,\mathrm{ell}}(\tilde{\mathrm{H}}_1)^{\mathrm{Aut}(\underline{\mathrm{H}})}$, we have, by [LMW18, Section 4.6, (5) and Lemma 3], an equality

$$\langle \mathbf{T}_{\underline{\mathrm{H}}}(\Theta_1^{\underline{\mathrm{H}}}), \mathbf{T}_{\underline{\mathrm{H}}}(\Theta_2^{\underline{\mathrm{H}}}) \rangle = c(\tilde{\mathrm{M}}, \underline{\mathrm{H}})^{-1} \langle \Theta_1^{\underline{\mathrm{H}}}, \Theta_2^{\underline{\mathrm{H}}} \rangle, \tag{18}$$

describing the behavior of the inner products we have defined with respect to $\mathbf{T}_{\underline{\mathrm{H}}}$; here, $c(\tilde{\mathrm{M}}, \underline{\mathrm{H}})$ is the constant from [LMW18, Section 4.6, just before Lemma 2].

Here, in view of Remark 3.4.6, let us add that the key point is the inner product formula of [MW16, Proposition I.4.17], from which one deduces a 'variant with central character' involving (16) (see [LMW18, Section 4.6, Lemma 2]), which in turn by duality gives the inner product formula of [LMW18, Section 4.6, (5)] for virtual characters.

(vii) It follows from (vi) above that the components $\mathbf{T}_{\underline{\mathrm{H}}}(SD_{\mathscr{Z}_1,\zeta_1,\mathrm{ell}}(\tilde{\mathrm{H}}_1)_{\mathrm{Aut}(\underline{\mathrm{H}})})$ of the decomposition of $D_{\mathscr{Z},\zeta,\mathrm{ell}}(\tilde{\mathrm{M}}, \omega)$ given by (17) are orthogonal to each other.

(viii) Let us recall the antilinear isomorphism $\iota_{\mathscr{Z},\zeta} : D_{\mathscr{Z},\zeta,\mathrm{ell}}(\tilde{\mathrm{M}}(F), \omega) \to \mathcal{I}_{\mathscr{Z},\zeta,\mathrm{cusp}}(\tilde{\mathrm{M}}, \omega)$ described in [LMW18, Section 4.6, between (2) and (3)], and denoted by '$\iota_{\mathscr{Z},\mu}$' in that reference (here, an antilinear isomorphism refers to a bijective additive map that is semilinear for complex conjugation). $\iota_{\mathscr{Z},\zeta}$ is defined to satisfy:

$$\int_{\mathscr{Z}\backslash\tilde{\mathrm{M}}(F)} \Theta(\gamma) f_2(\gamma)(d\gamma/dz) =: \Theta(f_2) = (\iota_{\mathscr{Z},\zeta}(\Theta), f_2)_{\mathscr{Z},\zeta,\mathrm{ell}}, \tag{19}$$

where the $(\cdot,\cdot)_{\mathscr{Z},\zeta,\mathrm{ell}}$ on the right-hand side refers to the inner product on $\mathcal{I}_{\mathscr{Z},\zeta,\mathrm{cusp}}(\tilde{\mathrm{M}}, \omega)$ as in [LMW18, Section 4.6, (2)]; thus, if h_1, h_2 map to f_1, f_2 under the obvious map $C^\infty_{\mathrm{cusp}}(\tilde{\mathrm{M}}(F)) \to C^\infty_{\zeta,\mathrm{cusp}}(\tilde{\mathrm{M}}(F)) \to \mathcal{I}_{\mathscr{Z},\zeta,\mathrm{cusp}}(\tilde{\mathrm{M}})$ (see the map $p_{\mathscr{Z},\mu}$ of [LMW18, page 315]), then we have a formula of the form:

$$(f_1, f_2)_{\mathscr{Z},\zeta,\mathrm{ell}} := \int_{\mathscr{Z}} \int_{\tilde{\mathrm{M}}(F)_{\mathrm{ell}}/\mathrm{conj}} D^{\tilde{\mathrm{M}}}(\gamma)\, \mathrm{meas}(\mathscr{Z}\backslash \mathrm{M}^\gamma(F))^{-1} \overline{O(\gamma, \omega, h_1)} O(z\gamma, \omega, h_2) \zeta(z)\, d\gamma dz, \tag{20}$$

where M^γ denotes the centralizer of γ. To cross check this formula, use [LMW18, the discussion shortly before (1) in Section 4.6, and (4) in Section 4.3], and take into account various slight differences in notation (such as the definition of orbital integrals, in particular our using the unnormalized ones) between us and [LMW18]. [LMW18] itself refers to [MW16, Section I.4.17] for some of the notation, such as the measure on the set

$\tilde{\mathrm{M}}(F)_{\mathrm{ell}}/\operatorname{conj}$ of $\mathrm{M}(F)$-conjugacy classes in $\tilde{\mathrm{M}}(F)_{\mathrm{ell}}$. Clearly, $\iota_{\mathscr{Z},\mu}$ depends on the chosen measures on $\mathrm{M}(F)$ and $\mathscr{Z}$.

(ix) The antilinear isomorphism $\iota_{\mathscr{Z},\zeta} : D_{\mathscr{Z},\zeta,\mathrm{ell}}(\tilde{\mathrm{M}},\omega) \to \mathcal{I}_{\mathscr{Z},\zeta,\mathrm{cusp}}(\tilde{\mathrm{M}},\omega)$ from (viii) above, we claim, is described as follows: $\iota_{\mathscr{Z},\zeta}(\Theta) = f_1$ if and only if for all $\gamma \in \tilde{\mathrm{M}}(F)_{\mathrm{ell}}$, we have:

$$\Theta(\gamma) = \overline{O(\gamma,\omega,f_1)}. \tag{21}$$

Write $f_1 = \iota_{\mathscr{Z},\zeta}(\Theta)$, and let h_1, h_2 map to f_1, f_2 under the obvious map $C^\infty_{\mathrm{cusp}}(\tilde{\mathrm{M}}(F)) \to C^\infty_{\zeta,\mathrm{cusp}}(\tilde{\mathrm{M}}(F)) \to \mathcal{I}_{\mathscr{Z},\zeta,\mathrm{cusp}}(\tilde{\mathrm{M}})$. Then (19) is equivalent to:

$$\Theta(h_2) = \int_{\mathscr{Z}} \int_{\tilde{\mathrm{M}}(F)_{\mathrm{ell}}/\operatorname{conj}} D^{\tilde{\mathrm{M}}}(\gamma)\,\mathrm{meas}(\mathscr{Z}\backslash \mathrm{M}^\gamma(F))^{-1}\overline{O(\gamma,\omega,h_1)}O(z\gamma,\omega,h_2)\zeta(z)\,d\gamma dz.$$

As far as the left-hand side is concerned, $\Theta(h_2)$, which is an integral over $\tilde{\mathrm{M}}(F)$, can be evaluated in terms of an integral over $\tilde{\mathrm{M}}(F)_{\mathrm{ell}}/\operatorname{conj}$ using an equality given in [MW16, Section I.4.17, shortly before (1)] (and keeping in mind that h_2 is a cuspidal function and that $\Theta(h_2)$ has an ω-equivariance on conjugation). On the right-hand side, we first replace γ by $z^{-1}\gamma$ and then change the order of integration and replace z by z^{-1}, and use the relation between h_1 and f_1 (and that $\zeta(z^{-1}) = \overline{\zeta(z)}$), to get that (19) is equivalent to:

$$\int_{\tilde{\mathrm{M}}(F)_{\mathrm{ell}}/\operatorname{conj}} D^{\tilde{\mathrm{M}}}(\gamma)m_{\mathscr{Z}}^{-1}\Theta(\gamma)O(\gamma,\omega,h_2)\,d\gamma = \int_{\tilde{\mathrm{M}}(F)_{\mathrm{ell}}/\operatorname{conj}} D^{\tilde{\mathrm{M}}}(\gamma)m_{\mathscr{Z}}^{-1}\overline{O(\gamma,\omega,f_1)}O(\gamma,\omega,h_2)\,d\gamma,$$

where $m_{\mathscr{Z}} = \mathrm{meas}(\mathscr{Z}\backslash \mathrm{M}^\gamma(F))$. From here, the claim involving (21) is easy to see.

(x) Suppose M is quasi-split and $\tilde{\mathrm{M}}$ has inner torsion, and assume that ω is trivial. In this case, one can view $\mathcal{SI}_{\mathscr{Z},\zeta,\mathrm{cusp}}(\tilde{\mathrm{M}}) = \mathcal{SI}_{\mathscr{Z},\zeta,\mathrm{cusp}}(\tilde{\mathrm{M}},\omega)$ as a subspace of $\mathcal{I}_{\mathscr{Z},\zeta,\mathrm{cusp}}(\tilde{\mathrm{M}})$, using the isomorphism (16) and noting that one has a principal endoscopic datum $\underline{\mathrm{H}}$ with $\mathrm{H} = \mathrm{M}$ and $\tilde{\mathrm{H}} = \tilde{\mathrm{M}}$, for which $\mathrm{Aut}(\underline{\mathrm{H}}) = 1$ and $\mathcal{SI}_{\mathscr{Z}_1,\zeta_1,\mathrm{cusp}}(\tilde{\mathrm{H}}_1)$ identifies with $\mathcal{SI}_{\mathscr{Z},\zeta,\mathrm{cusp}}(\tilde{\mathrm{M}})$. As in the discussion in [LMW18, Section 4.6, between Lemma 1 and Lemma 2], this identifies $\mathcal{SI}_{\mathscr{Z},\zeta,\mathrm{cusp}}(\tilde{\mathrm{M}})$ as the subspace of $\mathcal{I}_{\mathscr{Z},\zeta,\mathrm{cusp}}(\tilde{\mathrm{M}})$ consisting of all a such that $a(\gamma) = a(\gamma')$ (using the notation of Notation 3.4.4(iii)) whenever γ and γ' are stably conjugate. From this perspective, if $a \in \mathcal{SI}_{\mathscr{Z},\zeta,\mathrm{cusp}}(\tilde{\mathrm{M}}) \subset \mathcal{I}_{\mathscr{Z},\zeta,\mathrm{cusp}}(\tilde{\mathrm{M}})$ is the image of $f \in C^\infty_{\zeta,\mathrm{cusp}}(\tilde{\mathrm{M}}(F))$, then for all $\gamma \in \tilde{\mathrm{M}}(F)_{\mathrm{ell}}$ we have:

$$SO(\gamma,f) = \kappa(\gamma)a(\gamma), \tag{22}$$

where, like in Notation 3.4.1(iii), $\kappa(\gamma)$ denotes the number of conjugacy classes in the stable conjugacy class of γ, and $SO(\gamma,f)$ and $a(\gamma)$ are as in (ii) and (iii) above.

(xi) Consider the setting of (v) above, but assume for simplicity that we are in the situation of standard endoscopy. Fix $\underline{\mathrm{H}} \in \mathcal{E}(\tilde{\mathrm{M}},\mathbf{a})_\zeta = \mathcal{E}(\mathrm{M})_\zeta \subset \mathcal{E}(\mathrm{M})$, with associated data such as $\mathscr{Z}_1$ and ζ_1. The discussion of (x) above applies with $(\mathrm{M},\tilde{\mathrm{M}},\mathscr{Z},\zeta)$ replaced by $(\mathrm{H}_1,\tilde{\mathrm{H}}_1 = \mathrm{H}_1,\mathscr{Z}_1,\zeta_1)$, so we have $\mathcal{SI}_{\mathscr{Z}_1,\zeta_1,\mathrm{cusp}}(\mathrm{H}_1) \subset \mathcal{I}_{\mathscr{Z}_1,\zeta_1,\mathrm{cusp}}(\mathrm{H}_1)$. Then, using (iii) above and (22), the endoscopic transfer map $\mathcal{I}_{\mathscr{Z},\zeta,\mathrm{cusp}}(\mathrm{M}) \to \mathcal{SI}_{\mathscr{Z}_1,\zeta_1,\mathrm{cusp}}(\mathrm{H}_1)$ can now be described as follows: $a \mapsto b$ if and only if for all strongly M-regular $\delta_1 \in \mathrm{H}_1(F)_{\mathrm{ell}}$, we have:

$$b(\delta_1) = \kappa(\delta_1)^{-1}\sum_\gamma D^{\mathrm{H}_1}(\delta_1)^{-1/2}D^{\mathrm{M}}(\gamma)^{1/2}\Delta(\delta_1,\gamma)a(\gamma), \tag{23}$$

where γ runs over a set of representatives for conjugacy classes in the (possibly empty) stable conjugacy class in $\mathrm{M}(F)$ matching δ. Here, we recall that our orbital integrals are unnormalized, and we have used the convention wherein "Δ_{IV}" is not part of Δ, but is accounted for separately using the discriminant factors. Moreover, we have used the discussion on the normalization of measures in [LMW18, the top of page 317], to justify our use of the definition of orbital integrals in (ii) above without adding any extra normalizing constants. Since we are in the case of standard endoscopy, the factors "$d_\theta^{1/2}$" and "d_γ^{-1}" from [LMW18, Section 4.5, (2)] are trivial, and the local isomorphism "$\mathscr{Z}\backslash G_\gamma(F) \to \mathscr{Z}_1'\backslash G'_{1,\delta_1}(F)$" from [LMW18, page 317] is an isomorphism. Another standard fact we have used is that for strongly regular semisimple elements $\delta_1 \in \mathrm{H}_1(F)$ and $\gamma \in \mathrm{M}(F)$,

$\Delta(\delta_1,\gamma) \neq 0$ if and only if the stable conjugacy classes of δ_1 and γ match, in which case $\delta_1 \in \mathrm{H}_1(F)$ is elliptic if and only if $\gamma \in \mathrm{M}(F)$ is.

Lemma 3.4.8. *Let* $\underline{\mathrm{H}} \in \mathcal{E}(\mathrm{M})$ *(thus, we are considering standard endoscopy, not twisted endoscopy). Let* $\mathrm{H}_1 = \tilde{\mathrm{H}}_1, \mathscr{Z}, \zeta, \mathscr{Z}_1, \zeta_1$ *be as in Remark 3.4.7. Let* $\Theta \in D_{\mathscr{Z},\zeta,\mathrm{ell}}(\mathrm{M})$*, let* $\Theta^{\underline{\mathrm{H}}}$ *be its projection to* $SD_{\mathscr{Z}_1,\zeta_1,\mathrm{ell}}(\mathrm{H}_1)_{\mathrm{Aut}(\underline{\mathrm{H}})} = SD_{\mathscr{Z}_1,\zeta_1,\mathrm{ell}}(\mathrm{H}_1)^{\mathrm{Aut}(\underline{\mathrm{H}})}$ *as per* (17)*, and write* $\Theta^{\underline{\mathrm{H}},\mathrm{M}} := \mathbf{T}_{\underline{\mathrm{H}}}(\Theta^{\underline{\mathrm{H}}})$*. We have, for all* $\gamma \in \mathrm{M}(F)_{\mathrm{ell}}$*:*

$$\Theta^{\underline{\mathrm{H}},\mathrm{M}}(\gamma) = c(\mathrm{M},\underline{\mathrm{H}}) \sum_{\delta_1} \kappa(\delta_1)^{-1}\Delta(\delta_1,\gamma) \sum_{\gamma'} \overline{\Delta(\delta_1,\gamma')}\Theta(\gamma'),$$

where δ_1 *ranges over a set of representatives in* $\mathrm{H}_1(F)$ *for the* M*-regular stable conjugacy classes in* $\mathrm{H}(F)_{\mathrm{ell}}$*,* γ' *runs over a set of representatives for the* $\mathrm{M}(F)$*-conjugacy classes in* $\mathrm{M}(F)_{\mathrm{ell}}$*, and* $c(\mathrm{M},\underline{\mathrm{H}})$ *is as in* (18)*, i.e., as in* [MW16, Section I.4.17] *or equivalently* [LMW18, Section 4.6].

Proof. In this proof, any sum over δ_1 will range over representatives in $\mathrm{H}_1(F)$ for M-regular stable conjugacy classes in $\mathrm{H}(F)_{\mathrm{ell}}$, and any sum over γ' will range over representatives for $\mathrm{M}(F)$-conjugacy classes in $\mathrm{M}(F)_{\mathrm{ell}}$.
The first step is to study $\Theta^{\underline{\mathrm{H}}}$. We claim that for all $\delta_1 \in \mathrm{H}_1(F)_{\mathrm{ell}}$ we have:

$$\Theta^{\underline{\mathrm{H}}}(\delta_1) = c(\mathrm{M},\underline{\mathrm{H}})\kappa(\delta_1)^{-1} \cdot \sum_{\gamma'} D^{\mathrm{H}_1}(\delta_1)^{-1/2} D^{\mathrm{M}}(\gamma')^{1/2} \cdot \overline{\Delta(\delta_1,\gamma')}\Theta(\gamma'). \tag{24}$$

Consider the isomorphism $\iota_{\mathscr{Z}_1,\zeta_1} : D_{\mathscr{Z}_1,\zeta_1,\mathrm{ell}}(\mathrm{H}_1) \to \mathcal{I}_{\mathscr{Z}_1,\zeta_1,\mathrm{cusp}}(\mathrm{H}_1)$ analogous to $\iota_{\mathscr{Z},\zeta} : D_{\mathscr{Z},\zeta,\mathrm{ell}}(\mathrm{M}) \to \mathcal{I}_{\mathscr{Z},\zeta,\mathrm{cusp}}(\mathrm{M})$. It follows from Remark 3.4.7(ix) (specifically, (21)), and the discussion of Remark 3.4.7(x), that $\iota_{\mathscr{Z}_1,\zeta_1}$ carries $SD_{\mathscr{Z}_1,\zeta_1,\mathrm{ell}}(\mathrm{H}_1)$ to $\mathcal{SI}_{\mathscr{Z}_1,\zeta_1,\mathrm{cusp}}(\mathrm{H}_1) \subset \mathcal{I}_{\mathscr{Z}_1,\zeta_1,\mathrm{cusp}}(\mathrm{H}_1)$. More is true: if $\iota_{\mathscr{Z},\zeta}(\Theta) = a \in \mathcal{I}_{\mathscr{Z},\zeta,\mathrm{cusp}}(\mathrm{M})$ and $\iota_{\mathscr{Z}_1,\zeta_1}(\Theta^{\underline{\mathrm{H}}}) = b \in \mathcal{SI}_{\mathscr{Z}_1,\zeta_1,\mathrm{cusp}}(\mathrm{H}_1) \subset \mathcal{I}_{\mathscr{Z}_1,\zeta_1,\mathrm{cusp}}(\mathrm{H}_1)$, then, as explained in [LMW18, Section 4.6, slightly before (5)], then we have $b = c(\mathrm{M},\underline{\mathrm{H}})b'$, where b' is the image of a under the endoscopic transfer map $\mathcal{I}_{\mathscr{Z},\zeta,\mathrm{cusp}}(\mathrm{M}) \to \mathcal{SI}_{\mathscr{Z}_1,\zeta_1,\mathrm{cusp}}(\mathrm{H}_1)$. Therefore, we have, for all M-regular $\delta_1 \in \mathrm{H}_1(F)_{\mathrm{ell}}$:

$$\Theta^{\underline{\mathrm{H}}}(\delta_1) = \overline{b(\delta_1)} = c(\mathrm{M},\underline{\mathrm{H}})\overline{b'(\delta_1)} = c(\mathrm{M},\underline{\mathrm{H}})\kappa(\delta_1)^{-1} \sum_{\gamma'} \overline{\Delta(\delta_1,\gamma')D^{\mathrm{H}_1}(\delta_1)^{-1/2}D^{\mathrm{M}}(\gamma')^{1/2}a(\gamma')},$$

where we used Remark 3.4.7(ix) (specifically (21)) at the first step, and and (23) at the third. Noting that $\overline{a(\gamma')} = \Theta(\gamma')$ by (21), (24) follows.
The computation of $\Theta^{\underline{\mathrm{H}},\mathrm{M}} = \mathbf{T}_{\underline{\mathrm{H}}}(\Theta^{\underline{\mathrm{H}}})$ in terms of $\Theta^{\underline{\mathrm{H}}}$ can be done using [Art96, Lemma 2.3], analogously to how [Li13, Proposition 5.3.2] is proved from [Li13, Lemma 5.3.1], and is what is reflected in the 'character value' form of character identities found in, e.g., [Kal15, Theorem 6.6]; we merely state the result:

$$\Theta^{\underline{\mathrm{H}},\mathrm{M}}(\gamma) = \mathbf{T}_{\underline{\mathrm{H}}}(\Theta^{\underline{\mathrm{H}}})(\gamma) = \sum_{\delta_1} D^{\mathrm{H}_1}(\delta_1)^{1/2} D^{\mathrm{M}}(\gamma)^{-1/2}\Delta(\delta_1,\gamma)\Theta^{\underline{\mathrm{H}}}(\delta_1).$$

Thus, using (24), $\Theta^{\underline{\mathrm{H}}}(\gamma)$ equals:

$$c(\mathrm{M},\underline{\mathrm{H}}) \cdot \sum_{\delta_1} \kappa(\delta_1)^{-1} D^{\mathrm{H}_1}(\delta_1)^{1/2} D^{\mathrm{M}}(\gamma)^{-1/2}\Delta(\delta_1,\gamma) \sum_{\gamma'} D^{\mathrm{H}_1}(\delta_1)^{-1/2} D^{\mathrm{M}}(\gamma')^{1/2}\overline{\Delta(\delta_1,\gamma')}\Theta(\gamma').$$

Since the set of elements in $\mathrm{M}(F)_{\mathrm{srss}}$ that match a given $\delta_1 \in \mathrm{H}(F)_{\mathrm{srss}}$ form a single stable conjugacy class, we have $D^{\mathrm{M}}(\gamma')^{1/2} = D^{\mathrm{M}}(\gamma)^{1/2}$ for each γ' occurring in the above sum, so that the above expression equals the one given in the lemma. □

Proof of Lemma 3.4.5. We apply Lemma 3.4.8 with $\underline{\mathrm{H}}$ replaced by the principal endoscopic datum $\underline{\mathrm{M}}^*$ attached to M as in Notation 3.2.1(i). It is easy to compute that (every factor in the definition, in [MW16], of) $c(\mathrm{M},\underline{\mathrm{M}}^*)$ equals 1, and then, assuming transfer factors to be normalized as in Remark 3.2.2(i), that $\Theta_\sigma^{\mathrm{st}} = \Theta_\sigma^{\underline{\mathrm{M}}^*,\mathrm{M}} \in SD_{\mathrm{ell}}(\mathrm{M})$. It follows from Remark 3.4.7(vii), and the fact that $\mathbf{T}_{\underline{\mathrm{M}}^*}$ defines an isomorphism $SD_{\mathrm{ell}}(\mathrm{M}^*) \to SD_{\mathrm{ell}}(\mathrm{M})$ (see Corollary 3.2.7), that $\Theta_\sigma^{\underline{\mathrm{M}}^*,\mathrm{M}}$ is the projection of Θ_σ to $SD_{\mathrm{ell}}(\mathrm{M})$, and the lemma follows. □

The terminology in the following definition is ad hoc:

Definition 3.4.9. (i) By a discrete series L-packet for M equipped with an endoscopic decomposition, we refer to a finite set $\Sigma \subset \mathrm{Irr}_2(\mathrm{M})$ with the property that:

(a) There exists a nonzero complex number c_σ for each $\sigma \in \Sigma$, such that $\sum_{\sigma\in\Sigma} c_\sigma \Theta_\sigma$ is a stable distribution; and

(b) For each elliptic endoscopic datum $\underline{\mathrm{H}} \in \mathcal{E}(\mathrm{M})$, with underlying endoscopic group H, choosing auxiliary data and hence the 5-tuple $(\mathrm{H}_1 \to \mathrm{H}, \hat{\xi}_1, \tilde{\mathrm{H}}_1 \to \tilde{\mathrm{H}}, \mathrm{C}_1, \mu)$ as in Notation 3.1.2(iii), there exists a stable elliptic virtual character $\Theta^{\underline{\mathrm{H}}} \in SD_{\mu,\mathrm{ell}}(\mathrm{H}_1)$ on $\tilde{\mathrm{H}}_1(F) = \mathrm{H}_1(F)$, such that (letting $\underline{\mathrm{H}}$ vary in $\mathcal{E}(\mathrm{M})$ now) the following holds inside $D_{\mathrm{ell}}(\mathrm{M})$:

$$\sum_{\underline{\mathrm{H}}} \mathbb{C}\cdot \mathbf{T}_{\underline{\mathrm{H}}}(\Theta^{\underline{\mathrm{H}}}) = \sum_{\sigma\in\Sigma} \mathbb{C}\cdot\Theta_\sigma. \tag{25}$$

We will refer to (25) as an endoscopic decomposition for Σ.

