BERNSTEIN PROJECTORS FOR TAME SL_2 WITH SUPPORT IN A MOY-PRASAD G-DOMAIN

Abstract. Let G be the group SL_2 over a finite extension F of $\mathbb{Q}_p$, p an odd prime. For a fixed $r \geq 0$, we identify the elements of the Bernstein center of G supported in the Moy-Prasad G-domain G_{r+}, by characterizing them spectrally. We study the behavior of convolution with such elements on orbital integrals of functions in $C_c^\infty(\mathrm{G}(F))$, proving results in the spirit of semisimple descent. These are 'depth r versions' of results proved for general reductive groups by J.-F. Dat, R. Bezrukavnikov, A. Braverman and D. Kazhdan.

1. Introduction — Statement of Results

Throughout this work, we will deal with the reductive group $\mathrm{G} = \mathrm{SL}_2$ over a finite extension F of $\mathbb{Q}_p$, p odd. $\mathcal{Z}(\mathrm{G})$ will denote the Bernstein center of G.

Let $r \geq 0$ be a nonnegative real number, fixed for the rest of this work. We are interested in the $\mathbb{C}$-vector subspace $\mathcal{Z}_{r+}(\mathrm{G}) \subset \mathcal{Z}(\mathrm{G})$ consisting of distributions that are supported in the Moy-Prasad G-domain $\mathrm{G}_{r+} \subset \mathrm{G}(F)$ (in Section 2 we recall or give a reference for various standard notation that we will use freely in the introduction). One result we prove is that, in our very special case where G is a 'tame' SL_2, the following 'depth r' version of a conjecture of A. Braverman and D. Kazhdan ([BK16, Conjecture 1.5]) is satisfied:

Theorem 1. *$\mathcal{Z}_{r+}(\mathrm{G})$ is a subring of $\mathcal{Z}(\mathrm{G})$.*

This result and its proof are inspired by [BK16, Corollary 1.4], to recall which we temporarily introduce an arbitrary non-archimedean local field F' and a reductive group H over it. Then [BK16, Corollary 1.4] states that the $\mathbb{C}$-vector subspace $\mathcal{Z}_{\mathrm{comp}}(\mathrm{H})$ of its Bernstein center $\mathcal{Z}(\mathrm{H})$ consisting of distributions supported in the set H_c^0 of compact elements of the group $\mathrm{H}(F')$ of the rational points of H, is a subring of $\mathcal{Z}(\mathrm{H})$.

Note that [BK16, Corollary 1.4] is deduced from [BK16, Theorem 1.3], which spectrally characterizes $\mathcal{Z}_{\mathrm{comp}}(\mathrm{H})$ as the set of elements that, viewed as functions on the admissible dual $\hat{\mathrm{H}}_{\mathrm{adm}}$ of H, are constant on each Bernstein component. Naively, this could motivate the following informal question:

Question A: Assume that the local Langlands correspondence holds for H, and that the residue characteristic of F' is 'large enough'. Denote by $I^{r+}(F')$ the union of the upper ramification subgroups $I^s(F')$ of the Weil group $W_{F'}$ of F', as s ranges over real numbers greater than r. Suppose $z \in \mathcal{Z}(\mathrm{H})$ is such that $z(\pi_1) = z(\pi_2)$ whenever the Langlands parameters of π_1 and π_2 agree on $I^{r+}(F')$. Then is z supported in the Moy-Prasad H-domain H_{r+}?

We do not know an answer to the above question, and its converse is false; the patterns of failure in general might perhaps be related to L-indistinguishability. Nevertheless, let us mention a very special but striking case in which a result of R. Bezrukavnikov, D. Kazhdan and Y. Varshavsky from [BKV16] answers this question affirmatively. In cases where the local Langlands correspondence for H satisfies depth-preservation (e.g., when H is a quasi-split classical group and F' has large residue characteristic, see [Oi22]), a special case of the above question amounts to asking if the 'depth r projector' in $\mathcal{Z}(\mathrm{H})$, i.e., the element of $\mathcal{Z}(\mathrm{H})$ that acts as the identity (resp., zero) on irreducible admissible representations of $\mathrm{H}(F')$ of depth at most r (resp., depth greater than r), is supported on H_{r+}. This has been answered affirmatively in [BKV16].

For our case of 'tame' SL_2/F, we affirmatively answer Question A in Theorem 2 below, by proving, in the spirit of [BK16, Theorem 1.3], a sharper version whose converse is true as well.

Fix an additive character $\Lambda : F \to \mathbb{C}^\times$ of depth zero, i.e., such that Λ is nontrivial on the ring $\mathfrak{O} = \mathfrak{O}_F$ of integers of F but trivial on its maximal ideal $\mathfrak{p} = \mathfrak{p}_F$. For each maximal torus $\mathrm{T} \subset \mathrm{G}$ and for each smooth positive-depth character $\psi : \mathrm{T}(F) \to \mathbb{C}^\times$, we have a representation $\pi_\psi = \pi(\mathrm{T}, \psi)$ of $\mathrm{G}(F)$, defined as follows:

(i) if T is split, then we inflate ψ to the group $\mathrm{B}(F)$ of rational points of an arbitrarily chosen Borel subgroup $\mathrm{B} \supset \mathrm{T}$, and let $\pi_\psi := \pi(\mathrm{T}, \psi)$ be the parabolically induced representation $\mathrm{Ind}_{\mathrm{B}(F)}^{\mathrm{G}(F)} \psi$; and
(ii) if T is non-split, i.e., elliptic, then $\pi(\mathrm{T}, \psi)$ is the representation so denoted in [ADSS11], obtained from (T, ψ) using the Howe construction with our fixed choice of Λ.

In either case, $\pi(\mathrm{T}, \psi)$ is irreducible since ψ is not of depth zero; in case (i), $\pi(\mathrm{T}, \psi)$ is independent of the choice of B. For a nontrivial continuous complex character $\dot\psi \neq \mathbb{1}$ belonging to the group $\hat{\mathrm{T}}_{r+}$ of continuous characters of the Moy-Prasad filtration subgroup $\mathrm{T}_{r+} \subset \mathrm{T}(F)$, define $\Pi(\mathrm{T}, \dot\psi) \subset \hat{\mathrm{G}}_{\mathrm{adm}}$ ($\hat{\mathrm{G}}_{\mathrm{adm}}$ being the admissible dual of $\mathrm{G}(F)$) to be the set:

$$\{\pi(\mathrm{T}, \psi) \mid \psi : \mathrm{T}(F) \to \mathbb{C}^\times \text{ a continuous homomorphism}, \psi|_{\mathrm{T}_{r+}} = \dot\psi\}.$$

Note that each $\Pi(\mathrm{T}, \dot\psi)$ is a union of Bernstein components of G, as is the set $\hat{\mathrm{G}}_{\mathrm{adm}, \leq r}$ of all isomorphism classes of irreducible admissible representations of $\mathrm{G}(F)$ of depth $\leq r$. Further, it is easy to see from Remark 16 below that, together, they partition the admissible dual of $\mathrm{G}(F)$:

$$\hat{\mathrm{G}}_{\mathrm{adm}} = \hat{\mathrm{G}}_{\mathrm{adm}, \leq r} \cup \bigsqcup_{\{(\mathrm{T}, \dot\psi)\}/\sim} \Pi(\mathrm{T}, \dot\psi), \tag{1}$$

where $(\mathrm{T}, \dot\psi)$ runs over the equivalence classes of pairs consisting of a maximal torus $\mathrm{T} \subset \mathrm{G}$ and $\mathbb{1} \neq \dot\psi \in \hat{\mathrm{T}}_{r+}$, two such pairs $(\mathrm{T}^{(1)}, \dot\psi_1)$ and $(\mathrm{T}^{(2)}, \dot\psi_2)$ being equivalent if and only if there exists $g \in \mathrm{G}(F)$ such that $\mathrm{T}^{(2)} = \mathrm{Int}\, g^{-1}(\mathrm{T}^{(1)})$ and $\dot\psi_2 = \dot\psi_1 \circ \mathrm{Int}\, g$. Define $\mathcal{Z}'_{r+}(\mathrm{G})$ to be the set of elements of $\mathcal{Z}(\mathrm{G})$ such that $z(\pi_1) = z(\pi_2)$ whenever $\pi_1, \pi_2 \in \hat{\mathrm{G}}_{\mathrm{adm}}$ belong to the same partition in the above decomposition: i.e., either $\pi_1, \pi_2 \in \hat{\mathrm{G}}_{\mathrm{adm}, \leq r}$ or $\pi_1, \pi_2 \in \Pi(\mathrm{T}, \dot\psi)$ for some maximal torus $\mathrm{T} \subset \mathrm{G}$ and $\mathbb{1} \neq \dot\psi \in \hat{\mathrm{T}}_{r+}$.

Then we prove the following variant of [BK16, Theorem 1.3]:

Theorem 2. $\mathcal{Z}_{r+}(\mathrm{G}) = \mathcal{Z}'_{r+}(\mathrm{G})$.

For an explanation of why this theorem gives us, for our tame SL_2, (something slightly sharper than) an answer to Question A above, we refer to Remark 45.

The proof of Theorem 2 given below is an adaptation of ideas from [BK16] (some of which go back to [Dat03]), especially Theorem 2.2 of that paper, and goes via three steps:

(i) show that every element $z \in \mathcal{Z}'_{r+}(\mathrm{G})$ satisfies the property of Theorem 3 below (informally: for $f \in C_c^\infty(\mathrm{G}(F))$, the orbital integrals of $z * f$ at strongly regular semisimple elements of G_{r+} depend only on the orbital integrals of f at strongly regular semisimple elements of G_{r+});
(ii) show, using an easy Shalika germs argument, that any $z \in \mathcal{Z}(\mathrm{G})$ satisfying the property mentioned in (i) is supported in G_{r+} (this is done in Lemma 36); in particular, $\mathcal{Z}'_{r+}(\mathrm{G}) \subset \mathcal{Z}_{r+}(\mathrm{G})$;
(iii) (i) implies easily (see Corollary 42) that convolution with elements of $\mathcal{Z}'_{r+}(\mathrm{G})$ preserves $\mathcal{Z}_r(\mathrm{G})$, so one is reduced to determining elements of $\mathcal{Z}_{r+}(\mathrm{G})$ that, viewed as functions on $\hat{\mathrm{G}}_{\mathrm{adm}}$, are supported either in $\hat{\mathrm{G}}_{\mathrm{adm}, \leq r}$ or a specific $\Pi(\mathrm{T}, \dot\psi)$; one shows that such elements all belong to $\mathcal{Z}'_{r+}(\mathrm{G})$.

To state Theorem 3, recall the definition of orbital integrals, which are well-defined by a result of Ranga Rao: for $\gamma \in \mathrm{G}(F)$ and $f \in C_c^\infty(\mathrm{G}(F))$,

$$O(\gamma, f) = \int_{\mathrm{G}^\gamma(F) \backslash \mathrm{G}(F)} f(g^{-1}\gamma g)\, dg/dg_\gamma, \tag{2}$$

where G^γ is the centralizer of γ in $\mathrm{G}(F)$. We will work with choices of measures spelled out in Section 2.10.

Theorem 3. *Let $z \in \mathcal{Z}_{r+}(\mathrm{G})$. Then z satisfies the following property: if $f \in C_c^\infty(\mathrm{G}(F))$ is such that $O(\gamma, f) = 0$ for all strongly regular semisimple $\gamma \in \mathrm{G}_{r+}$, then $O(\gamma, z * f) = 0$ for all strongly regular semisimple $\gamma \in \mathrm{G}_{r+}$.*

Note that this property is the 'depth r analogue' of the condition '$\bar{\Phi}$ commutes with $\bar{1}_{G_c^0}$' from [BK16, Theorem 2.2].

Our next result, Theorem 4 below, is a generalization of Theorem 1, in the same way Bezrukavnikov's theorem [BK16, Theorem 1.8] is a generalization of [BK16, Corollary 1.4]. To state it, we need to introduce the analogue for our situation, of the partition in [BK16, (1.1)].

Note that, since G is simply connected, every element $\gamma \in \mathrm{G}(F) \setminus \mathrm{Z}_{\mathrm{G}}(F)\mathrm{G}_{r+} = \mathrm{G}(F) \setminus (\mathrm{G}_{r+} \cup -\mathrm{G}_{r+})$ is strongly regular semisimple (here $\mathrm{Z}_{\mathrm{G}}(F)$ is the center of $\mathrm{G}(F)$ and $-\mathrm{G}_{r+}$ is the translate of G_{r+} by the unique central non-identity element -1 of $\mathrm{G}(F)$). For any $\gamma \in \mathrm{G}(F) \setminus (\mathrm{G}_{r+} \cup -\mathrm{G}_{r+})$, define $\mathcal{U}_\gamma$ to be ${}^{\mathrm{G}(F)}(\gamma \cdot \mathrm{T}_{\gamma,r+})$, the union of all $\mathrm{G}(F)$-conjugates of $\gamma \cdot \mathrm{T}_{\gamma,r+}$, where $\mathrm{T}_{\gamma,r+} \subset \mathrm{T}_\gamma(F)$ is the '$r+$-indexed' Moy-Prasad filtration subgroup of the centralizer $\mathrm{T}_\gamma(F)$ of γ in $\mathrm{G}(F)$. Thanks to p being odd, $\mathrm{T}_\gamma(F) \cap \mathrm{G}_{r+} = \mathrm{T}_{\gamma,r+}$ and $\mathrm{T}_\gamma(F) \cap -\mathrm{G}_{r+} = -\mathrm{T}_{\gamma,r+}$ (use, e.g., [AD04, Lemma 2.2.9]), so that $\mathcal{U}_\gamma$ does not meet either G_{r+} or $-\mathrm{G}_{r+}$. It is readily verified that if $\gamma' \in \mathcal{U}_\gamma$, then $\mathcal{U}_\gamma = \mathcal{U}_{\gamma'}$. Therefore, we get a partition:

$$\mathrm{G}(F) = \bigsqcup_{\lambda \in I} \mathcal{U}_\lambda, \tag{3}$$

where for each $\lambda \in I$, $\mathcal{U}_\lambda$ is either G_{r+}, $-\mathrm{G}_{r+}$, or of the form ${}^{\mathrm{G}(F)}(\gamma \mathrm{T}_{\gamma,r+})$ with $\gamma \in \mathrm{G}(F) \setminus (\mathrm{G}_{r+} \cup -\mathrm{G}_{r+})$.

Each set in this partition, in addition to being invariant under $\mathrm{G}(F)$-conjugation, is open and closed in $\mathrm{G}(F)$ (see Section 2.12). Letting $J(\mathrm{G})$ denote the $\mathbb{C}$-vector space of ($\mathrm{G}(F)$-conjugation) invariant distributions on $\mathrm{G}(F)$ and J_λ the subspace of those supported on $\mathcal{U}_\lambda$, we get a decomposition

$$J(\mathrm{G}) = \prod_{\lambda \in I} J_\lambda, \tag{4}$$

analogous to [BK16, (1.2)].

Given $f \in C_c^\infty(\mathrm{G}(F))$ and a maximal torus $\mathrm{T} \subset \mathrm{G}$, let $\varphi_f = \varphi_f^{\mathrm{T}}$ be the function on $\mathrm{T}(F) \setminus \mathrm{Z}_{\mathrm{G}}(F)\mathrm{T}_{r+}$ that sends γ to $D(\gamma)^{1/2} O(\gamma, f)$, where D is the Weyl discriminant. Since the elements of $\mathrm{T}(F) \setminus \mathrm{Z}_{\mathrm{G}}(F)\mathrm{T}_{r+}$ are all regular, $\varphi_f \in C_c^\infty(\mathrm{T}(F) \setminus \mathrm{Z}_{\mathrm{G}}(F)\mathrm{T}_{r+}) \subset C_c^\infty(\mathrm{T}(F))$.

For z belonging to the subring $\mathcal{Z}^?(\mathrm{G}) \subset \mathcal{Z}(\mathrm{G})$ consisting of elements z' such that $z'(\pi_1) = z'(\pi_2)$ whenever $\pi_1, \pi_2 \in \hat{\mathrm{G}}_{\mathrm{adm},\leq 0}$, we have a well-defined element $\varphi_z = \varphi_z^{\mathrm{T}}$ in the Bernstein center $\mathcal{Z}(\mathrm{T})$ of $\mathrm{T}(F)$ such that for all ψ in the admissible dual $\hat{\mathrm{T}}_{\mathrm{adm}}$ of $\mathrm{T}(F)$, if ψ is of depth zero (resp., of positive depth), then $\varphi_z(\psi) = z(\pi)$, where $\pi \in \hat{\mathrm{G}}_{\mathrm{adm},\leq 0}$ (resp., $\pi = \pi(\mathrm{T}, \psi)$, which is irreducible as ψ has positive depth).

If T is split, note that this assignment $z \mapsto \varphi_z$ is simply the restriction to $\mathcal{Z}^?(\mathrm{G})$ of what is referred to as the Harish-Chandra homomorphism "i_{GM}^*" in [BDK86, Section 2.4], and denoted r_{MG} in [BK16].

(i) of the following theorem, analogous to [BK16, Theorem 1.8], strengthens Theorem 3 by showing that one could replace the 'G_{r+}' in it by any of the $\mathcal{U}_\lambda$:

Theorem 4. *Let $z \in \mathcal{Z}_{r+}(\mathrm{G})$, so that $z \in \mathcal{Z}'_{r+}(\mathrm{G})$ by Theorem 2, and φ_z is well-defined. Let $\mathrm{T} \subset \mathrm{G}$ be a maximal torus. Then:*

(i) Convolution with z preserves the decomposition (4).
(ii) $\varphi_z = \varphi_z^{\mathrm{T}} \in \mathcal{Z}_{r+}(\mathrm{T})$, i.e., φ_z is supported in T_{r+}. Moreover, for all $f \in C_c^\infty(\mathrm{G}(F))$ and for all $t \in \mathrm{T}(F) \setminus \mathrm{Z}_{\mathrm{G}}(F)\mathrm{T}_{r+}$, we have:

$$\varphi_{z*f}(t) = (\varphi_z * \varphi_f)(t).$$

Much of (i) is a consequence of (ii). (ii) seems to us to be reminiscent of results of Harish-Chandra on the behavior of the Harish-Chandra homomorphism across semisimple descent (see [HC57, Corollary to Theorem 3], [HC65, Corollary to Lemma 15]). Unlike in the real case, the elements of the Bernstein center that we consider are supported in G_{r+} for some $r \geq 0$ (instead of at the identity), and their compatibility with orbital integrals is only obtained in regions as "away from the identity" as to be in $\mathrm{G}(F) \setminus \mathrm{Z}_{\mathrm{G}}(F)\mathrm{G}_{r+}$ (something that reminds us of the fact that the character formulas for some depth-r-characters of $\mathrm{G}(F)$ outside $\mathrm{Z}_{\mathrm{G}}(F)\mathrm{G}_r$ bear some resemblance to character formulas for real groups).

It will also be interesting if Theorem 4, or some aspects of its proof, could be viewed as a form of the Schur orthogonality relations, in a sense related to the one invoked by Sally and Shalika in [SS84, Section 4]. See Remark 35.

We also find striking some of the parallels between this circle of ideas of Bezrukavnikov, Braverman and Kazhdan, and the ideas of T. Haines on the stable center conjecture (see [Hai14]): specifically, the proposal to prove that the subspace of the Bernstein center consisting of distributions satisfying a certain property B is a subring, by seeking a recharacterization of this subspace as the subspace of elements satisfying a stronger property C involving orbital integrals (where the property C is so designed that the elements of the Bernstein center that satisfy it manifestly form a subring).

Allowing ourselves even wilder speculation, let us remark that it would be nice if one could have a notion of the Harish-Chandra homomorphism or semisimple descent for elements of $\mathcal{Z}(\mathrm{G})$ that Haines calls 'geometric'. However, we do not even know how to formulate such a notion, even for our tame SL_2. While we do not know how far Theorem 4(ii) generalizes, we do believe that, with suitable assumptions on the residue characteristic, a statement of this nature is true for depth r projectors on more general reductive groups, and this is being pursued in an on-going project of ours with W.-W. Li and M. Oi.

It seems likely that most of those of the questions that we have raised here that will turn out to have an answer in general, were already considered by Bezrukavnikov, Braverman and Kazhdan.

In Section 2 we introduce (more) notation and recall prerequisites, mostly concerning structure theory for $\mathrm{G}(F)$. In Section 3 we recall from [ADSS11] some facts concerning the supercuspidal representations $\pi(\mathrm{T}, \psi)$ and their characters, review some notation and results from [SS84] concerning Fourier transforms of orbital integrals, and make preparations to adapt the strategy of [MT02] so as to study the effect of convolution with elements of $\mathcal{Z}(\mathrm{G})$ on orbital integrals. In Section 4 we prove that for each maximal torus $\mathrm{T} \subset \mathrm{G}$ and each nontrivial character $1 \neq \dot{\psi} \in \hat{\mathrm{T}}_{r+}$, the Bernstein projector $z_{\mathrm{T},\dot{\psi}}$ to $\Pi(\mathrm{T}, \dot{\psi})$ (a union of Bernstein components) satisfies (i) and (ii) of Theorem 4, as does the depth r Bernstein projector E_r. In Section 5 we prove the theorems stated above, using the results of Section 4 together with some additional work to characterize $\mathcal{Z}_{r+}(\mathrm{G})$ (by showing that it is contained in $\mathcal{Z}'_{r+}(\mathrm{G})$, see Lemma 43).

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2. Additional notation, and some review

Notation that we have already recalled or referenced in the introduction will continue to apply; we now add to those as well as review some basic facts that will be used, some of them repeatedly, in later sections. The paper [ADSS11] is a convenient reference for the facts reviewed here concerning the structure of $\mathrm{G}(F)$, as is [SS84].

2.1. **Notation concerning the field F.** $\mathfrak{O} = \mathfrak{O}_F$ denotes the ring of integers of F, and $\mathfrak{p} = \mathfrak{p}_F$ its maximal ideal. Let q denote the cardinality of the residue field $\mathbb{F}_q$ of F. Let $\bar{F}$ be a fixed algebraic closure of F and

$\mathrm{val} : \bar{F} \to \mathbb{Q} \cup \{\infty\}$ (resp., $|\cdot| : \bar{F} \to \mathbb{C}^\times$) the usual extension of the normalized discrete valuation (resp., normalized absolute value) on F. Fix a unit $\epsilon \in \mathfrak{O}^\times \setminus \mathfrak{O}^{\times 2}$ and a uniformizer $\varpi \in \mathfrak{O}$.

2.2. **Some standard maximal tori in** G. For $\theta \in \{\epsilon, \varpi, \epsilon\varpi\}$, we let T^θ be the unique (necessarily elliptic maximal) torus in $\mathrm{G} = \mathrm{SL}_2$ such that

$$\mathrm{T}^\theta(F) = \left\{ \begin{pmatrix} a & b \\ b\theta & a \end{pmatrix} \middle| a, b \in F, a^2 - \theta b^2 = 1 \right\} \cong E^1_{\mathrm{T}^\theta}, \tag{5}$$

where $E_{\mathrm{T}^\theta} = F[\sqrt{\theta}]$ is the (necessarily) unique quadratic extension of F in $\bar{F}$ splitting T^θ, and $E^1_{\mathrm{T}^\theta} \subset E^\times_{\mathrm{T}^\theta}$ is the subgroup of elements x such that $N_{E_{\mathrm{T}^\theta}/F}(x) = 1$.

One knows that every elliptic maximal torus T of G, which in our case means every non-split maximal torus of G, is $\mathrm{GL}_2(F)$-conjugate to T^θ for a unique $\theta \in \{\epsilon, \varpi, \epsilon\varpi\}$. For such T and θ, we let $E_\mathrm{T} = E_{\mathrm{T}^\theta} = F[\sqrt{\theta}]$, where $\theta \in \{\epsilon, \varpi, \epsilon\varpi\}$ is such that T is $\mathrm{GL}_2(F)$-conjugate to T^θ. Note that T is ramified (resp., unramified) if the extension E_T/F is ramified (resp., unramified), i.e., if $E_\mathrm{T} = F[\sqrt{\theta}]$ with $\theta \in \{\varpi, \epsilon\varpi\}$ (resp., $\theta = \epsilon$). An element $t \neq \pm 1$ belonging to a maximal torus T of G will be called split, unramified, ramified, elliptic etc. if T has this property.

We will let A be the diagonal maximal torus of G, and make the standard identifications $\mathrm{A} = \mathbb{G}_m$, $\mathrm{A}(F) = F^\times$.

