SOME COMMENTS ON THE STABLE BERNSTEIN CENTER

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Abstract. We study the stable Bernstein center of a connected reductive p-adic group G, following Vogan, Haines and others. We give a proof that some of its conjectured properties, such as the stable center conjecture, and its realization in the quasi-split case as the ring of regular functions on the variety of infinitesimal characters for G, follow from expected properties of the local Langlands correspondence, which are known when G is quasi-split classical. We then formulate a general form of Haines' $\mathcal{Z}$-transfer conjecture, which we prove to follow from expected properties of endoscopic transfer of tempered L-packets. Towards these results, we introduce a notion of atomically stable discrete series L-packets, and discuss criteria for detecting them. We also prove a weak but unconditional variant of Shahidi's constancy of the Plancherel μ-function in an L-packet, as well as of its transfer across inner forms. We show that these considerations can be used to give a weak form of the unrefined local Langlands correspondence for inner forms of classical and odd general spin groups.

1. Introduction

1.1. **The stable Bernstein center and three candidates for it.** Let G be a connected reductive group defined over a finite extension F of $\mathbb{Q}_p$, where p is a prime number. Let $\mathcal{Z}(\mathrm{G})$ and $\Omega(\mathrm{G})$ respectively denote the Bernstein center and the Bernstein variety of G (see, e.g., [BDK86, Hai14, BKV15, BKV16]), so that $\mathcal{Z}(\mathrm{G})$ identifies with the ring $\mathbb{C}[\Omega(\mathrm{G})]$ of regular functions on $\Omega(\mathrm{G})$.

The stable Bernstein center $\mathcal{Z}_{\mathrm{st}}(\mathrm{G})$ of G is a subring of $\mathcal{Z}(\mathrm{G})$ that seems to inform the study of the local Langlands conjectures and related topics such as stability and endoscopy. In fact, according to [BKV15, the introduction], studying it can provide both a 'supporting evidence' and a 'step in the proof of the local Langlands conjecture'. At the end of this subsection, we will attempt to add a brief commentary on this point.

Since the work of Vogan in [Vog93], which is the earliest reference on this topic that the author is aware of, several conjectural descriptions of what should deserve to be called the stable Bernstein center have emerged, only some of which are obviously contained in $\mathcal{Z}(\mathrm{G})$. In this paper, we will study the 'equality' of three such candidates, of which the first two are the harmonic analytically defined complex vector spaces $\mathcal{Z}_1(\mathrm{G}), \mathcal{Z}_2(\mathrm{G}) \subset \mathcal{Z}(\mathrm{G})$ below, and the third is the coordinate ring $\mathbb{C}[\Omega({}^L\mathrm{G})]$ of the variety of infinitesimal characters as defined by Vogan or Haines (see [Vog93], [Hai14]; we will follow the treatment of the latter):

Definition 1.1.1. (i) Let $\mathcal{Z}_1(\mathrm{G}) \subset \mathcal{Z}(\mathrm{G})$ be the *vector subspace* of elements z that, viewed as distributions on $\mathrm{G}(F)$, are stable.

(ii) Let $\mathcal{Z}_2(\mathrm{G}) \subset \mathcal{Z}(\mathrm{G})$ be the $\mathbb{C}$-*subalgebra* of elements z such that whenever $f \in C_c^\infty(\mathrm{G}(F))$ is unstable, so is $z * f$.

(iii) Let $\Omega({}^L\mathrm{G})$ denote the variety of infinitesimal characters for G, defined in [Hai14] (see towards the end of [Hai14, Section 5.3], where this variety is denoted $\mathfrak{Y}$), and let $\mathbb{C}[\Omega({}^L\mathrm{G})]$ denote its coordinate ring. (Here and henceforth, all references to the paper [Hai14] will follow the arxiv version of the paper, whose numbering system the author prefers).

Let us first discuss the relation between $\mathcal{Z}_1(\mathrm{G})$ and $\mathcal{Z}_2(\mathrm{G})$. We assume for the rest of this subsection (to keep this introduction simple) that G is quasi-split.

Remark 1.1.2. (i) While a priori $\mathcal{Z}_1(\mathrm{G})$ is only a $\mathbb{C}$-vector subspace of $\mathcal{Z}(\mathrm{G})$, $\mathcal{Z}_2(\mathrm{G})$ is a $\mathbb{C}$-subalgera of $\mathcal{Z}(\mathrm{G})$.