(ii) Suppose that the triple $(\mathrm{M}, \tilde{\mathrm{M}}, \mathbf{a})$ and the associated character $\omega : \mathrm{M}(F) \to \mathbb{C}^\times$ are as in Notation 3.1.1, and that they satisfy the hypotheses there. By a *discrete series L-packet for $(\tilde{\mathrm{M}}, \omega)$ equipped with an endoscopic decomposition*, we refer to a pair $(\Sigma, \tilde{\Sigma})$ such that:

(a) Σ is a discrete series L-packet for M together with an endoscopic decomposition, in the sense of (i);

(b) $\tilde{\Sigma}$ is a finite set of (isomorphism classes of) representations of $(\tilde{\mathrm{M}}(F), \omega)$ such that the map that takes an $\tilde{\mathrm{M}}(F)$-representation to its underlying $\mathrm{M}(F)$-representation defines an injection $\tilde{\Sigma} \hookrightarrow \Sigma$; and

(c) For each $\underline{\mathrm{H}} \in \mathcal{E}(\tilde{\mathrm{M}}, \mathbf{a})$ with underlying endoscopic group H, choosing auxiliary data and hence the 5-tuple $(\mathrm{H}_1 \to \mathrm{H}, \hat{\xi}_1, \tilde{\mathrm{H}}_1 \to \tilde{\mathrm{H}}, \mathrm{C}_1, \mu)$ as in Notation 3.1.2(iii), there exists a stable elliptic virtual character $\Theta^{\underline{\mathrm{H}}} \in SD_{\mu,\mathrm{ell}}(\tilde{\mathrm{H}}_1)$ on $\tilde{\mathrm{H}}_1(F)$, such that the following holds inside $D_{\mathrm{ell}}(\tilde{\mathrm{M}}, \omega)$:

$$\sum_{\underline{\mathrm{H}}} \mathbb{C}\cdot \mathbf{T}_{\underline{\mathrm{H}}}(\Theta^{\underline{\mathrm{H}}}) = \sum_{\tilde{\sigma}\in\tilde{\Sigma}} \mathbb{C}\cdot\Theta_{\tilde{\sigma}}. \tag{26}$$

Remark 3.4.10. By [Kal19, Theorem 6.3.4], it follows that when $p \gg 0$, the regular supercuspidal packets of Kaletha are equipped with an endoscopic decomposition in the sense of Definition 3.4.9(i). More generally, Kaletha-type results should give a large class of discrete series L-packets equipped with an endoscopic decomposition.

Now we prove that a discrete series L-packet equipped with an endoscopic decomposition is automatically unitarily stable.

Proposition 3.4.11. *Suppose that the triple $(\mathrm{M}, \tilde{\mathrm{M}}, \mathbf{a})$ and the associated unitary character $\omega : \mathrm{M}(F) \to \mathbb{C}^\times$ are as in Notation 3.1.1, and that they satisfy the hypotheses there.*

(i) Suppose $(\Sigma, \tilde{\Sigma})$ is a discrete series L-packet for $(\tilde{\mathrm{M}}, \omega)$ equipped with an endoscopic decomposition given by (26) (which in particular involves choosing auxiliary data, and involves some $\Theta^{\underline{\mathrm{H}}} \in SD_{\mu,\mathrm{ell}}(\tilde{\mathrm{H}}_1)$ for each $\underline{\mathrm{H}} \in \mathcal{E}(\tilde{\mathrm{M}}, \mathbf{a})$). Fix $\underline{\mathrm{H}}_0 \in \mathcal{E}(\tilde{\mathrm{M}}, \mathbf{a})$. Then any $\Theta \in D_{\mathrm{ell}}(\tilde{\mathrm{M}}, \omega)$ that belongs to the image $\mathbf{T}_{\underline{\mathrm{H}}_0}(SD_{\mu,\mathrm{ell}}(\tilde{\mathrm{H}}_{0,1})_{\mathrm{Aut}(\underline{\mathrm{H}}_0)})$ of $\mathbf{T}_{\underline{\mathrm{H}}_0}$ can be uniquely written as $c_0\mathbf{T}_{\underline{\mathrm{H}}_0}(\Theta^{\underline{\mathrm{H}}_0}) + c_\circ\Theta_\circ$, where $c_0, c_\circ \in \mathbb{C}$, and $\Theta_\circ$ is supported on a set of representations each of whose underlying $\mathrm{M}(F)$-representations lies outside the image of $\tilde{\Sigma} \hookrightarrow \Sigma$ (note that $\mathbf{T}_{\underline{\mathrm{H}}_0}(\Theta^{\underline{\mathrm{H}}_0})$ is supported in $\tilde{\Sigma}$ by definition).

(ii) Suppose Σ is a discrete series L-packet for M with an endoscopic decompostion (25). Then Σ is a unitarily stable discrete series L-packet.

Proof. Let us see that (ii) follows from (i). By definition, (Σ, Σ) can be viewed as a discrete series packet for $(\mathrm{M}, \mathbb{1})$ with an endoscopic decomposition. Fix $\underline{\mathrm{M}}^*$ and related objects as in Notation 3.2.1(i); thus, M^* is a quasi-split form of M. We also have $\Theta^{\underline{\mathrm{M}}^*} \in SD_{\mathrm{ell}}(\mathrm{M}^*)$ associated to Σ, as in (i), i.e., as in Definition 3.4.9(i). By the surjectivity assertion in Corollary 3.2.7, $SD_{\mathrm{ell}}(\mathrm{M}) = \mathbf{T}_{\underline{\mathrm{M}}^*}(SD_{\mathrm{ell}}(\mathrm{M}^*)) = \mathbf{T}_{\underline{\mathrm{H}}_0}(SD_{\mu,\mathrm{ell}}(\tilde{\mathrm{H}}_{0,1})_{\mathrm{Aut}(\underline{\mathrm{H}}_0)})$, where $\underline{\mathrm{H}}_0 = \underline{\mathrm{M}}^*$, for which we may and do take $\tilde{\mathrm{H}}_{0,1}$ to be $\mathrm{H}_{0,1}$ and μ to be trivial. Applying (i) to an arbitrary stable distribution

in $\sum \mathbb{C} \cdot \Theta_\sigma$ (the sum being over $\sigma \in \Sigma$), and using the linear independence of characters, we see that the space of stable distributions in $\sum \mathbb{C} \cdot \Theta_\sigma$ is at most one dimensional, and spanned by $\mathbf{T}_{\underline{\mathrm{M}}^*}(\Theta^{\underline{\mathrm{M}}^*})$ if it is nonzero. But the requirement in Definition 3.4.9(i) that there exists a stable distribution $\sum c_\sigma \Theta_\sigma$ supported in Σ, with each c_σ nonzero, then forces that this space is indeed one dimensional with $\Theta_\Sigma := \sum c_\sigma \Theta_\sigma$ as a basis, which is therefore a scalar multiple of $\mathbf{T}_{\underline{\mathrm{M}}^*}(\Theta^{\underline{\mathrm{M}}^*})$. Applying (i) once again to an arbitrary stable virtual character $\Theta \in SD_{\mathrm{ell}}(\mathrm{M}) = \mathbf{T}_{\underline{\mathrm{M}}^*}(SD_{\mathrm{ell}}(\mathrm{M}^*))$ then shows that Σ satisfies the conditions of Definition 3.3.2, i.e., that it is unitarily stable.
Thus, it remains to prove (i); we no longer have $\underline{\mathrm{H}}_0 = \underline{\mathrm{M}}^*$. We can write the given $\Theta \in \mathbf{T}_{\underline{\mathrm{H}}_0}(SD_{\mu,\mathrm{ell}}(\tilde{\mathrm{H}}_{0,1})_{\mathrm{Aut}(\underline{\mathrm{H}}_0)})$ of (26) as $c_0 \mathbf{T}_{\underline{\mathrm{H}}_0}(\Theta^{\underline{\mathrm{H}}_0}) + \Theta_\circ$, where $c_0 \in \mathbb{C}$, and $\Theta_\circ$ belongs to the image of $\mathbf{T}_{\underline{\mathrm{H}}_0}$ but is orthogonal in $D_{\mathrm{ell}}(\tilde{\mathrm{M}}, \omega)$ to $\mathbf{T}_{\underline{\mathrm{H}}_0}(\Theta^{\underline{\mathrm{H}}_0})$ (even if $\mathbf{T}_{\underline{\mathrm{H}}_0}(\Theta^{\underline{\mathrm{H}}_0})$ is 0). By the definition of the inner product on $D_{\mathrm{ell}}(\tilde{\mathrm{M}}, \omega)$ (see Notation 3.4.4), it is enough to see that $\Theta_\circ$ is orthogonal to $\Theta_{\tilde{\sigma}}$ for each $\tilde{\sigma} \in \tilde{\Sigma}$, or, equivalently by the decomposition (26), that it is orthogonal to each $\mathbf{T}_{\underline{\mathrm{H}}}(\Theta^{\underline{\mathrm{H}}})$ as $\underline{\mathrm{H}}$ varies through $\mathcal{E}(\tilde{\mathrm{M}}, \mathbf{a})$. By definition, this is so for $\underline{\mathrm{H}} = \underline{\mathrm{H}}_0$, while, by Remark 3.4.7(vii), this is so for all $\underline{\mathrm{H}} \neq \underline{\mathrm{H}}_0$. This proves (i), as desired. □

Corollary 3.4.12. *If $p \gg 0$, the regular supercuspidal L-packets constructed in* [Kal19] *are unitarily stable.*

Remark 3.4.13. If G is a quasi-split special orthogonal, symplectic or unitary group over F, so that Hypothesis 2.5.1 is satisfied by the work of Arthur and Mok ([Art13] and [Mok15]; see Proposition 5.1.2 below for more details), then Corollary 3.4.12, in view of of Lemma 3.3.8(iii), implies that regular supercuspidal L-packets for G in the sense of Kaletha are also L-packets in the sense of Arthur and Mok (though we have no result comparing the relevant Langlands parametrizations). An appropriately analogous comment applies with the work [Mg14] of Mœglin in place of [Art13] and [Mok15], provided one accounts for an outer automorphism in the even special orthogonal case.

Proof of Corollary 3.4.12. This follows from Remark 3.4.10 and Proposition 3.4.11. □

Remark 3.4.14. Lemma 3.4.8 is more involved than what was strictly needed to prove Lemma 3.4.5. The reason we went through Lemma 3.4.8 is to make the optimistic proposal that it might be possible in principle to start with the character of a single discrete series representation, and construct all the "unstable endoscopic characters" associated to the L-packet that contains it (for nice enough representations and packets). However, we do not know how far this can be used in practice to study, given the character of a single discrete series representation, the characters of the representations that belong to its L-packet.

4. The main technical results: Shahidi's μ-constancy argument and applications

4.1. Candidates for the stable Bernstein center, $\mathcal{Z}_1(\mathrm{G})$ and $\mathcal{Z}_2(\mathrm{G})$.

Notation 4.1.1. In this section, we will write $\Omega(\mathrm{G})$ for the Bernstein variety of cuspidal pairs for G: see [Hai14, Section 3.3]. We write $\underline{\Omega}(\mathrm{G})$ for its quotient under the action of $\mathcal{O} = \mathcal{O}_{\mathrm{G}}$ (which acts through a finite quotient). In this section, we will use the following facts about $\underline{\Omega}(\mathrm{G})$, and their simpler analogues for $\Omega(\mathrm{G})$. The $\mathbb{C}$-points of the variety $\underline{\Omega}(\mathrm{G})$ are the $\mathcal{O}_{\mathrm{G}}^+$-conjugacy classes of cuspidal pairs: these are the pairs (M, σ), where $\mathrm{M} \subset \mathrm{G}$ is a Levi subgroup and σ is a supercuspidal representation of $\mathrm{M}(F)$. Moreover, $\underline{\Omega}(\mathrm{G})$ is reduced as a variety, and a map $f : \underline{\Omega}(\mathrm{G}) \to \mathrm{X}(\mathbb{C})$, for a variety X over $\mathbb{C}$, is regular if and only if for each cuspidal pair (M, σ), the map $X^{\mathrm{unr}}(\mathrm{M}) \to \mathbb{C}$ given by $f \mapsto f(\underline{(\mathrm{M}, \sigma \otimes \chi)})$ is regular, where $\underline{(\mathrm{M}, \sigma \otimes \chi)} \in \underline{\Omega}(\mathrm{G})$ is the image of $(\mathrm{M}, \sigma \otimes \chi)$.

Notation 4.1.2. In this subsection, given $f \in C_c^\infty(\mathrm{G}(F))$, $f^\vee \in C_c^\infty(\mathrm{G}(F))$ will stand for the function $x \mapsto f(x^{-1})$. Given $g \in \mathrm{G}(F)$ and $f \in C_c^\infty(\mathrm{G}(F))$, we will let $l_g f(x) = f(g^{-1}x)$ and $r_g f(x) = f(xg)$. We fix a Haar measure on $\mathrm{G}(F)$, which will be used in the convolutions that follow.

We recall some facts on convolutions from [Hai14, Section 3.1]. For a distribution D on $\mathrm{G}(F)$ and $f \in C_c^\infty(\mathrm{G}(F))$, $D * f \in C^\infty(\mathrm{G}(F))$ is given by $g \mapsto D((r_g f)^\vee) = D(l_g f^\vee)$ — thus, this is defined so as to satisfy:

$$(D * r_g f) = r_g(D * f) \qquad \text{and} \qquad D(f) = (D * f^\vee)(1).$$

D is said to be essentially compact if $D * f \in C_c^\infty(\mathrm{G}(F))$ for all $f \in C_c^\infty(\mathrm{G}(F))$. If D' and D are distributions and D is essentially compact, we can convolve them by letting $(D' * D)(f) = D'((D * f^\vee)^\vee)$.

Now we recall the definition of the Bernstein center $\mathcal{Z}(\mathrm{G})$ of G.

Definition 4.1.3. The Bernstein center $\mathcal{Z}(\mathrm{G})$ of G is the $\mathbb{C}$-vector space of essentially compact invariant distributions on $C_c^\infty(\mathrm{G}(F))$, i.e., the space of ($\mathrm{G}(F)$-conjugation) invariant distributions $C_c^\infty(\mathrm{G}(F)) \to \mathbb{C}$ with the property that for all $f \in C_c^\infty(\mathrm{G}(F))$, $z * f \in C^\infty(\mathrm{G}(F))$ belongs to $C_c^\infty(\mathrm{G}(F))$. Convolution makes $\mathcal{Z}(\mathrm{G})$ into a commutative $\mathbb{C}$-algebra (see [Hai14, Corollary 3.1.2]), and the work of Bernstein gives the following alternate descriptions of the ring $\mathcal{Z}(\mathrm{G})$:

(i) Via $z \mapsto (f \mapsto z * f)$, $\mathcal{Z}(\mathrm{G})$ identifies with the ring of endomorphisms of $C_c^\infty(\mathrm{G}(F))$ that commute with left and right convolution.

(ii) One can uniquely make each $z \in \mathcal{Z}(\mathrm{G})$ act as an intertwining operator $\pi(z)$ on π, for each smooth representation π of $\mathrm{G}(F)$, such that:
- Denoting temporarily by l the left-regular representation of $\mathrm{G}(F)$ on $C_c^\infty(\mathrm{G}(F))$, we have $l(z)(f) = z * f$;
- $\pi \mapsto \pi(z)$ respects morphisms of representations.

$z \mapsto (\pi \mapsto \pi(z))$ defines a homomorphism from $\mathcal{Z}(\mathrm{G})$ to the ring of endomorphisms of the identity functor of the category of smooth representations of $\mathrm{G}(F)$, which Bernstein's work shows to be an isomorphism. The action of $\mathcal{Z}(\mathrm{G})$ on a smooth representation (π, V) can typically be computed using the following: given $v \in V$, we have a map $(l, C_c^\infty(\mathrm{G}(F))) \to (\pi, V)$ given by $f \mapsto \pi(f)v$, so that $\pi(z)(\pi(f)v) = \pi(l(z)f)(v) = \pi(z * f)(v)$. This also gives:

$$\pi(z * f) = \pi(z)\pi(f). \tag{27}$$

(iii) By (ii) and Schur's lemma, each $z \in \mathcal{Z}(\mathrm{G})$ acts on each irreducible admissible representation π of $\mathrm{G}(F)$ by multiplication by some scalar, which can be shown to depend only on the cuspidal support $(\mathrm{M}, \sigma)_{\mathrm{G}} \in \Omega(\mathrm{G})$ of π. We denote this scalar by:

$$\hat{z}(\pi) = \hat{z}(\mathrm{M}, \sigma) = \hat{z}((\mathrm{M}, \sigma)_{\mathrm{G}}).$$

More generally, for any Levi subgroup $\mathrm{M}' \subset \mathrm{G}$ and $\sigma' \in \mathrm{Irr}(\mathrm{M}')$ such that the cuspidal supports of σ' and π are $\mathrm{G}(F)$-conjugate, we will write $\hat{z}(\pi) = \hat{z}(\sigma') = \hat{z}((\mathrm{M}', \sigma')) = \hat{z}((\mathrm{M}', \sigma')_{\mathrm{G}})$ when there is no scope for confusion, where in turn $(\mathrm{M}', \sigma')_{\mathrm{G}}$ denotes the $\mathrm{G}(F)$-conjugacy class of (M, σ). By Bernstein's work, sending $z \in \mathcal{Z}(\mathrm{G})$ to $\hat{z} : \Omega(\mathrm{G}) \to \mathbb{C}$ gives an isomorphism of rings $\mathcal{Z}(\mathrm{G}) \to \mathbb{C}[\Omega(\mathrm{G})]$.

It is clear that $\mathcal{O}$ acts on $\mathcal{Z}(\mathrm{G})$; we now explicate this action. $\mathcal{O}$ acts on $C_c^\infty(\mathrm{G}(F))$ and $C^\infty(\mathrm{G}(F))$, and on the space of distributions on $\mathrm{G}(F)$: $(\beta \cdot f)(x) = f(\beta^{-1}(x))$ and $(\beta \cdot D)(f) = D(\beta^{-1} \cdot f)$. For $\beta \in \mathcal{O}$, one verifies the following equalities for each distribution D on $\mathrm{G}(F)$, $f \in C_c^\infty(\mathrm{G}(F))$ and $\beta \in \mathcal{O}$:

$$(\beta D)(\beta f) = D(f), \qquad \text{and} \qquad \beta D * \beta f = \beta(D * f). \tag{28}$$

It is now clear that the action of $\mathcal{O}$ on the space of distributions on $\mathrm{G}(F)$ preserves the subspace $\mathcal{Z}(\mathrm{G})$. The action of $\mathcal{O}$ on $\mathrm{Irr}(\mathrm{G})$, given by $\beta \cdot \pi = \pi \circ \beta^{-1}$, is related to the action of $\mathcal{O}$ on $\mathcal{Z}(\mathrm{G})$ as follows:

$$\widehat{\beta \cdot z}(\pi) = \hat{z}(\pi \circ \beta). \tag{29}$$

Indeed, using the identity $\pi(z * f) = \hat{z}(\pi)\pi(f)$ (which follows from (27)), the identity $(\pi \circ \beta)(f) = \pi(f \circ \beta^{-1})$, and (28), this follows from:

$$\hat{z}(\pi\circ\beta)\cdot(\pi\circ\beta)(f) = \pi\circ\beta(z*f) = \pi((z*f)\circ\beta^{-1}) = \pi(\beta\cdot(z*f)) = \pi((\beta\cdot z)*(\beta\cdot f)) = \widehat{\beta \cdot z}(\pi)\pi(f\circ\beta^{-1}) = \widehat{\beta \cdot z}(\pi)(\pi\circ\beta)(f).$$

From (28) and (29), the following is easy to deduce:

Lemma 4.1.4. *The isomorphism $\mathcal{Z}(\mathrm{G}) \to \mathbb{C}[\Omega(\mathrm{G})]$ given by $z \mapsto \hat{z}$ is $\mathcal{O}$-equivariant, and restricts to an isomorphism $\mathcal{Z}(\mathrm{G})^{\mathcal{O}} \to \mathbb{C}[\underline{\Omega}(\mathrm{G})]$, where the inclusion $\mathbb{C}[\underline{\Omega}(\mathrm{G})] \subset \mathbb{C}[\Omega(\mathrm{G})]$ comes from the quotient map $\Omega(\mathrm{G}) \to \underline{\Omega}(\mathrm{G})$ of varieties (see Notation 4.1.1). In particular, $\mathcal{Z}(\mathrm{G})^{\mathcal{O}} \subset \mathcal{Z}(\mathrm{G})$ is a subring.*

Notation 4.1.5. In this subsection, $\mathcal{I}(\mathrm{G})$ will denote the space of coinvariants for $\mathrm{G}(F)$-conjugation on $C_c^\infty(\mathrm{G}(F))$. We will consider the actions of $\mathcal{O}$ and $\mathcal{O}_{\mathrm{G}}^+$ on $\mathcal{I}(\mathrm{G})$ inherited from their actions on $C_c^\infty(\mathrm{G}(F))$. Note that the space of invariant distributions on $C_c^\infty(\mathrm{G}(F))$ identifies with $\mathrm{Hom}_{\mathbb{C}}(\mathcal{I}(\mathrm{G}), \mathbb{C})$.

Remark 4.1.6. Since $\mathrm{Int}\,\mathrm{G}(F)$ is of finite index in $\mathcal{O}_{\mathrm{G}}^+$ by Notation 2.4.1(iv) (see Lemma 2.4.3(ii)), $\mathcal{O}$ acts on $\mathcal{I}(\mathrm{G})$ through a finite quotient, and therefore, given $f \in C_c^\infty(\mathrm{G}(F))$, there exists $f' \in C_c^\infty(\mathrm{G}(F))$ such that:

- f' is a sum of finitely many $\mathcal{O}$-translates of f, and has $\mathcal{O}$-invariant image in $\mathcal{I}(\mathrm{G})$.

It follows from (28) that for any such f', and any $z \in \mathcal{Z}(\mathrm{G})^{\mathcal{O}}$, we have:

- $z * f'$ is a sum of finitely many $\mathcal{O}$-translates of $z * f$, and has $\mathcal{O}$-invariant image in $\mathcal{I}(\mathrm{G})$

(use the easy observation that, if $f'' \in C_c^\infty(\mathrm{G}(F))$ has $\mathcal{O}$-invariant image in $\mathcal{I}(\mathrm{G})$, then so does $z * f''$: this is because $f'' \mapsto z * f''$ is $\mathcal{O}$-equivariant by (28), and hence so is the map it induces from $\mathcal{I}(\mathrm{G})$ to itself). It will also help to note that for any such f', since $\mathcal{O}$ acts by algebraic automorphisms:

- If f (resp., $z * f$) is unstable, then so is f' (resp., $z * f'$).

Now we recall the spaces $\mathcal{Z}_1(\mathrm{G}), \mathcal{Z}_2(\mathrm{G}) \subset \mathcal{Z}(\mathrm{G})$ from the introduction.

Notation 4.1.7. (i) $\mathcal{Z}_1(\mathrm{G}) \subset \mathcal{Z}(\mathrm{G})$ is the (clearly $\mathcal{O}$-invariant) $\mathbb{C}$-vector subspace of $\mathcal{Z}(\mathrm{G})$ consisting of all $z \in \mathcal{Z}(\mathrm{G})$ that are stable as a distribution on $\mathrm{G}(F)$. We will also study the $\mathcal{O}$-fixed subspace $\mathcal{Z}_1(\mathrm{G})^{\mathcal{O}}$ of $\mathcal{Z}_1(\mathrm{G})$.

(ii)
- $\mathcal{Z}_2(\mathrm{G}) \subset \mathcal{Z}(\mathrm{G})$ is the $\mathbb{C}$-sublgebra of $\mathcal{Z}(\mathrm{G})$ consisting of all $z \in \mathcal{Z}(\mathrm{G})$ with the property that $z * f$ is unstable for every unstable function $f \in C_c^\infty(\mathrm{G}(F))$.
- More generally $\mathcal{Z}_{2,\mathcal{O}}(\mathrm{G}) \subset \mathcal{Z}_2(\mathrm{G})$ is the $\mathbb{C}$-subalgebra of $\mathcal{Z}(\mathrm{G})^{\mathcal{O}}$ consisting of all $z \in \mathcal{Z}(\mathrm{G})^{\mathcal{O}}$ such that for every unstable function $f \in C_c^\infty(\mathrm{G}(F))$ whose image in $\mathcal{I}(\mathrm{G})$ is fixed by $\mathcal{O}$, $z * f$ is unstable.

Note that if $\mathcal{O}$ is trivial, then $\mathcal{Z}_1(\mathrm{G}) = \mathcal{Z}_1(\mathrm{G})^{\mathcal{O}}$, and $\mathcal{Z}_2(\mathrm{G}) = \mathcal{Z}_{2,\mathcal{O}}(\mathrm{G})$.

Lemma 4.1.8. *We have $\mathcal{Z}_{2,\mathcal{O}}(\mathrm{G}) \subset \mathcal{Z}_1(\mathrm{G})^{\mathcal{O}}$. In particular, if $\mathcal{O}$ is trivial, then $\mathcal{Z}_2(\mathrm{G}) \subset \mathcal{Z}_1(\mathrm{G})$.*

Proof. Let $z \in \mathcal{Z}_{2,\mathcal{O}}(\mathrm{G}) \subset \mathcal{Z}(\mathrm{G})^{\mathcal{O}}$, and let $f \in C_c^\infty(\mathrm{G}(F))$ be unstable. It is enough to show that $z(f) := z * f^\vee(1)$ equals 0. Choose f' as in Remark 4.1.6. Then, by Remark 4.1.6, $f'^\vee$ is unstable. Since $z \in \mathcal{Z}_{2,\mathcal{O}}(\mathrm{G})$, we conclude that $z * f'^\vee$ is unstable, from which it follows that $z * f'^\vee(1) = 0$ (as $f'' \mapsto f''(1)$ is a stable distribution, by [Kot88, Proposition 1]).
Since $z * f'^\vee$ is a finite sum of $\mathcal{O}$-translates of $z * f^\vee$ (see Remark 4.1.6), and since the action of $\mathcal{O}_{\mathrm{G}}^+$ on $C_c^\infty(\mathrm{G}(F))$ preserves $f'' \mapsto f''(1)$, it follows that $z * f^\vee(1) = 0$, as desired. □

Proposition 4.1.9. *Let $z \in \mathcal{Z}(\mathrm{G})^{\mathcal{O}}$. Then the following are equivalent:*

(i) $z \in \mathcal{Z}_{2,\mathcal{O}}(\mathrm{G})$.
(ii) If D is a stable $\mathcal{O}$-invariant distribution on $\mathrm{G}(F)$, then the distribution $f \mapsto D(z * f)$ is stable.
(iii) If $D \in SD(\mathrm{G})^{\mathcal{O}}$, then the distribution $f \mapsto D(z * f)$ is stable.
(iv) If D is the $\mathcal{O}$-average of $\mathrm{Ind}_{\mathrm{M}}^{\mathrm{G}} \Theta'$, where $\mathrm{M} \subset \mathrm{G}$ is a Levi subgroup and $\Theta' \in SD_{\mathrm{ell}}(\mathrm{M})^{\mathcal{O}_{\mathrm{M}}}$, then the distribution $f \mapsto D(z * f)$ is stable.