2.3. **Notation and some facts related to algebraic groups and their rational points.** For an algebraic group denoted using a roman letter, the corresponding fraktur letter will denote its Lie algebra: $\mathfrak{g} = \mathrm{Lie}\,\mathrm{G}, \mathfrak{t} = \mathrm{Lie}\,\mathrm{T}$ etc. For a real number $s \geq 0$ and any reductive group H over F, H_s (resp., H_{s+}) will denote the union of the Moy-Prasad filtration subgroups $\mathrm{H}_{x,s}$ (resp., $\mathrm{H}_{x,s+}$) as x ranges over points in the Bruhat-Tits building of H, normalized as in the papers [AD02, AD04, ADSS11]. $\mathrm{H}_s, \mathrm{H}_{s+} \subset \mathrm{H}(F)$ are H-domains, i.e., open and closed subsets of $\mathrm{H}(F)$ that are closed under $\mathrm{H}(F)$-conjugation (see [AD02, Corollary 3.7.21]). For any algebraic group H over F, $\mathrm{Z_H}$ will denote its center. -1 will denote the nontrivial element of $\mathrm{Z_G}(F)$, and for $g \in \mathrm{G}(F)$ and $U \subset \mathrm{G}(F)$, $-g$ will denote $-1 \cdot g$ and $-U$ will denote $\{-1\} \cdot U$.

Remark 5. For any torus $\mathrm{T} \subset \mathrm{G}$ and any $s \geq 0$ (resp., $s \in \mathbb{R}$), T_s (resp., $\mathfrak{t}_s$) is simply the set of $t \in \mathrm{T}(F)$ (resp., $X \in \mathfrak{t}(F)$) such that $\mathrm{val}(\lambda - 1) \geq s$, for some eigenvalue (and hence both eigenvalues) of t (resp., X) acting on the standard representation of G (see [ADSS11, Section 3.2]).

Remark 6. A useful property is that for all $s \geq 0$, $\mathrm{G}_{s+} \cap \mathrm{T}(F) = \mathrm{T}_{s+}$ (see [AD04, Lemma 2.2.9]; this uses that p is odd as the extension 'E/F' of [AD04, Section 2.2] is required to be tamely ramified), and consequently we also have $-\mathrm{G}_{s+} \cap \mathrm{T}(F) = -\mathrm{T}_{s+}$.

Remark 7. The Cayley transform $\mathfrak{c}$ is a birational map $\mathfrak{g} \to \mathrm{G}$ given by the prescription $X \mapsto (1+X/2)(1-X/2)^{-1}$. One knows that for any $s > 0$, any maximal torus T of G and any x in the Bruhat-Tits building of G, $\mathfrak{c}$ maps $\mathfrak{g}_{x,s}$ homeomorphically to $\mathrm{G}_{x,s}$, $\mathfrak{t}_s$ homeomorphically to T_s, and $\mathfrak{g}_s$ homeomorphically to G_s. These standard facts are an easy consequence of, e.g., the lattice flag description of Moy-Prasad filtrations (see [LMS16, Theorem 4.4.1]); most of them also follow from [ADSS11, Lemma 2.3 and Lemma 5.4].

For a topological group H, $\hat{H}$ will denote its unitary dual. For a reductive group H over F (which for us will be G or a maximal torus thereof), $\hat{\mathrm{H}}_{\mathrm{adm}}$ and $\hat{\mathrm{H}}_t$ will denote the admissible dual and the tempered dual, respectively, of $\mathrm{H}(F)$. Each element of either of these sets is an isomorphism class of representations, but may by abuse of notation also stand for a representative of this class. Whenever we refer to a character of any topological group, we will implicitly assume the character to be continuous.

2.4. **Notation for parabolic induction.** Henceforth, B will denote the standard Borel subgroup of G and $\mathrm{B} = \mathrm{A}\,\mathrm{U}$ its Levi decomposition. For $\psi \in \widehat{\mathrm{A}(F)} = \widehat{\mathrm{B}(F)}$, we will denote by

$$\pi(\mathrm{A}, \psi) = \pi_\psi := \mathrm{Ind}_{\mathrm{B}}^{\mathrm{G}}\,\psi$$

the representation of $\mathrm{G}(F)$ obtained by normalized parabolic induction from ψ. This time $\pi(\mathrm{A}, \psi)$ may be reducible, but no inconsistency has been introduced into our notation.

2.5. **Notation concerning Harish-Chandra characters and the regular semisimple set.** Let $\mathrm{G}(F)_{\mathrm{reg}} \subset \mathrm{G}(F)$ be the open subset consisting of (necessarily strongly) regular semisimple elements of $\mathrm{G}(F)$, i.e., those elements whose centralizer is a maximal torus of G. For any finite-length smooth representation π of $\mathrm{G}(F)$, Θ_π will stand for the character of π, viewed either as a distribution on $\mathrm{G}(F)$ or as a locally constant function on $\mathrm{G}(F)_{\mathrm{reg}}$. Such a Θ_π is locally integrable on $\mathrm{G}(F)$. For $\mathcal{U} \subset \mathrm{G}(F)$, write $\mathcal{U}_{\mathrm{reg}}$ for $\mathcal{U} \cap \mathrm{G}(F)_{\mathrm{reg}}$.

2.6. **Notation concerning group actions and other miscellany.** For $g \in \mathrm{G}$, $\mathrm{Int}\, g$ and $\mathrm{Ad}\, g$ will denote the conjugation and adjoint actions of $g \in \mathrm{G}$ on G and $\mathfrak{g}$, respectively. Sometimes, we will write ${}^g x$ for $\mathrm{Int}\, g(x) = gxg^{-1}$. If an abstract group H acts on a set X and $x \in U \subset X$, ${}^H x$ and ${}^H U$ will denote the orbit of x under H and the union of the H-conjugates of U, respectively.

For a finite set X, $\#X$ will denote its cardinality. 'meas' will stand for measure. If U is a subset of a set X, which is understood from context, then the characteristic function of U will be denoted $\mathbb{1}_U$, and we will write $\mathbb{1}$ for $\mathbb{1}_X$.

2.7. **Review regarding conjugacy and stable conjugacy classes of tori in G, and their Weyl groups.** For a maximal torus $\mathrm{T} \subset \mathrm{G}$, E_{T} will denote the minimal extension of F in $\bar{F}$ over which T splits (this extends notation from Subsection 2.2). Further, W_{T} will denote the quotient by $\mathrm{T}(F)$ of the normalizer of $\mathrm{T}(F)$ in $\mathrm{G}(F)$. It is easy to see that $\#W_{\mathrm{T}}$ equals 2 or 1, the latter case occurring exactly when -1 does not belong to the multiplicative group $N_{E_{\mathrm{T}}/F}(E_{\mathrm{T}}^\times)$ of norms of elements of $E_{\mathrm{T}}^\times$ (or equivalently, exactly when E_{T} is a ramified quadratic extension of F and -1 is not a square in $\mathbb{F}_q^\times$); a reference is [ADSS11, Section 3.1]. We will frequently use the easy fact that for elliptic T, the number of $\mathrm{G}(F)$-conjugacy classes in the $\mathrm{GL}_2(F)$-conjugacy class of T (which is also the stable conjugacy class of T), is equal to $\#W_{\mathrm{T}}$; given the description of $\#W_{\mathrm{T}}$ above, this can also be seen from the discussion of [ADSS11, Section 3.1] (see slightly before Notation 3.2 there).

Remark 8. If $\mathrm{T} \subset \mathrm{G}$ is a maximal torus, $s > 0$ and $t \in \mathrm{T}(F) \setminus \mathrm{Z}_{\mathrm{G}}(F)\mathrm{T}_{s+} \subset \mathrm{G}(F)_{\mathrm{reg}}$, then it is easy to see that

$${}^{\mathrm{G}(F)}(t\mathrm{T}_{s+}) \cap \mathrm{T}(F) = W_{\mathrm{T}} \cdot (t\mathrm{T}_{s+}) = \bigsqcup_{w \in W_{\mathrm{T}}} \mathrm{Int}\, w(t\mathrm{T}_{s+}).$$

Here, the disjointness of the union follows from the fact that $t^2 \notin \mathrm{T}_{s+}$, so that $t\mathrm{T}_{s+} \cap t^{-1}\mathrm{T}_{s+} = \emptyset$.

2.8. **The sign characters.** For a quadratic extension E/F, denote by $\mathrm{sgn}_{E/F}$ the associated character on $F^\times$, which is the unique nontrivial quadratic character with kernel $N_{E/F}(F^\times)$. For $\theta \in \{\epsilon, \varpi, \epsilon\varpi\}$, set $\mathrm{sgn}_\theta = \mathrm{sgn}_{E_{\mathrm{T}^\theta}} = \mathrm{sgn}_{F[\sqrt{\theta}]/F}$.

2.9. **The Weyl discriminant and the functions ε and σ on the regular set.** For $t \in \mathrm{G}(F)$ regular semisimple, the Weyl discriminant $D(t)$ is (alternatively) defined as $|\lambda - \lambda^{-1}|^2$, where λ and λ^{-1} are the eigenvalues of t acting on the standard representation of G.

In [SS84, (2.3)], a function ε has been defined on regular F-rational elements of elliptic Cartan subgroups $\mathrm{T}^{\theta_1,\theta_2}(F)$ of $\mathrm{G}(F)$ of the form:

$$\left\{ t^{\theta_1,\theta_2}(a,b) := \begin{pmatrix} a & b\theta_1 \\ b\theta_2 & a \end{pmatrix} \mid a, b \in F \right\} \cap \mathrm{G}(F).$$

Here, the ellipticity just translates to the condition $\theta_1\theta_2 \notin F^{\times 2}$. When this is satisfied, $\mathrm{T}^{\theta_1,\theta_2}(F)$ is isomorphic to $\{x \in F[\sqrt{\theta_1\theta_2}]^\times \mid N_{F[\sqrt{\theta_1\theta_2}]/F}(x) = 1\}$, by the map taking $t^{\theta_1,\theta_2}(a,b)$ to $E := a + b\sqrt{\theta_1\theta_2}$. The function ε takes $t^{\theta_1,\theta_2}(a,b)$ to $\mathrm{sgn}_{F[\sqrt{\theta_1\theta_2}]/F}(b\theta_1)$.

For $\theta \in \{\epsilon, \varpi, \epsilon\varpi\}$, the group denoted by T_θ (resp., by $\mathrm{T}_\theta^\#$) in [SS84] equals, according to our notation, T^θ (resp., $\mathrm{Int}\, g_\theta(\mathrm{T}^\theta)$), for some diagonal matrix $g_\theta \in \mathrm{GL}_2(F)$ such that $\det g_\theta \notin N_{E_{\mathrm{T}^\theta}/F}(E_{\mathrm{T}^\theta}^\times)$. Using this, it is easy to see that on $\mathrm{T}^\theta(F)_{\mathrm{reg}}$:

$$\varepsilon \circ \mathrm{Int}_{g_\theta}|_{\mathrm{T}^\theta(F)_{\mathrm{reg}}} = -\varepsilon|_{\mathrm{T}^\theta(F)_{\mathrm{reg}}}. \tag{6}$$

Notation 9. We define a $\mathrm{G}(F)$-conjugation invariant function σ on $\mathrm{G}(F)_{\mathrm{reg}}$ as follows:

(i) On the set of elliptic elements in $\mathrm{G}(F)_{\mathrm{reg}}$, σ coincides with the $\mathrm{G}(F)$-conjugation invariant function so denoted around [SS84, (4.5)]; in particular, for $\theta \in \{\epsilon, \varpi, \epsilon\varpi\}$ and $t \in \mathrm{T}^\theta(F)_{\mathrm{reg}}$, $\sigma(t)$ equals $\varepsilon(t)D(t)^{-1/2}$ (resp., $-\mathrm{sgn}_{F[\sqrt{\theta}]/F}(-1)\varepsilon(t)D(t)^{-1/2}$) for $t \in \mathrm{T}^\theta(F)_{\mathrm{reg}}$ (resp., $t \in \mathrm{Int}\, g_\theta(\mathrm{T}^\theta(F))_{\mathrm{reg}}$).

(ii) If $t \in \mathrm{G}(F)_{\mathrm{reg}}$ is split, we let $\sigma(t) = D(t)^{-1/2}$.

Let $\theta \in \{\epsilon, \varpi, \epsilon\varpi\}$. If $\tilde{\mathrm{T}}^\theta(F)$ is the centralizer of $\mathrm{T}^\theta(F)$ in $\mathrm{GL}_2(F)$, then $\det(\tilde{\mathrm{T}}^\theta(F))$ is easily seen to be $N_{F[\sqrt{\theta}]/F}(F[\sqrt{\theta}]^\times)$. Hence $\sigma(gtg^{-1}) = \sigma(t)$ if $t \in \mathrm{T}^\theta(F)_{\mathrm{reg}}$ and $\det g \in N_{F[\sqrt{\theta}]/F}(F[\sqrt{\theta}]^\times)$. It is also easy to see from Equation (6) above that $\sigma(g_\theta t g_\theta^{-1}) = \mathrm{sgn}_{F[\sqrt{\theta}]/F}(-1)\sigma(t)$. Putting these two together, we get that for $t \in \mathrm{T}^\theta(F)$ and $g \in \mathrm{GL}_2(F)$, $\sigma(gtg^{-1})$ equals $\sigma(t)$ (resp., $\mathrm{sgn}_{F[\sqrt{\theta}]/F}(\det g)\cdot\sigma(t)$) if $\mathrm{sgn}_{F[\sqrt{\theta}]/F}(-1)$ equals 1 (resp., -1). Note that $\mathrm{sgn}_{F[\sqrt{\theta}]/F}(-1) = -1$ if and only if the following two conditions are met: $F[\sqrt{\theta}]/F$ is ramified and -1 is not a square in ${\mathbb{F}_q}^\times$. This is in turn equivalent, by the discussion of Subsection 2.7, to the stable conjugacy class of $\mathrm{T}^\theta(F)$ being a single $\mathrm{G}(F)$-conjugacy class.

Since every elliptic maximal torus of G is $\mathrm{GL}_2(F)$-conjugate to T^θ for some $\theta \in \{\epsilon, \varpi, \epsilon\varpi\}$, this means that for all $\gamma \in \mathrm{G}(F)_{\mathrm{reg}}$ and $g \in \mathrm{GL}_2(F)$, letting T_γ be the centralizer of γ in G,

$$\sigma(g\gamma g^{-1}) = \begin{cases} \sigma(\gamma), & \text{if } \mathrm{T}_\gamma \text{ is not ramified or } -1 \text{ is a square in } \mathbb{F}_q^\times, \text{ and} \\ \mathrm{sgn}_{E_{\mathrm{T}_\gamma}/F}(\det g)\sigma(\gamma), & \text{otherwise.} \end{cases} \tag{7}$$

Note that σ is entirely defined by Equation (7) together with its values on split elements and the requirement that for $\theta \in \{\epsilon, \varpi, \epsilon\varpi\}$, σ is given on $\mathrm{T}^\theta(F)$ by the formula:

$$t = \begin{pmatrix} a & b \\ b\theta & a \end{pmatrix} \mapsto \frac{\mathrm{sgn}_{E_{\mathrm{T}^\theta}/F}(b)}{D(t)^{1/2}} = \frac{1}{|\lambda - \lambda^{-1}|} \cdot \mathrm{sgn}_{E_{\mathrm{T}^\theta}}\left(\frac{\lambda - \lambda^{-1}}{2\sqrt{\theta}}\right), \tag{8}$$

where λ, λ^{-1} are the eigenvalues of t on the standard representation of G, taken in the order $a + b\sqrt{\theta}, a - b\sqrt{\theta}$.

σ and the Weyl discriminant enjoy the following well-known descent property.

Lemma 10. *σ is constant on each of the sets occurring in the partition of Equation* (3) *other than* G_{r+} *and* $-\mathrm{G}_{r+}$*, as is the Weyl discriminant.*

Proof. Because p is odd, any $\lambda' \in F^\times$ such that $\mathrm{val}(\lambda' - 1) > 0$ is a square. Note that for each $\theta \in \{\epsilon, \varpi, \epsilon\varpi\}$, the isomorphism $\mathrm{T}^\theta(F) \to E^1$ sending $\begin{pmatrix} a & b \\ b\theta & a \end{pmatrix}$ to $a + \sqrt{\theta}b$ takes T^θ_{r+} to $\{\lambda \in E^1 \mid \mathrm{val}(1 - \lambda) > r\}$.

Therefore, conjugating by $\mathrm{GL}_2(F)$ or $\mathrm{SL}_2(F)$ as required (the latter when T is ramified and $-1 \notin {\mathbb{F}_q^\times}^2$), the assertions (both regarding σ and the Weyl discriminant) reduce by the formula (8) to showing that for any quadratic extension E/F, and $\lambda_1, \lambda \in E^1 := \ker(N_{E/F} : E^\times \to F^\times)$, with $\mathrm{val}(1 + \lambda_1), \mathrm{val}(1 - \lambda_1) \le r$ and $\mathrm{val}(1 - \lambda) > r$, we have

$$0 < \mathrm{val}\left(\frac{\lambda_1\lambda - \lambda_1^{-1}\lambda^{-1}}{\lambda_1 - \lambda_1^{-1}} - 1\right) = \frac{(\lambda - 1)(\lambda_1 + \lambda_1^{-1}\lambda^{-1})}{\lambda_1 - \lambda_1^{-1}}.$$

Since $p \neq 2$, it is easy to see that $\mathrm{val}(\lambda_1 - \lambda_1^{-1}) \le r$, so the above expression indeed has positive valuation. □

Remark 11. By Lemma 10 and Equation (7), if $t_1, t_2 \in \mathrm{T}(F) \setminus (\mathrm{T}_{r+} \cup -\mathrm{T}_{r+})$ are such that $t_2 \in t_1 \cdot \mathrm{T}_{r+}$, then $\sigma(t_1)\sigma(t_2) = D(t_1)^{-1/2} D(t_2)^{-1/2}$.

2.10. **Measures and the Weyl integration formula.** Once and for all, we choose a Haar measure on $\mathrm{G}(F)$. If $\mathrm{T} \subset \mathrm{G}$ is a split (resp., elliptic) maximal torus, we give $\mathrm{T}(F)$ the Haar measure normalized so as to give $\mathrm{T}(\mathfrak{O})$ (resp., $\mathrm{T}(F)$) measure 1. For each such T, we give $\mathrm{T}(F)\backslash\mathrm{G}(F)$ the quotient measure, and use it to define the orbital integral $O(\gamma, f)$ of (2), for any $f \in C_c^\infty(\mathrm{G}(F))$ and $\gamma \in \mathrm{G}(F)$ regular semisimple. Thus, the following Weyl integration formula holds: for $f \in C^\infty(\mathrm{G}(F)) \cap L^1(\mathrm{G}(F))$ such that for each maximal torus $\mathrm{T} \subset \mathrm{G}$, the function $t \mapsto D(t)O(t, f)$ on $\mathrm{T}(F) \cap \mathrm{G}(F)_{\mathrm{reg}}$ belongs to $L^1(\mathrm{T}(F))$,

$$\int_{\mathrm{G}(F)} f(g)\, dg = \sum_{\mathrm{T} \in \mathcal{T}} \frac{1}{\# W_{\mathrm{T}}} \int_{\mathrm{T}(F)} D(t) \cdot O(t, f)\, dt,$$

where $\mathcal{T}$ is a set of representatives for the set of $\mathrm{G}(F)$-conjugacy classes of maximal tori in G.

For each maximal torus $\mathrm{T} \subset \mathrm{G}$ and each $f \in C_c^\infty(\mathrm{G}(F))$, the function $t \mapsto D(t)^{1/2} O(t, f)$ on $\mathrm{T}(F)_{\mathrm{reg}}$ is known to be locally bounded on $\mathrm{T}(F)$ ([HC70, Theorem 14]) and to be of relatively compact support ([HC70, Lemma 39]). From [HC70, Theorem 15] it follows that if Θ is a locally constant complex valued function on $\mathrm{G}(F)_{\mathrm{reg}}$ such that $\gamma \mapsto D(\gamma)^{1/2}\Theta(\gamma)$ is locally bounded on $\mathrm{G}(F)$, then Θ is locally integrable on $\mathrm{G}(F)$, i.e., $(g \mapsto \Theta(g)f(g)) \in L^1(\mathrm{G}(F))$ for all $f \in C_c^\infty(\mathrm{G}(F))$.

2.11. **The dual measure.** For a maximal torus $\mathrm{T} \subset \mathrm{G}$, we will give $\widehat{\mathrm{T}(F)}$ the dual of the Haar measure on $\mathrm{T}(F)$. We will use without further mention the following property of the dual measure: if $C \subset \mathrm{T}(F)$ is a compact open subgroup, then considering $\widehat{\mathrm{T}(F)/C}$ as a subgroup of $\widehat{\mathrm{T}(F)}$ in the obvious way:

$$\mathrm{meas}(C) \cdot \mathrm{meas}(\widehat{\mathrm{T}(F)/C}) = 1.$$

Note that this means that for elliptic maximal tori $\mathrm{T} \subset \mathrm{G}$, $\widehat{\mathrm{T}(F)}$ gets the counting measure. For split maximal tori $\mathrm{T} \subset \mathrm{G}$, choosing any identification $\mathrm{T} \cong \mathbb{G}_m$, so that $\mathrm{T}(F) = F^\times$, the measure on $\widehat{\mathrm{T}(F)}$ can be described as follows. For each $\dot{\psi} \in \widehat{\mathrm{T}(\mathfrak{O})}$, we have a homeomorphic homomorphism

$$\left\{\psi \in \widehat{\mathrm{T}(F)} \mid \psi|_{\mathrm{T}(\mathfrak{O})} = \dot{\psi}\right\} \stackrel{\cong}{\to} S^1,$$

where S^1 is the unit circle in the complex plane, given by $\psi \mapsto \psi(\varpi)$. Thus $\widehat{\mathrm{T}(F)}$ is the union of infinitely many copies of S^1, each of which gets a total measure of 1.

A consequence is that if $s > 0$, $\dot{\psi} \in \hat{\mathrm{T}}_{s+}$ and $t \in \mathrm{T}(F)$, for some maximal torus $\mathrm{T} \subset \mathrm{G}$, then:

$$\int_{\substack{\psi \in \widehat{\mathrm{T}(F)} \\ \psi|_{\mathrm{T}_{s+}} = \dot{\psi}}} \psi(t)\, d\psi = \begin{cases} \frac{1}{\mathrm{meas}\, \mathrm{T}_{s+}} \cdot \dot{\psi}(t), & \text{if } t \in \mathrm{T}_{s+}, \text{ and} \\ 0, & \text{otherwise.} \end{cases} \tag{9}$$

2.12. **The partion of Equation** (3)**.** Consider the partition of $\mathrm{G}(F)$ given in (3). It is well-known that G_{r+} is a G-domain, i.e., it is open and closed in $\mathrm{G}(F)$ (see [AD02, Corollary 3.7.21]). Hence so is $-\mathrm{G}_{r+}$. If λ is an eigenvalue of $\gamma \in \mathrm{T}(F)$ acting on the standard representation of G, then the adjoint action of γ on $\mathfrak{g}/\mathfrak{t}$ has eigenvalues $\lambda^{\pm 2}$. Thanks to Remark 5 and the fact that p is odd, if further $\gamma \in \mathrm{T}(F) \setminus (\mathrm{T}_{r+} \cup -\mathrm{T}_{r+})$, then these eigenvalues satisfy $\mathrm{val}(\lambda^2 - 1), \mathrm{val}(\lambda^{-2} - 1) \le r$, so that ${}^{\mathrm{G}(F)}(\gamma \mathrm{T}_{r+})$ is open (see, e.g., [AK07, Corollary 4.7 and Remark 7.2]). Thus, each set in the partion (3) is open in $\mathrm{G}(F)$, so that each of these sets is closed as well.

2.13. **Notation concerning the main Bernstein projectors of interest.** E_r will denote the depth r Bernstein projector for G (see [BKV16]), which acts as the identity on irreducible admissible representations in $\hat{\mathrm{G}}_{\mathrm{adm},\le r}$ and annihilates irreducible admissible representations in $\hat{\mathrm{G}}_{\mathrm{adm}} \setminus \hat{\mathrm{G}}_{\mathrm{adm},\le r}$. For a maximal torus $\mathrm{T} \subset \mathrm{G}$ and a nontrivial character $\mathbb{1} \neq \dot\psi \in \hat{\mathrm{T}}_{r+}$, $z_{\mathrm{T},\dot\psi}$ will denote the Bernstein projector for $\Pi(\mathrm{T},\dot\psi)$ — the unique element of $\mathcal{Z}(\mathrm{G})$ that acts as the identity on irreducible admissible representations in $\Pi(\mathrm{T},\dot\psi)$, and annihilates representations in $\hat{\mathrm{G}}_{\mathrm{adm}} \setminus \Pi(\mathrm{T},\dot\psi)$.

3. Some preliminaries and reductions

3.1. **Facts about the representations** $\pi(\mathrm{T},\psi)$**,** T **elliptic.** We will review some of the facts about the representations $\pi(\mathrm{T},\psi)$ (see shortly before (1)), where $\mathrm{T} \subset \mathrm{G}$ is an elliptic maximal torus and $\psi \in \widehat{\mathrm{T}(F)}$ is a positive depth character, some of which we will use in what follows without further mention.