Remark 4.1.10. Using the formula $D * z(f) = D\left((z * f^\vee)^\vee\right)$ (see just below Notation 4.1.2), one can show that each of the conditions (ii), (iii) and (iv) of the above proposition has an equivalent variant where the distribution $f \mapsto D(z * f)$ is replaced by the distribution $D * z$.

Proof of Proposition 4.1.9. Let us prove (i) $\Rightarrow$ (ii). Let $z \in \mathcal{Z}_{2,\mathcal{O}}(\mathrm{G}) \subset \mathcal{Z}(\mathrm{G})^{\mathcal{O}}$, and let us show that if D is an $\mathcal{O}$-invariant distribution on $\mathrm{G}(F)$, and $f \in C_c^\infty(\mathrm{G}(F))$ is unstable, then $D(z*f) = 0$. Let f' be as in Remark 4.1.6, so that f' is unstable, its image in $\mathcal{I}(\mathrm{G})$ is $\mathcal{O}$-invariant, and $z * f'$ is a finite sum of $\mathcal{O}$-translates of $z * f$. Therefore, $z * f'$ is unstable (by the definition of $\mathcal{Z}_{2,\mathcal{O}}(\mathrm{G})$), so that $D(z * f') = 0$, while by the $\mathcal{O}$-invariance of D, $D(z * f')$ is a nonzero integer multiple of $D(z * f)$. Therefore, $D(z * f) = 0$, and the implication (i) $\Rightarrow$ (ii) follows.
Now it is clear that (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) (for (iii) $\Rightarrow$ (iv), use that parabolic induction preserves stability of distributions, for which a nice reference is [KV16, Corollary 6.13]).
If $f \in C_c^\infty(\mathrm{G}(F))$ has $\mathcal{O}$-invariant image in $\mathcal{I}(\mathrm{G})$, then so does $z * f$ (we observed this in Remark 4.1.6). This fact together with Corollary 3.2.12 gives the implication (iv) $\Rightarrow$ (i). □

4.2. **Using Shahidi's argument on the Plancherel μ-function.** Let $\mathrm{M} \subset \mathrm{G}$ be a Levi subgroup, and $\zeta : \mathrm{A_M}(F) \to \mathbb{C}^\times$ a unitary character. One of the results that we will prove in this subsection is Corollary 4.2.11, part (i) of which says that the distribution $\sum d(\sigma)\mu(\sigma)\Theta_\sigma$ is stable, and more generally so is $\sum d(\sigma)\mu(\sigma)\hat{z}(\sigma)\Theta_\sigma$ for any $z \in \mathcal{Z}_1(\mathrm{G})$, where the sum ranges over the subset $\mathrm{Irr}_2(\mathrm{M})_\zeta \subset \mathrm{Irr}_2(\mathrm{M})$ of discrete series representations of $\mathrm{M}(F)$ whose central character restricts to ζ on $\mathrm{A_M}(F)$. Part (ii) of the corollary says that these distributions transfer well across inner forms. These are weaker but unconditional results in the spirit of the constancy of the Plancherel measure on L-packets as proved by Shahidi (see [Sha90, Section 9]), and that of the transfer of Plancherel measures across inner forms as one sees in the works of Choiy (see, e.g., [Cho14]) and Heiermann (see [Hei16, Appendix A]). We then use these results in Corollary 4.2.12 to show, in part (i) of the corollary, that the Plancherel measure or rather the μ-function, is constant on unitarily stable discrete series L-packets, and in part (ii) of the corollary that whenever a unitarily stable discrete series L-packet Σ on M transfers to a unitarily stable discrete series L-packet Σ^* on the quasi-split inner form M^* of M, we have $\mu(\sigma) = c\mu(\sigma^*)$ for all $\sigma \in \Sigma$ and $\sigma^* \in \Sigma^*$, where c is an explicit constant. We will use the Paley-Wiener theorem as stated in [Art96], so we begin by reviewing it.

4.2.1. *Review of the version of the Paley-Wiener theorem in* [Art96].

Notation 4.2.1.
(i) For this subsection, we fix a maximal split torus $\mathrm{A}_0 \subset \mathrm{G}$, and let M_0 be the minimal Levi subgroup of G obtained as the centralizer of A_0 in G. Further, let $\mathcal{L}$ denote the set of Levi subgroups of G that are semistandard, i.e., contain A_0, or equivalently, M_0.
(ii) Set $W_0 = W(\mathrm{M}_0)$ (i.e., the $W_\mathrm{G}(\mathrm{M}_0)$ in the sense of Subsubsection 2.1.7).
(iii) W_0 acts on $\mathcal{L}$, and we write $\mathcal{L}/W_0$ for the set-theoretic quotient. It is easy to see that each $\mathrm{G}(F)$-conjugacy class of Levi subgroups of G intersects $\mathcal{L}$ in a single W_0-orbit, so that $\mathcal{L}/W_0$ can be identified with the set of $\mathrm{G}(F)$-conjugacy classes of Levi subgroups of G.
(iv) We let the topological space $\tilde{T}(\mathrm{G})$, the topological space $\tilde{T}_{\mathrm{ell}}(\mathrm{M})$ for each Levi subgroup $\mathrm{M} \in \mathcal{L}$, and the decomposition

$$\tilde{T}(\mathrm{G}) = \bigsqcup_{\mathrm{M}\in\mathcal{L}/W_0} (\tilde{T}_{\mathrm{ell}}(\mathrm{M})/W(\mathrm{M})) \tag{30}$$

be as in [Art96, Section 4]. $\tilde{T}(\mathrm{G}) \supset \tilde{T}_{\mathrm{ell}}(\mathrm{G})$ is formed of certain tuples (L, σ, r), where L is a Levi subgroup of G and $\sigma \in \mathrm{Irr}_2(\mathrm{L})$; we will recall a few more details below. We will also occasionally use the quotients $T(\mathrm{G})$ and $T_{\mathrm{ell}}(\mathrm{G})$ of $\tilde{T}(\mathrm{G})$ and $\tilde{T}_{\mathrm{ell}}(\mathrm{G})$ as in [Art96, page 531]. For a smooth unitary character $\zeta : \mathrm{A_G}(F) \to \mathbb{C}^\times$, we also have subsets $\tilde{T}_\zeta(\mathrm{G}) \subset \tilde{T}(\mathrm{G}), \tilde{T}_{\zeta,\mathrm{ell}}(\mathrm{G}) \subset \tilde{T}_{\mathrm{ell}}(\mathrm{G}), T_\zeta(\mathrm{G}) \subset T(\mathrm{G})$ and $T_{\zeta,\mathrm{ell}}(\mathrm{G}) \subset T_{\mathrm{ell}}(\mathrm{G})$ represented by tuples (L, σ, r) such that the central character of σ restricts to ζ on $\mathrm{A_G}(F)$ (slightly differing in notation from [Art96, page 531]).
(v) If $\mathrm{M} \in \mathcal{L}$ and $\tau = (\mathrm{L}, \sigma, r) \in \tilde{T}(\mathrm{M})$, we let Θ_τ^M be the associated virtual character on $\mathrm{M}(F)$: for $\mathrm{M} = \mathrm{G}$, the 'normalized version' of Θ_τ^G, obtained by multiplying it by the discriminant factor $\gamma \mapsto |D(\gamma)|^{1/2}$ in the notation of [Art96], is what is denoted by $\gamma \mapsto I(\tau, \gamma)$ in [Art96, Section 4, near the top of page 532]. For $\tau \in \tilde{T}(\mathrm{G})$, let $\Theta_\tau = \Theta_\tau^\mathrm{G}$. If $\tau = (\mathrm{L}, \sigma, r) \in \tilde{T}_{\mathrm{ell}}(\mathrm{M})$,

where $\mathrm{M} \in \mathcal{L}$, and Θ_τ^{G} is defined using the image of τ in $\tilde{T}(\mathrm{G})$, one knows, and we will use without further comment in what follows, that $\Theta_\tau^{\mathrm{G}} = \mathrm{Ind}_{\mathrm{M}}^{\mathrm{G}} \Theta_\tau^{\mathrm{M}}$: use [MgW18, Lemma 2.10] (which works in the twisted case). One also knows that for each $\mathrm{M} \in \mathcal{L}$, the Θ_τ^{M} with $\tau \in \tilde{T}_{\mathrm{ell}}(\mathrm{M})$ running over a set of representatives for $T_{\mathrm{ell}}(\mathrm{M})$ form a basis for $D_{\mathrm{ell}}(\mathrm{M})$.

(vi) In this section too, we will write $\mathcal{I}(\mathrm{G})$ for the space of $\mathrm{Int}\,\mathrm{G}(F)$-coinvariants of $C_c^\infty(\mathrm{G}(F))$; it is also the quotient of $C_c^\infty(\mathrm{G}(F))$ by the subspace consisting of those functions whose strongly regular semisimple orbital integrals all vanish. $\mathcal{SI}(\mathrm{G})$ will denote the quotient of $\mathcal{I}(\mathrm{G})$ such that the kernel of $C_c^\infty(\mathrm{G}(F)) \to \mathcal{I}(\mathrm{G}) \to \mathcal{SI}(\mathrm{G})$ is the subspace of functions whose strongly regular semisimple orbital integrals all vanish. Thus, $\mathcal{I}(\mathrm{G})^* = \mathrm{Hom}(\mathcal{I}(\mathrm{G}), \mathbb{C})$ identifies with the space of invariant distributions on $\mathrm{G}(F)$, and $\mathcal{SI}(\mathrm{G})^* \subset \mathcal{I}(\mathrm{G})^*$ with the subspace of stable distributions. Of course, similar notation will apply with G replaced by a Levi subgroup M or a quasi-split form G^*, etc. According to the Paley-Wiener theorem, as stated in [Art96] and recalled in Remark 4.2.3 below, sending $f \in C_c^\infty(\mathrm{G}(F))$ to the function $\tilde{T}(\mathrm{G}) \to \mathbb{C}$ given by $\tau \mapsto \Theta_\tau(f)$, induces an isomorphism from $\mathcal{I}(\mathrm{G})$ to a concrete space of functions on $\tilde{T}(\mathrm{G})$. Once we describe this isomorphism in Remark 4.2.3, it will be thought of as an identification.

We now partially recall (slightly more than) what we need concerning the objects of Notation 4.2.1(iv). The second page of [Art96, Section 4] defines the set $\tilde{T}(\mathrm{G})$ as the set of W_0-orbits of certain triples (L, σ, r). For each such triple (L, σ, r), L is an element of $\mathcal{L}$, σ is a discrete series representation of $\mathrm{L}(F)$, and r is an element belonging to a certain central extension of the R-group of (L, σ) in G (we will not need the exact definition of this group, and hence refer the reader to [Art96] for more details). We refer to [Art96] for the definition of the subset $\tilde{T}_{\mathrm{ell}}(\mathrm{G}) \subset \tilde{T}(\mathrm{G})$ of elliptic elements, and the fact that we have a map from $\tilde{T}_{\mathrm{ell}}(\mathrm{M})$ (the set obtained by substituting M for G in the definition of $\tilde{T}_{\mathrm{ell}}(\mathrm{G})$) to $\tilde{T}(\mathrm{G})$, giving a decomposition of $\tilde{T}(\mathrm{G})$ as in (30).
For each $\mathrm{M} \in \mathcal{L}$, $X^{\mathrm{unr-uni}}(\mathrm{M})$ acts on $\tilde{T}_{\mathrm{ell}}(\mathrm{M})$, where the action of a unitary character $\chi : \mathrm{M}(F) \to \mathbb{C}^\times$ in $X^{\mathrm{unr-uni}}(\mathrm{M})$ sends each (L, σ', r) to $(\mathrm{L}, \sigma' \otimes \chi, r)$. This action makes each orbit into a torsor for a finite quotient of $X^{\mathrm{unr-uni}}(\mathrm{M})$, which being a compact torus topologizes the orbit. The orbits obtained this way partition $\tilde{T}_{\mathrm{ell}}(\mathrm{M})$, which we topologize by requiring this partition to be topological. Moreover, $\tilde{T}_{\mathrm{ell}}(\mathrm{M})/W(\mathrm{M})$ is then given the quotient topology. Allowing M to vary, this topologizes $\tilde{T}(\mathrm{G})$ by requiring the partition (30) to be topological.

Notation 4.2.2. For any $\mathrm{M} \in \mathcal{L}$, we have an injection $\mathrm{Irr}_2(\mathrm{M}) \hookrightarrow \tilde{T}_{\mathrm{ell}}(\mathrm{M})$ given by $\sigma \mapsto (\mathrm{M}, \sigma, 1)$, which will be thought of as an inclusion. Note that $\mathrm{Irr}_2(\mathrm{M}) \subset \tilde{T}_{\mathrm{ell}}(\mathrm{M})$ is a disjoint union of connected components of $\tilde{T}_{\mathrm{ell}}(\mathrm{M})$.

We will need to know Θ_τ, where $\tau \in \tilde{T}(\mathrm{G})$, only when it is the image of some $(\mathrm{M}, \sigma, 1) \in \mathrm{Irr}_2(\mathrm{M}) \subset \tilde{T}_{\mathrm{ell}}(\mathrm{M})$, where $\mathrm{M} \in \mathcal{L}$ and $\sigma \in \mathrm{Irr}_2(\mathrm{M})$. In such a situation, Θ_τ is simply the Harish-Chandra character of $\mathrm{Ind}_{\mathrm{M}}^{\mathrm{G}} \sigma$, i.e., of $\mathrm{Ind}_{\mathrm{P}}^{\mathrm{G}} \sigma$ for any parabolic subgroup $\mathrm{P} \subset \mathrm{G}$ with M as a Levi subgroup.

Remark 4.2.3. According to the trace Paley-Wiener theorem, as interpreted by Arthur in [Art96, page 532], the map $f \mapsto (\tau \mapsto \Theta_\tau^{\mathrm{G}}(f))$, from $C_c^\infty(\mathrm{G}(F))$ to some space of functions on $\tilde{T}(\mathrm{G})$, quotients to an isomorphism from $\mathcal{I}(\mathrm{G})$ to the space of functions $g : \tilde{T}(\mathrm{G}) \to \mathbb{C}$ satisfying the following three conditions:

(i) g is supported on finitely many connected components of $\tilde{T}(\mathrm{G})$ (Condition (i) on [Art96, page 532]);

(ii) For any $\mathrm{M} \in \mathcal{L}$ and any $\tau \in \tilde{T}_{\mathrm{ell}}(\mathrm{M})$, the map $X^{\mathrm{unr-uni}}(\mathrm{M}) \to \mathbb{C}$ given by $\chi \mapsto g(\overline{\chi \cdot \tau})$, where $\overline{\chi \cdot \tau}$ denotes the image of $\chi \cdot \tau \in \tilde{T}_{\mathrm{ell}}(\mathrm{M})$ in $\tilde{T}(\mathrm{G})$, is a finite complex linear combination of continuous characters of $X^{\mathrm{unr-uni}}(\mathrm{M})$ (Condition (ii) on [Art96, page 532]); and

(iii) Condition (iii) on [Art96, page 532], which only concerns the third component of a triple $\tau = (\mathrm{L}, \sigma, r)$, and is automatically satisfied for functions that are supported on the union over $\mathrm{M} \in \mathcal{L}/W_0$, of $\mathrm{Irr}_2(\mathrm{M})/W(\mathrm{M}) \subset \tilde{T}_{\mathrm{ell}}(\mathrm{M})/W(\mathrm{M}) \subset \tilde{T}(\mathrm{G})$.

As mentioned in Notation 4.2.1(vi), we will now start viewing $\mathcal{I}(\mathrm{G})$ also as the space of functions $\tilde{T}(\mathrm{G}) \to \mathbb{C}$ satisfying the three conditions above.

Remark 4.2.4. In fact, the original version of the trace Paley-Wiener theorem in [BDK86] was stated quite differently: it involved the set of cuspidal supports, rather than the triples $\tau = (\mathrm{L}, \sigma, r)$ above. It is the difference between these two formulations that necessitated the extra care taken in the proof of [Sha90, Proposition 9.3] (as Shahidi mentions in the remark after that Proposition), which the formulation of the Paley-Wiener theorem given by Arthur in [Art96] lets one avoid. According to [MgW18, Sections 6.1 and 6.2], the version we use follows from [LH17, Section 3.2] (which in fact handles the twisted case).

Notation 4.2.5. (i) We fix a Haar measure on $\mathrm{G}(F)$, and more generally on $\mathrm{M}(F)$ for each $\mathrm{M} \in \mathcal{L}$. For each $\mathrm{M} \in \mathcal{L}$, as in [MgW18, Section 1.2], we give $\mathrm{A_M}(F)$ and $X^{\mathrm{unr-uni}}(\mathrm{A_M})$ Haar measures such that $\mathrm{meas}(\mathrm{A_M}(F)_c)\,\mathrm{meas}(X^{\mathrm{unr-uni}}(\mathrm{A_M})) = 1$, where $\mathrm{A_M}(F)_c \subset \mathrm{A_M}(F)$ is the maximal compact subgroup. We give $\mathrm{M}(F)/\mathrm{A_M}(F)$ the quotient measure, and use it to define the formal degree $d(\sigma)$ for each $\sigma \in \mathrm{Irr}_2(\mathrm{M})$.

(ii) Unless otherwise stated, for any compact open subgroup $H \subset \mathrm{G}(F)$ and an algebraic subgroup $\mathrm{L} \subset \mathrm{G}$, H_{L} will denote $H \cap \mathrm{L}$.

(iii) Fix a maximal compact subgroup $K = K_{\mathrm{G}} \subset \mathrm{G}(F)$, which is the stabilizer of a special point belonging to the apartment of A_0 in the Bruhat-Tits building of G. We let $I = I_{\mathrm{G}} \subset K$ be an Iwahori subgroup of $\mathrm{G}(F)$ associated to a chamber in the same apartment. Thus, I has an Iwahori decomposition $I = I_{\mathrm{N}} I_{\mathrm{M}} I_{\mathrm{N}^-}$, whenever $\mathrm{M} \subset \mathrm{G}$ is a semistandard Levi subgroup, and N and N^- are unipotent radicals of opposite parabolic subgroups of G that have M as a common Levi subgroup.

(iv) To each semistandard Levi subgroup $\mathrm{M} \subset \mathrm{G}$, we attach constants $\gamma(\mathrm{G}|\mathrm{M}), \gamma'(\mathrm{G}|\mathrm{M})$ and $\gamma''(\mathrm{G}|\mathrm{M})$ (the latter two are, notationally, nonstandard and ad hoc) as follows. We choose opposite parabolic subgroups P and P^- having M as a common Levi subgroup, with N and N^- as their unipotent radicals, and let $\gamma(\mathrm{P}) = \gamma(\mathrm{G}|\mathrm{M})$ be as in [Wal03, page 241], using the choices of the measures as fixed in that reference. Moreover, we set (ad hoc and non-standard notation):

(31)
$$\gamma'(\mathrm{G}|\mathrm{M}) = \left(\prod_{\alpha} \gamma(\mathrm{M}_\alpha|\mathrm{M})^{-2}\right), \qquad \gamma''(\mathrm{G}|\mathrm{M}) = [K_{\mathrm{N}} : I_{\mathrm{N}}]^{-1}[K_{\mathrm{N}^-} : I_{\mathrm{N}^-}]^{-1}, \quad \text{and } \gamma'''(\mathrm{G}|\mathrm{M}) = \gamma'(\mathrm{G}|\mathrm{M})\gamma''(\mathrm{G}|\mathrm{M}),$$

where in the first product α runs over the set of reduced roots of $\mathrm{A_M}$ (outside M), taken up to a sign. It follows from [Wal03, Section I.1, (3)] that $\gamma(\mathrm{G}|\mathrm{M})$ and $\gamma'(\mathrm{G}|\mathrm{M})$ depend only on M, and not on P and P^-. That the same applies to $\gamma''(\mathrm{G}|\mathrm{M})$ and hence also to $\gamma'''(\mathrm{G}|\mathrm{M})$ follows from the relation

$$\gamma(\mathrm{G}|\mathrm{M}) = \frac{[K : H]}{[K_{\mathrm{N}} : H_{\mathrm{N}}][K_{\mathrm{M}} : H_{\mathrm{M}}][K_{\mathrm{N}^-} : H_{\mathrm{N}^-}]}, \tag{32}$$

which we claim holds for any compact open subgroup $H \subset \mathrm{G}(F)$ with an Iwahori decomposition $H = H_{\mathrm{N}} H_{\mathrm{M}} H_{\mathrm{N}^-}$ (and in particular for $H = I$, independently of P and P^-). The formula (32) follows from the latter equality of [Wal03, Section I.1, (2)], upon taking the f there to be the characteristic function of H, and noting that our analogues of the measures $dg, d\bar{u}, dm$ and du as in that equality are obtained by dividing arbitrarily chosen Haar measures on $\mathrm{G}(F), \mathrm{N}^-(F), \mathrm{M}(F)$ and $\mathrm{N}(F)$ respectively by $\mathrm{meas}(K), \mathrm{meas}(K_{\mathrm{N}^-}), \mathrm{meas}(K_{\mathrm{M}})$ and $\mathrm{meas}(K_{\mathrm{N}})$.

(v) For each Levi subgroup $\mathrm{M} \in \mathcal{L}$, let $\bar{E}_2(\mathrm{M})$ denote the set of connected components of $\mathrm{Irr}_2(\mathrm{M}) \subset \tilde{T}_{\mathrm{ell}}(\mathrm{M})$, and for each $\sigma \in \mathrm{Irr}_2(\mathrm{M})$, let

$$\mathscr{O}_\sigma := X^{\mathrm{unr-uni}}(\mathrm{M}) \cdot \sigma = X^{\mathrm{unr-uni}}(\mathrm{M}) \cdot (\mathrm{M}, \sigma) = X^{\mathrm{unr-uni}}(\mathrm{M}) \cdot (\mathrm{M}, \sigma, 1) \subset \mathrm{Irr}_2(\mathrm{M}) \subset \tilde{T}_{\mathrm{ell}}(\mathrm{M})$$

be the element of $\bar{E}_2(\mathrm{M})$ containing the image of $\sigma \in \mathrm{Irr}_2(\mathrm{M}) \subset \tilde{T}_{\mathrm{ell}}(\mathrm{M})$. As in [Wal03, pages 239 and 302], we give each $\mathscr{O}_\sigma$ the unique measure such that the restriction map $X^{\mathrm{unr-uni}}(\mathrm{M}) \to X^{\mathrm{unr-uni}}(\mathrm{A_M})$ and the obvious map $X^{\mathrm{unr-uni}}(\mathrm{M}) \to \mathscr{O}_\sigma$ locally preserve measures. In other words, the "$\mathrm{A_M}(F)$-central character" map from $\mathrm{Irr}_2(\mathrm{M})$ to the set $X^{\mathrm{uni}}(\mathrm{A_M})$ of unitary characters $\mathrm{A_M}(F) \to \mathbb{C}^\times$ is locally measure preserving, where

$X^{\mathrm{uni}}(\mathrm{A_M})$ is given the topology and measure such that each orbit map $X^{\mathrm{unr-uni}}(\mathrm{A_M}) \to X^{\mathrm{uni}}(\mathrm{A_M})$ is a measure preserving homeomorphism.

Here, the $\mathscr{O}$ in the orbit $\mathscr{O}_\sigma$ is not to be confused with the groups $\mathcal{O}_{\mathrm{M}}$ of automorphisms.

(vi) If $\mathrm{M} \in \mathcal{L}$ and $\sigma \in \mathrm{Irr}_2(\mathrm{L})$, we let $\mu(\sigma) = \mu^{\mathrm{G}}(\sigma)$ be the Harish-Chandra μ-function evaluated on σ, as in [Wal03, Section 5.2].

Now let us recall the Plancherel formula as stated in [Wal03, Theorem VIII.1.1(3)], but in terms of our different choice of measures, and in a form suited to our purposes:

Lemma 4.2.6. *Let $g : \tilde{T}(\mathrm{G}) \to \mathbb{C}$ be an element of $\mathcal{I}(\mathrm{G})$ (identified as in Remark 4.2.3), and let $f_g \in C_c^\infty(\mathrm{G}(F))$ have image g; in other words, $g(\tau) = \Theta_\tau(f_g)$ for all $\tau \in \tilde{T}(\mathrm{G})$. Let $f_g^\vee$ be as in Notation 4.1.2 (like what $\check{f}_g$ would be in the notation of* [Wal03, page 236]*). Then we have:*

$$f_g(1) = f_g^\vee(1) = \sum_{\mathrm{M} \in \mathcal{L}/W_0} \frac{\gamma'''(\mathrm{G}|\mathrm{M}) \,\mathrm{meas}(I_{\mathrm{M}})}{\mathrm{meas}(I) \cdot \# W(\mathrm{M})} \int_{\zeta \in X^{\mathrm{uni}}(\mathrm{A_M})} \left(\sum_{\sigma \in \mathrm{Irr}_2(\mathrm{M})_\zeta} \mu(\sigma) d(\sigma) g(\sigma) \right) d\zeta, \tag{33}$$

where $g(\sigma) = g((\mathrm{M}, \sigma))$ refers to the value of g on the image of $\sigma \in \mathrm{Irr}_2(\mathrm{M}) \subset \tilde{T}_{\mathrm{ell}}(\mathrm{M})$ in $\tilde{T}(\mathrm{G})$.