Remark 12. We remark that the proofs of the character formulas of [ADSS11] make use of [ADSS11, Hypothesis 1.4], requiring p to be at least $2e+3$ where e is the ramification degree of F over $\mathbb{Q}_p$. However, not only can that assumption be removed as they remark (e.g., due to the compatibility of their results with those of [SJS68], which only needs p to be odd), the use of the said hypothesis in [ADSS11] is only to treat depth zero representations, whereas we will need the results of [ADSS11] only for positive depth representations.

Let $\mathrm{T} \subset \mathrm{G}$ be an elliptic maximal torus, and $\psi \in \widehat{\mathrm{T}(F)}$ a character of positive depth s (which is written 'r' in [ADSS11], whereas we have fixed an r for a different purpose earlier).

We thank J. Adler and L. Spice for confirming that the following clarifications apply to the statement of the character formulas for $\pi(\mathrm{T},\psi)$ in the introduction of [ADSS11], along the lines articulated before the statements of their main formulas, for a few elements of $\mathrm{T}(F)$ in the case where T is ramified, as we describe below. For ramified T, the formula given in [ADSS11] for the character of $\pi(\mathrm{T},\psi)$ on $\mathrm{T}(F)\setminus \mathrm{Z}_{\mathrm{G}}(F)\mathrm{T}_s$, should be read as applying to $\gamma \in \mathrm{T}_{0+} \setminus \mathrm{Z}_{\mathrm{G}}(F)\mathrm{T}_s = \mathrm{T}_{0+} \setminus \mathrm{T}_s$. Since the central character of $\pi(\mathrm{T},\psi)$ is $\psi|_{\mathrm{Z}_{\mathrm{G}}(F)}$ (as is easy to see from the construction of [ADSS11], and as is mentioned shortly before the statement of the formulas in [ADSS11]), and since it is easy to see that

$$\mathrm{T}(F) \setminus \mathrm{Z}_{\mathrm{G}}(F)\mathrm{T}_s = (\mathrm{T}_{0+} \setminus \mathrm{Z}_{\mathrm{G}}(F)\mathrm{T}_s) \cdot \mathrm{Z}_{\mathrm{G}}(F)$$

for ramified tori, this determines the character of π on the whole of $\mathrm{T}(F)_{\mathrm{reg}}$.

To clarify this point a bit more, assume that T is of the form T^θ with $\theta \in \{\epsilon, \varpi, \epsilon\varpi\}$. The formula just referred to, valid on $\mathrm{T}_{0+} \setminus \mathrm{Z}_{\mathrm{G}}(F)\mathrm{T}_s$, has a factor of the form $\mathrm{sgn}_\theta(\mathrm{Im}_\theta(\gamma)) = \mathrm{sgn}_\theta\left((\gamma-\gamma^{-1})/2\sqrt{\theta}\right)$. Replacing $\gamma \in \mathrm{T}_{0+} \setminus \mathrm{T}_{s+}$ by $-\gamma \in \mathrm{T}(F) \setminus \mathrm{T}_{0+}$ multiplies this factor by $\mathrm{sgn}_\theta(-1)$, while multiplying the product of the remaining factors by $\psi(-1)$.

Since the central character of $\pi(\mathrm{T},\psi)$ is $\psi(-1)$, it is now easy to see (dropping the assumption that $\mathrm{T} = \mathrm{T}^\theta$) that the aforementioned formula for the character of $\pi(\mathrm{T},\psi)$ on $\mathrm{T}_{0+} \setminus \mathrm{Z}_{\mathrm{G}}(F)\mathrm{T}_{s+}$, actually applies on all of $\mathrm{T}(F) \setminus \mathrm{Z}_{\mathrm{G}}(F)\mathrm{T}_{s+}$, provided we replace ψ by $\psi\alpha$, where $\alpha = \alpha_{\mathrm{T}}$ is the trivial character of $\mathrm{T}(F)$ if T is unramified or if -1 is a square in $\mathbb{F}_q^\times$, and the unique nontrivial quadratic character in $\widehat{\mathrm{T}(F)/\mathrm{T}_{0+}} \subset \widehat{\mathrm{T}(F)}$ if T is ramified and -1 is not a square in $\mathbb{F}_q^\times$ (this matches what one would expect from the description of the rectifying character for GL_2 from [BH06], denoted $\Delta = \Delta_\xi$ in that reference).

Thus, henceforth, by the character formula for $\pi(\mathrm{T},\psi)$ from [ADSS11], we will actually refer to the expression furnished in the right-hand sides of the formulas for 'Θ_π' given in pages 24 and 25 of that reference, but with ψ replaced by $\psi\alpha$.

Remark 13. Now, the closing remark of the introduction in [ADSS11] should be interpreted as saying that the character formulas of [ADSS11] for $\pi(\mathrm{T},\psi)$, where $\mathrm{T} = \mathrm{T}^\theta$ for some $\theta \in \{\epsilon, \varpi, \epsilon\varpi\}$, agree with the character formulas of Sally and Shalika from [SJS68] for the representation that [SJS68] denotes as

$\pi(\Lambda'_\pi, \psi\alpha_{E_{\mathrm{T}^\theta}}, E_{\mathrm{T}^\theta})$; here $\alpha_{E_{\mathrm{T}^\theta}}$ is the trivial character of $E^1_{\mathrm{T}^\theta}$ if $\mathrm{sgn}_{E_{\mathrm{T}^\theta}/F}(-1) = 1$, and the unique nontrivial quadratic character of $E^1_{\mathrm{T}^\theta}$ otherwise (note the matching of central characters; the central characters of the representations of [SJS68] are discussed towards the end of page 1235 of that reference). It should be well-known that this representation should be one of the representations obtained by endoscopic transfer from the character ψ of $E^1_{\mathrm{T}^\theta}$, and hence Lemma 21 later below should be true even without requiring that ψ has positive depth, but since we are not able to find a convenient reference, we will quickly recall an argument in the special case of concern to us.

We continue to let $\mathrm{T} \subset \mathrm{G}$ be an elliptic maximal torus. To each supercuspidal representation $\pi(\mathrm{T}, \psi)$, where ψ is non-quadratic (a simplifying assumption as this case suffices for our purposes), [ADSS11] attaches in Notation 9.7, Notation 10.17 and Definition 14.1 of that reference an additive character $\Lambda' = \Lambda'_\pi$, starting from the fixed additive character Λ. In [ADSS11], this character is used to define an object denoted there as $H(\Lambda', k_\theta)$, which we will write as $H(\Lambda', E_{\mathrm{T}^\theta})$, and which enters into the character formulas for $\pi(\mathrm{T}, \psi)$ given in [ADSS11]. Since we need to analyze this object more closely in the ramified case, we recall some of the definition of Λ'_π, where $\pi = \pi(\mathrm{T}, \psi)$, with $\mathrm{T} = \mathrm{T}^\theta$ for some $\theta \in \{\varpi, \epsilon\varpi\}$ and ψ of positive depth, say s.

Since ψ is trivial on T_{s+}, as in [ADSS11, (10.14)] it gives an element of

$$\widehat{\mathrm{T}_{(s/2)+}/\mathrm{T}_{s+}} \stackrel{\circ\mathfrak{c}}{\to} \widehat{\mathfrak{t}_{(s/2)+}/\mathfrak{t}_{s+}} \to \mathfrak{t}_{-s}/\mathfrak{t}_{-(s/2)},$$

where $\circ\mathfrak{c}$ denotes the composition with the Cayley transform (see Remark 7). Write the resulting element of $\mathfrak{t}_{-s}/\mathfrak{t}_{-(s/2)}$ as $\varpi^{-\lceil s \rceil} X + \mathfrak{t}_{-(s/2)}$, where $X \in \mathfrak{t}_{\lceil s \rceil - s} = \mathfrak{t}_{1/2}$ (since $\pi(\mathrm{T}, \psi)$ has half-integral non-integral depth when T is ramified; this is easy to see from Remark 5 above and Remark 16(i) below). Thanks to the nontriviality of $\psi|_{\mathrm{T}_s}$, $X \in \mathfrak{t}_{(1/2)} \setminus \mathfrak{t}_{(1/2)+}$.

Using Remark 5 again, it is easy to see that the identification of F with $\mathfrak{t}^\theta$ by the prescription:

$$\iota : \beta \mapsto X_\beta := \begin{pmatrix} 0 & \beta \\ \beta\theta & 0 \end{pmatrix}$$

maps $\mathfrak{O}^\times$ bijectively to $\mathfrak{t}_{(1/2)} \setminus \mathfrak{t}_{(1/2)+}$. If β is the preimage of X under this identification (well-defined up to a positive power of $\mathfrak{p}$), then Λ'_π is defined to be $\Lambda_{\varpi^{-\lceil s \rceil}\beta\theta}$, that is to say, the additive character $x \mapsto \Lambda(\varpi^{-\lceil s \rceil}\beta\theta x)$.

Note that this agrees with the observation below [ADSS11, Definition 14.1], made for all positive depth $\pi(\mathrm{T}, \psi)$, that the depth of the additive character Λ'_π (i.e., the $d(\Lambda'_\pi)$ in the sense of [ADSS11, Definition 1.2], which is the smallest integer n such that Λ'_π is nontrivial on $\mathfrak{p}^n$) equals $s - (1/2)\,\mathrm{val}\,\theta$, namely s if $\theta = \epsilon$ and $s - (1/2)$ if $\theta = \varpi$ or $\theta = \epsilon\varpi$.

The above discussion (for ramified T) also shows the following: Suppose two characters $\psi_1, \psi_2 \in \widehat{\mathrm{T}(F)}$ of positive depth s are given, and elements $X_1, X_2 \in \mathfrak{t}_{(1/2)} \setminus \mathfrak{t}_{(1/2)+}$ and $\beta_1, \beta_2 \in \mathfrak{O}^\times$ are assigned as above. If ψ_1, ψ_2 agree on T_s, then X_1, X_2 are congruent modulo $\mathfrak{t}_{(1/2)+}$, and β_1, β_2 are congruent modulo the maximal ideal $\mathfrak{p}$ of $\mathfrak{O}$.

Lemma 14. *Let $\mathrm{T} = \mathrm{T}^\theta$ with $\theta \in \{\epsilon, \varpi, \epsilon\varpi\}$, and let $\psi_1, \psi_2 \in \widehat{\mathrm{T}(F)}$ be of positive depth s, such that $\psi_1|_{\mathrm{T}_s} = \psi_2|_{\mathrm{T}_s}$. Let $\Lambda'_i = \Lambda'_{\pi(\mathrm{T}, \psi_i)}$. Then $H(\Lambda'_1, E_{\mathrm{T}^\theta}) = H(\Lambda'_2, E_{\mathrm{T}^\theta})$, and this value is a fourth root of unity.*

Remark 15. Though we do not need it, we remark that the condition $\psi_1|_{\mathrm{T}_s} = \psi_2|_{\mathrm{T}_s}$ implies that $\pi(\mathrm{T}, \psi_1)$ and $\pi(\mathrm{T}, \psi_2)$ share unrefined minimal K-types.

Proof of Lemma 14. First consider the case where T is unramified, i.e., $\mathrm{T}^\theta = \mathrm{T}^\epsilon$. Then by [ADSS11, Lemma 4.2],

$$H(\Lambda'_1, E_{\mathrm{T}^\theta}) = H(\Lambda'_2, E_{\mathrm{T}^\theta}) = (-1)^{s+1},$$

since Λ'_1 and Λ'_2 have depth s.

If T is ramified, so that $\theta \in \{\varpi, \epsilon\varpi\}$, then with $\mathcal{G}(\cdot)$ denoting the Gauss sum defined in [ADSS11, Definition 4.1], recalling that Λ'_i has depth equal to $s - 1/2$, [ADSS11, Lemma 4.2] gives

$$H(\Lambda'_i, E_{\mathrm{T}^\theta}) = \mathcal{G}(\Lambda'_i) = q^{-1/2} \sum_{x\in\mathfrak{O}/\mathfrak{p}} \Lambda((-\varpi)^{s-(1/2)} \cdot \varpi^{-\lceil s\rceil}\beta_i\theta \cdot x^2) = q^{-1/2} \sum_{x\in\mathfrak{O}/\mathfrak{p}} \Lambda((-1)^{s-(1/2)}\varpi^{-1}\theta\beta_i x^2),$$

a fourth root of unity as mentioned just before Definition 4.1 in [ADSS11]. Since β_1 and β_2 are congruent modulo $\mathfrak{p}$ as observed earlier, and since Λ is of depth zero, and since $\varpi^{-1}\theta \in \mathfrak{O}^\times$, it follows that $H(\Lambda'_1, E_{\mathrm{T}^\theta}) = H(\Lambda'_2, E_{\mathrm{T}^\theta})$, as desired. □

Remark 16. Let $\mathrm{T} \subset \mathrm{G}$ be a maximal torus, and let $\psi \in \widehat{\mathrm{T}(F)}$ be such that ψ is of positive depth if T is elliptic, and such that $\mathrm{Ind}_{\mathrm{B}(F)}^{\mathrm{G}(F)} \psi$ is irreducible if T is split.

(i) $\pi(\mathrm{T}, \psi)$ is of depth equal to the depth of ψ; see [ADSS11, Remark 10.16] and [MP96, Theorem 5.2].
(ii) The central character of $\pi(\mathrm{T}, \psi)$ is the restriction of ψ to $\mathrm{Z_G}(F)$. We will abuse notation by writing in place of this central character just the element $\psi(-1) \in \{\pm 1\}$.
(iii) If T is elliptic and $\psi_1, \psi_2 \in \widehat{\mathrm{T}(F)}$ have the same positive depth, then $\pi(\mathrm{T}, \psi_1)$ and $\pi(\mathrm{T}, \psi_2)$ have the same formal degree (see [ADSS11, Lemma 14.4]; that lemma applies only to supercuspidal representations they call 'ordinary', but all positive depth supercuspidal representations are 'ordinary' — see shortly before Remark 10.16 in [ADSS11]).
(iv) For $g \in \mathrm{GL}_2(F)$, $\pi(\mathrm{T}, \psi) \circ \mathrm{Int}\, g = \pi(\mathrm{Int}\, g^{-1}(\mathrm{T}), \psi \circ \mathrm{Int}\, g)$ — this is easy if T is split; see Remark [ADSS11, Remark 10.19] when T is elliptic.
(v) $\pi(\mathrm{T}, \psi) \cong \pi(\mathrm{T}', \psi')$ if and only if there exists $g \in \mathrm{G}(F)$ such that $\mathrm{T}' = \mathrm{Int}\, g^{-1}(\mathrm{T})$ and $\psi' = \psi \circ \mathrm{Int}\, g$ — again, this is easy if T is split; use [ADSS11, Theorem 11.1] and the description of W_T reviewed in Subsection 2.7 if T is elliptic.

3.2. **Review on character values.** In this subsection, T need not be elliptic. Recall that when T is split, T is $\mathrm{G}(F)$-conjugate to A. Recall that in Section 3.1, we defined, for an elliptic maximal torus $\mathrm{T} \subset \mathrm{G}$, a quadratic character $\alpha = \alpha_\mathrm{T} \in \widehat{\mathrm{T}(F)}$. If T is split, we declare $\alpha = \alpha_\mathrm{T} \in \widehat{\mathrm{T}(F)}$ to be $\mathbb{1}$.

Remark 17. For $\xi \in \widehat{\mathrm{A}(F)} = \widehat{F^\times}$, recall the induced representation $\pi_\xi = \pi(\mathrm{A}, \psi) = \mathrm{Ind}_{\mathrm{B}(F)}^{\mathrm{G}(F)} \xi$. It character is known (see [vD72, Theorem 3]) to be given by:

$$\Theta_{\pi_\xi}(t) = D(t)^{-1/2} \cdot (\xi(t) + \xi(t^{-1})) \tag{10}$$

on the regular semisimple elements $t \in \mathrm{A}(F)_{\mathrm{reg}} = F^\times \setminus \{1, -1\}$, and to be 0 on regular semisimple elements without a $\mathrm{G}(F)$-conjugate in $\mathrm{A}(F)$.

As mentioned earlier, for T non-split, we will use [ADSS11] to get our character values for $\pi(\mathrm{T}, \psi)$ with $\psi \in \widehat{\mathrm{T}(F)}$. The following summarizes what we need of the character values of the $\pi(\mathrm{T}, \psi)$.

Lemma 18. *Let* T *be a maximal torus of* G*, split or non-split, and* $\mathbb{1} \neq \dot\psi \in \hat{\mathrm{T}}_{r+}$ *a nontrivial character. Let* $\gamma \in \mathrm{G}(F)$ *be regular semisimple.*

(i) If $\gamma = z\gamma'$*, where* $z \in \mathrm{Z_G}(F)$ *and* $\gamma' \in \mathrm{G}_{r+}$*, then for all characters* ψ_1, ψ_2 *of* $\mathrm{T}(F)$ *with* $\psi_1|_{\mathrm{T}_{r+}} = \psi_2|_{\mathrm{T}_{r+}} = \dot\psi$*, we have* $\Theta_{\pi(\mathrm{T},\psi_1)}(\gamma') = \Theta_{\pi(\mathrm{T},\psi_2)}(\gamma')$*, so that:*

$$\Theta_{\pi(\mathrm{T},\psi_1)}(\gamma) = (\psi_1\psi_2^{-1})(z) \cdot \Theta_{\pi(\mathrm{T},\psi_2)}(\gamma).$$

(ii) If γ *is an element of* $\mathrm{T}(F) \setminus \mathrm{Z_G}(F)\mathrm{T}_{r+}$*, then for any character* ψ *of* $\mathrm{T}(F)$ *with* $\psi|_{\mathrm{T}_{r+}} = \dot\psi$ *we have a formula:*

$$\Theta_{\pi(\mathrm{T},\psi)}(\gamma) = c_{\dot\psi}\sigma(\gamma) \cdot \sum_{w\in W_\mathrm{T}} (\psi\alpha \circ \mathrm{Int}\, w)(\gamma),$$

where $\alpha = \alpha_{\mathrm{T}}$ is as described just before Remark 13, and $c_{\dot\psi}$ is a fourth-root of unity not depending on γ or the specific ψ that extends $\dot\psi$, and which equals 1 if T *is split.*

(iii) If $\gamma \notin \mathrm{Z}_{\mathrm{G}}(F)\mathrm{G}_{r+} \cup {}^{\mathrm{G}(F)}\mathrm{T}(F)$ then for every character ψ of $\mathrm{T}(F)$ with $\psi|_{\mathrm{T}_{r+}} = \dot\psi$, $\Theta_{\pi(\mathrm{T},\psi)}(\gamma) = 0$.

Proof. For T split, all the three assertions follow from Remark 17, with $c_{\dot\psi} = 1$, so assume that T is not split.

Using Equation (7), and the fact that $\pi(\mathrm{T},\psi)\circ \mathrm{Int}\, g = \pi(\mathrm{Int}\, g^{-1}(\mathrm{T}), \psi\circ \mathrm{Int}\, g)$ (Remark 16(iv)), and noting that $\alpha_{\mathrm{Int}\, g^{-1}(\mathrm{T})} = \alpha_{\mathrm{T}} \circ \mathrm{Int}\, g$ for all $g \in \mathrm{GL}_2(F)$, we may and do replace (T,ψ) by $(\mathrm{Int}\, g^{-1}(\mathrm{T}), \psi \circ \mathrm{Int}\, g)$ for some $g \in \mathrm{GL}_2(F)$ (if T is not ramified or $-1 \in \mathbb{F}_q^{\times 2}$) or more narrowly some $g \in \mathrm{G}(F)$ (if T is ramified and $-1 \in \mathbb{F}_q^{\times 2}$), to assume without loss of generality that $\mathrm{T} = \mathrm{T}^\theta$ for some $\theta \in \{\epsilon, \varpi, \epsilon\varpi\}$.

$\dot\psi$ being nontrivial, the depth of π, being the smallest s such that $\dot\psi$ is trivial on T_{s+}, is *strictly greater* than r. Note also that for any ψ extending $\dot\psi$, when $\theta = \epsilon$, $[(-1)^{s+1} + H(\Lambda'_{\pi(\mathrm{T},\psi)}, E_{\mathrm{T}^\theta})]/2 = (-1)^{s+1}$ by [ADSS11, Lemma 4.2] (see the proof of Lemma 14). In each case, we take $c_{\dot\psi} = \Lambda'_{\pi(\mathrm{T},\psi)}$, which is a fourth root of unity independent of the ψ that extends $\dot\psi$, by Lemma 14. Moreover, for any ψ extending $\dot\psi$, the term $c_0(\pi(\mathrm{T},\psi))$ from the formulas of [ADSS11] equals either $-q^s$ (when $\theta = \epsilon$) or $-(1/2)(q+1)q^{s-1/2}$ (when $\theta \in \{\varpi, \epsilon\varpi\}$) — in each case, $c_0(\pi(\mathrm{T},\psi))$ depends only on s, which in turn depends only on $\dot\psi$.

Now all the three assertions of the lemma, except possibly for the assertions concerning the 'bad shell' character values on ${}^{\mathrm{GL}_2(F)}((\mathrm{T}^{\varpi}_s \setminus \mathrm{T}^{\varpi}_{s+}) \cup (\mathrm{T}^{\epsilon\varpi}_s \setminus \mathrm{T}^{\epsilon\varpi}_{s+}))$ when T is ramified, follows by inspection from the character formulas summarized in [ADSS11, Section 1], the above observations, and the values of σ from (8).

As far as the bad shell with T ramified is concerned, it remains to see that $\psi_1(\gamma') = \psi_2(\gamma')$, where $\psi_1, \psi_2 \in \widehat{\mathrm{T}(F)}$ extend $\dot\psi$, and γ' belongs to what [ADSS11] writes as $(C_\varpi)_{s:s+}$ (if $\mathrm{T} = \mathrm{T}^\varpi$) or $(C_{\epsilon\varpi})_{s:s+}$ (if $\mathrm{T} = \mathrm{T}^{\epsilon\varpi}$). Here, the notation $(C_\varpi)_{s:s+}$ and $(C_{\epsilon\varpi})_{s:s+}$ are as defined just before [ADSS11, Section 5.2], and the terms $\psi_1(\gamma')$ and $\psi_2(\gamma')$ are interpreted in the manner following the statement of Theorem 14.19 in [ADSS11], namely, transferring ψ_1, ψ_2 from $\mathrm{T}(F)$ to groups $C_\varpi = E^1_{\mathrm{T}^\varpi}$ or $C_{\epsilon\varpi} = E^1_{\mathrm{T}^{\epsilon\varpi}}$ via a particular isomorphism (the one that appeared in the proof of Lemma 10) between these groups. Then one can check that the equality $\psi_1(\gamma') = \psi_2(\gamma')$ follows from the fact that $\psi_1|_{\mathrm{T}_s} = \psi_2|_{\mathrm{T}_s} = \dot\psi|_{\mathrm{T}_s}$, together with Remark 5. □

Remark 19. Together, (i) and (ii) of Lemma 18 imply the following. Let $\mathrm{T} \subset \mathrm{G}$ be a maximal torus, $t \in {}^{\mathrm{G}(F)}\mathrm{T}(F) \cup \mathrm{G}_{r+} \cup -\mathrm{G}_{r+}$, and $\mathbb{1} \neq \dot\psi \in \hat{\mathrm{T}}_{r+}$. It will be convenient to make an artificial device of choosing $t' \in \mathrm{T}(F)$ to be $1, -1$ or an arbitrary element of ${}^{\mathrm{G}(F)}\{t\} \cap \mathrm{T}(F)$ depending on whether t belongs to $\mathrm{G}_{r+}, -\mathrm{G}_{r+}$ or ${}^{\mathrm{G}(F)}(\mathrm{T}(F) \setminus (\mathrm{T}_{r+} \cup -\mathrm{T}_{r+}))$, respectively. Then by Lemma 18 (i) and (ii), the function $\psi \mapsto \Theta_{\pi(\mathrm{T},\psi)}(t)$ on the set

$$\mathrm{T}_{\dot\psi} := \{\psi \in \widehat{\mathrm{T}(F)} \mid \psi|_{\mathrm{T}_{r+}} = \dot\psi\},$$

is a scalar multiple (depending on t) of the function:

$$\psi \mapsto \sum_{w \in W_{\mathrm{T}}} (\psi \circ \mathrm{Int}\, w)(t').$$

Remark 20. Let us record the following consequence. Lemma 18 gives that the elements of $\Pi(\mathrm{T}, \dot\psi)$ all have the same, nontrivial, character expansion (the lemma addresses only the tempered representations among them, but the nontempered case arises only when T is split, in which case the same proof applies). Hence, by [MW87], for a given Whittaker datum either they are all generic or they are all nongeneric. This last fact can also be seen from the equality of the various $H(\Lambda', E_{\mathrm{T}^\theta})$ that contribute (Lemma 14), together with the Labesse-Langlands character identities as discussed in Lemma 21 below.

Now let us deduce the result about the realization of $\pi(\mathrm{T},\psi)$ as a Weil representation, that was alluded to earlier.

Lemma 21. *Let* T *be an elliptic maximal torus of* G*, and* $\psi \in \widehat{\mathrm{T}(F)}$ *a character of positive depth* s*. Then, viewing* T *also as an endoscopic group of* G*,* $\pi(\mathrm{T}, \psi)$ *occurs as an endoscopic transfer from the character* $\psi\alpha$ *of* $\mathrm{T}(F)$*, where* $\alpha = \alpha_{\mathrm{T}}$ *is as mentioned just before Remark 13.*

Proof. We will be terse, since this result should be well-known thanks to the compatibility between the parametrizations of [ADSS11] and [SJS68] (see Remark 13), and since this result is needed only for the explanation in Remark 45.

Since the character of π vanishes on $\mathrm{T}'(F) \setminus \mathrm{T}'_{0+}$ for maximal tori $\mathrm{T}' \subset \mathrm{G}$ that are not isomorphic to T (by the formulas of [ADSS11]), we know that π arises by endoscopic transfer from some character λ of $\mathrm{T}(F)$ — here λ is not uniquely determined, but the set $\{\lambda, \lambda^{-1}\}$ is. We will show that λ can be chosen to equal $\psi\alpha_{\mathrm{T}}$. Write α for α_{T}.

Since L-packets are invariant under $\mathrm{GL}_2(F)$-conjugation, and since $\pi(\mathrm{T}, \psi) \circ \operatorname{Int} g = \pi(g^{-1}\mathrm{T}g, \psi \circ \operatorname{Int} g)$ (Remark 16(iv)), we may assume that $\mathrm{T} = \mathrm{T}^\theta$ for some $\theta \in \{\epsilon, \varpi, \epsilon\varpi\}$. Further, we identify $\mathrm{T}(F) = E^1$ with $E = E_{\mathrm{T}}$, sending $\begin{pmatrix} a & b \\ b\theta & a \end{pmatrix}$ to $a + \sqrt{\theta}b$. Since $\psi\alpha$ is of positive depth, one knows that the L-packet of $\pi = \pi(\mathrm{T}, \psi\alpha)$ has exactly two elements, and takes the form $\{\pi, \pi \circ \operatorname{Int} g\}$, where g is an element of $\mathrm{GL}_2(F)$ such that $\pi \circ \operatorname{Int} g \ncong \pi$. Since $\pi \circ \operatorname{Int} g = \pi(g^{-1}\mathrm{T}g, \psi \circ \operatorname{Int} g)$, it follows that we could either take g such that $g^{-1}\mathrm{T}g$ is nonconjugate to T (in case T is unramified or -1 is a square in $\mathbb{F}_q^\times$) or such that $g^{-1}\mathrm{T}g = \mathrm{T}$ and $\psi \circ \operatorname{Int} g = \psi^{-1}$ (if T is ramified and -1 is not a square in $\mathbb{F}_q^\times$). We fix such a g.

In either case the formulas from the introduction of [ADSS11] (with the modification involving the quadratic character α mentioned just before Remark 13) give us the following formula for $\Theta_\pi - \Theta_{\pi\circ\operatorname{Int} g}$ on any given $\gamma \in \mathrm{T}(F) \setminus \mathrm{T}_s(F) \subset \mathrm{T}(F) = E^1$:

$$(\Theta_\pi - \Theta_{\pi\circ\operatorname{Int} g})(\gamma) = c \cdot \frac{\operatorname{sgn}_\theta\left((\gamma - \gamma^{-1})/2\sqrt{\theta}\right)}{D(\gamma)^{1/2}} \cdot (\psi\alpha(\gamma) + \psi\alpha(\gamma)^{-1}), \tag{11}$$

for some constant c (consider the case where T is ramified and $-1 \notin \mathbb{F}_q{}^\times$ separately; in that situation, given the choice of g, one is simply computing $\Theta_\pi(\gamma) - \Theta_\pi(\gamma^{-1})$).

On the other hand, the Labesse-Langlands character identities (see [Lan80, Lemma 7.19]) give that for $\gamma \in \mathrm{T}(F) \setminus \{\pm 1\}$:

$$(\Theta_\pi - \Theta_{\pi\circ\operatorname{Int} g})(\gamma) = c' \frac{\operatorname{sgn}_\theta\left((\gamma - \gamma^{-1})/2\sqrt{\theta}\right)}{D(\gamma)^{1/2}} (\lambda(\gamma) + \lambda(\gamma)^{-1}) \tag{12}$$

for some constant c'.

One way now is to compute that (11) applies for all $\gamma \in \mathrm{T}(F)$, and then combine with (12) and the linear independence of characters, but we wiill give an argument that does not need this computation. By (11) and (12), $\gamma \mapsto \psi\alpha(\gamma) + \psi\alpha(\gamma^{-1})$ and $\gamma \mapsto \lambda(\gamma) + \lambda(\gamma)^{-1}$ are scalar multiples of each other on $\mathrm{T}(F) \setminus \mathrm{T}_s$. Varying γ over a nonidentity coset of T_s in T_0, and using the linear independence of characters on T_s and the fact that $\psi\alpha|_{\mathrm{T}_s}$ is nontrivial, we may assume without loss of generality that $\psi\alpha|_{\mathrm{T}_s} = \lambda|_{\mathrm{T}_s}$. Again by the linear independence of characters, this forces that for all $\gamma \in \mathrm{T}(F) \setminus \mathrm{T}_s$, we have $\psi\alpha\lambda^{-1}(\gamma) = \psi\alpha\lambda^{-1}(\gamma)^{-1}$, so that $\psi\alpha(\gamma) = \pm\lambda(\gamma)$. By (11) and (12), given $\gamma \in \mathrm{T}(F) \setminus \mathrm{T}_s$, $\psi\alpha(\gamma)$ equals $\lambda(\gamma)$ (resp., $-\lambda(\gamma)$) if and only if $c = c'$ (resp., $c = -c'$): in seeing this, one may translate γ by an element of the pro-p group T_s, on which $\psi\alpha$ is nontrivial and agrees with λ, to assume that $\psi\alpha(\gamma) \neq \pm\sqrt{-1}$, so that $\psi\alpha(\gamma) + \psi\alpha(\gamma^{-1}) \neq 0$. Thus, either $\psi\alpha(\gamma) = \lambda(\gamma)$ for all $\gamma \in \mathrm{T}(F) \setminus \mathrm{T}_s$, or $\psi\alpha(\gamma) = -\lambda(\gamma)$ for all $\gamma \in \mathrm{T}(F) \setminus \mathrm{T}_s$. In either case, this identity in fact holds for all $\gamma \in \mathrm{T}(F)$, since $\mathrm{T}(F)$ is generated by $\mathrm{T}(F) \setminus \mathrm{T}_s$. Evaluating at 1, we see that $\psi\alpha(\gamma) = \lambda(\gamma)$ for all $\gamma \in \mathrm{T}(F)$, as desired.

□

3.3. **Some notation and review of results from** [SS84]**.** We will use the kernel computation in [SS84, Theorem 4.6] and the expression for the Fourier transform of an elliptic orbital integral in [SS84, Theorem 5.1] very crucially. Some caveat regarding statements in [SS84] has been mentioned in [ADSS11, Remark 14.15]. However, that remark also explains that the relevant discrepancies do not affect either of the theorems.

We now review some notation, observations and results from [SS84], and make a few remarks that will be needed later.

For a quadratic extension E/F, we will denote by RPS_E the (two element) set of irreducible components of the principal series representation $\mathrm{Ind}_{\mathrm{B}(F)}^{\mathrm{G}(F)} \mathrm{sgn}_{E/F}$. For a non-split maximal torus $\mathrm{T} \subset \mathrm{G}$, write $\mathrm{RPS}_{\mathrm{T}} = \mathrm{RPS}_{E_{\mathrm{T}}}$.

Remark 22. The elements of $\mathrm{RPS}_{\mathrm{T}}$ and their characters have been discussed in (iii) of page 311 in [SS84, Section 3], as we recall now.

For each $\pi \in \mathrm{RPS}_{\mathrm{T}}$, there is a root of unity ζ (a Weil constant), such that for $t \in \mathrm{G}(F)_{\mathrm{reg}}$:

$$\Theta_\pi(t) = \begin{cases} \mathrm{sgn}_{E_{\mathrm{T}}}(t')D(t')^{-1/2}, & \text{if } t \in {}^{\mathrm{G}(F)}\{t'\} \text{ for some } t' \in \mathrm{A}(F)_{\mathrm{reg}} = F^\times \setminus \{1,-1\}, \\ \zeta\varepsilon(t)D(t)^{-1/2}, & \text{if } t \in \mathrm{T}^\theta(F) \cup \mathrm{Int}\, g_\theta(\mathrm{T}^\theta(F)) \text{ (see just before Notation 9)}, \\ 0, & \text{otherwise.} \end{cases}$$

By definition (see Notation 9), σ equals ε on $\mathrm{T}^\theta(F)_{\mathrm{reg}}$, and when T^θ has a non-conjugate stable conjugate, that is to say when $\mathrm{sgn}_{F[\sqrt{\theta}]/F}(-1) = 1$, σ equals $-\varepsilon$ on $\mathrm{Int}\, g_\theta(\mathrm{T}^\theta(F))_{\mathrm{reg}}$. Combining this with the fact that $\bar{\zeta}\zeta = 1$, it follows that for $t_1, t_2 \in \mathrm{G}(F)_{\mathrm{reg}}$ belonging to elliptic maximal tori $\mathrm{T}^{(1)}, \mathrm{T}^{(2)}$ of G, and for $\pi \in \mathrm{RPS}_{\mathrm{T}}$:

$$\overline{\Theta_\pi(t_1)}\Theta_\pi(t_2) = \begin{cases} \sigma(t_1)\sigma(t_2), & \text{if } \mathrm{T}^{(1)} \cong \mathrm{T}^{(2)} \cong \mathrm{T} \text{ and } {}^{\mathrm{G}(F)}\mathrm{T}^{(1)} = {}^{\mathrm{G}(F)}\mathrm{T}^{(2)}, \\ -\sigma(t_1)\sigma(t_2), & \text{if } \mathrm{T}^{(1)} \cong \mathrm{T}^{(2)} \cong \mathrm{T} \text{ but } {}^{\mathrm{G}(F)}\mathrm{T}^{(1)} \neq {}^{\mathrm{G}(F)}\mathrm{T}^{(2)}, \text{ and }. \\ 0, & \text{if one of } \mathrm{T}^{(1)}, \mathrm{T}^{(2)} \text{ is not isomorphic to T.} \end{cases} \tag{13}$$

Remark 23. For a maximal torus $\mathrm{T} \subset \mathrm{G}$, and a positive integer $d > 0$, a subgroup of $\mathrm{T}(F)$ that we shall call the 'conductor d subgroup' or the 'conductor d part' of $\mathrm{T}(F)$ has been defined between (3.3) and (3.4) in [SS84] (the notations used in that paper being $\mathrm{A}_d, (\mathrm{T}_\theta)_d, (\mathrm{T}_\theta^\sharp)_d$ etc.). It is easy to see that this subgroup equals T_d if T is (split or) split over an unramified extension, and $\mathrm{T}_{(d+1/2)}$ otherwise (to see this, compare the description of Remark 5 with the description of what Sally and Shalika call $\mathrm{C}_\theta^{(h)}$ in the beginning of Section 3 of [SS84], and what they define as U_n towards the beginning of that paper). Note that in either case, we can write this subgroup as $\mathrm{T}_{(d-1/2)+}$ (and we informally think of d as $r + (1/2)$). Thus, when we use a result from [SS84], what that paper writes as T_d will be, for us, $\mathrm{T}_{(d-1/2)+}$.

Remark 24. The above consideration applies when d is a positive integer. When $d = 0$ and T is ramified elliptic, [SS84] defines a group that they denote by $(\mathrm{T})_0$; this is our T_{0+}. If T is unramified, they do not define such a group, but in their Theorem 5.1, for unramified T, they denote by $(\mathrm{T})_0$ the set (not group) of elements of $\mathrm{T}(F)$ outside $-\mathrm{T}_{0+}$. Note that this does have some similarity with the terminology when T is ramified, since for ramified T we have $\mathrm{T}(F) = \mathrm{T}_{0+} \cup -\mathrm{T}_{0+}$. Thus, if T is elliptic — ramified or unramified — we will refer to $\mathrm{T}(F) \setminus -\mathrm{T}_{0+}$ as the 'conductor 0 part' of $\mathrm{T}(F)$. Given t_0 in this set, it will be a candidate for application of [SS84, Theorem 5.1].

For any elliptic maximal torus T of G, define θ_{T} to be the unique element $\theta \in \{\epsilon, \varpi, \epsilon\varpi\}$ such that $\mathrm{T} \cong \mathrm{T}^\theta$. Recalling that $|\cdot|$ is the extension of the normalized valuation on F to $\bar{F}$, we have that $|\theta_{\mathrm{T}}|$ equals 1 or q^{-1}, depending on whether or not T is unramified.

For a positive integer d, a function Δ_d is defined around [SS84, (4.4)]: on each Cartan subgroup $\mathrm{T}(F)$ of $\mathrm{G}(F)$, it is simply the characteristic function of the 'conductor d subgroup' $\mathrm{T}_{(d-1/2)+}$ of $\mathrm{T}(F)$. Thus, we will rather take Δ_d to be the characteristic function of $\mathrm{G}_{(d-1/2)+}$ — this will be consistent with [SS84] as we will only need its values on semisimple elements.

In the statement of [SS84, Lemma 2.5] is defined a constant κ_{T} for each elliptic maximal torus T of G: it equals $(q+1)/q$ if T is unramified, and $2q^{-1/2}$ if T is ramified. We will need to use the following relation: for every elliptic maximal torus $\mathrm{T} \subset \mathrm{G}$:

$$\kappa_{\mathrm{T}}|\theta_{\mathrm{T}}|^{-1/2}q^d = [\mathrm{T}(F) : \mathrm{T}_{(d-1/2)+}], \tag{14}$$

which, as observed towards the end of [SS84, Section 4], follows from [SS84, (3.1) and (3.2)].

Let $D \subset \hat{\mathrm{G}}_t$ denote the subset consisting of the discrete series representations. One can now check, using the description of the discrete series representations of $\mathrm{G}(F)$ given in (ii) of [SS84, Section 3], that for a positive integer d the function K_d given in [SS84, (4.1)] is such that for $t_1, t_2 \in \mathrm{G}(F)_{\mathrm{reg}}$:

$$K_d(t_1, t_2) = \sum_{\substack{\pi \in D \\ \mathrm{depth}(\pi) \leq d-1/2}} \Theta_\pi(t_1)\overline{\Theta_\pi(t_2)} \tag{15}$$

(this need a bit of careful checking; for instance, the factor 1/2 in [SS84, (4.1)] has been put in to account for the isomorphisms (2) in (iib) of [SS84, Section 3], and the discussion on ramified representations in [SS84, Section 3] only includes the positive depth ones as the depth zero ones are all accounted for among the unramified ones). Here, to relate the depth of a representation $\pi \in D$ to its conductor in the sense of [SS84], use Remark 16(i) together with the compatibility between the parametrizations of [ADSS11] and [SJS68] (see Remark 13; the characters α_{T} are always of depth zero); in [SS84], their notion of conductor has been spelled out shortly before (3.1) of that reference.

The following easy observations are [SS84, (4.2) and (4.3)]: for $t_1, t_2 \in \mathrm{G}(F)_{\mathrm{reg}}$:

$$K_d(-t_1, -t_2) = K_d(t_1, t_2), \qquad \text{and} \qquad K_d(t_2, t_1) = \overline{K_d(t_1, t_2)}. \tag{16}$$

We now recall [SS84, Theorem 5.1], which will be of much importance to us.

Theorem 25 (Theorem 5.1 from [SS84]). *Suppose* $\mathrm{T}^{(1)} \subset \mathrm{G}$ *is an elliptic maximal torus, and* $t_1 \in \mathrm{T}^{(1)}(F)_{\mathrm{reg}}$. *Assume* $t_1 \notin -\mathrm{T}^{(1)}_{0,+}$. *Let* h_1 *be the smallest positive integer such that* $t_1 \in \mathrm{T}^{(1)}_{(h_1-1/2)+}$, *if it exists, and* 0 *otherwise. Then for* $f \in C_c^\infty(\mathrm{G}(F))$:

$$\begin{aligned} O(t_1, f) = &\sum_{\pi \in D} \overline{\Theta_\pi(t_1)}\,\mathrm{tr}\,\pi(f) + \frac{1}{2}\sum_{\pi \in \mathrm{RPS}_{\mathrm{T}^{(1)}}} \overline{\Theta_\pi(t_1)}\,\mathrm{tr}\,\pi(f) \\ &- \frac{q+1}{2q}\,\mathrm{meas}(\mathrm{A}_1)\int_{\substack{\psi \in \widehat{F^\times} \\ \psi|_{\mathrm{A}_{h_1+1}} = \mathbb{1}}} |\Gamma(\psi)|^{-2}\,\mathrm{tr}\left(\mathrm{Ind}_B^{\mathrm{G}}\,\psi(f)\right)\,d\psi \\ &+ \frac{q}{2}\,\mathrm{meas}(\mathrm{A}_1)\kappa_{\mathrm{T}^{(1)}}|D(t_1)|^{-1/2}\int_{\substack{\psi \in \widehat{F^\times} \\ \psi|_{\mathrm{A}_{h_1+1}} = \mathbb{1}}} \mathrm{tr}\left(\mathrm{Ind}_B^{\mathrm{G}}\,\psi(f)\right)\,d\psi. \end{aligned}$$

Remark 26. To justify that we can use the above expression as such from [SS84], we need to check that the relevant choices of measures in [SS84] are consistent with ours (as fixed in Section 2.11). This involves checking three compatibilities. The first is that the definition of the term denoted '$I_f^{\mathrm{T}_0}(t_0)$' at the beginning of [SS84, Section 5] agrees with the definition of our orbital integral; this is satisfied as we give elliptic Cartan subgroups measure 1. The second is that changing the measure on $\mathrm{G}(F)$ multiplies all the terms above by the same constant, and hence doesn't affect the equation. The third is that, having chosen a measure on $\mathrm{G}(F)$, the right-hand side of the equation is entirely pinned down by the requirement that the measure used on $\widehat{F^\times} = \widehat{\mathrm{A}(F)}$ is dual to the measure used on $\mathrm{A}(F)$ — we are following this convention, and [SS84] has imposed it in their statement of the theorem, i.e., in [SS84, Theorem 5.1].

Remark 27. To proceed, let us make a small observation to eliminate certain cases related to Remark 24: if δ_{-1} is the Dirac delta measure at $-1 \in \mathrm{G}(F)$, then for all $z \in \mathcal{Z}(\mathrm{G})$ and $f \in C_c^\infty(\mathrm{G}(F))$, $O_{-\gamma}(z * \delta_{-1} * f) = O_\gamma(z * f)$. This has the following consequence. Let $\mathcal{U}$ be a part of the decomposition (3). Consider the following assertion: for all $f \in C_c^\infty(\mathrm{G}(F))$ such that the $O(t_2, f) = 0$ for all $t_2 \in \mathcal{U}_{\mathrm{reg}}$, we have $O(t_1, z*f) = 0$ for all $t_1 \in \mathcal{U}_{\mathrm{reg}}$. This assertion is equivalent to the assertion that for all $f \in C_c^\infty(\mathrm{G}(F))$ such that $O(t_2, f) = 0$ for all $t_2 \in -\mathcal{U}_{\mathrm{reg}}$, we have $O(t_1, z * f) = 0$ for all $t_1 \in -\mathcal{U}_{\mathrm{reg}}$.

3.4. **Review of facts, and some observations, concerning Fourier transforms of orbital integrals.**

Remark 28. We will be studying (Borel) measures on $\hat{\mathrm{G}}_t$, for which we will need to know a bit about the topology on $\hat{\mathrm{G}}_t$. Let us informally summarize (more than) what we need, referring those who wish a precise enunciation to work it out from [Tad82, Section 5]. Each discrete series representation in $\hat{\mathrm{G}}_t$ is both open and closed. The set of representations in $\hat{\mathrm{G}}_t$ that do not belong to the discrete series accepts a multivalued parabolic induction map, single-valued except at nontrivial quadratic characters, from $\widehat{F^\times} = \widehat{\mathrm{A}(F)}$. Recall from Section 2.11 that $\widehat{F^\times}$ was topologized by identifying it with a disjoint union of copies of S^1. Around those points where the map is single-valued, $\hat{\mathrm{G}}_t$ has the finest topology that makes this map continuous (and hence it is a local isomorphism around all nonquadratic nontrivial characters, and a quotient map around the trivial character). At each point where the map is two-valued, corresponding to a nontrivial quadratic character $\mathrm{sgn}_{E/F} \in \widehat{\mathrm{A}(F)}$, E a quadratic extension of F, we realize the image of the corresponding copy of S^1 as S^1 with a 'doubled point', the copies of the doubled point corresponding to the members of the two element set RPS_E defined just before Remark 22.

Remark 29. In what follows, we will abuse notation to mean, by a 'complex measure on $\hat{\mathrm{G}}_t$', a datum consisting of a complex Borel measure on each irreducible component of $\hat{\mathrm{G}}_t$; it may not be a complex measure on $\hat{\mathrm{G}}_t$ as it may not have finite total variation, but we will integrate against it only functions that are supported on finitely many components of $\hat{\mathrm{G}}_t$.

The following lemma is a variant of some of the results discussed in [MT02], such as [MT02, Theorem 2.5(ii)].

Lemma 30. *Let $t_1 \in \mathrm{G}(F)_{\mathrm{reg}}$. Then there exists a complex measure $d\mu_{t_1}$ on $\hat{\mathrm{G}}_t$ with the following properties.*
(a) For all $f \in C_c^\infty(\mathrm{G}(F))$, we have

$$O(t_1, f) = \int_{\pi \in \hat{\mathrm{G}}_t} \operatorname{tr} \pi(f) \cdot d\mu_{t_1}.$$

(b) If $z \in \mathcal{Z}(\mathrm{G})$ is supported on finitely many Bernstein components, then the function on $\mathrm{G}(F)_{\mathrm{reg}}$ given by

$$t_2 \mapsto D(t_2)^{1/2} \int_{\pi \in \hat{\mathrm{G}}_t} z(\pi) \Theta_\pi(t_2)\, d\mu_{t_1}$$

is locally bounded on $\mathrm{G}(F)$.
(c) If $z \in \mathcal{Z}(\mathrm{G})$ is supported on finitely many Bernstein components, then the function on $\mathrm{G}(F)_{\mathrm{reg}}$ given by

$$g \mapsto \int_{\pi \in \hat{\mathrm{G}}_t} z(\pi) \Theta_\pi(g)\, d\mu_{t_1}$$

is locally integrable on $\mathrm{G}(F)$, and represents the distribution $f \mapsto O(t_1, z * f)$.

Proof. The existence of a $d\mu_{t_1}$ satisfying (a) is well-known; in our case, it follows from Theorem 25 when t_1 is elliptic, and can be constructed using the constant term with respect to B and Pontrjagin duality on $\mathrm{A}(F)$ (more precisely, using (18) below) when t is split. Here, in the elliptic case, to make sure that Theorem 25 applies, note that we may replace t_1 by $-t_1$ if necessary, and make the considerations of Remark 27: the relation $O_{-t_1}(f) = O_{t_1}(\delta_{-1} * f)$ $(f \in C_c^\infty(\mathrm{G}(F)))$ implies that on $\hat{\mathrm{G}}_t$, the complex measure $d\mu_{-t_1}$ can be taken to be the product of the complex measure $d\mu_{t_1}$ with the function on $\hat{\mathrm{G}}_t$ that takes π to $\chi_\pi(-1)$, χ_π being the central character of π. This yields us a particular $d\mu_{t_1}$, and we will show below that it satisfies (b) and (c).

We will use the fact that the set of normalized tempered irreducible characters that belong to a finite set of Bernstein components on $\mathrm{SL}_2(F)$ is uniformly locally bounded: i.e., if Ω is a finite set of Bernstein components of $\mathrm{G}(F)$, then given any $g_0 \in \mathrm{G}(F)$, there exists a neighborhood U of g_0 and a constant $c > 0$ such that $|D(g)^{1/2}\Theta_\pi(g)| < c$ for all $g \in U_{\mathrm{reg}}$ and all $\pi \in \hat{\mathrm{G}}_t$ that belongs to Ω. To check this fact, it suffices to do so separately on tempered principal series characters, the Steinberg character, and supercuspidal characters. The cases of tempered principal series characters and the Steinberg character are easy using the formula for induced characters (Remark 17), while the supercuspidal case, where Bernstein components are singleton, is an immediate consequence of the general fact that normalized characters $g \mapsto D(g)^{1/2}\Theta_\pi(g)$ are locally bounded on $\mathrm{G}(F)$; in any case, one can see this directly from the character formulas of [ADSS11] or [SJS68].