Proof. Given the constraint $\mathrm{meas}(\mathrm{A_M}(F)_c)\,\mathrm{meas}(X^{\mathrm{unr-uni}}(\mathrm{A_M})) = 1$ (see Notation 4.2.5(i)), and because $d(\sigma)$ varies linearly with the measure on $\mathrm{A_M}(F)$, we may and do assume that $\mathrm{A_M}(F)_c$ and $X^{\mathrm{unr-uni}}(\mathrm{A_M})$ are given the normalized Haar measure, as in [Wal03].
Suppose we can prove:

$$f_g(1) = f_g^\vee(1) = \sum_{\mathrm{M} \in \mathcal{L}/W_0} \frac{\gamma'(\mathrm{G}|\mathrm{M})\gamma(\mathrm{G}|\mathrm{M}) \,\mathrm{meas}(K_{\mathrm{M}})}{\mathrm{meas}(K) \cdot \# W(\mathrm{M})} \sum_{\mathscr{O} \in \bar{E}_2(\mathrm{M})} \int_{(\mathrm{M},\sigma) \in \mathscr{O}} \mu(\sigma) d(\sigma) g(\sigma)\, d\sigma. \tag{34}$$

Since the "$\mathrm{A_M}(F)$-central character map" from $\mathrm{Irr}_2(\mathrm{M}) \subset \tilde{T}_{\mathrm{ell}}(\mathrm{M})$ to $X^{\mathrm{uni}}(\mathrm{A_M})$ preserves measures locally, the fiber measure on each fiber of this map is the counting measure. Moreover, the fibers have finite intersection with each $\mathscr{O} \in \bar{E}_2(\mathrm{M})$ (since $X^{\mathrm{unr}}(\mathrm{M}) \to X^{\mathrm{unr}}(\mathrm{A_M})$ is an isogeny). Using this and the Fubini theorem (justified by g being continuous and supported on finitely many connected components of $\tilde{T}(\mathrm{G})$, together with the finiteness of the map $\tilde{T}_{\mathrm{ell}}(\mathrm{M}) \to \tilde{T}(\mathrm{G})$), and the equality

$$\frac{\gamma(\mathrm{G}|\mathrm{M}) \,\mathrm{meas}(K_{\mathrm{M}})}{\mathrm{meas}(K)} = \frac{\gamma''(\mathrm{G}|\mathrm{M}) \,\mathrm{meas}(I_{\mathrm{M}})}{\mathrm{meas}(I)}$$

that follows from (32), it is easy to see that (33) follows from (34). Therefore, it now suffices to prove (34).
Thus, it is now enough to deduce (34) from the formula in [Wal03, Theorem VIII.1.1(3)]. In [Wal03, Theorem VIII.1.1(3)], the sum is over a set of associate classes of pairs $(\mathscr{O}, \mathrm{P})$ as defined in [Wal03, Remark VII.2.4], where $\mathrm{P} \subset \mathrm{G}$ is a semistandard parabolic subgroup and $\mathscr{O} \in \bar{E}_2(\mathrm{M})$, with M the unique semistandard Levi subgroup of P. Instead, we can clearly sum over pairs $(\mathrm{M}, \mathscr{O})$ with M running over (a set of representatives for) $\mathcal{L}/W_0$, and $\mathscr{O}$ running over elements of $\bar{E}_2(\mathrm{M})$ up to the action of $W(\mathrm{M})$ (which is the $W(\mathrm{G}|\mathrm{M})$ of [Wal03]): this is because, given pairs $(\mathrm{P}_1, \mathscr{O}_1)$ and $(\mathrm{P}_2, \mathscr{O}_2)$, where both P_1 and P_2 have the same $\mathrm{M} \in \mathcal{L}$ as their unique semistandard Levi subgroup, these pairs are associate if and only if $\mathscr{O}_1$ and $\mathscr{O}_2$ are conjugate under $W(\mathrm{M})$. It is then easy to check that the expression of [Wal03, Theorem VIII.1.1(3)] agrees with that on the right-hand side of (34), which adds the factors $\mathrm{meas}(K_{\mathrm{M}})$ and $\mathrm{meas}(K)$ to account for not fixing the measures on $\mathrm{G}(F)$ and $\mathrm{M}(F)$ as in [Wal03] (and we have also used our having normalized $\mathrm{meas}(\mathrm{A_M}(F)_c) = \mathrm{meas}(\mathrm{A_M}(F) \cap K)$ and $\mathrm{meas}(X^{\mathrm{unr-uni}}(\mathrm{A_M}))$ to 1). Note that the $c(G|M)^{-2}$ of [Wal03] equals our $\gamma(\mathrm{G}|\mathrm{M})^2\gamma'(\mathrm{G}|\mathrm{M})$. □

Remark 4.2.7. (i) We recall a decomposition of $\mathcal{I}(\mathrm{G})$ ((35) below) from the top of [Art96, page 533], to which we refer for more explanation. Recall the subspace $\mathcal{I}_{\mathrm{cusp}}(\mathrm{M}) \subset \mathcal{I}(\mathrm{M})$ defined to be the image of $C^\infty_{c,\mathrm{cusp}}(\mathrm{M}(F)) \subset C_c^\infty(\mathrm{M}(F))$ in $\mathcal{I}(\mathrm{M})$, as in the proof of Proposition 3.2.6 (and as in Remark 3.4.7(i)). One knows that $\mathcal{I}_{\mathrm{cusp}}(\mathrm{M})$ identifies via the Paley-Wiener theorem (i.e., as in Remark 4.2.3) with the space of those functions $\tilde{T}(\mathrm{M}) \to \mathbb{C}$ in $\mathcal{I}(\mathrm{M})$ that are supported on $\tilde{T}_{\mathrm{ell}}(\mathrm{M})$. Recall that for each Levi subgroup $\mathrm{M} \subset \mathrm{G}$, the map $\tilde{T}_{\mathrm{ell}}(\mathrm{M}) \to \tilde{T}(\mathrm{G})$ factors through an isomorphism from $\tilde{T}_{\mathrm{ell}}(\mathrm{M})/W(\mathrm{M})$ onto its image. Using this fact, we see that the trace Paley-Wiener theorem from [Art96] (recalled in Remark 4.2.3) gives us a decomposition for $\mathcal{I}(\mathrm{G})$ of the form:

$$\mathcal{I}(\mathrm{G}) = \bigoplus_{\mathrm{M}\in\mathcal{L}/W_0} (\mathcal{I}_{\mathrm{cusp}}(\mathrm{M}))^{W(\mathrm{M})}. \tag{35}$$

(ii) Concretely, given $g : \tilde{T}(\mathrm{G}) \to \mathbb{C}$ in $\mathcal{I}(\mathrm{G})$, its projection g_{M} to $\mathcal{I}_{\mathrm{cusp}}(\mathrm{M})^{W(\mathrm{M})}$ is the unique function $\tilde{T}(\mathrm{M}) \to \mathbb{C}$ that is supported on $\tilde{T}_{\mathrm{ell}}(\mathrm{M})$, and such that $\Theta_\tau(g_{\mathrm{M}}) = \Theta_{\mathrm{Ind}_{\mathrm{M}}^{\mathrm{G}}\tau}(g)$ for each $\tau \in D_{\mathrm{ell}}(\mathrm{M})$. This identifies $\mathcal{I}_{\mathrm{cusp}}(\mathrm{M})^{W(\mathrm{M})}$ with the subspace of $\mathcal{I}(\mathrm{G})$ consisting of the images of functions $f \in C_c^\infty(\mathrm{G}(F))$ such that $(\mathrm{Ind}_{\mathrm{L}}^{\mathrm{G}}\Theta)(f) = 0$ whenever L is not $\mathrm{G}(F)$-conjugate to M, and $\Theta \in D_{\mathrm{ell}}(\mathrm{L})$.

(iii) From (35), taking duals, we have a decomposition involving spaces of distributions:

$$\mathcal{I}(\mathrm{G})^* = \bigoplus_{\mathrm{M}\in\mathcal{L}/W_0} (\mathcal{I}_{\mathrm{cusp}}(\mathrm{M})^*)^{W(\mathrm{M})}. \tag{36}$$

Tautologically, the pairing of $\mathcal{I}(\mathrm{G})$ with $D(\mathrm{G})$, after using the identifications of (35) and (11), is obtained by taking a direct sum of the pairings between the $\mathcal{I}_{\mathrm{cusp}}(\mathrm{L})^{W(\mathrm{L})}$ and $D_{\mathrm{ell}}(\mathrm{L})^{W(\mathrm{L})}$, as L ranges over $\mathcal{L}/W_0$.

(iv) By Proposition 3.2.10, $\mathcal{SI}_{\mathrm{cusp}}(\mathrm{M})$ (resp., $\mathcal{SI}(\mathrm{G})$) is the quotient of $\mathcal{I}_{\mathrm{cusp}}(\mathrm{M})$ (resp., $\mathcal{I}(\mathrm{G})$) by its subspace consisting of elements on which elements of $SD_{\mathrm{ell}}(\mathrm{M})$ (resp., $SD(\mathrm{G})$) vanishes. Therefore (35) induces a decomposition

$$\mathcal{SI}(\mathrm{G}) = \bigoplus_{\mathrm{M}\in\mathcal{L}/W_0} (\mathcal{SI}_{\mathrm{cusp}}(\mathrm{M}))^{W(\mathrm{M})}. \tag{37}$$

This decomposition has a description analogous to that for (35) given in (ii) above. Taking duals, we get a decomposition

$$\mathcal{SI}(\mathrm{G})^* = \bigoplus_{\mathrm{M}\in\mathcal{L}/W_0} (\mathcal{SI}_{\mathrm{cusp}}(\mathrm{M})^{W(\mathrm{M})})^* \tag{38}$$

at the level of stable distributions, that extends (12), and clearly aligns with (36).

(v) Let us expand on (37) and (38), and their compatibility with (35) and (36). The terms of (37) identify with quotients of the corresponding terms of (35); the derivation of (37) tells us that (35) identifies the subspace of $\mathcal{I}(\mathrm{G})$ consisting of its unstable elements, with the direct sum, over $\mathrm{M} \in \mathcal{L}/W_0$, of the subspace of unstable elements of $\mathcal{I}_{\mathrm{cusp}}(\mathrm{M})^{W(\mathrm{M})}$. Quotienting (35) by this restricted isomorphism yields (37). Dualizing, (38) is a restriction of (36) in an obvious way. While $\mathcal{SI}(\mathrm{G})^*$ identifies with the space of stable distributions on $\mathrm{G}(F)$, we can also interpret $(\mathcal{I}_{\mathrm{cusp}}(\mathrm{M})^{W(\mathrm{M})})^*$ and $(\mathcal{SI}_{\mathrm{cusp}}(\mathrm{M})^{W(\mathrm{M})})^*$ as spaces of distributions on $\mathrm{M}(F)$, using $W(\mathrm{M})$-averaging and the analogues of (36) and (38) with G replaced by M. Thus, $(\mathcal{I}_{\mathrm{cusp}}(\mathrm{M})^{W(\mathrm{M})})^*$ can be identified with the space of $W(\mathrm{M})$-invariant functionals on $\mathcal{I}(\mathrm{M})$ (i.e., invariant distributions on $\mathrm{M}(F)$) that vanish on "$\mathcal{I}_{\mathrm{cusp}}(\mathrm{L})^{W_{\mathrm{M}}(\mathrm{L})}$" for each proper Levi subgroup $\mathrm{L} \subset \mathrm{M}$. A similar interpretation applies to $(\mathcal{SI}_{\mathrm{cusp}}(\mathrm{M})^{W(\mathrm{M})})^*$. Clearly, $(\mathcal{SI}_{\mathrm{cusp}}(\mathrm{M})^{W(\mathrm{M})})^*$ is precisely the subspace of $(\mathcal{I}_{\mathrm{cusp}}(\mathrm{M})^{W(\mathrm{M})})^*$ consisting of elements that when, viewed as distributions on $\mathrm{M}(F)$, are stable. Now it is easy to see the following: if, according to (36), $\Theta \in \mathcal{I}(\mathrm{G})^*$ has component $\Theta_{\mathrm{M}} \in (\mathcal{I}_{\mathrm{cusp}}(\mathrm{M})^{W(\mathrm{M})})^*$ for each $\mathrm{M} \in \mathcal{L}/W_0$, then the distribution Θ on $\mathrm{G}(F)$ is stable if and only if each $\Theta_{\mathrm{M}} \in (\mathcal{I}_{\mathrm{cusp}}(\mathrm{M})^{W(\mathrm{M})})^* \subset (\mathcal{I}(\mathrm{M})^{W(\mathrm{M})})^*$ is stable as a distribution on $\mathrm{M}(F)$.

(vi) The compatibility between parabolic induction and endoscopic transfer (Remark 3.2.2(iii)) admits a slight generalization involving more general distributions than virtual characters, as we now review in the case of transfer to the quasi-split inner form; this can perhaps be viewed as an 'endoscopic version' of (v) above. Let G^* be a quasi-split inner form of G underlying an endoscopic datum $\underline{\mathrm{G}}^*$ for G, as in Notation 3.2.1(i). Let $\mathcal{L}^*$ and $\mathcal{L}^*/W_0^*$ be analogues, for G^*, of $\mathcal{L}$ and $\mathcal{L}/W_0$. Choosing representatives, we identify $\mathcal{L}/W_0$ and

$\mathcal{L}^*/W_0^*$ with subsets of $\mathcal{L}$ and $\mathcal{L}^*$. Notation 3.2.1(v) gives us an injection $\mathcal{L}/W_0 \hookrightarrow \mathcal{L}^*/W_0^*$. For each $\mathrm{M} \in \mathcal{L}/W_0 \subset \mathcal{L}$ and its image $\mathrm{M}^* \in \mathcal{L}^*/W_0^* \subset \mathcal{L}^*$, M^* is a quasi-split form of M, and more precisely, a choice of 'Levi subgroup matching data' as in Notation 3.2.1(vi) gives an endoscopic datum $\underline{\mathrm{M}}^*$ realizing M^* as endoscopic to M. Now we make two easy but useful observations:

(a) Let $\mathrm{M} \in \mathcal{L}/W_0 \subset \mathcal{L}$, and consider the corresponding $\mathrm{M}^* \in \mathcal{L}^*/W_0^* \subset \mathcal{L}^*$. Via (35) and the analogue of (37) for G^*, the endoscopic transfer map $\mathcal{I}(\mathrm{G}) \to \mathcal{SI}(\mathrm{G}^*)$ along $\underline{\mathrm{G}}^*$ takes $\mathcal{I}_{\mathrm{cusp}}(\mathrm{M})^{W(\mathrm{M})}$ to $\mathcal{SI}_{\mathrm{cusp}}(\mathrm{M}^*)^{W(\mathrm{M}^*)}$, and moreover, the resulting map $\mathcal{I}_{\mathrm{cusp}}(\mathrm{M})^{W(\mathrm{M})} \to \mathcal{SI}_{\mathrm{cusp}}(\mathrm{M}^*)^{W(\mathrm{M}^*)}$ is obtained by restricting the endoscopic transfer map $\mathcal{I}(\mathrm{M}) \to \mathcal{SI}(\mathrm{M}^*)$ along $\underline{\mathrm{M}}^*$. Indeed, using the concrete description in (ii) and its analogue for (37) (as applied to G^*), both these assertions follow from the compatibility between parabolic induction and endoscopic transfer (Remark 3.2.2(iii)), together with the fact that the endoscopic transfer maps $\mathcal{I}(\mathrm{G}) \to \mathcal{SI}(\mathrm{G}^*)$ and $\mathcal{I}(\mathrm{M}) \to \mathcal{SI}(\mathrm{M}^*)$ are uniquely determined as dual to the endsocopic transfer maps $SD(\mathrm{G}^*) \to SD(\mathrm{G})$ and $SD(\mathrm{M}^*) \to SD(\mathrm{M})$, by the density of characters in [Art96, Lemma 6.3] (or as recalled in Proposition 3.2.10).
(b) The map $\mathcal{SI}(\mathrm{G}^*)^* \to \mathcal{I}(\mathrm{G})^*$, via (36) and the analogue of (38) for G^*, restricts as follows to each $(\mathcal{SI}_{\mathrm{cusp}}(\mathrm{M}^*)^{W(\mathrm{M}^*)})^*$: If $\mathrm{M}^* \in \mathcal{L}^*/W_0^*$ is not the image of any element of $\mathcal{L}/W_0$, then this restriction is zero; if not, say M^* is the image of $\mathrm{M} \in \mathcal{L}/W_0$, it is a map $(\mathcal{SI}_{\mathrm{cusp}}(\mathrm{M}^*)^{W(\mathrm{M}^*)})^* \to (\mathcal{I}_{\mathrm{cusp}}(\mathrm{M})^{W(\mathrm{M})})^* \subset \mathcal{I}(\mathrm{G})^*$ obtained as the restriction of the endoscopic transfer map $\mathcal{SI}(\mathrm{M}^*)^* \to \mathcal{I}(\mathrm{M})^*$ along $\underline{\mathrm{M}}^*$. This observation follows by dualizing the observation (a) above (applied with M replaced by each $\mathrm{L} \in \mathcal{L}/W_0 \subset \mathcal{L}$).

4.2.2. *Stability of certain distributions, and their transfer to inner forms.*

Proposition 4.2.8. *Suppose $\Theta \in \mathcal{I}(\mathrm{G})^*$, and that for each $\mathrm{L} \in \mathcal{L}$, $\mu_\Theta = \mu_{\Theta,\mathrm{L}} : \mathrm{Irr}_2(\mathrm{L}) \to \mathbb{C}$ is a continuous function that is invariant under $W(\mathrm{L})$. Suppose that for all $g : \tilde{T}(\mathrm{G}) \to \mathbb{C}$ in $\mathcal{I}(\mathrm{G})$ we have:*

$$\Theta(g) = \sum_{\mathrm{L} \in \mathcal{L}/W_0} \int_{\zeta \in X^{\mathrm{uni}}(\mathrm{A}_\mathrm{L})} \Theta_{\mathrm{L},\zeta}(g)\, d\zeta, \tag{39}$$

where $\Theta_{\mathrm{L},\zeta} \in \mathcal{I}_{\mathrm{cusp}}(\mathrm{L})^* \subset \mathcal{I}(\mathrm{L})^*$ is a distribution of the form:

$$\Theta_{\mathrm{L},\zeta} = \sum_{\sigma \in \mathrm{Irr}_2(\mathrm{L})_\zeta} \mu_\Theta(\sigma) d(\sigma) \Theta_\sigma^{\mathrm{L}}, \tag{40}$$

and in (39) *$\Theta_{\mathrm{L},\zeta}(g)$ refers to $\Theta_{\mathrm{L},\zeta}(g_\mathrm{L})$, $g_\mathrm{L} \in \mathcal{I}_{\mathrm{cusp}}(\mathrm{L})^{W(\mathrm{L})}$ being the projection of g via* (35)*. Suppose that $\Theta \in \mathcal{I}(\mathrm{G})^*$ is stable, and let $\mathrm{M} \in \mathcal{L}$. Then $\Theta_{\mathrm{M},\zeta} \in \mathcal{I}(\mathrm{M})^*$, for each $\zeta \in X^{\mathrm{uni}}(\mathrm{A}_\mathrm{M})$.*

Proof. By the compatibility between (36) and (38), the projection Θ_M of $\Theta \in \mathcal{SI}(\mathrm{G})^*$ to $(\mathcal{I}_{\mathrm{cusp}}(\mathrm{M})^*)^{W(\mathrm{M})} \subset \mathcal{I}_{\mathrm{cusp}}(\mathrm{M})^* \subset \mathcal{I}(\mathrm{M})^*$ under (36) belongs to $(\mathcal{SI}_{\mathrm{cusp}}(\mathrm{M})^*)^{W(\mathrm{M})}$.

Note that for any $g : \tilde{T}(\mathrm{M}) \to \mathbb{C}$ in $\mathcal{I}(\mathrm{M})$, $\zeta \mapsto \Theta_{\mathrm{M},\zeta}(g)$ is the push-forward of $\sigma \mapsto \mu_\Theta(\sigma) d(\sigma) g(\sigma)$ along the $\mathrm{A}_\mathrm{M}(F)$-central character map $\tilde{T}_{\mathrm{ell}}(\mathrm{M}) \supset \mathrm{Irr}_2(\mathrm{M}) \to X^{\mathrm{uni}}(\mathrm{A}_\mathrm{M})$, which is a local homeomorphism, so that $\zeta \mapsto \Theta_{\mathrm{M},\zeta}(g)$ is continuous (use that g is supported on finitely many connected components of $\mathrm{Irr}_2(\mathrm{M}) \subset \tilde{T}_{\mathrm{ell}}(\mathrm{M})$). We claim that for all $f \in C_c^\infty(\mathrm{M}(F))$, we have:

$$\Theta_\mathrm{M}(f) = \int_{\zeta \in X^{\mathrm{uni}}(\mathrm{A}_\mathrm{M})} \Theta_{\mathrm{M},\zeta}(f)\, d\zeta. \tag{41}$$

The right-hand side of (41) represents a distribution in f that belongs to $\mathcal{I}_{\mathrm{cusp}}(\mathrm{M})^* \subset \mathcal{I}(\mathrm{M})^*$, since the $\Theta_{\mathrm{M},\zeta}$ are "supported" in discrete series representations. Therefore, (41) is tautological once we see that the right-hand side of (41) is a $W(\mathrm{M})$-invariant distribution in $f \in C_c^\infty(\mathrm{M}(F))$, which in turn follows from the $W(\mathrm{M})$-invariance of $\sigma \mapsto \mu_\Theta(\sigma)$ (by hypothesis) and that of $\sigma \mapsto d(\sigma)$.
Using (41) and the stability of Θ_M, let us show that $\Theta_{\mathrm{M},\zeta}$ is stable for each $\zeta \in X^{\mathrm{uni}}(\mathrm{A}_\mathrm{M})$. For all $h \in C_c^\infty(\mathrm{A}_\mathrm{M}(F))$ and $f \in C_c^\infty(\mathrm{M}(F))$, let $h * f \in C_c^\infty(\mathrm{M}(F))$ denote the left-regular action of h on f. It is easy to see that $\sigma(h * f) = \hat{h}(\zeta_\sigma)\sigma(f)$ for all unitary representations $\sigma \in \mathrm{Irr}(\mathrm{M})$,

where ζ_σ is the $\mathrm{A_M}(F)$-central character of σ and $\hat{h} \in C_0(X^{\mathrm{uni}}(\mathrm{A_M}))$ is the Fourier transform of h (C_0 stands for functions that 'vanish at ∞'). This implies that $\Theta_{\mathrm{M},\zeta}(h * f) = \hat{h}(\zeta)\Theta_{\mathrm{M},\zeta}(f)$, for all $h \in C_c^\infty(\mathrm{A_M}(F))$ and $f \in C_c^\infty(\mathrm{M}(F))$.
It is immediately verified that if $f \in C_c^\infty(\mathrm{M}(F))$ is unstable, then so is $h * f$ for all $C_c^\infty(\mathrm{A_M}(F))$. It follows from the stability of Θ_M that for unstable functions $f \in C_c^\infty(\mathrm{M}(F))$:

$$\int_{\zeta \in X^{\mathrm{uni}}(\mathrm{A_M})} \hat{h}(\zeta)\Theta_{\mathrm{M},\zeta}(f)\, d\zeta = 0.$$

Since the image of $C_c^\infty(\mathrm{A_M}(F))$ in $C_0(X^{\mathrm{uni}}(\mathrm{A_M})) \cap L^2(X^{\mathrm{uni}}(\mathrm{A_M}))$ under the Fourier transform is dense in $L^2(X^{\mathrm{uni}}(\mathrm{A_M}))$ by Pontrjagin duality, it follows that $(\zeta \mapsto \Theta_{\mathrm{M},\zeta}(f)) \in C_c(X^{\mathrm{uni}}(\mathrm{A_M}))$ vanishes as an element of $L^2(X^{\mathrm{uni}}(\mathrm{A_M}))$, and hence as an element of $C_c(X^{\mathrm{uni}}(\mathrm{A_M}))$. Since this is true for all unstable $f \in C_c^\infty(\mathrm{M}(F))$, the stability of $\Theta_{\mathrm{M},\zeta}$ follows. □

Remark 4.2.9. (i) The above proof can probably be adapted to prove a more general version of the proposition, where $\mu_\Theta = \mu_{\Theta,\mathrm{M}}$ is allowed to be any continuous function on the larger space $\tilde{T}_{\mathrm{ell}}(\mathrm{M}) \supset \mathrm{Irr}_2(\mathrm{M})$ that has an equivariance property opposite to that in the [Art96, page 532, condition (iii)] (whose articulation we omitted from Remark 4.2.3). The sum defining $\Theta_{\mathrm{M},\zeta}$ will then have to be over a set $T_{\mathrm{ell}}(\mathrm{M})_\zeta \supset \mathrm{Irr}_2(\mathrm{M})_\zeta$, the '$\zeta$-part' of the quotient $T_{\mathrm{ell}}(\mathrm{M})$ of $\tilde{T}_{\mathrm{ell}}(\mathrm{M})$ as in [Art96, just before (4.2)].

(ii) The argument of the proof can be adapted to deduce a 'version with central character' of the Plancherel formula: if $\zeta : \mathrm{A_G}(F) \to \mathbb{C}^\times$ is a smooth unitary character, and $f \in C^\infty_{\mathrm{A_G}(F),\zeta}(\mathrm{G}(F))$, then for an appropriate choice of a measure on $X^{\mathrm{uni}}(\mathrm{A_M}/\mathrm{A_G})$ we have a formula analogous to that in Lemma 4.2.6:

$$f(1) = \sum_{\mathrm{M}\in\mathcal{L}/W_0} \frac{\gamma'''(\mathrm{G}|\mathrm{M})\,\mathrm{meas}(I_\mathrm{M})}{\mathrm{meas}(I)\cdot \#W(\mathrm{M})} \int_{\zeta' \in X^{\mathrm{uni}}(\mathrm{A_M}/\mathrm{A_G})} \left(\sum_{\sigma\in \mathrm{Irr}_2(\mathrm{M})_{\zeta\zeta'}} \mu(\sigma)d(\sigma)\Theta^{\mathrm{G}}_\sigma(f)\right) d\zeta'. \tag{42}$$

Proposition 4.2.10. *Let G^* be an inner form of G. Fix an endoscopic datum $\underline{\mathrm{G}}^*$ for G with underlying group G^*, as in Notation 3.2.1(i). Let $\mathrm{M} \subset \mathrm{G}$ be a Levi subgroup in $\mathcal{L}$, and $\mathrm{M}^* \subset \mathrm{G}^*$ a Levi subgroup in an analogous set $\mathcal{L}^*$ defined using a maximal split torus $\mathrm{A}_0^* \subset \mathrm{G}^*$. Assume that some choice of 'Levi subgroup matching data' as in Notation 3.2.1(vi) matches M and M^*, giving an endoscopic datum $\underline{\mathrm{M}}^*$ for M with underlying group M^*. In particular, we have identifications $\mathrm{A_{M^*}} = \mathrm{A_M}$ and $\mathrm{A_{G^*}} = \mathrm{A_G}$. Let $\Theta = \Theta_\mathrm{G}$, the Θ_L, the $\mu_\Theta = \mu_{\Theta,\mathrm{L}}$ and the $\Theta_{\mathrm{L},\zeta}$ be as in Proposition 4.2.8. Suppose that $\Theta^* = \Theta_{\mathrm{G}^*}$ the Θ_{L^*}, the $\mu_{\Theta^*} = \mu_{\Theta^*,\mathrm{L}^*}$ and the*

$$\Theta_{\mathrm{L}^*,\zeta} = \sum_{\sigma^*\in\mathrm{Irr}_2(\mathrm{L}^*)_\zeta} d(\sigma^*)\mu_{\Theta^*}(\sigma^*)\Theta^{\mathrm{L}^*}_{\sigma^*}$$

are analogous objects associated to G^*; in particular, Θ and Θ^* are stable, and the $\mu_{\Theta,\mathrm{L}}$ and the $\mu_{\Theta^*,\mathrm{L}^*}$ are invariant under the $W(\mathrm{L})$ and the $W(\mathrm{L}^*)$, respectively. Assume that Θ^* has image Θ under the endoscopic transfer map $\mathcal{SI}(\mathrm{G}^*)^* \to \mathcal{SI}(\mathrm{G})^*$. Then for each $\zeta \in X^{\mathrm{uni}}(\mathrm{A_M}) = X^{\mathrm{uni}}(\mathrm{A_{M^*}})$, the image of $\Theta_{\mathrm{M}^*,\zeta}$ under the endoscopic transfer map $\mathcal{SI}(\mathrm{M}^*)^* \to \mathcal{SI}(\mathrm{M})^*$ equals $\Theta_{\mathrm{M},\zeta}$.

Proof. We follow the proof of Proposition 4.2.8. As we saw in that proof, the projection Θ_M of Θ along $\mathcal{I}(\mathrm{G})^* \to (\mathcal{I}_{\mathrm{cusp}}(\mathrm{M})^{W(\mathrm{M})})^*$ is stable, and and similarly we get the projection $\Theta_{\mathrm{M}^*} \in (\mathcal{SI}_{\mathrm{cusp}}(\mathrm{M}^*)^{W(\mathrm{M}^*)})^* \subset \mathcal{SI}(\mathrm{G}^*)^*$ of $\Theta^* = \Theta_{\mathrm{G}^*}$. By Proposition 4.2.8, $\Theta_{\mathrm{M},\zeta}$ and $\Theta_{\mathrm{M}^*,\zeta}$ are stable for each $\zeta \in X^{\mathrm{uni}}(\mathrm{A_M}) = X^{\mathrm{uni}}(\mathrm{A_{M^*}})$. We have (41) expressing Θ_M in terms of the $\Theta_{\mathrm{M},\zeta}$, and a similar equation relates Θ_{M^*} to the $\Theta_{\mathrm{M}^*,\zeta}$. Remark 4.2.7 (vi)(b) gives us the following claim:
Claim. The restriction of the endoscopic transfer map $\mathcal{SI}(\mathrm{G}^*)^* \to \mathcal{SI}(\mathrm{G})^* \subset \mathcal{I}(\mathrm{G})^*$ to $(\mathcal{SI}_{\mathrm{cusp}}(\mathrm{M}^*)^{W(\mathrm{M}^*)})^*$ is a map $(\mathcal{SI}_{\mathrm{cusp}}(\mathrm{M}^*)^{W(\mathrm{M}^*)})^* \to (\mathcal{I}_{\mathrm{cusp}}(\mathrm{M})^{W(\mathrm{M})})^*$, obtained by restricting the endoscopic transfer map $\mathcal{SI}(\mathrm{M}^*)^* \to \mathcal{I}(\mathrm{M})^*$.
By this claim, Θ_{M^*} has image Θ_M under $\mathcal{SI}(\mathrm{M}^*)^* \to \mathcal{SI}(\mathrm{M})^*$. We then identify $C_c^\infty(\mathrm{A_{M^*}}(F))$ and $C_c^\infty(\mathrm{A_M}(F))$ with each other, and consider their left-regular actions on $C_c^\infty(\mathrm{M}^*(F))$ and $C_c^\infty(\mathrm{M}(F))$, as well as the induced actions on associated spaces such as $\mathcal{I}(\mathrm{M}^*)$ and $\mathcal{I}(\mathrm{M})$ and

$\mathcal{SI}(\mathrm{M}^*)$ and $\mathcal{SI}(\mathrm{M})$. It is easy to see, using the arguments in the proof of Lemma 3.2.3(i), that this action respects the map $\mathcal{SI}(\mathrm{M}^*) \to \mathcal{SI}(\mathrm{M})$.

Now assume that $f \in C_c^\infty(\mathrm{M}(F))$ and $f^* \in C_c^\infty(\mathrm{M}^*(F))$ have matching orbital integrals. We need to show that $\Theta_{\mathrm{M}^*,\zeta}(f^*) = \Theta_{\mathrm{M},\zeta}(f)$ for all $\zeta \in X^{\mathrm{uni}}(\mathrm{A_M})$. For all $h \in C_c^\infty(\mathrm{A_M}(F)) = C_c^\infty(\mathrm{A_{M^*}}(F))$, we have that $h * f$ and $h * f^*$ have matching orbital integrals, and (as in the proof of Proposition 4.2.8) that $\Theta_{\mathrm{M}^*,\zeta}(h * f^*) = \hat{h}(\zeta)\Theta_{\mathrm{M}^*,\zeta}(f^*)$, and that $\Theta_{\mathrm{M},\zeta}(h * f) = \hat{h}(\zeta)\Theta_{\mathrm{M},\zeta}(f)$. Therefore,

$$\int_{X^{\mathrm{uni}}(\mathrm{A_{M^*}})=X^{\mathrm{uni}}(\mathrm{A_M})} \hat{h}(\zeta)\Theta_{\mathrm{M}^*,\zeta}(f^*)\, d\zeta = \Theta_{\mathrm{M}^*}(h*f^*) = \Theta_{\mathrm{M}}(h*f) = \int_{X^{\mathrm{uni}}(\mathrm{A_{M^*}})=X^{\mathrm{uni}}(\mathrm{A_M})} \hat{h}(\zeta)\Theta_{\mathrm{M},\zeta}(f)\, d\zeta.$$

Using Pontrjagin duality on $X^{\mathrm{uni}}(\mathrm{A_{M^*}}) = X^{\mathrm{uni}}(\mathrm{A_M})$ as in the proof of Proposition 4.2.8, it is now easy to see that $\Theta_{\mathrm{M}^*,\zeta}(f^*) = \Theta_{\mathrm{M},\zeta}(f)$ for each $\zeta \in X^{\mathrm{uni}}(\mathrm{A_M})$, as desired. □

Corollary 4.2.11. *(i) Let* $\mathrm{M} \subset \mathrm{G}$ *be a Levi subgroup. Then for each* $\zeta \in X^{\mathrm{uni}}(\mathrm{A_M})$ *and* $z \in \mathcal{Z}_1(\mathrm{G})$*, the distribution*

$$\sum_{\sigma \in \mathrm{Irr}_2(\mathrm{M})_\zeta} d(\sigma)\mu(\sigma)\hat{z}(\sigma)\Theta_\sigma^{\mathrm{M}} \in \mathcal{I}(\mathrm{M}^*)$$

is stable. In particular, $\sum_{\sigma \in \mathrm{Irr}_2(\mathrm{M})_\zeta} d(\sigma)\mu(\sigma)\Theta_\sigma^{\mathrm{M}}$ *is stable.*

(ii) Let G^* *be a quasi-split inner form of* G*, and let* $\underline{\mathrm{G}}^*$ *be as in Notation 3.2.1(i). As in Proposition 4.2.10, let* $\mathrm{M}^* \subset \mathrm{G}^*$ *and* $\mathrm{M} \subset \mathrm{G}$ *be 'compatible Levi subgroups', i.e., related by an endoscopic datum* $\underline{\mathrm{M}}^*$ *obtained using 'Levi subgroup matching data' as in Notation 3.2.1(vi). Assume that the measures on* $\mathrm{M}^*(F)$ *and* $\mathrm{M}(F)$ *are compatible in the sense explained in* [Kot88, page 631]*, and that the identification* $\mathrm{A_{M^*}}(F) = \mathrm{A_M}(F)$ *is measure preserving. Let* $\mathbf{T}_{\underline{\mathrm{M}}^*}$ *denote the endoscopic transfer map* $\mathcal{SI}(\mathrm{M}^*)^* \to \mathcal{SI}(\mathrm{M})^*$*. Assume that* $z^* \in \mathcal{Z}_1(\mathrm{G}^*)$ *and* $z \in \mathcal{Z}_1(\mathrm{G})$ *are related by* $\mathbf{T}_{\underline{\mathrm{G}}^*}(z^*) = e(\mathrm{G})z$*, where* $\mathbf{T}_{\underline{\mathrm{G}}^*}$ *is the endoscopic transfer map* $\mathcal{SI}(\mathrm{G}^*)^* \to \mathcal{SI}(\mathrm{G})^*$*, normalized using compatible measures on* $\mathrm{G}^*(F)$ *and* $\mathrm{G}(F)$ *as in* [Kot88, page 631]*. Then for each* $\zeta \in X^{\mathrm{uni}}(\mathrm{A_{M^*}}) = X^{\mathrm{uni}}(\mathrm{A_M})$ *we have that, and* $\mathcal{SI}(\mathrm{G}^*)^* \to \mathcal{SI}(\mathrm{G})^*$ *we have:*

$$\gamma'''(\mathrm{G}^*|\mathrm{M}^*)\cdot\mathbf{T}_{\underline{\mathrm{M}}^*}\left(\sum_{\sigma^* \in \mathrm{Irr}_2(\mathrm{M}^*)_\zeta} d(\sigma^*)\mu(\sigma^*)\hat{z^*}(\sigma^*)\Theta_{\sigma^*}^{\mathrm{M}^*}\right) = e(\mathrm{G})\gamma'''(\mathrm{G}|\mathrm{M})\left(\sum_{\sigma \in \mathrm{Irr}_2(\mathrm{M})_\zeta} d(\sigma)\mu(\sigma)\hat{z}(\sigma)\Theta_\sigma^{\mathrm{M}}\right).$$

In particular, we have

$$\gamma'''(\mathrm{G}^*|\mathrm{M}^*)\cdot\mathbf{T}_{\underline{\mathrm{M}}^*}\left(\sum_{\sigma^* \in \mathrm{Irr}_2(\mathrm{M}^*)_\zeta} d(\sigma^*)\mu(\sigma^*)\Theta_{\sigma^*}^{\mathrm{M}^*}\right) = e(\mathrm{G})\gamma'''(\mathrm{G}|\mathrm{M})\left(\sum_{\sigma \in \mathrm{Irr}_2(\mathrm{M})_\zeta} d(\sigma)\mu(\sigma)\Theta_\sigma^{\mathrm{M}}\right).$$

Proof. In (i), the latter assertion (i.e., the one starting with 'In particular') can be deduced from the former, by letting $z \in \mathcal{Z}(\mathrm{G})$ be the Dirac delta distribution at the identity, which ensures that $\hat{z}(\sigma) = 1$ for all $\sigma \in \mathrm{Irr}_2(\mathrm{M})$. A similar comment applies to (ii): if we take $z^* \in \mathcal{Z}(\mathrm{G}^*)$ to be the Dirac delta measure at the identity, then by [Kot88, Proposition 2] (which assumes the compatibility of measures between $\mathrm{G}^*(F)$ and $\mathrm{G}(F)$), we can take z to be the Dirac delta at the identity too (this is the reason for adding the Kottwitz sign $e(\mathrm{G})$ in the condition $\mathbf{T}_{\underline{\mathrm{G}}^*}(z^*) = e(\mathrm{G})z$). Therefore, in both (i) and (ii), we will only prove the former assertion.

To prove (i), we first note that Lemma 4.2.6 and the equality $\Theta_\tau^{\mathrm{G}}(z * f) = \hat{z}(\sigma)\Theta_\tau^{\mathrm{G}}(f)$ for $\mathrm{L} \in \mathcal{L}$ and $\tau = (\mathrm{L}, \sigma, 1) \in \mathrm{Irr}_2(\mathrm{L}) \subset \tilde{T}(\mathrm{G})$ (use (27)) imply:

(43)
$$z(f^\vee) = z * f(1) = \sum_{\mathrm{L} \in \mathcal{L}/W_0} \frac{\gamma'''(\mathrm{G}|\mathrm{L})\,\mathrm{meas}(I_\mathrm{L})}{\mathrm{meas}(I)\cdot \#W(\mathrm{L})} \int_{\zeta \in X^{\mathrm{uni}}(\mathrm{A_L})} \left(\sum_{\sigma \in \mathrm{Irr}_2(\mathrm{L})_\zeta} \mu(\sigma)d(\sigma)\hat{z}(\sigma)\Theta_\sigma^{\mathrm{G}}(f)\right) d\zeta.$$

Let $\Theta = \Theta_\mathrm{G} \in \mathcal{I}(\mathrm{G})^*$ be given by $f \mapsto z(f^\vee) = z * f(1)$. We claim that the hypotheses of Proposition 4.2.8 are satisfied for Θ, if we take, for each $\mathrm{L} \in \mathcal{L}/W_0$ and $\sigma \in \mathrm{Irr}_2(\mathrm{L})$:

$$\mu_\Theta(\sigma) = \mu_{\Theta,\mathrm{L}}(\sigma) = \frac{\gamma'''(\mathrm{G}|\mathrm{L})\,\mathrm{meas}(I_\mathrm{L})}{\mathrm{meas}(I)\cdot \#W(\mathrm{L})}\hat{z}(\sigma)\mu(\sigma). \tag{44}$$

By Proposition 4.2.8 (and using the expression (40)), proving this claim will yield (i). Given (43), using that Θ is stable (since $z \in \mathcal{Z}_1(\mathrm{G})$), this follows from the following three observations applied to each $\mathrm{L} \in \mathcal{L}$:

- If $f \in C_c^\infty(\mathrm{G}(F))$ maps to g in $\mathcal{I}(\mathrm{G})$, then for each $\sigma \in \mathrm{Irr}_2(\mathrm{L})$, $\Theta^{\mathrm{G}}_\sigma(f) = \Theta^{\mathrm{G}}_\sigma(g) = \Theta^{\mathrm{L}}_\sigma(g_{\mathrm{L}})$, where g_{L} is, as in Proposition 4.2.8, the projection of g to $\mathcal{I}_{\mathrm{cusp}}(\mathrm{L})^{W(\mathrm{L})}$ as per (35).
- $\mu_{\Theta,\mathrm{L}}$ is continuous, since for each $\sigma \in \mathrm{Irr}_2(\mathrm{L})$, $\chi \mapsto \hat{z}(\sigma \otimes \chi)$ and $\chi \mapsto \mu(\sigma \otimes \chi)$ are rational functions on $X^{\mathrm{unr}}(\mathrm{L})$ that are regular on $X^{\mathrm{unr-uni}}(\mathrm{L})$ (for the latter, see [Wal03, Lemma V.2.1]).
- $\mu_{\Theta,\mathrm{L}}$ is $W(\mathrm{L})$-invariant, since $\sigma \mapsto \hat{z}(\sigma)$ and $\sigma \mapsto \mu(\sigma)$ are (for the latter, again use [Wal03, Lemma V.2.1]).

This gives (i). The proof of (ii) will implicitly use the observations made in the proof of (i). Without loss of generality, M^* belongs to the set $\mathcal{L}^*$ analogous to $\mathcal{L}$, defined using a chosen split maximal torus $\mathrm{A}_0^* \subset \mathrm{G}^*$. We apply Proposition 4.2.10 with the $\Theta^* = \Theta_{\mathrm{G}^*} \in \mathcal{SI}(\mathrm{G}^*)$ of that proposition taken to be the distribution $f^* \mapsto z^*((f^*)^\vee) = (z^* * f^*)(1)$. It is easy to see from Remark 3.2.2(i) that $f^\vee$ and $(f^*)^\vee$ have matching orbital integrals whenever f and f^* do, so that the image $\Theta := \Theta_{\mathrm{G}} := \mathbf{T}_{\underline{\mathrm{G}}^*}(\Theta^*)$ of Θ^* under endoscopic transfer with respect to $\underline{\mathrm{G}}^*$ equals $f \mapsto e(\mathrm{G})z(f^\vee) = e(\mathrm{G})(z * f)(1)$.
Combining Proposition 4.2.10 with (44) and its analogue with G replaced by G^*, we get

$$\frac{\gamma'''(\mathrm{G}^*|\mathrm{M}^*)\,\mathrm{meas}(I_{\mathrm{M}^*})}{\mathrm{meas}(I_{\mathrm{G}^*})\cdot \#W(\mathrm{M}^*)} \cdot \mathbf{T}_{\underline{\mathrm{M}}^*}\left(\sum_{\sigma^* \in \mathrm{Irr}_2(\mathrm{M}^*)_\zeta} d(\sigma^*)\mu(\sigma^*)\hat{z^*}(\sigma^*)\Theta^{\mathrm{M}^*}_{\sigma^*}\right)$$
$$= \frac{\gamma'''(\mathrm{G}|\mathrm{M})\,\mathrm{meas}(I_{\mathrm{M}})}{\mathrm{meas}(I_{\mathrm{G}})\cdot \#W(\mathrm{M})} \cdot \left(\sum_{\sigma \in \mathrm{Irr}_2(\mathrm{M})_\zeta} d(\sigma)\mu(\sigma)\cdot e(\mathrm{G})\hat{z}(\sigma)\Theta^{\mathrm{M}}_{\sigma}\right),$$

for each $\zeta \in X^{\mathrm{uni}}(\mathrm{A}_{\mathrm{M}}) = X^{\mathrm{uni}}(\mathrm{A}_{\mathrm{M}^*})$, where $I_{\mathrm{G}} = I$ is as in Notation 4.2.5(iii), I_{G^*} is the analogous subgroup of $\mathrm{G}^*(F)$. Note that $I_{\mathrm{M}} = I \cap \mathrm{M}(F)$ and $I_{\mathrm{M}^*} = I_{\mathrm{G}^*} \cap \mathrm{M}^*(F)$ is are Iwahori subgroups of $\mathrm{M}(F)$ and $\mathrm{M}^*(F)$. This much is what we get without imposing any compatibility between the measures on $\mathrm{G}^*(F)$ and $\mathrm{G}(F)$, and between the ones on $\mathrm{M}^*(F)$ and $\mathrm{M}(F)$. Since $\#W(\mathrm{M}^*) = \#W(\mathrm{M})$ by the discussion of Notation 3.2.1(viii), (ii) will follow if we show that, for our choices of measures, we have:

$$\frac{\mathrm{meas}(I_{\mathrm{M}})}{\mathrm{meas}(I_{\mathrm{G}})} = \frac{\mathrm{meas}(I_{\mathrm{M}^*})}{\mathrm{meas}(I_{\mathrm{G}^*})}. \tag{45}$$

As in [Kot88, page 632], we may and do choose the measures on $\mathrm{G}(F), \mathrm{M}(F), \mathrm{G}^*(F)$ and $\mathrm{M}^*(F)$ to be integral and with nonzero reduction for the integral models of the parahoric group scheme structures associated to $I_{\mathrm{G}}, I_{\mathrm{M}}, I_{\mathrm{G}^*}$ and I_{M^*}. It is then enough to show that $\mathrm{meas}(I_{\mathrm{G}}) = \mathrm{meas}(I_{\mathrm{M}})$; for then we will similarly have $\mathrm{meas}(I_{\mathrm{G}^*}) = \mathrm{meas}(I_{\mathrm{M}^*})$, and (45) will follow. But this equality is an easy consequence of the discussion in [Kot88, page 633]; one has a formula $|\mathrm{S}_1(\mathbb{F}_q)|q^{-\dim \mathrm{S}_1}$ describing both $\mathrm{meas}(I_{\mathrm{G}})$ and $\mathrm{meas}(I_{\mathrm{M}})$, where $\mathrm{S}_1 \subset \mathrm{M}$ is an F-torus that becomes a maximal split torus over the maximal unramified extension of F in $\bar{F}$ (this does not need that G is simply connected, and is implicitly used for general reductive groups in a discussion in [Gro97, page 295, near (4.11)]). □

4.2.3. *Consequences for unitarily stable packets.*

Corollary 4.2.12. *(i) Let* $\mathrm{M} \subset \mathrm{G}$ *be a Levi subgroup. Let* $\mathcal{O}'_{\mathrm{M}} \subset \mathrm{Aut}(\mathrm{M})$ *be a subgroup that acts trivially on* A_{M}*, consists of elements that extend to automorphisms of* G*, and has finite image in* $\mathrm{Out}(\mathrm{M})$*. Let* Σ *be an* $\mathcal{O}'_{\mathrm{M}}$*-unitarily stable discrete series L-packet on* $\mathrm{M}(F)$*. Then* μ *is constant on* Σ*, and for all* $z \in \mathcal{Z}_1(\mathrm{G})$ *such that* $\sigma \mapsto \hat{z}(\sigma)$ *is* $\mathcal{O}'_{\mathrm{M}}$*-invariant on* $\mathrm{Irr}_2(\mathrm{M})$*,* $\sigma \mapsto \hat{z}(\sigma)$ *is constant on* Σ*.*

(ii) Suppose we are in the situation of Corollary 4.2.11(ii), with various measures chosen as in that corollary. Assume that for some subgroups $\mathcal{O}'_{\mathrm{M}} \subset \mathrm{Aut}(\mathrm{M})$ *and* $\mathcal{O}'_{\mathrm{M}^*} \subset \mathrm{Aut}(\mathrm{M}^*)$ *that act trivially on* $\mathrm{A}_{\mathrm{M}} = \mathrm{A}_{\mathrm{M}^*}$ *and consist of elements that extend to automorphisms of*

G and G^*, respectively, the images $\bar{\mathcal{O}}'_{\mathrm{M}}$ of $\mathcal{O}'_{\mathrm{M}}$ and $\bar{\mathcal{O}}'_{\mathrm{M}^*}$ of $\mathcal{O}'_{\mathrm{M}^*}$ in $\mathrm{Out}(\mathrm{M}) = \mathrm{Out}(\mathrm{M}^*)$ are finite and equal. Assume that $\mathcal{O}'_{\mathrm{M}}$ and $\mathcal{O}'_{\mathrm{M}^*}$ in Σ is an $\mathcal{O}'_{\mathrm{M}}$-unitarily stable discrete series L-packet on $\mathrm{M}(F)$ and Σ^* is an $\mathcal{O}'_{\mathrm{M}^*}$-unitarily stable discrete series L-packet on $\mathrm{M}^*(F)$. Assume that Σ is a transfer of Σ^*, in the sense that some nonzero virtual character $\Theta_{\Sigma^*} \in SD_{\mathrm{ell}}(\mathrm{M}^*)^{\mathcal{O}'_{\mathrm{M}^*}}$ supported on Σ^* transfers to a virtual character $\Theta_\Sigma \in SD_{\mathrm{ell}}(\mathrm{M})^{\mathcal{O}'_{\mathrm{M}}}$ supported on Σ. Then for all $\sigma^* \in \Sigma^*$ and $\sigma \in \Sigma$, we have:

$$\gamma'''(\mathrm{G}^*|\mathrm{M}^*)\mu(\sigma^*) = \gamma'''(\mathrm{G}|\mathrm{M})\mu(\sigma).$$

Moreover, for all $z^* \in \mathcal{Z}_1^*(\mathrm{G}^*)$ and $z \in \mathcal{Z}_1(\mathrm{G})$ such that z^* maps to $e(\mathrm{G})z$ under $\mathcal{SI}(\mathrm{G}^*)^* \to \mathcal{SI}(\mathrm{G})^*$, and such that $\sigma \mapsto \hat{z}(\sigma)$ is $\mathcal{O}'_{\mathrm{M}}$-invariant on $\mathrm{Irr}_2(\mathrm{M})$ and $\sigma^* \mapsto \hat{z^*}(\sigma^*)$ is $\mathcal{O}'_{\mathrm{M}^*}$-invariant on $\mathrm{Irr}_2(\mathrm{M})^*$, we have $\hat{z^*}(\sigma^*) = \hat{z}(\sigma)$.