Assume that there exists a finite measure space $(X, d\mu')$ and a finite-to-one measurable function $\iota : X \to \hat{\mathrm{G}}_t$, with image contained in finitely many Bernstein components, such that the complex measure $z(\pi)d\mu_{t_1}$ on $\hat{\mathrm{G}}_t$ is the push forward of $x d\mu'$ with respect to ι, for some bounded measurable complex function $x \in L^1(X, \mu')$. Under this assumption, using the 'uniform local boundedness' result recalled in the above paragraph, it is easy to see that the integral of (b) is (absolutely) dominated by one that converges to a locally bounded function on $\mathrm{G}(F)$. The discussion of Section 2.10 will then yield the first assertion of (c), and given these estimates, we will be done by the observation that for $z \in \mathcal{Z}(\mathrm{G})$, we have:
(17)

$$O(t_1, z * f) = \int_{\pi\in\hat{\mathrm{G}}_t} \mathrm{tr}\,\pi(z * f) \cdot d\mu_{t_1} = \int_{\pi\in\hat{\mathrm{G}}_t} z(\pi)\,\mathrm{tr}\,\pi(f) \cdot d\mu_{t_1} = \int_{\pi\in\hat{\mathrm{G}}_t} z(\pi) \cdot \left(\int_{\mathrm{G}(F)} \Theta_\pi(g) f(g)\,dg\right) d\mu_{t_1}.$$

Thus, it is enough to prove the existence of $X, d\mu', \iota : X \to \hat{\mathrm{G}}_t$ and x as above.

First suppose t_1 belongs to a split maximal torus of $\mathrm{G}(F)$, which we may and do assume to be A. Let $f^{(\mathrm{B})} \in C_c^\infty(\mathrm{A}(F))$ denote the constant term of any $f \in C_c^\infty(\mathrm{G}(F))$ along B (e.g., see [vD72], just before Lemma 9, where the constant term of f along a parabolic subgroup $P = MN$ is denoted by g_f). Applying [vD72, Lemma 9] and [vD72, Theorem 2], we get:

$$D(t_1)^{1/2} O(t_1, f) = f^{\mathrm{B}}(t_1) = \int_{\widehat{F^\times}} \psi(f^{(\mathrm{B})})\overline{\psi(t_1)}\,d\psi = \int_{\widehat{F^\times}} (\mathrm{tr}\,\mathrm{Ind}_{\mathrm{B}}^{\mathrm{G}}\,\psi)(f) \cdot \overline{\psi(t_1)}\,d\psi \tag{18}$$

(alternatively we could use the Weyl integration formula as in [MT05, (4.2a)]). One may restrict this integral so as to be over only nonquadratic ψ, so that $\mathrm{Ind}_{\mathrm{B}}^{\mathrm{G}}\,\psi = \pi_\psi$ is irreducible. Note that our $d\mu_{t_1}$ is just the push forward, with respect to ι, of $D(t_1)^{-1/2}\overline{\psi(t_1)}$ under the partially defined map $\widehat{F^\times} \to \hat{\mathrm{G}}_t$ given by $\psi \mapsto \mathrm{Ind}_{\mathrm{B}}^{\mathrm{G}}\,\psi =: \pi_\psi$ (ignoring the quadratic ψ). One takes X to be the set of nonquadratic characters $\psi \in \widehat{F^\times}$ such that z does not annihilate the Bernstein component of $\mathrm{Ind}_{\mathrm{B}}^{\mathrm{G}}\,\psi$. $\iota : X \to \hat{\mathrm{G}}_t$ is defined to be $\psi \mapsto \pi_\psi$. Then, since z is supported on finitely many Bernstein components, X is a cofinite subset of a union of finitely many copies of S^1 (where S^1 is the unit circle in the complex plane), each of which we give the usual (normalized Haar) measure from S^1. This determines $d\mu'$. Finally, set $x(\psi) = D(t_1)^{-1/2} z(\pi_\psi)\overline{\psi(t_1)}$. Then x is continuous and bounded. This takes care of the case when t_1 is split.

Now suppose that t_1 belongs to an elliptic maximal torus of $\mathrm{G}(F)$. Recall that in this case we replace t_1 by $-t_1$ if necessary so that $t_1 \notin -\mathrm{G}_{0+}$, so that Theorem 25 applies to it; if we could attach a tuple $(X, d\mu', \iota, x)$ as above to $d\mu_{t_1}$, then $(X, d\mu', \iota, x')$ does the job for $d\mu_{-t_1}$, where x' is the product of x with the continuous bounded map $\chi_{\iota(\cdot)}(-1)$ on X, so that x' is again bounded and in $L^1(X, \mu')$.

Let us then give the prescriptions for $X, d\mu', \iota$ and x using notation from [SS84] reviewed earlier.

Let Ω_z be the set of finite length representations of $\mathrm{G}(F)$ that belong to Bernstein components that are not annihilated by z. Let $\mathrm{T}^{(1)}$ be the centralizer of t_1. Let h_1 be the smallest positive integer such that $t_1 \in \mathrm{T}^{(1)}_{(h_1-1/2)+}$, if it exists, and 0 if such a positive integer does not exist. Thus, h_1 takes the role of what it denotes in Theorem 25 (see Remarks 23 and 24). Define X to be:

$$(D \cap \Omega_z) \sqcup (\mathrm{RPS}_{\mathrm{T}^{(1)}} \cap \Omega_z) \sqcup Y_1 \sqcup Y_2,$$

where Y_1, Y_2 are copies of

$$Y = \{\psi \in \widehat{F^\times} \mid \psi|_{\mathrm{A}_{h_1+1}} = 1, \psi^2 \neq 1, \mathrm{Ind}_{\mathrm{B}}^{\mathrm{G}} \psi \in \Omega_z\}$$

(recall that A_{h_1+1} is the h_1+1-indexed Moy-Prasad filtration subgroup). Note that Y is a cofinite subset of a disjoint union of copies of S^1, each cofinite subset of a copy of S^1 consisting of characters whose restrictions to A_0 coincide. The map $\iota : X \to \hat{\mathrm{G}}_t$ takes each $\pi \in (D \cap \Omega_z) \sqcup (\mathrm{RPS}_{\mathrm{T}^{(1)}} \cap \Omega_z)$ to $\pi \in \hat{\mathrm{G}}_t$, and each ψ belonging to Y_1 or Y_2 to $\mathrm{Ind}_{\mathrm{B}}^{\mathrm{G}} \psi$. Note that ι is continuous.

$d\mu'$ then is defined to restrict to the counting measure on $(D \cap \Omega_z) \sqcup (\mathrm{RPS}_{\mathrm{T}^{(1)}} \cap \Omega_z)$, and to the normalized Haar measure on each copy of S^1 mentioned above. One sets $x(\pi)$ to be $z(\pi)\overline{\Theta_\pi(t_1)}$ if $\pi \in D \cap \Omega_z$ and to be $(1/2)z(\pi)\overline{\Theta_\pi(t_1)}$ for $\pi \in \mathrm{RPS}_{\mathrm{T}^{(1)}}$. One sets

$$x(\psi) = \begin{cases} -z(\pi_\psi) \cdot \frac{q+1}{2q} \cdot \mathrm{meas}(\mathrm{A}_1)|\Gamma(\psi)|^{-2}, & \text{if } \psi \in Y_1, \text{ and} \\ z(\pi_\psi) \cdot \frac{q}{2} \,\mathrm{meas}(\mathrm{A}_1)\kappa_{\mathrm{T}_1}|D(t_1)|^{-1/2}, & \text{if } \psi \in Y_2. \end{cases}$$

That x is bounded and continuous is immediate, once one uses the fact that $\psi \mapsto |\Gamma(\psi)|^{-2}$ is a continuous function on $\widehat{F^\times}$ thanks to [ST66, Theorem 1] (the function Γ of [SS84], used in Theorem 25 above, was sourced from [ST66]). Now Theorem 25 tells us that (X, μ', ι, x) does the job.

□

Remark 31. Henceforth, for any $t_1 \in \mathrm{G}(F)_{\mathrm{reg}}$, $d\mu_{t_1}$ will stand for the complex measure on $\hat{\mathrm{G}}_t$ constructed in the proof of the above lemma. Note that for all $g \in \mathrm{G}(F)$, $d\mu_{t_1} = d\mu_{gt_1g^{-1}}$.

Remark 32. The above lemma (or at least a variant that suffices for us), for a general reductive group in place of our SL_2, should follow from some of the results in [Art94], though one will need to take care of a small complication, one that does not arise while dealing with distributions $f \mapsto z(f)$ ($z \in \mathcal{Z}(\mathrm{G})$) in place of $f \mapsto O_{t_1}(z * f)$ — the 'uniform local boundedness' will need to apply to all irreducible tempered representations belonging to a given Bernstein component, and not just those that are fully induced from a discrete series representation. In the SL_2-case being considered here, this is not a problem since there are only finitely many (isomorphism classes) of reducible principal series representations $\mathrm{RPS}_{\mathrm{T}^{(1)}}$.

Lemma 30 has the following corollary:

Corollary 33. *Suppose $\mathcal{U} \subset \mathrm{G}(F)$ is $\mathrm{G}(F)$-conjugation invariant, open and closed.*

(i) Suppose $z \in \mathcal{Z}(\mathrm{G})$ is supported on only finitely many Bernstein components and satisfies the property that for all $t_1 \in \mathcal{U}_{\mathrm{reg}}$, the function on $\mathrm{G}(F)_{\mathrm{reg}}$ given by

$$t_2 \mapsto \int_{\pi \in \hat{\mathrm{G}}_t} z(\pi)\Theta_\pi(t_2)\, d\mu_{t_1} \tag{19}$$

is supported on $\mathcal{U}_{\mathrm{reg}}$. Then, for all $f \in C_c^\infty(\mathrm{G}(F))$ such that $O(t, f) = 0$ for all $t \in \mathcal{U}_{\mathrm{reg}}$, $O(t, z * f) = 0$ for all $t \in \mathcal{U}_{\mathrm{reg}}$.

(ii) Assume further that $\mathcal{U}$ is of the form ${}^{\mathrm{G}(F)}(\gamma \mathrm{T}_{r+})$, for some $\gamma \in \mathrm{T}(F) \setminus (\mathrm{T}_{r+} \cup -\mathrm{T}_{r+})$, where $\mathrm{T} \subset \mathrm{G}$ is a maximal torus. Suppose that for all $t_1 \in \mathrm{T}(F) \cap \mathcal{U}_{\mathrm{reg}}$, the function on $\mathrm{G}(F)_{\mathrm{reg}}$ given by (19) is supported on $\mathcal{U}_{\mathrm{reg}}$. Define $\varphi \in C_c^\infty(\mathrm{T}_{r+}) \subset C_c^\infty(\mathrm{T}(F))$ by requiring that the restriction of (19)

to $t_1 \cdot \mathrm{T}_{r+} \subset \mathrm{T}(F)_{\mathrm{reg}}$ *is given by* $t_2 \mapsto D(t_2)^{-1/2} D(t_1)^{-1/2} \varphi(t_2^{-1} t_1)$. *Then for all* $t_1 \in \mathcal{U} \cap \mathrm{T}(F) = \mathcal{U}_{\mathrm{reg}} \cap \mathrm{T}(F)$ *and* $f \in C_c^\infty(\mathcal{U})$:

$$\varphi_{z*f}(t_1) = (\varphi * \varphi_f)(t_1),$$

where φ_f *and* φ_{z*f} *are defined as in the introduction (sending* $t \in \mathrm{T}(F) \backslash (\mathrm{T}_{r+} \cup -\mathrm{T}_{r+})$ *to* $D(t)^{1/2} O(t, f)$ *and* $D(t)^{1/2} O(t, z * f)$*, respectively).*

Proof. As in (17) (which is in the proof of Lemma 30), we have:
(20)

$$O(t_1, z * f) = \int_{\pi \in \hat{\mathrm{G}}_t} z(\pi) \operatorname{tr} \pi(f) \cdot d\mu_{t_1} = \int_{\pi \in \hat{\mathrm{G}}_t} z(\pi) \left(\sum_{\mathrm{T} \in \mathcal{T}} \frac{1}{\# W_{\mathrm{T}}} \int_{\mathrm{T}(F)_{\mathrm{reg}}} D(t_2) \cdot \Theta_\pi(t_2) O(t_2, f) \, dt_2 \right) d\mu_{t_1}$$

(use the local boundedness of normalized characters and the discussion of Section 2.10). In the proof of Lemma 30, we showed that for any maximal torus $\mathrm{T} \subset \mathrm{G}$ and any $t \in \mathrm{T}(F)_{\mathrm{reg}}$, the integral in (b) of that lemma was dominated by an integral that, as a function of t, converged to an expression locally bounded on $\mathrm{T}(F)$. Since $t \mapsto D(t)^{1/2} O(t, f)$ is also locally bounded on $\mathrm{T}(F)$, and with relatively compact support, it follows that the integral over $\hat{\mathrm{G}}_t$ in the right-most term of (20) can be moved inside both the sum over T and the integral over $\mathrm{T}(F)_{\mathrm{reg}}$, so that we get:

$$O(t_1, z * f) = \sum_{\mathrm{T} \in \mathcal{T}} \frac{1}{\# W_{\mathrm{T}}} \int_{\mathrm{T}(F)_{\mathrm{reg}}} \left(\int_{\pi \in \hat{\mathrm{G}}_t} D(t_2)^{1/2} \cdot z(\pi) \Theta_\pi(t_2) d\mu_{t_1} \right) \cdot D(t_2)^{1/2} O(t_2, f) \, dt_2.$$

From this, (i) follows immediately.

We move to (ii), where T is fixed, and $\mathcal{U} \subset {}^{\mathrm{G}(F)}\mathrm{T}(F)_{\mathrm{reg}}$. For $w \in W_{\mathrm{T}}$, since $d\mu_{t_1} = d\mu_{\operatorname{Int} w(t_1)}$ (see Remark 31), the expression (19) does not change if t_1 is replaced by $\operatorname{Int} w(t_1)$. Since $\gamma \in \mathrm{T}(F) \setminus (\mathrm{T}_{r+} \cup -\mathrm{T}_{r+})$, $\mathcal{U} \cap \mathrm{T}(F) = \mathcal{U}_{\mathrm{reg}} \cap \mathrm{T}(F)$ is the disjoint union of the $w(\gamma \mathrm{T}_{r+})$ as w runs over W_{T}. Further, φ is invariant under W_{T} (as seen by simultaneously replacing t_1 and t_2 by their w-conjugates in the definition of φ). Hence $\varphi_{z*f}(t_1)$ equals:

$$\begin{aligned} D(t_1)^{1/2} O(t_1, z * f) &= \frac{1}{\# W_{\mathrm{T}}} \cdot \sum_{w \in W_{\mathrm{T}}} \int_{\operatorname{Int} w(t_1) \mathrm{T}_{r+}} \left(\int_{\pi \in \hat{\mathrm{G}}_t} D(t_2)^{1/2} D(t_1)^{1/2} \cdot z(\pi) \Theta_\pi(t_2) d\mu_{\operatorname{Int} w(t_1)} \right) D(t_2)^{1/2} O(t_2, f) \, dt_2 \\ &= \frac{1}{\# W_{\mathrm{T}}} \cdot \sum_{w \in W_{\mathrm{T}}} \int_{t_1 \mathrm{T}_{r+}} \varphi(\operatorname{Int} w(t_2^{-1} t_1)) D(\operatorname{Int} w(t_2))^{1/2} O(\operatorname{Int} w(t_2), f) \, dt_2 \\ &= (\varphi * \varphi_f)(t_1). \end{aligned}$$

□

Remark 34. The theme alluded to in (a) is treated using a slightly different language in [Dat03] and [BK16], as we now explain. Note that the condition being discussed in (a) amounts to asserting that in the cocenter of G, i.e., in the $\mathbb{C}$-vector space of coinvariants of $C_c^\infty(\mathrm{G}(F))$ under the conjugation action of $\mathrm{G}(F)$, the following endomorphisms commute: the one induced by convolution with z on $C_c^\infty(\mathrm{G}(F))$, and the one induced by the "multiplication with $\mathbb{1}_{\mathcal{U}}$" self-map of $C_c^\infty(\mathrm{G}(F))$. This explains the connection between the question above and, say, one of the considerations of [BK16, Theorem 2.2].

Remark 35. When π is a discrete series representation and t_1 is an element of an elliptic Cartan subgroup of $\mathrm{G}(F)$, $d\mu_{t_1}$ evaluated on $\{\pi\}$ equals, up to a constant, $\Theta_{\pi^\vee}(t_1) = \overline{\Theta_\pi(t_1)}$ (this follows from the Selberg principle; alternatively, Theorem 25 recalled from [SS84] which gives much more). Arthur's beautiful paper [Art94] gives a variant when π does not belong to the discrete series, in terms of the invariant distribution associated to a weighted character of π. In addition, the elements $z \in \mathcal{Z}(\mathrm{G})$ of interest to us are projectors. Hence, it would be interesting if there is an interpretation for the vanishing of (19) as a form of of Schur's orthogonality relations adapted to special elements of the Bernstein center that we will be considering, noting the use of this terminology by Sally and Shalika at the beginning of [SS84, Section 4].

3.5. **Support and behavior on orbital integrals.** The following lemma addresses (ii) of the strategy that was described below the statement of Theorem 2 (for the proof one could also use the density of regular semisimple orbital integrals in place of Shalika germs).

Lemma 36. *Suppose $z \in \mathcal{Z}(\mathrm{G})$ satisfies the property of Theorem 3: for all $f \in C_c^\infty(\mathrm{G}(F))$ such that $O(\gamma, f) = 0$ for $\gamma \in \mathrm{G}_{r+,\mathrm{reg}}$, $O(z * \gamma, f) = 0$ for all $\gamma \in \mathrm{G}_{r+,\mathrm{reg}}$. Then, as a distribution on $\mathrm{G}(F)$, z is supported on G_{r+}.*

Proof. This is a standard 'Shalika germs' argument. If $O(\gamma, f) = 0$ for all $\gamma \in \mathrm{G}_{r+,\mathrm{reg}}$, then $O(\gamma, \check{f}) = 0$ for all $\gamma \in \mathrm{G}_{r+,\mathrm{reg}}$, where $\check{f}$ is defined by $g \mapsto f(g^{-1})$. The property tells us that the orbital integral of $z * \check{f}$ at each $\gamma \in \mathrm{G}_{r+,\mathrm{reg}}$ is zero. Since the Shalika germs of regular semisimple elliptic elements near 1 for the trivial unipotent orbit are nonvanishing by [Rog81], it follows that $0 = z * \check{f}(1) = z(f)$. □

3.6. **Analogue of Theorem 4(i) for groups anisotropic modulo center.** We make an observation slightly more general than is necessary but hardly needs any extra effort:

Lemma 37. *Let H be a connected reductive group over F that is anisotropic modulo center, and suppose z is an element of its Bernstein center $\mathcal{Z}(\mathrm{H})$ that is supported in H_{r+}. Let $\gamma \in \mathrm{H}(F)$, and let $\mathcal{U} = {}^{\mathrm{H}(F)}(\gamma \mathrm{H}_{r+})$. Suppose $f \in C_c^\infty(\mathrm{G}(F))$ is such that $O(t, f) = 0$ for all regular semisimple $t \in \mathcal{U}$. Then $O(z * t, f) = 0$ for all regular semisimple $t \in \mathcal{U}$.*

Proof. Since H is anisotropic modulo its center, H_{r+} is a normal subgroup of $\mathrm{H}(F)$ — it is $\mathrm{H}_{x,r+}$ where x is the unique point of the reduced Bruhat-Tits building of H. It follows that every element of $\mathcal{Z}(\mathrm{H})$ supported in H_{r+} preserves $C_c^\infty(\gamma' \mathrm{H}_{r+}) \subset C_c^\infty(\mathrm{H}(F))$, for every $\gamma' \in \mathrm{H}(F)$.

Since $\mathcal{U}$ is a union of cosets of H_{r+}, this proves the lemma in the special case where f is identically zero on $\mathcal{U}$. The general case follows from the special case once we can show the existence of some $\varphi_1 \ldots, \varphi_r \in C_c^\infty(\mathrm{H}(F))$ and $h_1, \ldots, h_r \in \mathrm{H}(F)$ such that f equals $\sum(\varphi_i - \varphi_i \circ \operatorname{Int} h_i)$ on $\mathcal{U}$. Since both $\mathcal{U}$ and $\mathrm{H}(F) \setminus \mathcal{U}$ are open in $\mathrm{H}(F)$ (being unions of translates of H_{r+}), this follows from the well-known density of regular semisimple orbital integrals (see [KV16, Theorem B.1]). □

3.7. **Some consequences of character values for Fourier transforms of orbital integrals.**

Lemma 38. *Let $\mathrm{T} \subset \mathrm{G}$ be a maximal torus and $\mathbb{1} \neq \dot\psi \in \hat{\mathrm{T}}_{r+}$ a nontrivial character. Then the map $\iota_{\dot\psi} : \psi \mapsto \pi(\mathrm{T}, \psi)$ defines a homeomorphism from $\hat{\mathrm{T}}_{\dot\psi} := \{\psi \in \widehat{\mathrm{T}(F)} \mid \psi|_{\mathrm{T}_{r+}} = \dot\psi\}$ to $\Pi(\mathrm{T}, \dot\psi) \cap \hat{\mathrm{G}}_t$. Write $d\psi$ for the measure on $\widehat{\mathrm{T}(F)}$, and let it also denote its own restriction to the open subset $\hat{\mathrm{T}}_{\dot\psi} \subset \widehat{\mathrm{T}(F)}$. Choose $t_1' \in \mathrm{T}(F)$ as in Remark 19, i.e., to be $1, -1$ or an arbitrary element of ${}^{\mathrm{G}(F)}\{t_1\} \cap \mathrm{T}(F)$ in the cases where $t_1 \in \mathrm{G}_{r+}, t_1 \in -\mathrm{G}_{r+}$, and $t_1 \in {}^{\mathrm{G}(F)}(\mathrm{T}(F) \setminus (\mathrm{T}_{r+} \cup -\mathrm{T}_{r+}))$, respectively. Then the restriction of $d\mu_{t_1}$ to $\Pi(\mathrm{T}, \dot\psi) \cap \hat{\mathrm{G}}_t$ is the push forward along $\iota_{\dot\psi}$ of the following measure on $\hat{\mathrm{T}}_{\dot\psi}$:*

(i) the zero measure, if $t_1 \in \mathrm{G}(F) \setminus (\mathrm{G}_{r+} \cup -\mathrm{G}_{r+} \cup {}^{\mathrm{G}(F)}\mathrm{T}(F))$;

(ii) a constant (not necessarily nonzero) multiple of $(\psi \mapsto \overline{\sum_{w \in W_\mathrm{T}} \psi \circ \operatorname{Int} w(t_1')}) \cdot d\psi$, if $t_1 \in \mathrm{G}_{r+} \cup -\mathrm{G}_{r+} \cup \mathrm{T}(F)$; this constant being $\overline{c_{\dot\psi} \sigma(t_1) \alpha_\mathrm{T}(t_1)}$ if $t_1 \in \mathrm{T}(F) \setminus \mathrm{T}_{r+}$.

Proof. Note that $\dot\psi \neq \dot\psi^{-1}$, since $p \neq 2$. Then the assertion about $\psi \mapsto \pi(\mathrm{T}, \psi)$ being a bijection, and hence a homeomorphism (see Remark 28), from $\hat{\mathrm{T}}_{\dot\psi}$ to $\Pi(\mathrm{T}, \dot\psi) \cap \hat{\mathrm{G}}_t$, follows from the fact that $\pi(\mathrm{T}, \psi) \not\cong \pi(\mathrm{T}, \psi')$ unless possibly if $\psi' = \psi \circ \operatorname{Int} w$ ($= \psi$ or ψ^{-1}) for some $w \in W_\mathrm{T}$ — see Remark 16(v).

When T is split, $\psi \mapsto \pi(\mathrm{T}, \psi)$ is a well-defined map from the set of nonquadratic characters in $\hat{\mathrm{T}}$ to $\hat{\mathrm{G}}_t$, and the above consideration shows that the full preimage of $\Pi(\mathrm{T}, \dot\psi) \cap \hat{\mathrm{G}}_t$ under this map equals $W_\mathrm{T} \cdot \hat{\mathrm{T}}_{\dot\psi}$, where $w \cdot \psi = \psi \circ \operatorname{Int} w^{-1}$ for $\psi \in \widehat{\mathrm{T}(F)}, w \in W_{\mathrm{T}}$. Therefore, from the definition of $d\mu_{t_1}$ around Equation (18) (if t_1 is split) or the fact that $d\mu_{t_1}(\pi) = \overline{\Theta_\pi(t_1)}$ for discrete series representations π (if t_1 is elliptic), we can verify that the pull back of $d\mu_{t_1}$ to $\hat{\mathrm{T}}_{\dot\psi}$ equals:

$$\begin{cases} D(t_1)^{-1/2} \cdot \left(\sum_{w \in W_{\mathrm{T}}} \overline{(\psi \circ \operatorname{Int} w)(t_1)}\right) d\psi, & \text{if T is split and } t_1 \in \mathrm{T}(F), \\ 0, & \text{if T is not split but } t_1 \text{ is}, \\ \overline{\Theta_{\pi(\mathrm{T},\psi)}(t_1)}\, d\psi, & \text{if neither T nor } t_1 \text{ is split} \end{cases}$$

— here, we have left out the case where T is split and t_1 is elliptic, which will be treated after dealing with the others.