Proof. Let us first prove (i). Since $\mathcal{O}'_{\mathrm{M}}$ acts trivially on A_{M}, Proposition 3.3.6 implies that the elements of Σ have a common $\mathrm{A}_{\mathrm{M}}(F)$-central character, which we denote by $\zeta \in X^{\mathrm{uni}}(\mathrm{A}_{\mathrm{M}})$.
It is easy to see from the definition (see [Wal03, Section V.2]) that the μ-function on $\mathrm{Irr}_2(\mathrm{M})$ is invariant under automorphisms of M that extend to automorphisms of G, and hence that $\sigma \mapsto d(\sigma)\mu(\sigma)\hat{z}(\sigma)$ is $\mathcal{O}'_{\mathrm{M}}$-invariant on $\mathrm{Irr}_2(\mathrm{M})$. By Corollary 4.2.11(i) and Proposition 3.3.6(iii), it follows that $\sigma \mapsto \mu(\sigma)\hat{z}(\sigma)$ is constant on Σ. Applying this with z replaced by the Dirac delta measure z_0 at 1, so that $\hat{z}_0(\sigma) = 1$ for all σ, we get the constancy of $\sigma \mapsto \mu(\sigma)$ on Σ. If $\mu(\sigma) \neq 0$ for $\sigma \in \Sigma$, the constancy of $\sigma \mapsto \hat{z}(\sigma)$ follows from that of $\sigma \mapsto \mu(\sigma)\hat{z}(\sigma)$. In general, since $\chi \mapsto \mu(\sigma \otimes \chi)\hat{z}(\sigma \otimes \chi)$ is meromorphic on $X^{\mathrm{unr}}(\mathrm{M})$ and holomorphic at points of $X^{\mathrm{unr-uni}}(\mathrm{M})$ by [Wal03, Lemma V.2.1], and since $\chi \mapsto \mu(\sigma \otimes \chi)$ is not identically zero on $X^{\mathrm{unr-uni}}(\mathrm{A}_{\mathrm{M}})$ (otherwise it would be so on $X^{\mathrm{unr}}(\mathrm{A}_{\mathrm{M}})$, contradicting that intertwining operators are holomorphic on a dense open subset of the vector space on which they are defined), it suffices to show that for all $\chi \in X^{\mathrm{unr-uni}}(\mathrm{M})$, $\sigma \mapsto \mu(\sigma \otimes \chi)\hat{z}(\sigma \otimes \chi)$ is constant on Σ. This in turn follows from applying the above considerations with Σ replaced by $\Sigma \otimes \chi$, which is an $\mathcal{O}'_{\mathrm{M}}$-unitarily stable discrete series L-packet by Lemma 3.3.8, and the fact that $\mathcal{O}'_{\mathrm{M}}$ acts trivially on $X^{\mathrm{unr}}(\mathrm{M})$ (since it does so on $X^{\mathrm{unr}}(\mathrm{A}_{\mathrm{M}})$ and hence on $X^{\mathrm{unr}}(\mathrm{S}_{\mathrm{M}})$, which surjects to $X^{\mathrm{unr}}(\mathrm{M})$).
Now let us prove (ii). By Lemma 3.2.3(i), the common A_{M}-central character $\zeta \in X^{\mathrm{uni}}(\mathrm{A}_{\mathrm{M}}) = X^{\mathrm{uni}}(\mathrm{A}_{\mathrm{M}^*})$ of the elements of Σ is also the common $\mathrm{A}_{\mathrm{M}^*}$-central character of the elements of Σ^*. This time, one similarly has the $\mathcal{O}'_{\mathrm{M}}$-invariance of $\sigma \mapsto d(\sigma)\mu(\sigma)\hat{z}(\sigma)$ on $\mathrm{Irr}_2(\mathrm{M})$ and the $\mathcal{O}'_{\mathrm{M}^*}$-invariance of $\sigma^* \mapsto d(\sigma^*)\mu(\sigma^*)\hat{z}(\sigma^*)$ on $\mathrm{Irr}_2(\mathrm{M}^*)$. Thus, we apply Corollary 4.2.11(ii) and Proposition 3.3.7(iii) to get:

$$e(\mathrm{M})\gamma'''(\mathrm{G}^*|\mathrm{M}^*)\mu(\sigma^*)\hat{z}^*(\sigma^*) = e(\mathrm{G})\gamma'''(\mathrm{G}|\mathrm{M})\mu(\sigma)\hat{z}(\sigma). \tag{46}$$

Applying this with z^* replaced by the Dirac measure at the identity, which transfers to $e(\mathrm{G})$ times the Dirac measure at the identity by [Kot88, Proposition 2], we get $\gamma'''(\mathrm{G}^*|\mathrm{M}^*)\mu(\sigma^*) = e(\mathrm{G})e(\mathrm{M})^{-1}\gamma'''(\mathrm{G}|\mathrm{M})\mu(\sigma)$, which gives the first assertion of (ii), since $e(\mathrm{G}) = e(\mathrm{M})$ (see [Kot83, Corollary, (6)]). If $\mu(\sigma^*) \neq 0$, so that $\mu(\sigma) \neq 0$ as well, the remaining assertion of (ii) follows from (46). The case where we allow $\mu(\sigma^*)$ to be 0 then follows by twisting by various $\chi \in X^{\mathrm{unr-uni}}(\mathrm{M}^*) = X^{\mathrm{unr-uni}}(\mathrm{M})$, as in the proof of (i). □

Corollary 4.2.13. *For some $p \gg 0$ depending on G, the following holds.*

(i) Let $\mathrm{M} \subset \mathrm{G}$ be a Levi subgroup. Let Σ be a regular supercuspidal packet on $\mathrm{M}(F)$, in the sense of Kaletha. Then μ is constant on Σ.

(ii) Let G^* be a quasi-split reductive group over F, and assume that there exists an inner twist from G^* to G that restricts to an inner twist from a Levi subgroup $\mathrm{M}^* \subset \mathrm{G}^*$ to $\mathrm{M} \subset \mathrm{G}$. Let Σ^* be a regular supercuspidal packet on $\mathrm{M}^*(F)$ with the same Langlands parameter as Σ. Then for all $\sigma \in \Sigma$ and $\sigma^* \in \Sigma^*$, we have

$$\gamma'''(\mathrm{G}^*|\mathrm{M}^*)\mu(\sigma^*) = \gamma'''(\mathrm{G}|\mathrm{M})\mu(\sigma).$$

Proof. Since $p \gg 0$, Σ and Σ^* are unitarily stable (Corollary 3.4.12). By Kaletha's endoscopic character identities, Σ is a transfer of Σ^*. Now the corollary is a special case of Corollary 4.2.12. □

4.3. **Two applications.**

4.3.1. *Normalizing intertwining operators using Langlands-Shahidi L-functions.*

Lemma 4.3.1. *Let* $\mathrm{G}, \mathrm{M}, \mathrm{G}^*, \mathrm{M}^*, \Sigma, \Sigma^*$ *be as in Corollary 4.2.12(ii); in particular we used the discussion of (i) and (vi) of Notation 3.2.1 to fix inner twists* ψ_{G^*} *from* G^* *to* G *and* ψ_{M^*} *from* M^* *to* M*, using fixed parabolic subgroups, say* $\mathrm{P}^* \subset \mathrm{G}^*$ *and* $\mathrm{P} \subset \mathrm{G}$ *(the analogues of* Q^* *and* Q *in Notation 3.2.1(vi)), with* M^* *and* M *respectively as Levi subgroups. Without loss of generality, we assume that* ψ_{M^*} *is a restriction of* ψ_{G^*} *(and not just a restriction of an element of* $\operatorname{Int}\mathrm{G}(\bar{F})\circ\psi_{\mathrm{G}^*}$*). Let* $\mathrm{P}^{*,-} \subset \mathrm{G}^*$ *and* $\mathrm{P}^- \subset \mathrm{G}$ *be parabolic subgroups that are opposite to* P^* *and* P *and contain* M^* *and* M*. Let* $\mathrm{N}^*, \mathrm{N}^{*,-}, \mathrm{N}$ *and* N^- *be the unipotent radicals of* $\mathrm{P}^*, \mathrm{P}^{*,-}, \mathrm{P}$ *and* P^-*. Note that* ψ_{G^*} *takes* $\mathrm{N}^*_{\bar{F}}$ *to* $\mathrm{N}_{\bar{F}}$ *and* $\mathrm{N}^{*,-}_{\bar{F}}$ *to* $\mathrm{N}^-_{\bar{F}}$*, letting us transfer top-degree differential forms (defined over* $\bar{F}$*) between these groups, and therefore lets us transfer Haar measures from* $\mathrm{N}^*(F)$ *to* $\mathrm{N}(F)$ *and* $\mathrm{N}^{*,-}(F)$ *to* $\mathrm{N}^-(F)$ *(using either the absolute value on* $\bar{F}$ *or a nontrivial continuous additive character* $\psi_F : F \to \mathbb{C}^\times$*). We choose Haar measures on* $\mathrm{N}^*(F)$ *and* $\mathrm{N}^{*,-}(F)$*, and transfer them to* $\mathrm{N}(F)$ *and* $\mathrm{N}^-(F)$ *using* ψ_{G^*}*, as just explained. For* $\sigma^* \in \Sigma^*$ *and* $\sigma \in \Sigma$*, let the intertwining operators* $J_{\mathrm{P}^{*,-}|\mathrm{P}^*}(\sigma^*), J_{\mathrm{P}^*|\mathrm{P}^{*,-}}(\sigma^*), J_{\mathrm{P}^-|\mathrm{P}}(\sigma)$ *and* $J_{\mathrm{P}|\mathrm{P}^-}(\sigma)$ *be defined as in* [Art89, around (1.1)] *or equivalently as in* [Wal03, just before Theorem IV.1.1]*, but using the choices of measures just fixed. Then, as meromorphic functions in* $\chi \in X^{\mathrm{unr}}(\mathrm{M}^*) = X^{\mathrm{unr}}(\mathrm{M})$*, we have:*

$$J_{\mathrm{P}^*|\mathrm{P}^{*,-}}(\sigma^* \otimes \chi) \circ J_{\mathrm{P}^{*,-}|\mathrm{P}^*}(\sigma^* \otimes \chi) = J_{\mathrm{P}|\mathrm{P}^-}(\sigma \otimes \chi) \circ J_{\mathrm{P}^-|\mathrm{P}}(\sigma \otimes \chi) \tag{47}$$

— here, the operators on either side are scalar multiplications, and hence viewed as complex numbers, for a dense subset of $\chi \in X^{\mathrm{unr-uni}}(\mathrm{M}^*) = X^{\mathrm{unr-uni}}(\mathrm{M})$*, which is automatically Zariski dense in* $X^{\mathrm{unr}}(\mathrm{M}^*) = X^{\mathrm{unr}}(\mathrm{M})$*.*

Proof. Recall $K = K_{\mathrm{G}}, I = I_{\mathrm{G}}, K_{\mathrm{M}}, K_{\mathrm{N}}, K_{\mathrm{N}^-}, I_{\mathrm{M}}, I_{\mathrm{N}}$ and I_{N^-} from Notation 4.2.5(iii); here $H_{\mathrm{L}} = H \cap \mathrm{L}(F)$ for each compact open subgroup $H \subset \mathrm{G}(F)$ and algebraic subgroup $\mathrm{L} \subset \mathrm{G}$. We choose analogous objects for G^*: $K^* = K_{\mathrm{G}^*}, I^* = I_{\mathrm{G}^*}, K_{\mathrm{M}^*} = K^* \cap \mathrm{M}^*(F), K_{\mathrm{N}^*}, K_{\mathrm{N}^{*,-}}, I_{\mathrm{M}^*}, I_{\mathrm{N}^*}$ and $I_{\mathrm{N}^{*,-}}$. We give $\mathrm{M}^*(F)$ and $\mathrm{M}(F)$ Haar measures that are compatible under ψ_{M^*}. We give $\mathrm{G}^*(F)$ and $\mathrm{G}(F)$ the unique Haar measures such that the multiplication maps $\mathrm{N}^*(F) \times \mathrm{M}^*(F) \times \mathrm{N}^{*,-}(F) \to \mathrm{G}^*(F)$ and $\mathrm{N}(F) \times \mathrm{M}(F) \times \mathrm{N}^-(F) \to \mathrm{G}(F)$ are measure preserving near the identity. It is then easy to see that $\mathrm{G}^*(F)$ and $\mathrm{G}(F)$ have measures that are compatible under ψ_{G^*}. Therefore, the equality (45) proved in the proof of Corollary 4.2.11 holds, and gives:

$$\operatorname{meas}(I_{\mathrm{N}})\operatorname{meas}(I_{\mathrm{N}^-}) = \frac{\operatorname{meas}(I)}{\operatorname{meas}(I_{\mathrm{M}})} = \frac{\operatorname{meas}(I^*)}{\operatorname{meas}(I_{\mathrm{M}^*})} = \operatorname{meas}(I_{\mathrm{N}^*})\operatorname{meas}(I_{\mathrm{N}^{*,-}}). \tag{48}$$

By the definitions in [Wal03, Sections IV.3 and V.2] and (31), the reciprocal of the left-hand side (resp., the reciprocal of the right-hand side) of (47) equals

$$\gamma'(\mathrm{G}^*|\mathrm{M}^*)\mu(\sigma^*\otimes\chi)\operatorname{meas}(K_{\mathrm{N}^*})^{-1}\operatorname{meas}(K_{\mathrm{N}^{*,-}})^{-1} = \gamma'''(\mathrm{G}^*|\mathrm{M}^*)\mu(\sigma^*\otimes\chi)\operatorname{meas}(I_{\mathrm{N}^*})^{-1}\operatorname{meas}(I_{\mathrm{N}^{*,-}})^{-1}$$

(resp., $\gamma'(\mathrm{G}|\mathrm{M})\mu(\sigma \otimes \chi)\operatorname{meas}(K_{\mathrm{N}})^{-1}\operatorname{meas}(K_{\mathrm{N}^-})^{-1} = \gamma'''(\mathrm{G}|\mathrm{M})\mu(\sigma \otimes \chi)\operatorname{meas}(I_{\mathrm{N}})^{-1}\operatorname{meas}(I_{\mathrm{N}^-})^{-1}$).

Now the lemma follows from (48) and Corollary 4.2.12(ii), the latter applied with Σ^* and Σ replaced by $\Sigma^* \otimes \chi$ and $\Sigma \otimes \chi$, as justified by Lemmas 3.3.8 and 3.2.3(i) (and the fact that $\mathcal{O}'_{\mathrm{M}}, \mathcal{O}'_{\mathrm{M}^*}$ act trivially on $\mathrm{A}_{\mathrm{M}}, \mathrm{A}_{\mathrm{M}^*}$). □

Remark 4.3.2. In Lemma 4.3.1, it is an easy exercise to see that replacing ψ_{G^*} by a different inner twist, while yielding different measures on $\mathrm{N}(F)$ and $\mathrm{N}^-(F)$, yields the same product measure on $\mathrm{N}(F) \times \mathrm{N}^-(F)$, and hence does not change the right-hand side of (47).

Remark 4.3.3. Assume that we are in the setting of Lemma 4.3.1, and assume for simplicity that $\mathcal{O}'_{\mathrm{M}}$ and $\mathcal{O}'_{\mathrm{M}^*}$ are trivial. Let r_i^* denote the representations of ${}^L\mathrm{M}^* = {}^L\mathrm{M}$ associated to $\mathrm{M}^* \subset \mathrm{P}^* \subset \mathrm{G}^*$ as in [Sha90]. Assume also that Σ^* contains a generic representation σ^*. Thus, the definition of the Langlands-Shahidi L-functions and ϵ-factors extend to representations $\sigma \in \Sigma$, as explained in [Sha90, shortly before Theorem 9.5] (with the difference that we are stopping at discrete series packets and not invoking Langlands classification):

$$L(s, \sigma, r_i) = L(s, \sigma^*, r_i^*),$$

and for any continuous nontrivial additive character $\psi_F : F \to \mathbb{C}^\times$,

$$\epsilon(s, \sigma, r_i, \psi_F) = \epsilon(s, \sigma^*, r_i^*, \psi_F).$$

It should be possible to use Lemma 4.3.1 to deduce from [Sha90] that these L-functions and ϵ-factors give a normalization of intertwining operators as in [Art89, Theorem 2.1]. We will skip exploring the precise details.

4.3.2. *Consequences for depth preservation.*

Corollary 4.3.4. *Suppose the residue characteristic p of F is a very good prime for* G *in the sense of* [BKV16, Section 8.10]*. Let* $\mathrm{M} \subset \mathrm{G}$ *be a Levi subgroup. Let* $\mathcal{O}'_{\mathrm{M}} \subset \mathrm{Aut}(\mathrm{M})$ *be a subgroup that acts trivially on* $\mathrm{A_M}$*, consists of elements that extend to automorphisms of* G*, and has finite image in* $\mathrm{Out}(\mathrm{M})$.

(i) The elements of Σ all have the same depth in the sense of Moy and Prasad (see [MP96]*).*

(ii) Assume that $\mathrm{G}^*, \mathcal{O}'_{\mathrm{M}}, \mathcal{O}'_{\mathrm{M}^*}, \Sigma$ and Σ^* are as in the situation of Corollary 4.2.12(ii). Assume additionally that there exists a nice bilinear form B on $\mathfrak{g}$, in the sense of Definition 4.3.6(iii) below. Then for each $\sigma \in \Sigma$ and $\sigma^* \in \Sigma^*$, we have

$$\mathrm{depth}(\sigma) = \mathrm{depth}(\sigma^*).$$

The proof of (ii) of the corollary will use:

Proposition 4.3.5. *Suppose p is a very good prime for* G *in the sense of* [BKV16, Section 8.10]*, and that there exists a nice bilinear form on $\mathfrak{g}$, in the sense of Definition 4.3.6(iii) below. Let* G^* *be a quasi-split inner form for* G*, underlying an endoscopic datum* $\underline{\mathrm{G}}^*$ *for* G *defined using an inner twist* ψ_{G^*} *as in Notation 3.2.1(i). Let $r \geq 0$, and let $E_r \in \mathcal{Z}(\mathrm{G})$ and $E_r^* \in \mathcal{Z}(\mathrm{G}^*)$ be the depth r projectors in the sense of* [BKV16]*. Then the distribution E_r on* $\mathrm{G}(F)$ *and the distribution E_r^* on* $\mathrm{G}^*(F)$ *are stable. Moreover, E_r^* transfers to the distribution* $e(\mathrm{G})E_r$ *on* $\mathrm{G}(F)$*, provided* $\mathrm{G}(F)$ *and* $\mathrm{G}^*(F)$ *are given measures that are compatible with respect to* ψ_{G^*}.

Now we make some preparations for the proof of Proposition 4.3.5.

Definition 4.3.6. (i) For a finite extension F_1/F, we will denote by $\mathcal{B}(\mathrm{G}/F_1)$ the reduced Bruhat-Tits building of G_{F_1}, and abbreviate $\mathcal{B}(\mathrm{G}/F)$ to $\mathcal{B}(\mathrm{G})$. F^{unr} will denote the maximal unramified extension of F in $\bar{F}$, and for each extension F_1 of F in F^{unr}, $\mathcal{B}(\mathrm{G}/F_1)$ will be realized as $\mathcal{B}(\mathrm{G}/F^{\mathrm{unr}})^{\mathrm{Gal}(F^{\mathrm{unr}}/F_1)}$. As usual, the notation that follows will be adapted to more general groups and fields in place of G and F.

(ii) For $x \in \mathcal{B}(\mathrm{G})$ and $r \geq 0$ (resp., $r \in \mathbb{R}$), we have the Moy-Prasad filtration subgroups $\mathrm{G}(F)_{x,r}, \mathrm{G}(F)_{x,r+} \subset \mathrm{G}(F)$, and the Moy-Prasad filtration lattices $\mathfrak{g}(F)_{x,r}, \mathfrak{g}(F)_{x,r+} \subset \mathfrak{g}(F)$ and $\check{\mathfrak{g}}(F)_{x,r}, \check{\mathfrak{g}}(F)_{x,r+} \subset \check{\mathfrak{g}}(F)$, where $\check{\mathfrak{g}}$ is the dual vector space of $\mathfrak{g}$, which is given the coadjoint action. We also have the Moy-Prasad G-domains $\mathrm{G}(F)_r = \bigcup_x \mathrm{G}(F)_{x,r}, \mathrm{G}(F)_{r+} = \bigcup_x \mathrm{G}(F)_{x,r+} \subset \mathrm{G}(F)$, $\mathfrak{g}(F)_r = \bigcup_x \mathfrak{g}(F)_{x,r}, \mathfrak{g}(F)_{r+} = \bigcup_x \mathfrak{g}(F)_{x,r+} \subset \mathfrak{g}(F)$ and $\check{\mathfrak{g}}_r = \bigcup_x \check{\mathfrak{g}}(F)_{x,r}, \check{\mathfrak{g}}_{r+} = \bigcup_x \check{\mathfrak{g}}(F)_{x,r+} \subset \check{\mathfrak{g}}(F)$, where each of these unions is over $x \in \mathcal{B}(\mathrm{G})$. Thus, $\mathrm{G}(F)_{x,r+} = \mathrm{G}(F)_{x,r+\epsilon}$ for all small enough $\epsilon > 0$, and similarly with $\mathfrak{g}(F)_{x,r+}, \check{\mathfrak{g}}(F)_{x,r+}$, $\mathrm{G}(F)_{r+}, \mathfrak{g}(F)_{r+}$ and $\check{\mathfrak{g}}(F)_{r+}$. Here and in the rest of this subsection, we will often write $\mathfrak{g}(F)$ despite it being also denoted by $\mathfrak{g}$, to distinguish it from $\mathfrak{g}(F_1)$ for another field F_1.

(iii) A bilinear form $B : \mathfrak{g}(F) \times \mathfrak{g}(F) \to F$ will be called nice if it is symmetric, nondegenerate, $\mathrm{Ad}\,\mathrm{G}$-invariant and identifies each Moy-Prasad filtration lattice $\mathfrak{g}(F)_{x,r}$ in $\mathfrak{g}(F)$ with $\check{\mathfrak{g}}(F)_{x,r}$; this translates to requiring that for all $x \in \mathcal{B}(\mathrm{G})$ and $r \in \mathbb{R}$ we have

$$\{X \in \mathfrak{g}(F) \mid B(X, \mathfrak{g}(F)_{x,(-r)+}) \subset \varpi\mathfrak{O}_F\} = \mathfrak{g}(F)_{x,r}. \tag{49}$$

(iv) For the rest of this subsection G^* will denote a quasi-split inner form of G, and ψ_{G^*} and the endoscopic datum $\underline{\mathrm{G}}^*$ will be as in Notation 3.2.1(i). Note that $\mathfrak{g}^* = \mathrm{Lie}\,\mathrm{G}^*$ should not be confused with $\check{\mathfrak{g}}$.

(v) If B is an $\mathrm{Ad}\,\mathrm{G}$-invariant bilinear form on $\mathfrak{g}(F)$, then its transport by ψ_{G^*} refers to the bilinear form B^* on $\mathfrak{g}^*$ such that for all $X^*, Y^* \in \mathfrak{g}^*(\bar{F})$, $B^*(X^*, Y^*) = B(\psi_{\mathrm{G}^*}(X^*), \psi_{\mathrm{G}^*}(Y^*))$: that this prescription descends to a bilinear form on $\mathfrak{g}^*$ follows from the $\mathrm{Ad}\,\mathrm{G}$-invariance of B and the fact that ψ_{G^*} is an inner twisting. In fact, B^* is the transfer of B to $\mathfrak{g}^*$ via the endoscopic datum $\underline{\mathrm{G}}^*$ as in [Wal95, Section VIII.6] (see also the discussion after the proof of Remark 2 of that reference).

(vi) For this subsection, given $r \geq 0$, E_r (resp., E_r^*) will denote the depth r projector for G (resp., G^*).

Lemma 4.3.7. *Assume that* G *is not 'bad' in the sense of* [BKV16, Section 3.13]*, i.e., either* p *is odd, or* $(\mathrm{G}_{\mathrm{sc}})_{F^{\mathrm{unr}}}$ *does not have a restriction of scalars of an odd special unitary group over* F^{unr} *as a factor. Given a symmetric nondegenerate* $\mathrm{Ad}\,\mathrm{G}$*-invariant bilinear form on* $\mathfrak{g}$*, the following are equivalent:*

(i) B *is nice.*
(ii) For some $x \in \mathcal{B}(\mathrm{G})$ *and all* $r \in \mathbb{R}$*,* (49) *holds.*
(iii) For some finite unramified extension F_1 *of* F*, the base-change of* B *to* F_1 *is nice with respect to* G_{F_1}*.*

Proof. (i) $\Rightarrow$ (ii) is immediate. Let us prove (ii) $\Rightarrow$ (i); we refer to the condition in (ii) as x-nice. For this, it is enough to show that if $x, y \in \mathcal{B}(\mathrm{G})$, and if B is x-nice, then B is y-nice as well. Choose an apartment in $\mathcal{B}(\mathrm{G})$ containing x and y, associated to some split maximal torus S in G, and let M_0 be the centralizer of S in G. Then we have (see [BKV16, Proposition 3.10(b)]) an expansion:

$$\mathfrak{g}(F)_{x,r} = \mathfrak{m}_0(F)_r \oplus \bigoplus_{\alpha} \mathfrak{u}_\alpha(F)_{x,r},$$

where α runs over the set of roots of S in G, and $\mathfrak{u}_\alpha(F)_{x,r}$ is the union of the affine root lattices $\mathfrak{u}_\psi \subset \mathfrak{u}_\alpha(F)$ as ψ runs over the affine roots associated to the apartment of S that have gradient α and satisfy $\psi(x) \geq r$. We have a similar expression for $\mathfrak{g}(F)_{x,r+}$, where the definition of $\mathfrak{u}_\alpha(F)_{x,r+}$ involves the condition $\psi(x) > r$.

Thus, it is clear, using the S-equivariance of B, that B is x-nice if and only if the following two conditions are satisfied:

- $\{X \in \mathfrak{m}_0 \mid B(X, \mathfrak{m}_0(F)_{(-r)+}) \subset \varpi\mathcal{O}\} = \mathfrak{m}_0(F)_r$ for all $r \in \mathbb{R}$; and
- $\{X \in \mathfrak{u}_\alpha(F) \mid B(X, \mathfrak{u}_{-\alpha}(F)_{x,(-r)+}) \subset \varpi\mathcal{O}\} = \mathfrak{u}_\alpha(F)_{x,r}$, for each root α of S in G and each $r \in \mathbb{R}$.