In all the cases except the one where t_1 is elliptic and T is split, one checks that each of the remaining assertions follow from the above expressions together with Lemma 18 (compare with Remark 19) and the fact that the central character of $\pi(\mathrm{T}, \psi)$ equals $\psi(-1)$, for any $\psi \in \hat{\mathrm{T}}_{\dot\psi}$ (the constant multiple of (ii) is zero if T_1 is not split but t_1 is, even if $t_1 \in \mathrm{G}_{r+} \cup -\mathrm{G}_{r+}$).

Thus, assume that t_1 is elliptic and T is split. Let $\mathrm{T}^{(1)}$ be the centralizer of t_1. In this case, we appeal to Theorem 25. We have already noted that $d\mu_{t_1}(\{\pi\}) = \chi_\pi(-1) \cdot d\mu_{-t_1}(\{\pi\})$, where $\chi_\pi(-1)$ is the central character of π. Since the central character of $\pi(\mathrm{T}, \psi)$ is $\psi(-1)$, and since the claimed expression for the pull back too gets multiplied by $\psi(-1)$ when t_1 gets replaced by $-t_1$, we may assume without loss of generality that $t_1 \notin -\mathrm{T}^{(1)}_{0+}$; this enables us to apply Theorem 25.

Choose h_1 as in Theorem 25, i.e., it is the smallest positive integer such that $t_1 \in \mathrm{T}^{(1)}_{(h_1-(1/2))+}$, if it exists, and it equals 0 otherwise.

Assume first that $t_1 \notin \mathrm{G}_{r+}$, i.e., $t_1 \notin \mathrm{T}^{(1)}_{r+}$. Since $t_1 \notin -\mathrm{T}^{(1)}_{0+}$, this implies that $t_1 \notin \pm\mathrm{T}^{(1)}_{r+}$. Thus, $r \geq h_1$ if $\mathrm{T}^{(1)}$ is unramified, and $r \geq h_1 + (1/2)$ if $\mathrm{T}^{(1)}$ is ramified. By Theorem 25, the principal series representations supporting $d\mu_{t_1}$ have depth at most h_1, while for any $\psi \in \hat{\mathrm{T}}_{\dot\psi}$, $\pi(\mathrm{T}, \psi)$ has depth strictly greater than r. Hence, in these cases, the restriction of $d\mu_{t_1}$ to $\Pi(\mathrm{T}, \dot\psi) \cap \hat{\mathrm{G}}_t$ is zero, as desired.

Now assume $t_1 \in \mathrm{G}_{r+}$. By Theorem 25, together with the full preimage of $\Pi(\mathrm{T}, \dot\psi)$ in $\hat{\mathrm{T}}$ under $\psi \mapsto \pi(\mathrm{T}, \psi)$ being $W_{\mathrm{T}} \cdot \hat{\mathrm{T}}_{\dot\psi}$, we get $d\mu_{t_1}|_{\Pi(\mathrm{T},\dot\psi) \cap \hat{\mathrm{G}}_t}$ to be the push forward of the measure on $\hat{\mathrm{T}}_{\dot\psi}$ given by:

$$\sum_{w \in W_{\mathrm{T}}} \left(-\frac{q+1}{2q} \operatorname{meas}(\mathrm{A}_1) \left|\Gamma(\psi \circ \operatorname{Int} w^{-1})\right|^{-2} + \frac{q}{2} \operatorname{meas}(\mathrm{A}_1) \kappa_{\mathrm{T}^{(1)}} |D(t_1)|^{-1/2}\right) \cdot \mathbb{1}_{\operatorname{depth}(\psi) \leq h_1} \cdot d\psi.$$

Since $t_1' = 1$, it suffices to see that the coefficient of $d\psi$ in the above expression is independent of $\psi \in \hat{\mathrm{T}}_{\dot\psi}$. However, the only contributions in the above expression that a priori possibly depend nontrivially on ψ are $\operatorname{depth}(\psi)$ and the $|\Gamma(\psi \circ \operatorname{Int} w^{-1})|^{-2}$. The former, being the smallest s such that $\dot\psi|_{\mathrm{T}_{s+}} = \mathbb{1}$, depends on ψ only through $\dot\psi$ (and is hence constant on $\hat{\mathrm{T}}_{\dot\psi}$). The latter, by [ST66, Theorem 1(i)], equals $q^{(\operatorname{depth}(\psi)-1)}$, which again depends on ψ only through $\dot\psi$. This finishes the proof. □

4. $z_{\mathrm{T},\dot\psi}$ AND E_r SATISFY (I) AND (II) OF THEOREM 4

4.1. $z_{\mathrm{T},\dot\psi}$ satisfies (i) and (ii) of Theorem 4.

Lemma 39. *For any maximal torus* $\mathrm{T} \subset \mathrm{G}$ *and any nontrivial character* $\mathbb{1} \neq \dot\psi \in \hat{\mathrm{T}}_{r+}$, $z_{\mathrm{T},\dot\psi}$ *satisfies both (i) and (ii) of Theorem 4.*

Proof. Write $z = z_{\mathrm{T},\dot\psi}$. Let $\mathrm{T}^{(1)}$ be the centralizer of a regular semisimple element $t_1 \in \mathrm{G}(F)$, belonging to a component $\mathcal{U}_1$ of the partition (3) of $\mathrm{G}(F)$. Recall that $\varphi_z^{\mathrm{T}^{(1)}}$ is the element of $\mathcal{Z}(\mathrm{T}^{(1)})$ that acts by 1 (resp., 0) on characters ψ_1 such that $\pi(\mathrm{T}^{(1)}, \psi_1) \in \Pi(\mathrm{T}, \dot\psi)$ (resp., $\pi(\mathrm{T}^{(1)}, \psi_1) \notin \Pi(\mathrm{T}, \dot\psi)$). As in Lemma 38, let $\hat{\mathrm{T}}_{\dot\psi}$ be the set of characters in $\widehat{\mathrm{T}(F)}$ that restrict to $\dot\psi$ on T_{r+}.

The proof will make use of the following claims:

Claim 1. $\varphi_z^{\mathrm{T}^{(1)}}$ is represented by the following function on $\mathrm{T}^{(1)}(F)$:

$$\varphi_z^{\mathrm{T}^{(1)}} = \begin{cases} 0, & \text{if } \mathrm{T}^{(1)} \text{ and } \mathrm{T} \text{ are not } \mathrm{G}(F)\text{-conjugate, and} \\ \frac{1}{\operatorname{meas} \mathrm{T}_{r+}} \cdot \sum_{w \in W_{\mathrm{T}}} \overline{(\dot\psi \circ \operatorname{Int} w)} \in C_c^\infty(\mathrm{T}_{r+}) \subset C_c^\infty(\mathrm{T}(F)), & \text{if } \mathrm{T}^{(1)} = \mathrm{T}. \end{cases}$$

Claim 2. For $t_2 \in \mathrm{G}(F)_{\mathrm{reg}}$ belonging to a component $\mathcal{U}_2$ of the partition (3), if $\mathcal{U}_1 \neq \mathcal{U}_2$ or if $t_1 \in \mathrm{G}(F) \setminus (\mathrm{G}_{r+} \cup -\mathrm{G}_{r+} \cup {}^{\mathrm{G}(F)}\mathrm{T}(F))$ or $t_2 \in \mathrm{G}(F) \setminus (\mathrm{G}_{r+} \cup -\mathrm{G}_{r+} \cup {}^{\mathrm{G}(F)}\mathrm{T}(F))$ then:

$$\int_{\hat{\mathrm{G}}_t} z(\pi) \cdot \Theta_\pi(t_2) d\mu_{t_1} = 0.$$

Claim 3. Suppose $t_1, t_2 \in \mathrm{T}(F) \setminus (\mathrm{T}_{r+} \cup -\mathrm{T}_{r+})$ and $t = t_2^{-1} t_1 \in \mathrm{T}_{1,r+}$. Then:

$$D(t_1)^{1/2} D(t_2)^{1/2} \cdot \int_{\hat{\mathrm{T}}_{\dot\psi}} \Theta_{\pi(\mathrm{T},\psi)}(t_2) \cdot \overline{c_{\dot\psi} \sigma(t_1) \alpha_{\mathrm{T}}(t_1) \cdot \sum_{w \in W_{\mathrm{T}}} (\psi \circ \operatorname{Int} w)(t_1)} d\psi = \frac{1}{\operatorname{meas} \mathrm{T}_{r+}} \sum_{w \in W_{\mathrm{T}}} \overline{(\dot\psi \circ \operatorname{Int} w)(t)}.$$

Once Claim 2 is proved, Corollary 33(i) will give the condition of Theorem 4(i) for z (a few more details on this deduction will be seen later in a special case; see the proof Corollary 42 below). Further, Claim 1 and the "$t_1 \in \mathrm{G}(F) \setminus (\mathrm{G}_{r+} \cup -\mathrm{G}_{r+} \cup {}^{\mathrm{G}(F)}\mathrm{T}(F))$" case of Claim 2 together will also give the condition of Theorem 4(ii) for those maximal tori that are not $\mathrm{G}(F)$-conjugate to T (we apologize for the variance in notation: our $\mathrm{T}^{(1)}$ assumes the role of what is denoted T in the statement of Theorem 4).

If in addition Claim 3 is also proved, then, by Lemma 38 and the fact that z is the projector onto $\Pi(\mathrm{T}, \dot\psi)$, we get the following assertion: if $t_1 \in \mathrm{T}(F) \setminus \mathrm{T}_{r+}$ (so $\mathrm{T}^{(1)} = \mathrm{T}$) and $t_2 \in t_1 \cdot \mathrm{T}_{r+}$, then:

$$\int_{\hat{\mathrm{G}}_t} z(\pi) \Theta_\pi(t_2)\, d\mu_{t_1} = D(t_1)^{-1/2} D(t_2)^{-1/2} \cdot \frac{1}{\operatorname{meas} \mathrm{T}_{r+}} \sum_{w \in W_{\mathrm{T}}} \overline{(\dot\psi \circ \operatorname{Int} w)(t_2^{-1} t_1)},$$

which equals $D(t_1)^{-1/2} D(t_2)^{-1/2} \varphi_z^{\mathrm{T}}(t_2^{-1} t_1)$, once Claim 1 is proved. Then Corollary 33(ii) will yield the condition of Theorem 4(ii) for the maximal torus T and hence for all of its $\mathrm{G}(F)$-conjugates too.

Thus, let us prove the claims. Since $\dot\psi \neq \dot\psi^{-1}$ (as $p \neq 2$ and $\dot\psi$ is of positive depth), Claim 1 is equivalent to establishing the following characterization of φ_z^{T}: that it acts as 1 on those characters of $\mathrm{T}(F)$ that are W_{T}-conjugate to some $\psi \in \widehat{\mathrm{T}(F)}$ with $\psi|_{\mathrm{T}_{r+}} = \dot\psi$, and as 0 on the remaining characters. This is immediate from the definition of $\Pi(\mathrm{T}, \dot\psi)$, and the fact that for $\psi, \psi' \in \widehat{\mathrm{T}(F)}$ of positive depth, $\pi(\mathrm{T}, \psi) = \pi(\mathrm{T}, \psi')$ if and only if ψ' is in the W_{T}-orbit of ψ (see Remark 16(v)).

Now let us prove Claim 3, so assume for now that $t_1 \in \mathrm{T}(F) \setminus (\mathrm{T}_{r+} \cup -\mathrm{T}_{r+})$ and $t_2 \in t_1 \cdot \mathrm{T}_{r+}$. Write $\alpha = \alpha_{\mathrm{T}}$. Applying Lemma 18(ii) in the second step below, we get that the left-hand side of the equation of

Claim 3 equals:

$$D(t_1)^{1/2}D(t_2)^{1/2}\int_{\hat{\mathrm{T}}_{\dot\psi}}\sum_{w_1\in W_{\mathrm{T}}}\Theta_{\pi(\mathrm{T},\psi)}(t_2)\cdot\overline{c_{\dot\psi}\sigma(w_1t_1w_1^{-1})(\psi\alpha\circ\operatorname{Int}w_1)(t_1)}\,d\psi$$
$$=D(t_1)^{1/2}D(t_2)^{1/2}\int_{\hat{\mathrm{T}}_{\dot\psi}}\left(c_{\dot\psi}\sigma(t_2)\sum_{w_2\in W_{\mathrm{T}}}(\psi\alpha\circ\operatorname{Int}w_2)(t_2)\right)\cdot\overline{c_{\dot\psi}}\overline{\sigma(t_1)}\left(\sum_{w_1\in W_{\mathrm{T}}}\overline{(\psi\alpha\circ\operatorname{Int}w_1)(t_1)}\right)d\psi$$
$$\overset{(*)}{=}\int_{\hat{\mathrm{T}}_{\dot\psi}}\sum_{w_1,w_2\in W_{\mathrm{T}}}\overline{\psi(w_2t_2^{-1}w_2^{-1}\cdot w_1t_1w_1^{-1})}\,d\psi$$
$$\overset{(**)}{=}\frac{1}{\operatorname{meas}\mathrm{T}_{r+}}\sum_{w\in W_{\mathrm{T}}}\overline{(\dot\psi\circ\operatorname{Int}w)(t)},$$

where:

- in the step marked (*), we used that $c_{\dot\psi}\overline{c_{\dot\psi}}=1$, that $\sigma(t_2)\overline{\sigma(t_1)}=D(t_1)^{-1/2}D(t_2)^{-1/2}$ (a consequence of Lemma 10), and that $\psi\mapsto\psi\alpha$ is a measure preserving self-bijection of $\hat{\mathrm{T}}_{\dot\psi}$, and
- in the step marked (**), we used Remark 8 and Equation (9).

Now let us prove Claim 2. We transfer the integral into one over the set $\hat{\mathrm{T}}_{\dot\psi}$ of Lemma 38. Let $\mathrm{T}^{(2)}$ be the centralizer of t_2.

If $t_1\in\mathrm{G}(F)\setminus(\mathrm{G}_{r+}\cup-\mathrm{G}_{r+}\cup{}^{\mathrm{G}(F)}\mathrm{T}(F))$ (resp., if $t_2\in\mathrm{G}(F)\setminus(\mathrm{G}_{r+}\cup-\mathrm{G}_{r+}\cup{}^{\mathrm{G}(F)}\mathrm{T}(F))$), then by Lemma 38 (resp., Lemma 18(iii)), the integral of Claim 2 is zero. Thus, assume that $t_1,t_2\in\mathrm{G}_{r+}\cup-\mathrm{G}_{r+}\cup{}^{\mathrm{G}(F)}\mathrm{T}(F)$, so that one can assign t_1',t_2' to t_1,t_2 as in Remark 19. Now, by Lemma 38 (for handling t_1) and by Remark 19, the integral of Claim 2 is a scalar multiple (the multiple depending on t_1 and t_2) of:

$$\int_{\hat{\mathrm{T}}_{\dot\psi}}\sum_{w_1,w_2\in W_{\mathrm{T}}}\overline{\psi(\operatorname{Int}w_1(t_1'))}\cdot\psi(\operatorname{Int}w_2(t_2'))\,d\psi,$$

which is easily seen to be zero unless $t_1'\in\operatorname{Int}w(t_2')\cdot\mathrm{T}_{r+}$ for some $w\in W_{\mathrm{T}}$, which is in turn the case if and only $\mathcal{U}_1=\mathcal{U}_2$.

□

Remark 40. We have not assumed anywhere in the above proofs that $\mathrm{G}_r\neq\mathrm{G}_{r+}$. Thus, if in the above situation $\mathrm{T}_r\neq\mathrm{T}_{r+}$, then one can as well replace r by $r+0.9$ (since for any rank one torus T, whenever $\mathrm{T}_r\neq\mathrm{T}_{r+}$, $\mathrm{T}_{r+}=\mathrm{T}_{(r+l)+}$ for all l such that $0\le l<1$) so that $z_{\mathrm{T},\dot\psi}$ is actually supported in $\mathrm{G}_{(r+0.9)+}$, and satisfies the analogue of Theorem 4 with r replaced by $r+0.9$.

4.2. E_r satisfies (i) and (ii) of Theorem 4.

Lemma 41. *The depth r projector E_r satisfies the conditions of both (i) and (ii) of Theorem 4.*

Proof. If r is not a half-integer, we know that $\mathrm{T}_r=\mathrm{T}_{r+}$ for all maximal tori $\mathrm{T}\subset\mathrm{G}$, and that $\mathrm{G}_r=\mathrm{G}_{r+}$. Hence the partition of Equation (3) remains unchanged if we replace r by $\lfloor 2r\rfloor/2$. It is now easy to see that we may replace r by $\lfloor 2r\rfloor/2$, and assume r to be a half-integer.

First we reduce to assuming that r is a half-integer which is not an integer, so assume this case is known. Let r be an integer. Thus, we know that the depth-$(r+1/2)$-projector $E_{r+(1/2)}$ is supported on $\mathrm{G}_{(r+1/2)+}$. Moreover, $E_{r+(1/2)}-E_r$ is a sum of terms of the form $z_{\mathrm{T},\dot\psi}$, where T is a maximal torus of G and $\dot\psi$ is a nontrivial character of $\mathrm{T}_{(r+1/2)}=\mathrm{T}_{(r-1/2)+}$ which is trivial on $\mathrm{T}_{(r+1/2)+}$ (note that every such torus T is ramified; ignore $\mathrm{T}_{(r-1/2)+}$ if $r=0$). We know from Lemma 39 that $z_{\mathrm{T},\dot\psi}$ satisfies the conditions (i) and (ii) of Theorem 4, with r replaced by any s such that $\mathrm{T}_{(r+1/2)}=\mathrm{T}_{s+}$, and in particular with $s=r$. Since $E_{r+(1/2)}$ does too (because the partition (3) becomes finer, though of course not always strictly, with increasing r), under the assumption that the lemma is known for half-integers, the lemma for general r would then follow.

Thus, we now assume r to be a half-integer which is not an integer. Set $d = r + (1/2)$ for the rest of this proof.

The analogue of Claim 1 from the proof of Lemma 39 is easy: for a maximal torus $\mathrm{T} \subset \mathrm{G}$, E_r acts as 1 on $\pi(\mathrm{T}, \psi)$ for a positive depth character $\psi \in \widehat{\mathrm{T}(F)}$ if and only if $\pi(\mathrm{T}, \psi)$, or equivalently by Remark 16(i) ψ, has depth $\leq r$. Using the definition of $\varphi^{\mathrm{T}}_{E_r}$ from the introduction, it is now easy to see that $\varphi^{\mathrm{T}}_{E_r} \in \mathcal{Z}(\mathrm{T})$ is represented by the locally integrable function:

$$\varphi^{\mathrm{T}}_{E_r} = \frac{1}{\operatorname{meas} \mathrm{T}_{r+}} \mathbb{1}_{\mathrm{T}_{r+}} \in C_c^\infty(\mathrm{T}(F)).$$

Let $t_1, t_2 \in \mathrm{G}(F)_{\mathrm{reg}}$ belong to parts $\mathcal{U}_1, \mathcal{U}_2$, respectively, of the decomposition (3), and let $\mathrm{T}^{(i)}$ be the centralizer of t_i for $i = 1, 2$. By (i) and (ii) of Corollary 33, it is enough to prove the following claim.

Claim. Write $\hat{\mathrm{G}}_{t,\leq r} := \hat{\mathrm{G}}_t \cap \hat{\mathrm{G}}_{\mathrm{adm},\leq r}$. We have:

(21)

$$D(t_1)^{1/2} D(t_2)^{1/2} \int_{\pi \in \hat{\mathrm{G}}_{t,\leq r}} \Theta_\pi(t_2) d\mu_{t_1} = \begin{cases} 0, & \text{if } \mathcal{U}_1 \neq \mathcal{U}_2, \text{ and} \\ \frac{1}{\operatorname{meas} \mathrm{T}_{r+}} \mathbb{1}_{\mathrm{T}_{r+}}(t_2^{-1} t_1), & \text{if } \mathrm{T}^{(2)} = \mathrm{T}^{(1)} =: \mathrm{T},\ t_1 \notin \pm \mathrm{T}_{r+}, \text{ and } t_2 \in t_1 \mathrm{T}_{r+}. \end{cases}$$

Case 1. t_1 is conjugate to an element of $\mathrm{A}(F)$. Without loss of generality, suppose $t_1 \in \mathrm{A}(F)$. Recall from around Equation (18) that in this case $d\mu_{t_1}$ was defined as the pushforward of $\psi \mapsto D(t_1)^{-1/2} \overline{\psi(t_1)} d\psi$ with respect to the parabolic induction map $\widehat{F^\times} \to \hat{\mathrm{G}}_t$ (ignoring the quadratic characters).

Thus, in this case, by the formula for induced characters (Remark 17), the left-hand side of (21) equals 0 if $t_2 \notin {}^{\mathrm{G}(F)} \mathrm{A}(F)$, and

$$D(t_1)^{1/2} D(t_2)^{1/2} \int_{\substack{\psi \in \widehat{F^\times} \\ \operatorname{depth}(\psi) \leq r}} \frac{\psi(t_2) + \psi(t_2^{-1})}{D(t_2)^{1/2}} \cdot D(t_1)^{-1/2} \overline{\psi(t_1)} d\psi,$$

if $t_2 \in \mathrm{A}(F)$, since the contribution from quadratic characters is zero. Assume now, without loss of generality, that $t_2 \in \mathrm{A}(F)$. By Equation (9), this expression equals $\#(\{t_1 t_2, t_2^{-1} t_1\} \cap \mathrm{A}_{r+}) / \operatorname{meas}(\mathrm{A}_{r+})$. Further, this cardinality is zero unless $\mathcal{U}_1 = \mathcal{U}_2$, and equals at most one if $t_1 \in \mathrm{A}(F) \setminus (\mathrm{A}_{r+} \cup -\mathrm{A}_{r+})$ (see Remark 8). This gives the claim when t_1 is split.

Case 2. t_1 is elliptic (i.e., $\mathrm{T}^{(1)}$ is). Now onwards, we denote the eigenvalues of t_i as $\lambda_i, \lambda_i^{-1}$ (in some order). Applying Remark 27, and noting that the right hand side of Equation (21) too is invariant under replacing (t_1, t_2) by $(-t_1, -t_2)$, we may carry this replacement out if necessary, to assume that $1 + \lambda_1, 1 + \lambda_1^{-1}$ are units. This allows us to choose an integer $h_1 \geq 0$ such that t_1 belongs to the 'conductor h_1 part' of $\mathrm{T}^{(1)}(F)$, but not to its 'conductor $h_1 + 1$ part' — see Remark 24. Recall that in this case, $d\mu_{t_1}$ was obtained from Theorem 25 which said that for $f \in C_c^\infty(\mathrm{G}(F))$:

$$\begin{aligned} O(t_1, f) = \sum_{\pi \in D} \overline{\Theta_\pi(t_1)} \operatorname{tr} \pi(f) + \frac{1}{2} \sum_{\pi \in \mathrm{RPS}_{\mathrm{T}^{(1)}}} \overline{\Theta_\pi(t_1)} \operatorname{tr} \pi(f) - \frac{q+1}{2q} \operatorname{meas}(\mathrm{A}_1) \int_{\substack{\xi \in \widehat{F^\times} \\ \xi|_{\mathrm{A}_{h_1+}} = 1}} |\Gamma(\xi)|^{-2} \operatorname{tr} \pi_\xi(f) d\xi \\ + \frac{q}{2} \operatorname{meas}(\mathrm{A}_1) \kappa_{\mathrm{T}^{(1)}} |D(t_1)|^{-1/2} \int_{\substack{\xi \in \widehat{F^\times} \\ \xi|_{\mathrm{A}_{h_1+}} = 1}} \operatorname{tr} \pi_\xi(f)\, d\xi. \end{aligned}$$

Thus, the expression on the left-hand side of (21) is the product of $D(t_1)^{1/2}D(t_2)^{1/2}$ and:

$$t_2 \mapsto \sum_{\substack{\pi\in D\\ \text{depth } \pi\le r}} \overline{\Theta_\pi(t_1)}\Theta_\pi(t_2) + \frac{1}{2}\sum_{\pi\in \mathrm{RPS}_{\mathrm{T}^{(1)}}} \overline{\Theta_\pi(t_1)}\Theta_\pi(t_2) - \frac{q+1}{2q}\operatorname{meas}(\mathrm{A}_1)\int_{\substack{\xi\in\widehat{F^\times}\\ \text{depth } \xi\le\min(h_1,d-1)}} |\Gamma(\xi)|^{-2}\Theta_{\pi_\xi}(t_2)d\xi$$

(22)

$$+\frac{q}{2}\operatorname{meas}(\mathrm{A}_1)\kappa_{\mathrm{T}^{(1)}}|D(t_1)|^{-1/2}\int_{\substack{\xi\in\widehat{F^\times}\\ \text{depth } \xi\le\min(h_1,d-1)}} \Theta_{\pi_\xi}(t_2)\,d\xi$$

(since the condition $\operatorname{depth}(\xi)\le r$ is equivalent to the condition $\operatorname{depth}(\xi)\le d-1$).