Since similar considerations apply to y, it suffices to show that the above conditions are satisfied if and only if they are satisfied with x replaced by y. This is clear since the first condition is x-agnostic, while if $y = x + \lambda$ with $\lambda \in X_*(\mathrm{S}) \otimes \mathbb{R}$, then it is easy to see that $\mathfrak{u}_{-\alpha}(F)_{y,(-r)+} = \mathfrak{u}_{-\alpha}(F)_{x,-(r-\langle\alpha,\lambda\rangle)+}$ and $\mathfrak{u}_\alpha(F)_{y,r} = \mathfrak{u}_\alpha(F)_{x,r-\langle\alpha,\lambda\rangle}$.

This gives the equivalence of (i) and (ii). Given this, for the equivalence of either of these notions with (iii), assuming without loss of generality that $F_1 \subset F^{\mathrm{unr}}$ so that $\mathcal{B}(\mathrm{G}) \subset \mathcal{B}(\mathrm{G}/F_1) \subset \mathcal{B}(\mathrm{G}/F_{\mathrm{unr}})$, it suffices to check that for some $x \in \mathcal{B}(\mathrm{G})$ and each $r \in \mathbb{R}$, (49) holds if and only if it does with F replaced by F_1. In turn, this is easy to see using 'dual bases' with respect to B if we can show that for each $r \in \mathbb{R}$, $\mathfrak{g}(F)_{x,r} \otimes_{\mathfrak{O}_F} \mathfrak{O}_{F_1} = \mathfrak{g}(F_1)_{x,r}$, where $\mathfrak{O}_F$ and $\mathfrak{O}_{F_1}$ are the rings of integers of F and F_1, respectively. Since $\mathfrak{g}(F)_{x,r}$ is the $\mathrm{Gal}(F^{\mathrm{unr}}/F)$-fixed points of $\mathfrak{g}(F_1)_{x,r}$ (see [Adl98, Proposition 1.4.1] and [BKV16, Lemma 3.14] and use that G is not 'bad'), this should in turn follow from some sort of unramified descent.

Being naive about this sort of descent, let us give an elementary argument instead; it is enough to show that for each finitely generated $\mathfrak{O}_{F_1}$-lattice L with a semilinear action of $\mathrm{Gal}(F_1/F)$, the map $L^{\mathrm{Gal}(F_1/F)} \otimes_{\mathfrak{O}_F} \mathfrak{O}_{F_1} \to L$ is surjective. Let $\mathrm{Gal}(F_1/F) = \{\sigma_1 = \mathrm{id}, \ldots, \sigma_n\}$, and let $a_1, \ldots, a_n$ be an $\mathfrak{O}_F$-basis for $\mathfrak{O}_{F_1}$. The matrix $A = [\sigma_i(a_j)]_{1 \leq i,j \leq n}$ has determinant in $\mathfrak{O}_{F_1}^\times$, since $\mathrm{tr}_{F_1/F}(a_i a_j)$ is the (i,j)-th entry of ${}^tAA$, and $\mathrm{tr}_{F_1/F}$ is a perfect pairing $\mathfrak{O}_{F_1} \times \mathfrak{O}_{F_1} \to \mathfrak{O}_F$. Therefore there exist $b_1, \ldots, b_n \in \mathfrak{O}_{F_1}$ such that $\sum_{l=1}^n \sigma_i(a_l) b_l$ equals $\delta_{1,i}$ for each i, i.e., 1 if $i = 1$, and 0 otherwise. Thus, given $v \in L$,

$$v = \sigma_1(v) = \sum_{i=1}^n \delta_{1,i}\sigma_i(v) = \sum_{i=1}^n \left(\sum_{l=1}^n b_l \sigma_i(a_l)\right)\sigma_i(v) = \sum_{l=1}^n b_l \cdot \left(\sum_{i=1}^n \sigma_i(a_l v)\right),$$

which lies in the image of $L^{\mathrm{Gal}(F_1/F)} \otimes_{\mathfrak{O}_F} \mathfrak{O}_{F_1} \to L$. $\square$

Corollary 4.3.8. *If B is a nice bilinear form on $\mathfrak{g}$, then the associated bilinear form B^* on $\mathfrak{g}^*$ (obtained by transporting B via ψ_{G^*}) is nice as well.*

Proof. We reduce to the situation where ψ_{G^*} is defined over the maximal unramified extension F^{unr} of F. Recall that the inner twists of G^* are parameterized by $H^1(F, \mathrm{G}^*_{\mathrm{ad}})$. By the inflation-restriction sequence and the theorem of Steinberg which says that $H^1(F_{\mathrm{unr}}, (\mathrm{G}^*_{\mathrm{ad}})_{F^{\mathrm{unr}}})$ is trivial, the class of the inner twist ψ_{G^*} arises from an element of $H^1(\mathrm{Gal}(F^{\mathrm{unr}}/F), \mathrm{G}^*_{\mathrm{ad}}(F^{\mathrm{unr}}))$. This has the consequence that we may modify ψ_{G^*} to ensure that it is defined over F^{unr}, and hence over a finite extension F_1 of F contained in F^{unr}. Since the condition 'niceness' behaves well under isomorphisms of algebraic groups, it follows that the base-change of B to F_1 is nice if and only if the base-change of B^* to F_1 is. Therefore, the corollary follows from Lemma 4.3.7. □

From [BKV16] we have:

Lemma 4.3.9. *Suppose $\mathrm{G}_1 \to \mathrm{G}_2$ is an isogeny of connected reductive groups over F, whose degree is prime to p. Then it induces analytic isomorphisms $\mathrm{G}_1(F)_{x,r+} \to \mathrm{G}_2(F)_{x,r+}$ and $\mathrm{G}_1(F)_{r+} \to \mathrm{G}_2(F)_{r+}$ for all $x \in \mathcal{B}(\mathrm{G}_1) = \mathcal{B}(\mathrm{G}_2)$ and $r \geq 0$. Moreover, the depth r projector for G_2, which (as a distribution) is supported on $\mathrm{G}_2(F)_{r+}$, when pulled back along $\mathrm{G}_1(F)_{r+} \to \mathrm{G}_2(F)_{r+}$, equals the depth r projector for G_1, which is supported on $\mathrm{G}_1(F)_{r+}$.*

Proof. The first assertion is [BKV16, Lemma 8.12]. The second assertion is implicit in the proof of [BKV16, Corollary 8.13], and follows from the first assertion together with the Euler-Poincare formula for the depth r projector given in [BKV16, Corollary 1.9]. □

Lemma 4.3.10. *Let $r > 0$ (resp., $r \in \mathbb{R}$). If strongly regular semisimple elements $\gamma^* \in \mathrm{G}^*(F)$ and $\gamma \in \mathrm{G}(F)$ (resp., regular semisimple elements $X^* \in \mathfrak{g}^*(F)$ and $X \in \mathfrak{g}(F)$) match with respect to $\underline{\mathrm{G}}^*$, we have $\gamma^* \in \mathrm{G}^*(F)_r$ if and only if $\gamma \in \mathrm{G}(F)_r$ (resp., $X^* \in \mathfrak{g}^*(F)_r$ if and only if $X \in \mathfrak{g}(F)_r$). Consequently, for $r \in \mathbb{R}$, the stable distribution $\mathbb{1}_{\mathfrak{g}^*(F)_r}$ transfers to $\mathbb{1}_{\mathfrak{g}(F)_r}$ under endoscopic transfer with respect to $\underline{\mathrm{G}}^*$ (a similar assertion involving the $\mathbb{1}_{\mathrm{G}^*(F)_r}$ and the $\mathbb{1}_{\mathrm{G}(F)_r}$ holds, but we will not need it).*

Proof. By [BKV16, Lemma B.3], $\mathfrak{g}(F)_r \subset \mathfrak{g}(F)$ and $\mathfrak{g}^*(F)_r \subset \mathfrak{g}^*(F)$ are stable, justifying the stability of $\mathbb{1}_{\mathfrak{g}^*(F)_r}$. Given the simple description of transfer factors for $\underline{\mathrm{G}}^*$ (Remark 3.2.2(i)), and since $\mathfrak{g}(F)_r \subset \mathfrak{g}(F)$ and $\mathfrak{g}^*(F)_r \subset \mathfrak{g}^*(F)$ are open and closed, the latter assertion of the lemma follows from the former, which is a "$\underline{\mathrm{G}}^*$-endoscopic" form of the stability of the $\mathrm{G}(F)_r$ and the $\mathfrak{g}(F)_r$. Thus, we will adapt the proof of the stability assertion in [BKV16, Lemma B.3].

We will prove the assertion involving γ and γ^*; the proof of the assertion involving X and X^* is similar. As in the proof of Corollary 4.3.8, we may and do assume that ψ_{G^*} is defined over a finite extension F_1 of F contained in F^{unr}. We have $\gamma = \mathrm{Ad}\, g(\psi_{\mathrm{G}^*}(\gamma^*))$ for some $g \in \mathrm{G}(\bar{F})$. Since $\gamma, \delta := \psi_{\mathrm{G}^*}(\gamma^*) \in \mathrm{G}(F_1)$, letting T be the centralizer of δ in G_{F_1} and using that $H^1(\mathrm{Gal}(F^{\mathrm{unr}}/F_1), \mathrm{T}(F^{\mathrm{unr}})) \to H^1(F_1, \mathrm{T})$ is an isomorphism (by Steinberg's theorem that $\mathrm{H}^1(F^{\mathrm{unr}}, \mathrm{T}_{F^{\mathrm{unr}}})$ is trivial) and that $H^1(\mathrm{Gal}(F^{\mathrm{unr}}/F_1), \mathrm{G}(F^{\mathrm{unr}})) \to H^1(F_1, \mathrm{G}_{F_1})$ is injective, we may assume without loss of generality that $g \in \mathrm{G}(F^{\mathrm{unr}})$. Thus, there exists a finite extension F_2 of F_1 (and hence of F) in F^{unr} such that $g \in \mathrm{G}(F_2)$. It follows that $\gamma^* \in \mathrm{G}^*(F_2)_r$ if and only if $\gamma \in \mathrm{G}(F_2)_r$.

Using [AD04, Lemma 2.2.3] and the fact that the finite extension F_2/F is unramified, we have $\mathrm{G}^*(F_2)_r \cap \mathrm{G}^*(F) = \mathrm{G}^*(F)_r$ and $\mathrm{G}(F_2)_r \cap \mathrm{G}(F) = \mathrm{G}(F)_r$, so that $\gamma^* \in \mathrm{G}^*(F)_r$ if and only if $\gamma \in \mathrm{G}(F)_r$, as desired. Let us remark that the assertion involving X and X^* uses the Lie algebra version of [AD04, Lemma 2.2.3] (for arbitrary r), which can be proved similarly (it is a simple application of the Bruhat-Tits fixed point theorem also found in the proof of [BKV16, Lemma B.3]). □

Proof of Proposition 4.3.5. Since p is a very good prime for G and hence also for G^*, the isogenies $\mathrm{Z}^0_{\mathrm{G}} \times \mathrm{G}_{\mathrm{sc}} \to \mathrm{G}$ and $\mathrm{Z}^0_{\mathrm{G}^*} \times \mathrm{G}^*_{\mathrm{sc}} \to \mathrm{G}^*$ have degrees prime to p, as observed in [BKV16, the proof of Theorem 1.23]. Therefore, by Lemma 4.3.9, $\mathrm{Z}^0_{\mathrm{G}} \times \mathrm{G}_{\mathrm{sc}} \to \mathrm{G}$ induces an isomorphism from $C_c^\infty(\mathrm{Z}_{\mathrm{G}^0}(F)_r \times \mathrm{G}_{\mathrm{sc}}(F)_r) \subset C_c^\infty(\mathrm{Z}_{\mathrm{G}^0}(F) \times \mathrm{G}_{\mathrm{sc}}(F))$ to $C_c^\infty(\mathrm{G}(F)_r) \subset C_c^\infty(\mathrm{G}(F))$, which clearly preserves stable orbital integrals (for compatible choices of measures). A similar comment applies to the isogeny $Z^0_{G^*} \times G^*_{sc} \to G^*$. Since ψ_{G^*} induces an isomorphism $Z^0_{G^*} \to Z^0_G$ as well as determines a unique inner twist from G^*_{sc} to G_{sc}, and since the transfer factors all have a simple description in our setting, it is now easy to reduce, using Lemma 4.3.9 and Lemma 4.3.10, to the case where G is simply connected.

Moreover, as in the proof of Corollary 4.3.8, we may and do assume that ψ_{G^*} is defined over a finite extension F_1 of F contained in F^{unr}.

Since G and G^* are simply connected, and since p is a very good prime for G, it follows from [BKV16, Corollary 8.11] that G admits an r-logarithm in the sense of [BKV16, Section 1.21], which is an $\mathrm{Ad}\, G(F)$-equivariant homeomorphism $G(F)_{r+} \to \mathfrak{g}(F)_{r+}$ restricting to a homeomorphism $G(F)_{x,r+} \to \mathfrak{g}(F)_{x,r+}$ for each $x \in \mathcal{B}(G)$. The same applies to G^*. Moreover, by [BKV16, Corollary 1.9(b)], E_r and E^*_r are supported in $G(F)_{r+}$ and $G^*(F)_{r+}$, respectively, and by [BKV16, Corollary 1.22], the push-forwards of E_r and E^*_r to $\mathfrak{g}(F)_{r+} \subset \mathfrak{g}(F)$ and $\mathfrak{g}^*(F)_{r+} \subset \mathfrak{g}^*(F)$ are the restrictions of the Lie algebra versions $\mathcal{E}_r$ and $\mathcal{E}^*_r$ of the depth r projectors to $\mathfrak{g}(F)_{r+}$ and $\mathfrak{g}^*(F)_{r+}$, respectively. Again using Lemma 4.3.10 and the fact that the transfer factors are particularly simple in our situation, it suffices to show that the transfer of the distribution $\mathcal{E}^*_r$ on $\mathfrak{g}^*(F)$ equals the distribution $\mathcal{E}_r$ on $\mathfrak{g}(F)$, where we use measures on $\mathfrak{g}^*$ and $\mathfrak{g}$ that are compatible via ψ_{G^*} (which, being an innner twist, even induces an F-rational map at the level of top-degree differential forms).

By [BKV16, (b) of Section 1.19], $\mathcal{E}_r$ is the inverse Fourier transform of the characteristic function of $\check{\mathfrak{g}}(F)_{-r}$, where the Fourier transform is defined as in [BKV16, Section 1.18], using a fixed additive character $\Lambda : F \to \mathbb{C}^\times$ that is nontrivial on the ring $\mathfrak{O}_F$ of integers of F but trivial on the maximal ideal $\mathfrak{p}_F$ of $\mathfrak{O}_F$. We identify $\mathfrak{g}$ with $\check{\mathfrak{g}}$ using the nice bilinear form B, so that the Fourier transform is a map from the space of distributions on $\mathfrak{g}(F)$ to itself, and $\mathcal{E}_r$ is the inverse Fourier transform of the characteristic function $\mathbb{1}_{\mathfrak{g}(F)_{-r}}$ of $\mathfrak{g}(F)_{-r}$. The transfer B^* of B to $\mathfrak{g}^*$ is nice by Corollary 4.3.8, using which we similarly realize $\mathcal{E}^*_r$ as the inverse Fourier transform of the characteristic function $\mathbb{1}_{\mathfrak{g}^*(F)_{-r}}$. Moreover, we may and do use measures on $\mathfrak{g}(F)$ and $\mathfrak{g}^*(F)$ that are self-dual for $\Lambda \circ B$ and $\Lambda \circ B^*$: this is because B^* is the transfer of B, and hence this use of self-dual measures satisfies the constraint that the measure on $\mathfrak{g}^*(F)$ is the transfer of the measure on $\mathfrak{g}(F)$ via ψ_{G^*}.

Given this choice of measures, one knows the commutativity of endoscopic transfer and Fourier transform (see [Wal95, Conjecture 1], which has been proved since, as explained in [KV12, Theorem 4.1.3]): if a distribution Θ^* on $\mathfrak{g}^*(F)$ transfers to a distribution Θ on $\mathfrak{g}(F)$, then the Fourier transform (resp., the inverse Fourier transform) of Θ^* transfers to $\gamma_\Lambda(B)/\gamma_\Lambda(B^*)$ times the Fourier transform of Θ (resp., $\gamma_\Lambda(B^*)/\gamma_\Lambda(B)$ times the inverse Fourier transform of Θ), where $\gamma_\Lambda(B)$ and $\gamma_\Lambda(B^*)$ are Weil constants as in [Wal95, Section VIII.1].

Now we are reduced to showing that the distribution $\mathbb{1}_{\mathfrak{g}^*(F)_{-r}}$ on $\mathfrak{g}^*(F)$ transfers to the distribution $e(G)\gamma_\Lambda(B) \cdot \gamma_\Lambda(B^*)^{-1} \cdot \mathbb{1}_{\mathfrak{g}(F)_{-r}}$ on $\mathfrak{g}(F)$, or equivalently to $\mathbb{1}_{\mathfrak{g}(F)_{-r}}$, since one knows that $e(G)\gamma_\Lambda(B^*)/\gamma_\Lambda(B) = 1$: see [KV12, Proposition 4.2.2], whose restrictions are unnecessary in our setting as mentioned in the first sentence of [KV12, Section 4.2.9, the proof of Proposition 4.2.2], or use [Kal15, Lemma 4.8 and Proposition 4.3].

It remains to note that $\mathbb{1}_{\mathfrak{g}^*(F)_r}$ transfers to $\mathbb{1}_{\mathfrak{g}(F)_r}$, which we already know from Lemma 4.3.10. □

Proof of Corollary 4.3.4. Let us prove (i). Since the depth of a representation is the same as that of its cuspidal support by [MP96, Theorem 5.2(1)], it follows that for each $\sigma \in \mathrm{Irr}(M)$ and $r \geq 0$, $\hat{E}_r(\sigma) = \hat{E}_r((M, \sigma))$ equals 1 if the depth of σ is at most r, and 0 otherwise. Thus, (i) follows if we prove that for each $r \geq 0$, $\hat{E}_r$ takes the same value on each element of Σ. But since $E_r \in \mathcal{Z}_1(G)$ by [BKV16, Theorem 1.23] and the hypothesis that p is a very good prime for G, and since $\sigma \mapsto \hat{E}_r(\sigma)$ is $\mathcal{O}'_M$-invariant on $\mathrm{Irr}_2(M)$ (since the action of Aut(M) on Irr(M) preserves depth), this follows from Corollary 4.2.12(i).

For (ii), by the observation at the beginning of (i), it suffices to prove that for all $r \geq 0$, $\sigma \in \Sigma$ and $\sigma^* \in \Sigma^*$, we have $\hat{E}_r(\sigma) = \hat{E}^*_r(\sigma^*)$. But using the invariance of $\sigma \mapsto \hat{E}_r(\sigma)$ and $\sigma^* \mapsto \hat{E}^*_r(\sigma^*)$ under $\mathcal{O}'_M$ and $\mathcal{O}'_{M^*}$, this follows from combining Proposition 4.3.5 with Corollary 4.2.12(ii). □

4.4. **Consequences for $\mathcal{Z}_1(\mathrm{G})$ and $\mathcal{Z}_2(\mathrm{G})$.** Let us now deduce from Corollary 4.2.12(i) that $\mathcal{Z}_1(\mathrm{G})^{\mathcal{O}} = \mathcal{Z}_{2,\mathcal{O}}(\mathrm{G})$ (see Notation 4.1.7) when the hypothesis on the existence of tempered L-packets (Hypothesis 2.5.1) is satisfied. Let us begin by restating Corollary 4.2.12(i) in the special case that concerns us here.

Corollary 4.4.1. *Suppose Σ is an $\mathcal{O}_{\mathrm{M}}$-unitarily stable discrete series L-packet (see Definition 3.3.2). Then:*

(i) The Plancherel measure $\sigma \mapsto \mu(\sigma)$ ([Wal03, Section V.2]*) is constant on Σ.*
(ii) For all $z \in \mathcal{Z}_1(\mathrm{G})^{\mathcal{O}}$ and $\sigma_1, \sigma_2 \in \Sigma$, and any parabolic subgroup P of G with M as a Levi subgroup, z acts by the same scalar on $\mathrm{Ind}_{\mathrm{P}}^{\mathrm{G}} \sigma_1$ and $\mathrm{Ind}_{\mathrm{P}}^{\mathrm{G}} \sigma_2$. In other words, if (M_1, σ_1) and (M_2, σ_2) are cuspidal supports of elements of Σ, then $\hat{z}((\mathrm{M}_1, \sigma_1)_{\mathrm{G}}) = \hat{z}((\mathrm{M}_2, \sigma_2)_{\mathrm{G}})$.

Proof. Each $\mathcal{O}_{\mathrm{M}}$ from Notation 2.4.1 satisfies the hypotheses of Corollary 4.2.12(i), by the conditions imposed in (iv) of Notation 2.4.1, and Lemma 2.4.3(ii). This also gives that for each $z \in \mathcal{Z}(\mathrm{G})^{\mathcal{O}}$, $\sigma \mapsto \hat{z}(\sigma)$ is $\mathcal{O}_{\mathrm{M}}$-invariant in $\mathrm{Irr}_2(\mathrm{M})$. Thus, the corollary follows from Corollary 4.2.12(i). □

We now prove Theorem 1.2.5, after restating it in a slightly more convenient way.

Theorem 4.4.2. *Assume the hypothesis on the existence of tempered L-packets (Hypothesis 2.5.1). Then for $z \in \mathcal{Z}(\mathrm{G})$, the following are equivalent:*

(i) $\hat{z}$ is constant on each $\Sigma \in \Phi_{\mathrm{temp}}(\mathrm{G})$ (see Notation 2.5.6 for the definition of $\Phi_{\mathrm{temp}}(\mathrm{G})$).
(ii) $z \in \mathcal{Z}_{2,\mathcal{O}}(\mathrm{G})$.
(iii) $z \in \mathcal{Z}_1(\mathrm{G})^{\mathcal{O}}$.