Recall from Equation (15) that the first of these terms is simply $\overline{K_d(t_1,t_2)} = K_d(t_2,t_1)$, since $d = r+(1/2)$. In the rest of the proof, we will be using this a lot of the notation introduced in Section 3.3 — $\kappa_{\mathrm{T}}, \theta_{\mathrm{T}}, \Delta_d$.

Case 2a. $\mathrm{T}^{(2)}$ is elliptic.

Recall the convention that elliptic Cartan subgroups get measure 1. Since t_2 is elliptic, $\Theta_{\pi_\xi}(t_2)=0$ for all $\xi\in\widehat{F^\times}$, so only the first two terms in (22) survive. Set $e_{\mathrm{T}^{(1)},\mathrm{T}^{(2)}}$ to be 1 if $\mathrm{T}^{(1)}$ and $\mathrm{T}^{(2)}$ are $\mathrm{G}(F)$-conjugate, and 0 otherwise. Set $e'_{\mathrm{T}^{(1)},\mathrm{T}^{(2)}}$ to be 1 if these tori are isomorphic but not $\mathrm{G}(F)$-conjugate, and 0 otherwise. Let $\delta_r(t_1,t_2)$ equal 1 if t_1,t_2 belong to the same part of the partition (3), and let it equal 0 otherwise. Since we have assumed without loss of generality that $t_1\notin -\mathrm{G}_{r+}$, we have the following: when $\mathrm{T}^{(1)}$ and $\mathrm{T}^{(2)}$ are conjugate under $\mathrm{G}(F)$, $\delta_r(t_1,t_2)=1$ if and only if either $\Delta_d(t_1)\Delta_d(t_2)=1$, or if, after conjugating t_1,t_2 into the same torus, they satisfy:

$$\Delta_d(t_1t_2^{-1}) + (\#W_{\mathrm{T}^{(1)}}-1)\Delta_d(t_1t_2)=1.$$

Using this fact, and the fact that $1+\lambda_1, 1+\lambda_1^{-1}$ are units, [SS84, Theorem 4.6](i)-(iii) now gives the following expression for $\overline{K_d(t_1,t_2)} = K_d(t_2,t_1)$:

(23)

$$K_d(t_2,t_1)=\overline{K_d(t_1,t_2)} = \left(\frac{q+1}{q}\right)q^{3d}\Delta_d(t_1)\Delta_d(t_2) + e'_{\mathrm{T}^{(1)},\mathrm{T}^{(2)}}\sigma(t_1)\sigma(t_2)(1-|\theta_{\mathrm{T}^{(1)}}|^{-1/2}\kappa_{\mathrm{T}^{(1)}}\Delta_d(t_1)\Delta_d(t_2))$$
$$-e_{\mathrm{T}^{(1)},\mathrm{T}^{(2)}}\sigma(t_1)\sigma(t_2)(1-|\theta_{\mathrm{T}^{(1)}}|^{-1/2}\kappa_{\mathrm{T}^{(1)}}q^d\delta_r(t_1,t_2)).$$

As regards the second term of (22), since there are exactly two reducible principal series representations associated to each isomorphism class of elliptic maximal tori in G, Equation (13) gives:

$$\frac{1}{2}\sum_{\pi\in\mathrm{RPS}_{\mathrm{T}^{(1)}}}\overline{\Theta_\pi(t_1)}\Theta_\pi(t_2) = -e'_{\mathrm{T}^{(1)},\mathrm{T}^{(2)}}\sigma(t_1)\sigma(t_2) + e_{\mathrm{T}^{(1)},\mathrm{T}^{(2)}}\sigma(t_1)\sigma(t_2). \tag{24}$$

Adding (23) and (24), the expression of (22) equals:

$$\begin{cases} 0, & \text{if } \delta_r(t_1,t_2)=0,\\ q^d e_{\mathrm{T}^{(1)},\mathrm{T}^{(2)}}\sigma(t_1)\sigma(t_2)|\theta_{\mathrm{T}^{(1)}}|^{-1/2}\kappa_{\mathrm{T}^{(1)}}, & \text{else if } t_1,t_2\notin \mathrm{G}_{r+}, \text{ and}\\ \left(\frac{q+1}{q}\right)\cdot q^{3d} + (-e'_{\mathrm{T}^{(1)},\mathrm{T}^{(2)}} + q^d e_{\mathrm{T}^{(1)},\mathrm{T}^{(2)}})\sigma(t_1)\sigma(t_2)|\theta_{\mathrm{T}^{(1)}}|^{-1/2}\kappa_{\mathrm{T}^{(1)}}, & \text{else if } t_1,t_2\in\mathrm{G}_{r+}\end{cases}$$

(note that we have already excluded the possibility where $t_1\in-\mathrm{G}_{r+}$, and also used that if $\delta_r(t_1,t_2)=0$ then automatically $\Delta_d(t_1)\Delta_d(t_2)=0$).

To finish Case 2a by proving (21) for that case, it suffices to show that if $\delta_r(t_1,t_2)\ne0$ and $t_1,t_2\notin\mathrm{G}_{r+}$, then the second of the three expressions above equals $D(t_1)^{-1/2}D(t_2)^{-1/2}(\operatorname{meas}\mathrm{T}^{(1)}_{r+})^{-1}$ — this would handle the second case of Equation (21). For this, note that in this situation $e_{\mathrm{T}^{(1)},\mathrm{T}^{(2)}}=1, e'_{\mathrm{T}^{(1)},\mathrm{T}^{(2)}}=0$, $\sigma(t_1)\sigma(t_2) = D(t_1)^{-1/2}D(t_2)^{-1/2}$ by Remark 11, and $|\theta_{\mathrm{T}^{(1)}}|^{-1/2}\kappa_{\mathrm{T}^{(1)}}q^d$ equals $[\mathrm{T}^{(1)}(F):\mathrm{T}^{(1)}_{r+}] = (\operatorname{meas}\mathrm{T}^{(1)}_{r+})^{-1}$ by Equation (14), since $\mathrm{T}^{(1)}(F)$ has been given measure 1.

Case 2b. $\mathrm{T}^{(2)}$ is split. Without loss of generality, we assume that $\mathrm{T}^{(2)} = \mathrm{A}$. Recall that $D(t_i) = |\lambda_i - \lambda_i^{-1}|^2$ for each i. The expression of Equation (22) is in an obvious way a sum of four terms; denote these as I_1, I_2, I_3 and I_4.

Consider the first of these terms, $I_1 = \overline{K_d(t_1, t_2)} = K_{r+(1/2)}(t_2, t_1)$. In case $t_1 \notin \mathrm{T}^{(1)}_{r+}$ (which is equivalent to requiring that $h_1 < d$; note also that our t_2 is the t_1 of the reference just quoted and vice versa, and our h_1 is the h_2 of the reference just quoted), Theorem [SS84, 4.6(iv)] gives us that I_1 equals:

$$I_1 = \begin{cases} \frac{-|\lambda_2|}{|1-\lambda_2|^2 D(t_2)^{1/2}} & \text{if } h_1 < d \text{ and } t_2 \in \mathrm{A}(F) \setminus \mathrm{A}_{h_1+}, \\ \frac{-1}{D(t_1)D(t_2)^{1/2}}, & \text{if } h_1 < d \text{ and } t_2 \in \mathrm{A}_{h_1+}. \end{cases}$$

Even if possibly $h_1 \geq d$, in the case where $t_2 \notin \mathrm{A}_{h_1+} \cup \mathrm{A}_{(d-1)+} = \mathrm{A}_{\min(h_1,d-1)+}$, we have $\Theta_\pi(t_2) = 0$ for every supercuspidal representation π of depth at least d (as follows from the character formulas of [ADSS11]). Therefore, it follows that $K_d(t_2, t_1) = K_{\max(h_1,d-1)+1}(t_2, t_1)$, so that the expression in the first of the two cases above continues to give the value for $I_1 = K_d(t_2, t_1)$. This allows us to expand the scope of the above equation as:

$$I_1 = \begin{cases} \frac{-|\lambda_2|}{|1-\lambda_2|^2 D(t_2)^{1/2}} & \text{if } t_2 \in \mathrm{A}(F) \setminus \mathrm{A}_{\min(h_1,d-1)+}, \\ \frac{-1}{D(t_1)D(t_2)^{1/2}}, & \text{if } h_1 < d \text{ and } t_2 \in \mathrm{A}_{h_1+} = \mathrm{A}_{\min(h_1,d-1)+}. \end{cases} \tag{25}$$

Now we come to I_2. $\mathrm{RPS}_{\mathrm{T}^{(1)}}$ has exactly two elements, say π_1 and π_2, and we know that $\Theta_{\pi_1}(t_2) = \Theta_{\pi_2}(t_2)$ because t_2 is split (see Remark 22) while $\Theta_{\pi_1}(t_1) = -\Theta_{\pi_2}(t_1)$ (because $\Theta_{\pi_1} + \Theta_{\pi_2}$ being the induced character $\mathrm{Ind}_{\mathrm{B}(F)}^{\mathrm{G}(F)} \mathrm{sgn}_{\mathrm{T}^{(1)}}$ is supported on split elements by Remark 17). Therefore:

$$I_2 = 0. \tag{26}$$

As for I_3, the only difficulty is in computing the integral involving the function Γ, but the possible values of this integral are given across two cases on page 318 of [SS84], as a part of and after the sentence that begins with 'As stated in...'. Using this, we get:

$$I_3 = \begin{cases} \dfrac{|\lambda_2|}{D(t_2)^{1/2}|1-\lambda_2|^2}, & \text{if } t_2 \in \mathrm{A}(F) \setminus \mathrm{A}_{\min(h_1,d-1)+}, \text{ and} \\ -\dfrac{q^{2\min(h_1,d-1)+1}}{D(t_2)^{1/2}}, & \text{otherwise.} \end{cases} \tag{27}$$

On the other hand, using Equation (9) and the fact that $(\operatorname{meas} \mathrm{A}_1)(\operatorname{meas} \mathrm{A}_{\min(h_1,d-1)+})^{-1} = q^{\min(h_1,d-1)}$, we calculate from the formula for induced characters (Remark 17) that the fourth term equals:

$$I_4 = \begin{cases} 0, & \text{if } t_2 \notin \mathrm{A}_{\min(h_1,d-1)+}, \text{ and} \\ q^{\min(h_1,d-1)+1}\kappa_{\mathrm{T}^{(1)}} D(t_1)^{-1/2} D(t_2)^{-1/2}, & \text{otherwise.} \end{cases} \tag{28}$$

Recall that what remains to prove is only that the left-hand side of Equation (21), namely $I_1 + I_2 + I_3 + I_4$, vanishes if $\mathcal{U}_1 \neq \mathcal{U}_2$, since the second case on the right-hand side of that equation excludes the current case of t_1 being elliptic and t_2 being split.

In all the cases where $t_2 \notin \mathrm{A}_{\min(h_1,d-1)+}$, it is immediate that $I_3 = -I_1$ while $I_2 = I_4 = 0$, so that $I_1 + I_2 + I_3 + I_4 = 0$, as desired. Thus, we only need to focus on the cases where $t_2 \in \mathrm{A}_{\min(h_1,d-1)+}$.

We will consider the cases where $t_1 \in \mathrm{G}_{r+}$ and $t_1 \notin \mathrm{G}_{r+}$ separately. Suppose $t_1 \in \mathrm{G}_{r+}$ and $\mathcal{U}_1 \neq \mathcal{U}_2$, so that $h_1 \geq d$ and $t_2 \notin \mathrm{G}_{r+}$. Hence $t_2 \notin \mathrm{A}_{r+} = \mathrm{A}_{\min(h_1,d-1)+}$, a case we had already taken care of.

Hence, assume that $t_1 \notin \mathrm{G}_{r+}$, i.e., $h_1 < d$. Recall that we now only need to consider the case where $t_2 \in \mathrm{A}_{\min(h_1,d-1)+} = \mathrm{A}_{h_1+}$. In this case, $I_1 + I_2 + I_3 + I_4 = I_3 + I_1 + I_4$ equals:

$$D(t_2)^{-1/2} \cdot \left(- \left(q^{2h_1+1} + D(t_1)^{-1} \right) + q^{h_1+1}\kappa_{\mathrm{T}^{(1)}} D(t_1)^{-1/2} \right),$$

which vanishes by the last equation before the start of Section 6 in [SS84], the h_0 and t_0 of that reference being our h_1 and t_1. □

4.3. **A corollary.**

Corollary 42. *Suppose D is an invariant (i.e., $\mathrm{G}(F)$-conjugation invariant) distribution on $\mathrm{G}(F)$ supported in G_{r+}. Suppose $z_0 \in \mathcal{Z}(\mathrm{G})$ equals either E_r, or $z_{\mathrm{T},\dot\psi}$ for some maximal torus $\mathrm{T} \subset \mathrm{G}$ and a nontrivial smooth character $\mathbb{1} \neq \dot\psi \in \hat{\mathrm{T}}_{r+}$. Then $D * z_0$ is also supported on G_{r+}.*

Proof. For $f \in C_c^\infty(\mathrm{G}(F))$, we write $\check f = (g \mapsto f(g^{-1})) \in C_c^\infty(\mathrm{G}(F))$, so that we have $z(f) = (z * \check f)(1)$ for all $f \in C_c^\infty(\mathrm{G}(F))$. Note that $D * z_0$ is the distribution $f \mapsto D(g \mapsto (z_0 * \check f)(g^{-1}))$. [1]

Suppose $f \in C_c^\infty(\mathrm{G}(F))$ is supported outside G_{r+}. Then so is $\check f$. Hence for all $\gamma \in \mathrm{G}_{r+}$, $O(\gamma, z_0 * \check f) = 0$, by Lemma 41 (in case $z_0 = E_r$) or Lemma 39 (in case $z_0 = z_{\mathrm{T},\dot\psi}$ for a suitable $\mathrm{T}, \dot\psi$). Therefore:

$$D * z_0(f) = D(g \mapsto (z_0 * \check f)(g^{-1})) = D\left(\mathbb{1}_{\mathrm{G}_{r+}} \cdot (g \mapsto (z_0 * \check f)(g^{-1}))\right) = 0,$$

where the penultimate equality follows from D being supported in G_{r+}, and the last equality follows from the density of regular semisimple orbital integrals ([KV16, Theorem B.1]), finishing the argument that $D * z_0$ is supported in G_{r+}. □

5. Proof of the main theorems

5.1. **Proof of the main theorems, assuming that $\mathcal{Z}_{r+}(\mathrm{G}) \subset \mathcal{Z}'_{r+}(\mathrm{G})$.**

Lemma 43. *$\mathcal{Z}_{r+}(\mathrm{G}) \subset \mathcal{Z}'_{r+}(\mathrm{G})$.*

Before proving this lemma we note that it yields proofs of Theorems 1, 2, 3 and 4:

Proof of Theorems 1, 2, 3 and 4, assuming Lemma 43. Since $\mathcal{Z}'_{r+}(\mathrm{G})$ is clearly a subring of $\mathcal{Z}(\mathrm{G})$, Theorem 1 follows from Theorem 2. Further, Theorem 3 is a special case of Theorem 4. So it suffices to prove Theorems 2 and 4.

Lemma 43 gives the inclusion $\mathcal{Z}_{r+}(\mathrm{G}) \subset \mathcal{Z}'_{r+}(\mathrm{G})$. Suppose we can show that any given $z \in \mathcal{Z}'_{r+}(\mathrm{G})$ satisfies (i) and (ii) of Theorem 4. Then, in particular, any $z \in \mathcal{Z}'_{r+}(\mathrm{G})$ would satisfy Theorem 3, and then we would have $z \in \mathcal{Z}_{r+}(\mathrm{G})$ by Lemma 36. This would yield $\mathcal{Z}'_{r+}(\mathrm{G}) \subset \mathcal{Z}_{r+}(\mathrm{G})$, giving Theorem 2. Hence it remains to show that any given $z \in \mathcal{Z}'_{r+}(\mathrm{G})$ satisfies (i) and (ii) of Theorem 4.

By Lemma 41, this requirement is satisfied if $z = E_r$, and by Lemma 39, it is satisfied if $z = z_{\mathrm{T},\dot\psi}$ for some maximal torus $\mathrm{T} \subset \mathrm{G}$ and some nontrivial character $\mathbb{1} \neq \dot\psi : \mathrm{T}_{r+} \to \mathbb{C}^\times$. Hence, finite $\mathbb{C}$-linear combinations of E_r and such $z_{\mathrm{T},\dot\psi}$ satisfy (i) and (ii) of Theorem 4 as well.

Now given a general $z \in \mathcal{Z}'_{r+}(\mathrm{G})$, we can think of it as a sort of 'weak limit' of the $z * E_s$: more precisely, given $f \in C_c^\infty(\mathrm{G}(F))$, we choose s such that:

(i) $s > r$;
(ii) $E_s * f = f$, where E_s is the depth-s projector; and
(iii) for any maximal torus $\mathrm{T} \subset \mathrm{G}$, writing $E_{\mathrm{T},s} = \varphi^{\mathrm{T}}_{E_s}$ for the depth s projector on T,

$$E_{\mathrm{T},s} * \varphi^{\mathrm{T}}_f = \varphi^{\mathrm{T}}_f$$

($\varphi^{\mathrm{T}}_f \in C_c^\infty(\mathrm{T}(F))$ is as in the introduction; that $E_{\mathrm{T},s}$ equals $\varphi^{\mathrm{T}}_{E_s}$ was seen in the proof of Lemma 41).

Here, (iii) can be ensured because there are only finitely many $\mathrm{G}(F)$-conjugacy classes of maximal tori in G.

(ii) gives that $z * f$ equals $z * E_s * f$. Since any two representations belonging to the same part of the decomposition (1) either both have depth at most r or have the same depth, (i) gives that $z * E_s \in \mathcal{Z}'_{r+}(\mathrm{G})$.

[1] It seems inelegant to use 'widecheck' from the mathabx package.

Further, $z*E_s$ is supported on finitely many Bernstein components, and is hence a finite $\mathbb{C}$-linear combination of E_r and such $z_{\mathrm{T},\dot\psi}$ as above. Hence $z*E_s$ satisfies (i) and (ii) of Theorem 4 as well.

Thus, if the regular semisimple orbital integrals of f are supported outside a part $\mathcal{U}_\lambda$ of the decomposition (3), then so are those of $z*E_s*f = z*f$, so that z satisfies the condition of Theorem 4(i). Moreover, using that $z*E_s$ satisfies Theorem 4(ii), we get that for any maximal torus $\mathrm{T}\subset\mathrm{G}$ and $t\in\mathrm{T}(F)\setminus(\mathrm{T}_{r+}\cup -\mathrm{T}_{r+})$:

$$\varphi^{\mathrm{T}}_{z*f}(t) = \varphi^{\mathrm{T}}_{z*E_s*f}(t) = (\varphi^{\mathrm{T}}_{z*E_s}*\varphi^{\mathrm{T}}_f)(t) = \varphi^{\mathrm{T}}_z*\varphi^{\mathrm{T}}_{E_s}*\varphi^{\mathrm{T}}_f(t) = \varphi^{\mathrm{T}}_z*E_{\mathrm{T},s}*\varphi^{\mathrm{T}}_f(t) = \varphi^{\mathrm{T}}_z*\varphi^{\mathrm{T}}_f(t)$$

by the condition (iii) on s above. Here, we used that $z\in\mathcal{Z}'_{r+}(\mathrm{G})\subset\mathcal{Z}^?(\mathrm{G})$, $\varphi^{\mathrm{T}}_{z*E_s} = \varphi^{\mathrm{T}}_z*\varphi^{\mathrm{T}}_{E_s} = \varphi^{\mathrm{T}}_z*E_{\mathrm{T},s}$, since $z\mapsto\varphi^{\mathrm{T}}_z$ is a homomorphism $\mathcal{Z}^?(\mathrm{G})\to\mathcal{Z}(\mathrm{T})$, and since $\varphi^{\mathrm{T}}_{E_s} = E_{\mathrm{T},s}$. □

5.2. **Proof that** $\mathcal{Z}_{r+}(\mathrm{G})\subset\mathcal{Z}'_{r+}(\mathrm{G})$.

Proof of Lemma 43. Assume that z is supported on G_{r+}, and that π_1,π_2 either both belong to $\hat{\mathrm{G}}_{\le r}$ or both belong to $\Pi(\mathrm{T},\dot\psi)$ for some maximal torus $\mathrm{T}\subset\mathrm{G}$ and a character $\dot\psi\ne\mathbb{1}$ of T_{r+}. We need to show that $z(\pi_1)=z(\pi_2)$. Let z_0 be E_r if $\pi_1,\pi_2\in\hat{\mathrm{G}}_{\le r}$, and let z_0 be $z_{\mathrm{T},\dot\psi}$ if π_1,π_2 belong to $\Pi(\mathrm{T},\dot\psi)$. Then we know that $z(\pi_i) = z*z_0(\pi_i)$ for $i=1,2$. By Corollary 42, the distribution $z*z_0$ is supported on G_{r+}, so that we may and do replace z by $z*z_0$ for the rest of the proof, and assume that z is supported on $\hat{\mathrm{G}}_{\le r}$ or on $\Pi(\mathrm{T},\dot\psi)$ for a suitable $\mathrm{T},\dot\psi$.

Case 1. z is supported on $\Pi(\mathrm{T},\dot\psi)$ for a suitable $\mathrm{T},\dot\psi$. In case T is elliptic, any representation $\pi = \pi(\mathrm{T},\psi)\in\Pi(\mathrm{T},\dot\psi)$ is supercuspidal, and any element of $\mathcal{Z}(\mathrm{G})$ supported in the singleton Bernstein component $\{\pi\}=\{\pi(\mathrm{T},\psi)\}$ is unique up to scalars and is a scalar multiple of the projector z_ψ onto this component, which (see, e.g., [MT02]) is represented by the locally integrable function $d(\pi)\Theta_{\pi^\vee}$, $\pi^\vee$ being the contragredient of π and $d(\pi)$ its formal degree (with respect to our fixed choice of Haar measure). Since $\dot\psi$ is nontrivial, all elements of $\Pi(\mathrm{T},\dot\psi)$ have the same, positive, depth and the same formal degree (see Remark 16(iii)).

In case T is split, we may assume without loss of generality (see Remark 44 below) that z is a $\mathbb{C}$-linear combination of the projectors onto the Bernstein components in $\Pi(\mathrm{T},\dot\psi)$, which are indexed by characters $\hat\psi\in\widehat{\mathrm{T}(\mathfrak{O})}$ such that $\hat\psi|_{\mathrm{T}_{r+}}=\dot\psi$. Further, the projector $z_{\hat\psi}$ onto the Bernstein component indexed by $\hat\psi$ is computed in [MT02, (3.4.3)], and shown to be represented by the locally integrable function that is supported in ${}^{\mathrm{G}(F)}(\mathrm{T}(\mathfrak{O}))$, and on $\mathrm{T}(\mathfrak{O})$ given by a formula:

$$\varsigma(\dot\psi)D(t)^{-1/2}\sum_{w\in W_{\mathrm{T}}}(\hat\psi\circ\operatorname{Int}w)(t), \tag{29}$$

where $\varsigma(\dot\psi)$ is a nonzero constant that depends only on $\dot\psi$ and not on its extension $\hat\psi$. Recall that $\sigma(t) = D(t)^{-1/2}$ for split t.

To treat the two cases (of T being split and elliptic) uniformly, write T_c for the maximal compact subgroup of $\mathrm{T}(F)$, which equals $\mathrm{T}(\mathfrak{O})$ (resp., $\mathrm{T}(F)$) if T is split (resp., elliptic). A typical character of T_c extending $\dot\psi$ will be denoted $\hat\psi$.

We have just seen that we can write:

$$z = \sum_{\substack{\hat\psi\in\hat{\mathrm{T}}_c\\ \hat\psi|_{\mathrm{T}_{r+}}=\dot\psi}} c(\hat\psi)z_{\hat\psi}.$$

What we need to show is that $c(\hat\psi_1)=c(\hat\psi_2)$ for any two $\hat\psi_1,\hat\psi_2\in\hat{\mathrm{T}}_c$ extending $\dot\psi$.

We have already seen that the central character of any representation in the Bernstein component corresponding to $\hat\psi\in\hat{\mathrm{T}}_c$ equals $\hat\psi(-1)$ (see Remark 16(ii)). Therefore, if $\tilde\psi$ is any extension of ψ to $\mathrm{Z}_{\mathrm{G}}(F)\mathrm{T}_{r+}$, then by the linear independence of characters on $\mathrm{Z}_{\mathrm{G}}(F)$, replacing the condition '$\hat\psi|_{\mathrm{T}_{r+}}=\dot\psi$' in the above sum by the condition '$\hat\psi|_{\mathrm{Z}_{\mathrm{G}}(F)\mathrm{T}_{r+}}=\tilde\psi$' gives a distribution supported in $\mathrm{Z}_{\mathrm{G}}(F)\mathrm{G}_{r+}$.

Combining this observation with the formula (29) (if T is split) or Lemma 18(ii) (if T is elliptic), it follows that for $t \in \mathrm{T}_c \setminus \mathrm{Z}_{\mathrm{G}}(F)\mathrm{T}_{r+}$:

$$\sum_{\substack{\hat{\psi}\in\hat{\mathrm{T}}_c \\ \hat{\psi}|_{\mathrm{Z}_{\mathrm{G}}(F)\mathrm{T}_{r+}}=\tilde{\psi}}} c(\hat{\psi}) \cdot \sum_{w\in W_{\mathrm{T}}} (\hat{\psi}\circ \operatorname{Int} w)(t) = 0$$

(in the elliptic case, use that $\sigma(t), \alpha_{\mathrm{T}}(t)$ and $c_{\dot{\psi}}$ do not depend on the specific $\hat{\psi} = \psi$ extending $\dot{\psi}$). [2] If $w \in W_{\mathrm{T}}$ is not the identity, then $\dot{\psi} \circ \operatorname{Int} w \neq \dot{\psi}$ as p is odd, so averaging against $\dot{\psi}^{-1}$ we have that for all $t \in \mathrm{T}_c \setminus \mathrm{Z}_{\mathrm{G}}(F)\mathrm{T}_{r+}$:

$$\sum_{\substack{\hat{\psi}\in\hat{\mathrm{T}}_c \\ \hat{\psi}|_{\mathrm{Z}_{\mathrm{G}}(F)\mathrm{T}_{r+}}=\tilde{\psi}}} c(\hat{\psi}) \cdot \hat{\psi}(t) = 0.$$

This forces $\sum c(\hat{\psi})\hat{\psi}$, where the sum runs over $\hat{\psi} \in \hat{\mathrm{T}}_c$ such that $\hat{\psi}|_{\mathrm{Z}_{\mathrm{G}}(F)\mathrm{T}_{r+}}$ equals $\tilde{\psi}$, to be a scalar multiple of

$$\operatorname{Ind}_{\mathrm{Z}_{\mathrm{G}}(F)\mathrm{T}_{r+}}^{\mathrm{T}_c} \tilde{\psi} = \sum_{\substack{\hat{\psi}\in\hat{\mathrm{T}}_c \\ \hat{\psi}|_{\mathrm{Z}_{\mathrm{G}}(F)\mathrm{T}_{r+}}=\tilde{\psi}}} \hat{\psi}.$$

By linear independence of characters, $c(\hat{\psi}_1) = c(\hat{\psi}_2)$ whenever $\psi_1|_{\mathrm{Z}_{\mathrm{G}}(F)\mathrm{T}_{r+}} = \psi_2|_{\mathrm{Z}_{\mathrm{G}}(F)\mathrm{T}_{r+}} = \tilde{\psi}$; denote this complex number by $c_{\tilde{\psi}}$. Denote the extensions of $\dot{\psi}$ to $\mathrm{Z}_{\mathrm{G}}(F)\mathrm{T}_{r+}$ by $\tilde{\psi}_1$ and $\tilde{\psi}_2$, and assume without loss of generality that $\tilde{\psi}_1(-1) = 1, \tilde{\psi}_2(-1) = -1$. In other words, representations in the Bernstein component associated to a $\hat{\psi} \in \hat{\mathrm{T}}_c$ have trivial (resp., nontrivial) central character if and only $\hat{\psi}|_{\mathrm{Z}_{\mathrm{G}}(F)\mathrm{T}_{r+}} = \tilde{\psi}_1$ (resp., $\tilde{\psi}_2$). To finish Case 1, it suffices to show that $c(\tilde{\psi}_1) = c(\tilde{\psi}_2)$.

Now, since the locally integrable function representing z vanishes at $-1\cdot\gamma$ for regular semisimple elements γ near 1 we get that for such γ:

$$c(\tilde{\psi}_1) \sum_{\hat{\psi}|_{\mathrm{Z}_{\mathrm{G}}(F)\mathrm{T}_{r+}}=\tilde{\psi}_1} \Theta_{\hat{\psi}}(\gamma) = c(\tilde{\psi}_2) \sum_{\hat{\psi}|_{\mathrm{Z}_{\mathrm{G}}(F)\mathrm{T}_{r+}}=\tilde{\psi}_2} \Theta_{\hat{\psi}}(\gamma),$$

where $\Theta_{\hat{\psi}}$ equals $\Theta_{\pi(\mathrm{T},\hat{\psi})}$ if T is elliptic, and the locally integrable function of Equation (29) representing $z_{\hat{\psi}}$ if T is split. Now use that there exist regular semisimple γ arbitrarily close to 1 such that the $\Theta_{\hat{\psi}_1}(\gamma) = \Theta_{\hat{\psi}_2}(\gamma) \neq 0$ for all $\hat{\psi}_1, \hat{\psi}_2$ extending $\dot{\psi}$ (as seen by inspection from Equation (29) if T is split, and from Lemma 18(ii) or the character formulas of [ADSS11] if T is elliptic), and that $\tilde{\psi}_1$ admits as many extensions to T_c as does $\tilde{\psi}_2$; this forces $c(\tilde{\psi}_1) = c(\tilde{\psi}_2)$, as desired.

Case 2: z is supported on $\hat{\mathrm{G}}_{\mathrm{adm},\leq r}$. Note that, since z is supported on finitely many Bernstein components, z is given by an invariant locally integrable function on $\mathrm{G}(F)$ (see [MT02], (2.4.3) and the proof of Theorem 2.5 there), which we still denote by z, given by the formula:

$$z(g) = \int_{\hat{\mathrm{G}}_t} z(\pi)\Theta_{\pi^\vee}(g)\, d\pi, \tag{30}$$

where $d\pi$ is the Plancherel measure on the tempered dual $\hat{\mathrm{G}}_t$ of G.

We first claim that z is stable as a distribution on $\mathrm{G}(F)$. Suppose not.

Then there exists $\mathbb{1} \neq \kappa \in \widehat{F^\times/F^{\times 2}}$, such that:

$$0 \neq z^\kappa := \sum_{a\in F^\times/F^{\times 2}} \kappa(a) \cdot {}^{g_a}z,$$

[2]Note that when T is ramified and $r = 0$, no such t will exist, but this does not affect the arguments.

where we have written g_a for a choice of an element of $\mathrm{GL}_2(F) \supset \mathrm{G}(F)$ with determinant in the class of a, and where ${}^{g_a}z \in \mathcal{Z}(\mathrm{G})$ is such that for all $f \in C_c^\infty(\mathrm{G}(F))$, ${}^{g_a}z(f) = z(f \circ \mathrm{Int}\, g_a)$. Note that, G_{r+} being invariant under $\mathrm{GL}_2(F)$-conjugation, z^κ is supported in G_{r+} too. We have that for $\pi \in \hat{\mathrm{G}}_{\mathrm{adm}}$, $z^\kappa(\pi) = \sum_a \kappa(a) z(\pi_a)$, where the sum runs over $a \in F^\times/F^{\times 2}$, and π_a is the well-defined representation $\pi \circ \mathrm{Int}\, g_a$. This implies that for $b \in F^\times/F^{\times 2}$, and any $\pi \in \hat{\mathrm{G}}_{\mathrm{adm}}$, $z^\kappa(\pi_b) = \kappa(b^{-1}) z^\kappa(\pi)$. This also forces $z^\kappa(\pi)$ to be zero for each nonsupercuspidal irreducible admissible representation π of $\mathrm{G}(F)$ — to see this, use that conjugation by $\mathrm{GL}_2(F)$ preserves the isomorphism class of each (not necessarily irreducible) principal series representation of $\mathrm{G}(F)$, and that all irreducible components of a given principal series representation map to the same point on the Bernstein variety.

Thus, z^κ is supported on supercuspidal representations of $\mathrm{G}(F)$. As in Case 1, we can write down a character theoretic expression for z^κ. Recall that supercuspidal L-packets on $\mathrm{G}(F)$ are precisely ones of the form $\{\pi_a \mid a \in F^\times/F^{\times 2}\}$, with π a supercuspidal representation of $\mathrm{G}(F)$. It is well-known (and in any case easy to see from the preceding sentence) that the formal degree is constant on each L-packet on $\mathrm{G}(F)$. We have recalled in Case 1 the fact that the Bernstein projector onto a (necessarily singleton) supercuspidal Bernstein component $\{\pi\}$ is $d(\pi)\Theta_{\pi^\vee}$, where $d(\pi)$ is the formal degree of π and $\pi^\vee$ is the contragredient of π. We conclude that the locally integrable function representing z^κ, which we will denote by z^κ as well, is a linear combination of 'unstable character sums':

(31)
$$z^\kappa(g) = \sum_{\Pi} \sum_{\pi' \in \Pi} z^\kappa(\pi') d(\pi') \Theta_{\pi'^\vee}(g) = \sum_{\Pi} z^\kappa(\pi) \cdot d(\Pi) \cdot n_\Pi^{-1} \sum_{a \in F^\times/F^{\times 2}} \kappa(a)^{-1} \Theta_{\pi_a^\vee}(g) = \sum_{\Pi} z^\kappa(\pi) \cdot d(\Pi) \Theta^\kappa_{\Pi^\vee}(g),$$

where the missing notation is as follows: Π runs over supercuspidal L-packets on $\mathrm{G}(F)$, $d(\Pi)$ is the common formal degree of the representations in Π, we have used π to denote an arbitrary but fixed element of Π (suppressing the dependence of π on Π from the notation) and let n_Π be the cardinality of $\{a \in F^\times/F^{\times 2} \mid \pi_a \cong \pi\}$, $\Pi^\vee$ stands for $\{\pi'^\vee \mid \pi' \in \Pi\}$, and for a supercuspidal L-packet Π we have written:

$$\Theta^\kappa_{\Pi^\vee} = n_\Pi^{-1} \sum_{a \in F^\times/F^{\times 2}} \kappa(a)^{-1} \Theta_{\pi_a^\vee}.$$

The work of Labesse and Langlands (see [LL79]) tells us that the set of *supercuspidal* L-packets Π for which Θ^κ_Π is nonzero is in bijection with equivalence classes of *nontrivial* characters $\mathbb{1} \neq \psi \in \widehat{\mathrm{T}_\kappa(F)}$, where T_κ is a one-dimensional anisotropic torus split over the quadratic extension E_κ of F determined by the nontrivial character κ, and where the equivalence relation is determined by the prescription $\psi \sim \psi^{-1}$. Further, if Π_ψ is the L-packet defined in this manner by $\psi \in \widehat{\mathrm{T}_\kappa(F)}$, then we have an endoscopic character identity, which says in particular that for $\delta \in \mathrm{G}(F)_{\mathrm{reg}}$ whose eigenvalues lie in E_κ, choosing any $\gamma \in \mathrm{T}_\kappa(F)$ such that some embedding $\mathrm{T}_\kappa \hookrightarrow \mathrm{G}(F)$ maps γ to δ:

$$\Theta^\kappa_{\Pi_\psi}(\delta) = c_0(\psi) \cdot \Delta(\gamma, \delta) \cdot D(\delta)^{-1/2} (\psi(\gamma) + \psi(\gamma^{-1})),$$

where (the nonzero complex number) $\Delta(\gamma, \delta)$ is what is called a transfer factor (we are following the convention of getting the "Δ_{IV}" transfer factor accounted for through normalization of orbital integrals), and $c_0(\psi)$ is a nonzero constant to account for various normalizations and choices (e.g., we have fixed a representative for Π_ψ arbitrarily) and the constant n_Π^{-1}.

Fixing an arbitrary such γ and δ, and using that $\Theta_{\pi^\vee}(g) = \overline{\Theta_\pi(g)}$ for any unitary representation π of $\mathrm{G}(F)$, we find, thanks to Equation (31), that the locally integrable function z^κ satisfies:

$$z^\kappa(\delta) = \overline{\Delta(\gamma, \delta) D(\delta)^{-1/2}} \cdot \sum_{\left\{\mathbb{1} \neq \psi \in \widehat{\mathrm{T}_\kappa(F)}\right\}/\psi \sim \psi^{-1}} z^\kappa(\pi_\psi) \cdot d(\Pi_\psi) \cdot \overline{c_0(\psi)(\psi(\gamma) + \psi(\gamma^{-1}))}, \tag{32}$$

where $d(\Pi_\psi)$ is the common formal degree of any element of Π_ψ and π_ψ is the member of Π_ψ fixed arbitrarily above (we also recall that the sum above is finite as z^κ is supported on finitely many Bernstein components).

Note that in Equation (32), $z^\kappa(\pi_\psi) = 0$ unless ψ is trivial on $\mathrm{T}_{\kappa,r+}$ — this is because the depth of π_ψ is equal to the depth of ψ, thanks to the depth preservation property being satisfied by the local Langlands correspondences for tame SL_2 and tame tori (much more general versions of these results having been proved in [ABPS16] and [Yu09] respectively), together with the fact that the endoscopic transfer under consideration is compatible with the local Langlands correspondence.

Since δ is the image of γ under an embedding $\mathrm{T}_\kappa \hookrightarrow \mathrm{G}$, $\gamma \in \mathrm{T}_{\kappa,r+}$ if and only if $\delta \in \mathrm{G}_{r+}$ (see Remark 6). Since $z \in \mathcal{Z}_{r+}(\mathrm{G})$, the right-hand side of Equation (32) vanishes for every $\gamma \in \mathrm{T}_\kappa(F) \setminus \mathrm{T}_{\kappa,r+}$. The linear independence of characters on $\mathrm{T}_\kappa(F)/\mathrm{T}_{\kappa,r+}$ then forces that all characters of $\mathrm{T}_\kappa(F)/\mathrm{T}_{\kappa,r+}$ contribute equally to (32). Since the summation in (32) is only over nontrivial characters ψ, and since the constants $\Delta(\gamma,\delta), D(\delta)^{-1/2}, d(\Pi_\psi), c_0(\psi)$ are all nonzero, it follows that $z^\kappa(\pi_\psi) = 0$ for all these ψ, and in turn that $z^\kappa = 0$, a contradiction. Thus, we conclude that z is stable.

By assumption, $z(\pi)$ is zero unless π is of depth less than or equal to r. Since $\mathrm{G} = \mathrm{SL}_2$ and since p is odd, given any irreducible admissible representation π of $\mathrm{G}(F)$ of depth at most r, its character expansion holds on G_{r+}, as mentioned in [ADSS11, Section 14.5.2]. The formula of [MT02, (2.4.3)] and the proof of [MT02, Theorem 2.5] give an expression of the following form for the locally integrable function representing z (also denoted z):

$$a(\mathrm{G}|\mathrm{G}) \sum_{\substack{\pi \in D \\ \mathrm{depth}(\pi) \leq r}} z(\pi) d(\pi) \Theta_{\pi^\vee}(g) + a(\mathrm{G}|\,\mathrm{A}) \int_{\substack{\xi \in \widehat{\mathrm{A}(F)} \\ \mathrm{depth}(\xi) \leq r}} z\left(\mathrm{Ind}_{\mathrm{B}}^{\mathrm{G}}\, \xi\right) \Theta_{\left(\mathrm{Ind}_{\mathrm{B}}^{\mathrm{G}}\, \xi\right)^\vee}(g) d(\xi) \mu(\xi) d\xi,$$

ignoring the negligible points of reducibility of the parabolic induction as usual, where the constants $a(\mathrm{G}|\mathrm{G})$ and $a(\mathrm{G}|\,\mathrm{A})$ and the functions $\mu(\xi)$ are easy and explicit but do not concern us, and where superscripting with '$\vee$' stands for taking the contragredient. Note that passing to the contragredient does not change the depth. From this it is easy to see that z has a character expansion at the identity that is valid on G_{r+} (the principal series characters that contribute all agree on G_{r+}, by the formula for induced characters). However, z is also supported on G_{r+}.

To explicate these conditions on z, let us introduce some notation. For each nilpotent $\mathrm{G}(F)$-orbit $\mathcal{O}$ in $\mathfrak{g}(F)$, we know that there exists a $\mathrm{G}(F)$-invariant measure $\nu_\mathcal{O}$ on $\mathcal{O}$ that is unique up to scaling and extends to a distribution on $\mathfrak{g}(F)$, still denoted $\nu_\mathcal{O}$. Choosing a symmetric nondegenerate $\mathrm{Ad}\,\mathrm{G}(F)$-invariant bilinear form on $\mathfrak{g}(F)$ and using our fixed additive character $\Lambda : F \to \mathbb{C}^\times$, we can define a Fourier transform $f \mapsto \hat{f}$ on $C_c^\infty(\mathfrak{g}(F))$. Let $\hat{\nu}_\mathcal{O}$ be the Fourier transform of $\nu_\mathcal{O}$, that is to say, the distribution $\nu_\mathcal{O} \circ (f \mapsto \hat{f})$ on $\mathfrak{g}(F)$. Let $\hat{J}(\mathcal{N})$ denote the linear span of the $\hat{\nu}_\mathcal{O}$, $\mathcal{O}$ running over nilpotent $\mathrm{G}(F)$-orbits in $\mathfrak{g}(F)$. Recall the Cayley transform from Remark 7. For a distribution T on (any open and closed subset of) $\mathfrak{g}(F)$, let $\mathfrak{c}_*(T) = \mathfrak{c}_{*,r}(T)$ denote the distribution on $\mathrm{G}(F)$ that is supported on $\mathrm{G}_{r+}(F)$, and on it given by $f \mapsto T(f \circ \mathfrak{c})$.

Thus, $\mathfrak{c}_*(\hat{J}(\mathcal{N})|_{\mathfrak{g}_{r+}})$ denotes the $\mathbb{C}$-vector space of distributions on $\mathrm{G}(F)$ that are supported on $\mathrm{G}_{r+}(F)$, and such that, on G_{r+}, they are given by expressions of the form:

$$f \mapsto \sum_{\mathcal{O}} c_\mathcal{O} \hat{\nu}_\mathcal{O}(f \circ \mathfrak{c}), \tag{33}$$

where the sum runs over nilpotent $\mathrm{G}(F)$-orbits $\mathcal{O}$ in $\mathfrak{g}(F)$. Then the conclusion of the above paragraph is that, as a distribution on $\mathrm{G}(F)$, we have $z \in \mathfrak{c}_*(\hat{J}(\mathcal{N})|_{\mathfrak{g}_{r+}})$.

Since z belongs to the subspace $\mathcal{Z}_{\mathrm{st}}(\mathrm{G}) \subset \mathcal{Z}(\mathrm{G})$ consisting of stable distributions, it now suffices to show that:

$$\dim_\mathbb{C} \mathfrak{c}_* \left(\hat{J}(\mathcal{N})|_{\mathfrak{g}_{r+}}\right) \cap \mathcal{Z}_{\mathrm{st}}(\mathrm{G}) = 1.$$

Since for $\mathrm{G}(F)$ and $\mathfrak{g}(F)$, stability is equivalent to $\mathrm{GL}_2(F)$-invariance, and since $\mathfrak{c} \circ \mathrm{Ad}\, g = \mathrm{Int}\, g \circ \mathfrak{c}$ for all $g \in \mathrm{GL}_2(F)$, this is equivalent to showing:

$$\dim_{\mathbb{C}} \left(\mathfrak{c}_* \left(\hat{J}(\mathcal{N})|_{\mathfrak{g}_{r+}} \right) \right)^{\mathrm{GL}_2(F)} \cap \mathcal{Z}(\mathrm{G}) = 1$$

(where superscripting with $\mathrm{GL}_2(F)$ stands for taking invariants under the action induced by conjugation by $\mathrm{GL}_2(F)$).

Since all nonzero nilpotent elements of $\mathrm{G}(F)$ are $\mathrm{GL}_2(F)$-conjugate, and since the Fourier transform on $\mathfrak{g}(F)$ is equivariant for $\mathrm{GL}_2(F)$-conjugation, it is easy to see that

$$\dim_{\mathbb{C}} \mathfrak{c}_* \left(\hat{J}(\mathcal{N})|_{\mathfrak{g}_{r+}} \right)^{\mathrm{GL}_2(F)} = 2.$$

Thus, we are reduced to showing:

$$\mathfrak{c}_*(\hat{J}(\mathcal{N})|_{\mathfrak{g}_{r+}})^{\mathrm{GL}_2(F)} \not\subset \mathcal{Z}(\mathrm{G}).$$

But if this condition were not satisfied, then, considering $\mathfrak{c}_*(\hat{\nu}_0|_{\mathfrak{g}_{r+}})$ where 0 is the zero nilpotent orbit, we get that the characteristic function $\mathbb{1}_{\mathrm{G}_{r+}}$ of G_{r+} represents an element of $\mathcal{Z}(\mathrm{G})$. However, this is manifestly seen to not be the case, since for any compact open subgroup K of $\mathrm{G}(F)$, $\mathbb{1}_{\mathrm{G}_{r+}} * \mathbb{1}_K$ has in its support all of $\mathrm{G}_{r+} \cdot K$, contradicting essential compactness.

□

Remark 44. In Case 1 of the proof of Lemma 43, when T is split, we assumed z to be a $\mathbb{C}$-linear combination of the Bernstein projectors associated to the Bernstein components determined by the $\hat{\psi}$, as $\hat{\psi}$ runs over characters of $\mathrm{T}(\mathfrak{O})$ extending $\dot{\psi}$. Since z vanished on all but finitely many Bernstein components and had support in the set of compact elements of $\mathrm{G}(F)$, this assumption is justified according to [BK16, Theorem 1.3]. In our simple case when G is a tame SL_2 and z is an element of $\mathcal{Z}(\mathrm{G})$ supported on $\Pi(\mathrm{T}, \dot{\psi})$, this can also be seen more elementarily, directly from the computation of [MT02, (3.4.2)], using the fact that the multiset of characters of $\mathrm{T}(\mathfrak{O})$ given by:

$$\{\hat{\psi} \in \widehat{\mathrm{T}(\mathfrak{O})} \mid \hat{\psi}|_{\mathrm{T}_{r+}} = \dot{\psi}\} \cup \{\hat{\psi}^{-1} \in \widehat{\mathrm{T}(\mathfrak{O})} \mid \hat{\psi}|_{\mathrm{T}_{r+}} = \dot{\psi}\}$$

is actually a set, and hence its members are linearly independent.

Remark 45. We now sketch an explanation of how Theorem 2 answers 'Question A' in the introduction. Thus, let $z \in \mathcal{Z}(\mathrm{G})$ be such that $z(\pi_1) = z(\pi_2)$ whenever the Langlands parameters of π_1 and π_2 agree on the upper ramification subgroup $I^{r+}(F)$ of the Weil group W_F of F. We need to show that $z(\pi_1) = z(\pi_2)$ whenever π_1 and π_2 either both belong to $\hat{\mathrm{G}}_{\mathrm{adm}, \leq r}$, or they both belong to $\Pi(\mathrm{T}, \dot{\psi})$ for a maximal torus $\mathrm{T} \subset \mathrm{G}$ and $\mathbb{1} \neq \dot{\psi} \in \hat{\mathrm{T}}_r$. Thus, it suffices to show that, if π_1, π_2 both belong to $\hat{\mathrm{G}}_{\mathrm{adm}, \leq r}$ or some $\Pi(\mathrm{T}, \dot{\psi})$, then their Langlands parameters agree on $I^{r+}(F)$. If $\pi_1, \pi_2 \in \hat{\mathrm{G}}_{\mathrm{adm}, \leq r}$, then this follows from the fact that their Langlands parameters are trivial on $I^{r+}(F)$, thanks to the depth preservation property of the local Langlands correspondence for tame SL_2 (a much more general result is proved in [ABPS16]). Thus, let us consider $\Pi(\mathrm{T}, \dot{\psi})$ for some maximal torus $\mathrm{T} \subset \mathrm{G}$ and a character $\mathbb{1} \neq \dot{\psi} \in \hat{\mathrm{T}}_r$. We need to show that the elements of $\Pi(\mathrm{T}, \dot{\psi})$ all have Langlands parameters with the same restriction to $I^{r+}(F)$, or equivalently, that they arise by parabolic induction (if T is split) or endoscopic transfer (if T is elliptic) from characters of $\mathrm{T}(F)$ that have the same restriction to T_{r+} (to see this equivalence, use the fact that the local Langlands corresondence for tame tori preserves depth ([Yu09]), and is a homomorphism of abelian groups $\widehat{\mathrm{T}(F)} \cong \mathrm{Hom}_{\mathrm{admissible}}(W_F, {}^L\mathrm{T}) \cong H^1(W_F, \hat{\mathrm{T}})$). This is obvious if T is split. If T is elliptic, since the representations in $\Pi(\mathrm{T}, \dot{\psi})$ all have positive depth (even if $r = 0$), this follows from Lemma 21.

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