Proof. Let us assume (i) and prove (ii). Since $\Phi_{\mathrm{temp}}(\mathrm{G})$ partitions $\mathrm{Irr}_{\mathrm{temp}}(\mathrm{G})$ (see Lemma 2.5.7(i)), and since each $\Sigma \in \Phi_{\mathrm{temp}}(\mathrm{G})$ is stable under the action of $\mathcal{O}$, it follows that $\hat{z}(\pi) = \hat{z}(\pi \circ \beta^{-1})$, for all $\beta \in \mathcal{O}$ and $\pi \in \mathrm{Irr}_{\mathrm{temp}}(\mathrm{G})$. Thus, $\hat{z} \in \mathbb{C}[\Omega(\mathrm{G})]$ takes the same value on (M, σ) and $\beta \cdot (\mathrm{M}, \sigma)$ whenever (M, σ) is a cuspidal pair such that σ is unitary. But the images of such cuspidal pairs in $\Omega(\mathrm{G})$ is Zariski dense (since $X^{\mathrm{unr-uni}}(\mathrm{M}) \subset X^{\mathrm{unr}}(\mathrm{M})$ is Zariski dense), so that $\hat{z}$ factors through $\Omega(\mathrm{G}) \to \underline{\Omega}(\mathrm{G})$ (use the discussion of Notation 4.1.1). From this and Lemma 4.1.4, we get that $z \in \mathcal{Z}(\mathrm{G})^{\mathcal{O}}$.
Now if $\mathrm{M} \subset \mathrm{G}$ is a Levi subgroup and $\Sigma \in \Phi_2(\mathrm{M})$, then the constancy of $\hat{z}$ on $\Sigma^{\mathrm{G}} \in \Phi_{\mathrm{temp}}(\mathrm{G})$ (see (ii) and (iii) of Notation 2.5.6) implies that $f \mapsto \Theta_{\Sigma^{\mathrm{G}}}(z * f)$ is a scalar multiple of $\Theta_{\Sigma^{\mathrm{G}}}$ (where $\Theta_{\Sigma^{\mathrm{G}}} = \mathrm{Avg}_{\mathcal{O}_{\mathrm{G}}}(\mathrm{Ind}_{\mathrm{M}}^{\mathrm{G}} \Theta_\Sigma)$ as in Notation 2.5.6(iv), and we use the identity $\Theta_\pi(z * f) = \mathrm{tr}\, \pi(z * f) = \hat{z}(\pi)\Theta_\pi(f)$), which is stable as seen in Proposition 3.2.8. Hence by Hypothesis 2.5.1, if D is the $\mathcal{O}$-average of $\mathrm{Ind}_{\mathrm{M}}^{\mathrm{G}} \Theta'$, where $\mathrm{M} \subset \mathrm{G}$ is a Levi subgroup and $\Theta' \in SD_{\mathrm{ell}}(\mathrm{M})^{\mathcal{O}_{\mathrm{M}}}$, then $f \mapsto D(z * f)$ is stable. Therefore, by the implication (iv) ⇒ (i) of Proposition 4.1.9 (which applies as $z \in \mathcal{Z}(\mathrm{G})^{\mathcal{O}}$), we get $z \in \mathcal{Z}_{2,\mathcal{O}}(\mathrm{G})$, as desired.
The implication (ii) ⇒ (iii) is Lemma 4.1.8.
For any Levi subgroup $\mathrm{M} \subset \mathrm{G}$, Hypothesis 2.5.1 implies that the elements of $\Phi_2(\mathrm{M})$ are all $\mathcal{O}_{\mathrm{M}}$-unitarily stable (use Lemma 2.5.3(i)). Therefore, the implication (iii) ⇒ (i) is immediate from Corollary 4.4.1(ii) and the $\mathcal{O}$-invariance of z (the latter is used to account for the fact that the description of $\Phi_{\mathrm{temp}}(\mathrm{G})$ as given by (ii) and (iii) of Notation 2.5.6 involves taking a union of $\mathcal{O}$-orbits of quotients of parabolically induced representations). □

Lemma 4.4.3. *Assume Hypothesis 2.5.1. Suppose $z \in \mathcal{Z}(\mathrm{G})$ is such that $\hat{z}(\pi_1) = \hat{z}(\pi_2)$ whenever π_1, π_2 are irreducible subquotients of $\mathrm{Ind}_{\mathrm{M}}^{\mathrm{G}} \sigma_1, \mathrm{Ind}_{\mathrm{M}}^{\mathrm{G}} \sigma_2$, respectively, for some Levi subgroup $\mathrm{M} \subset \mathrm{G}$ and representations σ_1, σ_2 that belong to the same element of $\Phi_2(\mathrm{M})$. Then $\hat{z}(\pi_1) = \hat{z}(\pi_2)$ whenever π_1, π_2 are irreducible subquotients of $\mathrm{Ind}_{\mathrm{M}}^{\mathrm{G}} \sigma_1, \mathrm{Ind}_{\mathrm{M}}^{\mathrm{G}} \sigma_2$, respectively, for some Levi subgroup $\mathrm{M} \subset \mathrm{G}$ and representations σ_1, σ_2 that belong to the same element of $\Phi_2^+(\mathrm{M})$.*

Proof. The proof is similar to the first step in that of the implication (i) ⇒ (ii) of Theorem 4.4.2. Suppose $\mathrm{M} \subset \mathrm{G}$ is a Levi subgroup, and σ_1, σ_2 belong to the same element $\Sigma \in \Phi_2^+(\mathrm{M})$. Since z factors through the cuspidal support map, it is easy to see that $\chi \mapsto \hat{z}(\sigma_1 \otimes \chi)$ and $\chi \mapsto \hat{z}(\sigma_2 \otimes \chi)$ are regular on $X^{\mathrm{unr}}(\mathrm{M})$, where for $i = 1, 2$, $\hat{z}(\sigma_i \otimes \chi)$ is the scalar with which z acts on any irreducible subquotient of $\mathrm{Ind}_{\mathrm{M}}^{\mathrm{G}} \sigma_i \otimes \chi$. By hypothesis, we have $\hat{z}(\sigma_1 \otimes \chi) = \hat{z}(\sigma_2 \otimes \chi)$ whenever $\Sigma \otimes \chi$ is unitary (and hence belongs to $\Phi_2(\mathrm{M}) \subset \Phi_2^+(\mathrm{M})$). Since the set of such χ is a coset of $X^{\mathrm{unr-uni}}(\mathrm{M})$ in $X^{\mathrm{unr}}(\mathrm{M})$, and is hence Zariski dense in $X^{\mathrm{unr}}(\mathrm{M})$, it follows that $\hat{z}(\sigma_1 \otimes \chi) = \hat{z}(\sigma_2 \otimes \chi)$ for all $\chi \in X^{\mathrm{unr}}(\mathrm{M})$. In particular $\hat{z}(\sigma_1) = \hat{z}(\sigma_2)$. □

Corollary 4.4.4. *Assume Hypothesis 2.5.1. If z belongs to $\mathcal{Z}_1(\mathrm{G})^{\mathcal{O}}$ or $\mathcal{Z}_{2,\mathcal{O}}(\mathrm{G})$, $\mathrm{M} \subset \mathrm{G}$ is a Levi subgroup and σ, σ' belong to the same element of $\Phi_2^+(\mathrm{M})$, then z acts by the same scalar on any irreducible subquotient of $\mathrm{Ind}_{\mathrm{M}(F)}^{\mathrm{G}(F)} \sigma$ as it does on any irreducible subquotient of $\mathrm{Ind}_{\mathrm{M}(F)}^{\mathrm{G}(F)} \sigma'$.*

Proof. This follows from Lemma 4.4.3, whose hypothesis is satisfied by either the implication (ii) $\Rightarrow$ (i) (if $z \in \mathcal{Z}_{2,\mathcal{O}}(\mathrm{G})$) or (iii) $\Rightarrow$ (i) (if $z \in \mathcal{Z}_1(\mathrm{G})^{\mathcal{O}}$) of Theorem 4.4.2. □

Theorem 4.4.2 has the following corollary.

Corollary 4.4.5. *Assume $z \in \mathcal{Z}(\mathrm{G})$. Denote by z_{M} the image of z under what is called the Harish-Chandra homomorphism $\mathcal{Z}(\mathrm{G}) \to \mathcal{Z}(\mathrm{M})$ in* [BDK86, Section 2.4]*, i.e., the homomorphism of $\mathbb{C}$-algebras that is dual to the obvious finite morphism $\Omega(\mathrm{M}) \to \Omega(\mathrm{G})$ induced by inclusion at the level of cuspidal supports. If Hypothesis 2.5.1 holds and $z \in \mathcal{Z}_1(\mathrm{G})^{\mathcal{O}}$, then $z_{\mathrm{M}} \in \mathcal{Z}_1(\mathrm{M})^{\mathcal{O}_{\mathrm{M}}}$.*

Proof. If $\mathrm{L} \subset \mathrm{M}$ is a Levi subgroup and $\upsilon_1, \upsilon_2 \in \Upsilon$ for some $\Upsilon \in \Phi_2(\mathrm{L})$, then

$$\hat{z}_{\mathrm{M}}((\mathrm{L}, \upsilon_1)_{\mathrm{M}}) = \hat{z}((\mathrm{L}, \upsilon_1)_{\mathrm{G}}) = \hat{z}((\mathrm{L}, \upsilon_2)_{\mathrm{G}}) = \hat{z}_{\mathrm{M}}((\mathrm{L}, \upsilon_2)_{\mathrm{M}}),$$

where the middle equality holds by the implication (iii) $\Rightarrow$ (i) of Theorem 4.4.2. Here, as usual, $\hat{z}_{\mathrm{M}}((\mathrm{L}, \upsilon_1)_{\mathrm{M}})$ refers to $\hat{z}_{\mathrm{M}}((\mathrm{L}', \upsilon_1')_{\mathrm{M}})$, where $(\mathrm{L}', \upsilon_1')$ is a cuspidal support of (L, υ_1), and the other terms are similar. Therefore, the corollary follows from the implication (i) $\Rightarrow$ (iii) of Theorem 4.4.2 applied with $(\mathrm{M}, \{\mathcal{O}_{\mathrm{L}}\}_{\mathrm{L}})$ in place of $(\mathrm{G}, \{\mathcal{O}_{\mathrm{L}}\}_{\mathrm{L}})$ (L ranging over the set of Levi subgroups of M or G, as appropriate): this application is justified by Lemma 2.4.3(iii), which ensures the validity Hypothesis 2.5.1 for $(\mathrm{M}, \{\mathcal{O}_{\mathrm{L}}\}_{\mathrm{L}})$. □

Remark 4.4.6. As the phrasing of Corollary 4.4.5 above indicates, we are not able to prove that $z \mapsto z_{\mathrm{M}}$ sends $\mathcal{Z}_1(\mathrm{G})$ to $\mathcal{Z}_1(\mathrm{M})$, without using Hypothesis 2.5.1. In contrast it is easy to show that if $z \in \mathcal{Z}_2(\mathrm{G})$, then its image $z_{\mathrm{M}} \in \mathcal{Z}(\mathrm{M})$ under the Harish-Chandra homomorphism belongs to $\mathcal{Z}_2(\mathrm{M})$. We do not prove this assertion, since we will not use it anywhere.

5. The case of quasi-split classical groups

5.1. The stable center conjecture and depth preservation for many 'classical' groups.

The following proposition describes how to go from Arthur's endoscopic classification, which is stated in terms of discrete series and tempered L-packets, to the statement of Hypothesis 2.5.1, which is stated in terms of elliptic representations.

Proposition 5.1.1. *Suppose, for each Levi subgroup $\mathrm{M} \subset \mathrm{G}$, we are given a partition $\Phi_{\mathrm{temp}}(\mathrm{M})$ of $\mathrm{Irr}_{\mathrm{temp}}(\mathrm{M})$ by $\mathcal{O}_{\mathrm{M}}$-invariant subsets, such that the following properties are satisfied:*

(a) Some subset $\Phi_2(\mathrm{M}) \subset \Phi_{\mathrm{temp}}(\mathrm{M})$ partitions $\mathrm{Irr}_2(\mathrm{M})$. Moreover, for each Levi subgroup $\mathrm{M} \subset \mathrm{G}$ and each $\Sigma \in \Phi_{\mathrm{temp}}(\mathrm{M})$, there exists a Levi subgroup $\mathrm{L} \subset \mathrm{M}$ and some $\Upsilon \in \Phi_2(\mathrm{L})$, such that Σ equals Υ^{M} as defined in Notation 2.5.6(ii), i.e., Σ is the union of the sets of $\mathcal{O}_{\mathrm{M}}$-conjugates of the irreducible constituents of the unitary representation $\mathrm{Ind}_{\mathrm{L}}^{\mathrm{M}} \upsilon$, as υ ranges over Υ.

(b) Let $\mathrm{M} \subset \mathrm{G}$ be a Levi subgroup and $\Sigma \in \Phi_{\mathrm{temp}}(\mathrm{M})$, and choose (L, Υ) as in (a). Then:
- *(Compare with Definition 3.4.9(i)) For each (not necessarily elliptic) relevant endoscopic datum $\underline{\mathrm{H}}$, with underlying endoscopic group H, choosing auxiliary data and hence the 5-tuple $(\mathrm{H}_1 \to \mathrm{H}, \hat{\xi}_1, \tilde{\mathrm{H}}_1 = \mathrm{H}_1 \to \tilde{\mathrm{H}} = \mathrm{H}, \mathrm{C}_1, \mu)$ as in Notation 3.1.2(iii), there exists a stable tempered virtual character $\Theta^{\underline{\mathrm{H}}} \in SD_{\mu}(\mathrm{H}_1)$ on $\tilde{\mathrm{H}}_1(F) = \mathrm{H}_1(F)$, such that the following holds inside $D(\mathrm{M})$:*

$$\sum_{\underline{\mathrm{H}}} \mathbb{C} \cdot \mathbf{T}_{\underline{\mathrm{H}}}(\Theta^{\underline{\mathrm{H}}}) = (\sum_{\sigma \in \Sigma} \mathbb{C} \cdot \Theta_{\sigma})^{\mathcal{O}_{\mathrm{M}}}. \tag{50}$$

- *In* (50)*, the contribution* $\mathbf{T}_{\underline{\mathrm{M}}^*}(\Theta^{\underline{\mathrm{M}}^*})$ *from the 'principal' endoscopic datum* $\underline{\mathrm{M}}^*$ *as in Notation 3.2.1(i) equals the* $\mathcal{O}_{\mathrm{M}}$*-average* $\mathrm{Avg}_{\mathcal{O}_{\mathrm{M}}}(\mathrm{Ind}_{\mathrm{L}}^{\mathrm{M}}\Theta_{\Upsilon})$ *of a character induced from some* $\Theta_{\Upsilon} \in SD(\mathrm{L})^{\mathcal{O}_{\mathrm{L}}}$*.*

Then the hypothesis on the existence of tempered L-packets, namely Hypothesis 2.5.1, is satisfied.

Proof. Let $\mathrm{M} \subset \mathrm{G}$ be a Levi subgroup. For any $\Sigma \in \Phi_2(\mathrm{M})$, note that any (L, Υ) as in (a) equals (M, Σ), and that the $\Theta_\Sigma := \Theta_\Upsilon \in SD(\mathrm{M})$ as in (b) is supported on Σ. It suffices to show that the Θ_Σ, Σ varying over $\Phi_2(\mathrm{M})$, form a basis for $SD_{\mathrm{ell}}(\mathrm{M})^{\mathcal{O}_{\mathrm{M}}}$. Each such Θ_Σ is clearly contained in $SD(\mathrm{M}) \cap D_{\mathrm{ell}}(\mathrm{M}) = SD_{\mathrm{ell}}(\mathrm{M})$, and it is also clear that the Θ_Σ form a linearly independent set as Σ varies over $\Phi_2(\mathrm{M})$. It remains to show that their span $SD_{\mathrm{ell}}(\mathrm{M})'$, which is contained in $SD_{\mathrm{ell}}(\mathrm{M})^{\mathcal{O}_{\mathrm{M}}}$, equals all of $SD_{\mathrm{ell}}(\mathrm{M})^{\mathcal{O}_{\mathrm{M}}}$.

Let $SD(\mathrm{M})' \subset SD(\mathrm{M})^{\mathcal{O}_{\mathrm{M}}}$ denote the span of the contributions $\mathbf{T}_{\underline{\mathrm{M}}^*}(\Theta^{\underline{\mathrm{M}}^*})$ from $\underline{\mathrm{M}}^*$ in (50) as Σ varies over $\Phi_{\mathrm{temp}}(\mathrm{M})$. Write $D_{\text{non-ell}}(\mathrm{M}) \subset D(\mathrm{M})$ for the span of tempered virtual characters fully induced from proper Levi subgroups of M, and $D_{\text{ell,non-st}}(\mathrm{M})$ for the span of the $\mathbf{T}_{\underline{\mathrm{H}}}(SD_{\mu,\mathrm{ell}}(\mathrm{H}_1))$ as $\underline{\mathrm{H}}$ varies over the elliptic endoscopic data for M distinct from $\underline{\mathrm{M}}^*$. It follows from the second condition of (b) that:

$$SD(\mathrm{M})' \subset SD_{\mathrm{ell}}(\mathrm{M})' + D_{\text{non-ell}}(\mathrm{M}). \tag{51}$$

The first condition of (b) gives us an expression of the form:

$$D(\mathrm{M})^{\mathcal{O}_{\mathrm{M}}} \subset SD(\mathrm{M})' + \sum_{\underline{\mathrm{H}} \neq \underline{\mathrm{M}}^*} \mathbf{T}_{\underline{\mathrm{H}}}(SD_\mu(\mathrm{H}_1)), \tag{52}$$

where $\underline{\mathrm{H}}$ runs over a set of relevant endoscopic data for M (taken up to isomorphism), and where for each $\underline{\mathrm{H}}$ we have implicitly chosen and fixed auxiliary data including μ and $\mathrm{H}_1 = \tilde{\mathrm{H}}_1$.

For any given endoscopic datum $\underline{\mathrm{H}}$ for M, using (12), together with the compatibility between endoscopic transfer and parabolic induction in the form of Remark 3.1.4(i) (if $\underline{\mathrm{H}}$ is elliptic) or Remark 3.1.4(ii) (otherwise), we have that :

$$\mathbf{T}_{\underline{\mathrm{H}}}(SD_\mu(\mathrm{H}_1)) \subset D_{\text{non-ell}}(\mathrm{M}) + D_{\text{ell,non-st}}(\mathrm{M}).$$

Combining this with (51) and 52, we get:

$$SD_{\mathrm{ell}}(\mathrm{M})^{\mathcal{O}_{\mathrm{M}}} \subset D(\mathrm{M})^{\mathcal{O}_{\mathrm{M}}} \cap SD_{\mathrm{ell}}(\mathrm{M}) \subset (SD_{\mathrm{ell}}(\mathrm{M})' + D_{\text{non-ell}}(\mathrm{M}) + D_{\text{ell,non-st}}(\mathrm{M})) \cap SD_{\mathrm{ell}}(\mathrm{M}) = SD_{\mathrm{ell}}(\mathrm{M})',$$

where the last equality uses (11) and (8), as desired. □

Proposition 5.1.2. *Suppose* G *is a quasi-split symplectic, special orthogonal, unitary, general symplectic, even general special orthogonal or odd general spin group:* $\mathrm{Sp}_{2n}, \mathrm{SO}_n, \mathrm{U}_n, \mathrm{GSp}_{2n}, \mathrm{GSO}_{2n}$ *or* GSpin_{2n+1}*. Except in the* GSO_{2n} *case, assume that each* $\mathcal{O}_{\mathrm{M}}$ *is trivial. When* $\mathrm{G} = \mathrm{GSO}_{2n}(F)$*, for any Levi subgroup* $\mathrm{M} \subset \mathrm{G}$ *with the 'GSO-part' equal to* G_{M}*, if* G_{M} *is nonabelian (resp., abelian), assume that* $\mathcal{O}_{\mathrm{M}}$ *is a two element group contained in the restriction of* $\mathrm{Int}\,\mathrm{O}_{2n}(F)$ *to* M*, with nontrivial image in* $\mathrm{Out}(\mathrm{G}_{\mathrm{M}})$ *(resp., that* G_{M} *is trivial). Then the hypothesis on the existence of tempered L-packets (Hypothesis 2.5.1) is satisfied.*

Proof. We will consider multiple cases, but these cases will overlap.

First assume that G is symplectic, special orthogonal, or unitary; in these cases we will use [Art13] (in the special orthogonal and symplectic cases) or [Mok15] (in the unitary case). In these cases, partitions $\Phi(\mathrm{M})$ as in Proposition 5.1.1 have been constructed by Arthur and Mok in [Art13] and [Mok15]. Here, for the case of even special orthogonal groups, see [Art13, Theorem 8.4.1]. To see that the latter assertions in (a) and (b) of Proposition 5.1.1 are satisfied, note that the tempered L-packets on these groups and their endoscopic decompositions are defined in [Art13] or [Mok15] starting from the discrete series case, using parabolic induction and the local intertwining relation: see [Art13, the proof of Proposition 2.4.3 and Sections 6.5, 6.6 and 8.4] and [Mok15, the proof of Proposition 3.4.4 and Section 7.6]. Thus, we are done in these cases by Proposition 5.1.1.

Now suppose that G equals GSp_{2n} or GSO_{2n} (and is quasi-split). Then partitions $\Phi(\mathrm{M})$ as in Proposition 5.1.1 have been constructed by Xu in [Xu18], associated to the given collection $\{\mathcal{O}_{\mathrm{M}}\}_{\mathrm{M}}$ (the analogue of $\mathcal{O}_{\mathrm{G}}$ for [Xu18] is the group Σ_0 of [Xu18, Introduction, page 73]). Here, to see that the latter assertions in (a) and (b) of the proposition are satisfied, note that the tempered

L-packets on these groups and their endoscopic decompositions are defined in [Xu18] starting from the discrete series case, using parabolic induction and the local intertwining relation: see [Xu18, Lemma 4.10 and Section 6.4]. Thus, we are done in these cases by Proposition 5.1.1.
Thirdly, assume that G is odd special orthogonal, symplectic, unitary or odd general spin, i.e., $\mathrm{SO}_{2n+1}, \mathrm{Sp}_{2n}, \mathrm{U}_n$, or GSpin_{2n+1}; in these cases, we will use [Mg14]. For use in a later paper, we will also allow G to be an even special orthogonal group $\mathrm{SO}(V,q)$, but take $\mathcal{O}_{\mathrm{G}}$ to be $\mathrm{Int}\,\mathrm{O}(V,q)(F)$ and each $\mathcal{O}_{\mathrm{M}}$ to be the set of elements of $\mathcal{O}_{\mathrm{G}}^{+}$ that preserve M and act trivially on its center. Each Levi subgroup $\mathrm{M} \subset \mathrm{G}$ can be written as $\mathrm{GL}_{\mathrm{M}} \times \mathrm{G}_{\mathrm{M}}$, where GL_{M} is isomorphic to a product of groups of the form $\mathrm{Res}_{E/F}\,\mathrm{GL}_m$ for some trivial or quadratic extension E/F, and G_{M} is a group of the same type as G but of smaller rank (these groups can be possibly trivial). Hypothesis 2.5.1 is trivial for GL_{M}, because it is standard that $SD_{\mathrm{ell}}(\mathrm{GL}_{\mathrm{M}})$ equals $D_{\mathrm{ell}}(\mathrm{GL}_{\mathrm{M}})$ and is spanned by the characters of discrete series representations of $\mathrm{GL}_{\mathrm{M}}(F)$. Therefore, the construction of $\Phi_2(\mathrm{M})$ as in Hypothesis 2.5.1 reduces to such a construction for $\Phi_2(\mathrm{G}_{\mathrm{M}})$. The latter construction is trivial if G_{M} is abelian, while if G_{M} is nonabelian it follows from [Mg14, Corollary 4.11], noting that what is denoted $I^{\underline{G}}_{cusp,st}$ in that corollary is also what is noted $I^{G}_{cusp,st}$ in that reference, since the group Aut of automorphisms of the endoscopic datum $\underline{G}$ of that corollary is trivial in the cases that we are currently considering. □

Remark 5.1.3. In Proposition 5.1.2, we have avoided discussing the case of quasi-split even general spin groups GSpin_{2n}. This is because, for our arguments to work in this case, we need the relevant transfer factors $\Delta(\cdot,\cdot)$ to be invariant under the conjugation action of $\mathrm{O}_{2n}(F)$ on the first factor, so that the action of the group 'Aut' on the space noted $I^{G}_{cusp,st}$ in [Mg14, Section 2.3], where $G = \mathrm{GSpin}_{2n}(F)$, can be defined simply through its action on G, without the more complicated involvement of transfer factors as in [MW16, Section I.2.6]. It seems to us that the relevant invariance property is likely to hold, and hence should give Hypothesis 2.5.1 up to the action of the obvious outer automorphism group, but we have not verified it.

Corollary 5.1.4. *If* G *is a quasi-split symplectic, special orthogonal, unitary, general symplectic or odd general spin group (*$\mathrm{SO}_n, \mathrm{Sp}_{2n}, \mathrm{U}_n, \mathrm{GSp}_{2n}$ *or* GSpin_{2n+1}*), then* G *satisfies the stable center conjecture, i.e.,* $\mathcal{Z}_1(\mathrm{G}) = \mathcal{Z}_2\mathrm{G}$*). If* G *is a quasi-split even general special orthogonal group* GSO_{2n}*, then we have a weaker equality* $\mathcal{Z}_1(\mathrm{G})^{\mathcal{O}} = \mathcal{Z}_{2,\mathcal{O}}(\mathrm{G})$*, where* $\mathcal{O} = \mathcal{O}_{\mathrm{G}}$ *is as in Proposition 5.1.2.*

Proof. This follows from combining Theorem 4.4.2 with Proposition 5.1.2. □

Now we address questions related to depth preservation:

Proposition 5.1.5. *Let* G *be as in Proposition 5.1.2, i.e.,* G *is quasi-split and is of the form* $\mathrm{Sp}_{2n}, \mathrm{SO}_n, \mathrm{U}_n, \mathrm{GSp}_{2n}, \mathrm{GSO}_{2n}$ *or* GSpin_{2n+1}*. Let* $\Sigma \in \Phi_2(\mathrm{G})$ *be an* $\mathcal{O}_{\mathrm{G}}$*-stable discrete series packet as in Proposition 5.1.2, with* $\mathcal{O}_{\mathrm{G}}$ *nontrivial in the* GSO_{2n}*-case, but not in any of the other cases including the* SO_{2n}*-case. Assume that* $p > 2$ *and that, in the unitary case,* p *is greater than the rank of* G*. Then for each* $\sigma_1, \sigma_2 \in \Sigma$*, we have* $\mathrm{depth}(\sigma_1) = \mathrm{depth}(\sigma_2)$*.*

Proof. In each case, the assumptions on p imply that p is a very good prime for G in the sense of [BKV16, Section 8.10], so the proposition follows from Corollary 4.3.4(i), since its hypotheses are satisfied by Proposition 5.1.2. □

The work of M. Oi ([Oi22]) allows us to deduce the following corollary:

Corollary 5.1.6. *Let* G *be a quasi-split symplectic, special orthogonal or unitary group. There exists a constant* $N_{\mathrm{G}} > 2$*, depending only on the absolute root datum of* G*, such that the following holds if* $p > N_{\mathrm{G}}$*. Let* σ *be a discrete series representation of* $\mathrm{G}(F)$*, and let* φ_σ *be its Langlands parameter (possibly well-defined only up to an outer automorphism). Then:*

$$\inf\{r \geq 0 \mid \dot{\varphi}_\sigma|_{I_F^{r+}} = s|_{I_F^{r+}} \text{ for a preferred section } s : W_F \to {}^L\mathrm{G}\} =: \mathrm{depth}\,\varphi_\sigma = \mathrm{depth}\,\sigma,$$

where $\dot{\varphi}_\sigma : W_F \to {}^L\mathrm{G}$ *is a representative for* φ_σ*.*

Proof. If Σ is the packet in $\Phi_2(\mathrm{G})$ (in the sense of Proposition 5.1.2) containing σ, then [Oi22, Theorem 1.2] gives:

$$\max\{\mathrm{depth}\,\sigma' \mid \sigma' \in \Sigma\} = \mathrm{depth}\,\varphi_\sigma.$$

Therefore, the corollary follows from Proposition 5.1.5. We also remark that if G is a unitary group, the corollary follows from [Oi22, Theorem 1.4] and [Oi21, Theorem 1.3] (without any need for Proposition 5.1.5), and that the precise bounds for N_{G} are given in [Oi21] and [Oi22]. □

Remark 5.1.7. Suppose that $F = \mathbb{Q}_p$, and that G is a unitary group in an odd number of variables, associated to an unramified extension E/F. In this case, the work [MHN22] of Bertoloni Meli, Hamann and Nguyen proves that the local Langlands correspondence for G constructed in [Mok15] and [KMSW14] agrees with the local Langlands correspondence constructed by Fargues and Scholze in [FS21]. It seems to us that for such a G, combining [MHN22] with Proposition 5.1.2 and Theorem 4.4.2 should show that the elements of $\mathcal{Z}(\mathrm{G})$ constructed by Fargues and Scholze using excursion operators belong to the stable Bernstein center, i.e., to $\mathcal{Z}_2(\mathrm{G})$. Given the work of Hamann in [Ham21], it might also be interesting to ask a similar question when G is an inner form of GSp_4.

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