(ii) $\mathcal{Z}_2(\mathrm{G}) \subset \mathcal{Z}_1(\mathrm{G})$: since $z(f) = z * \check{f}(1)$, this follows from the fact that $f \mapsto f(1)$ is a stable distribution (see [Kot88, Proposition 1]).

A weak form of the stable center conjecture, namely [BKV15, Conjecture 3.1.4(a)], says:

Conjecture 1.1.3. $\mathcal{Z}_1(\mathrm{G}) \subset \mathcal{Z}(\mathrm{G})$ is a subalgebra.

By Remark 1.1.2, this conjecture follows from the following conjecture, which is thus a stronger form of the stable center conjecture:

Conjecture 1.1.4. $\mathcal{Z}_2(\mathrm{G}) = \mathcal{Z}_1(\mathrm{G})$.

As mentioned in the introduction of [BKV15], one expects $\mathcal{Z}_1(\mathrm{G})$ to be the set of elements in $\mathcal{Z}(\mathrm{G})$ with the property that, if π_1, π_2 are tempered representations of $\mathrm{G}(F)$ belonging to the same L-packet, then z acts as multiplication by the same scalar on π_1 and π_2. Note that this would transparently yield Conjecture 1.1.3 as well. It is easy to turn this comment into an easy proof of the stronger Conjecture 1.1.4, for those groups for which tempered L-packets have been defined and shown to satisfy the appropriate stability properties.
Namely, according to the formalism of Langlands and Arthur, one expects that the set $\mathrm{Irr}_{\mathrm{temp}}(\mathrm{G})$ of isomorphism classes of irreducible tempered representations of $\mathrm{G}(F)$ can be partitioned into finite subsets, called tempered L-packets, such that each such packet Σ supports a nonzero stable virtual character Θ_Σ, and such that the Θ_Σ form a basis for the space of stable tempered virtual characters on $\mathrm{G}(F)$. One makes a slightly more precise requirement: for each Levi subgroup M of G, one asks for a partition of the set $\mathrm{Irr}_2(\mathrm{M})$ of isomorphism classes of irreducible unitary square-integrable (modulo center) representations of $\mathrm{M}(F)$ into 'discrete series L-packets' Σ each supporting a nonzero stable virtual character Θ_Σ, and one asks for these Θ_Σ to constitute a basis for the space of stable elliptic virtual characters on $\mathrm{M}(F)$. If this condition is satisfied, we will say that G satisfies the existence of tempered L-packets (thus implicitly assumed to have the appropriate stability properties). In the body of the present paper, this requirement is stated as Hypothesis 2.7.1, and referred to as the existence of tempered L-packets. Let us remark that in the body of the paper, including in Hypothesis 2.7.1, we work with a system $\{\mathcal{O}_\mathrm{M}\}_\mathrm{M}$ of automorphisms of Levi subgroups of G (essentially to deal with outer automorphisms of groups such as SO_{2n} or GSO_{2n}), but to keep the introduction simple, we will assume all these groups to be trivial.
Thus, one of the aims of this paper is to show the following fact, which is probably known to many experts but which the author cannot find in literature:

Theorem 1.1.5. *If* G *satisfies the existence of tempered L-packets, then Conjecture 1.1.4 is true, i.e.,* $\mathcal{Z}_2(\mathrm{G}) = \mathcal{Z}_1(\mathrm{G})$ *(and hence, so is Conjecture 1.1.3).*

This theorem is contained in the more precisely stated Theorem 5.4.2.
As is well-known, in the case of quasi-split orthogonal and symplectic groups, the monumental work of Arthur in [Art13] gives us such a description of tempered L-packets as well as a proof of character identities satisfied by them, while the work of Mok [Mok15] adapts the work of Arthur to quasi-split unitary groups. In [Xu18], Bin Xu proves analogous results for quasi-split general symplectic groups, and a weaker version for even general special orthogonal groups involving an outer automorphism. On the other hand, the work [Mg14] of Mœglin deals with quasi-split general spin groups in addition to the quasi-split classical groups considered by Arthur and Mok, and, proves a slightly weakened form of the character theoretic properties that these packets are expected to satisfy; her results for even special orthogonal and even general spin groups too involve an outer automorphism (and as far as the author understands, the results of [Mg14] do not depend on the twisted weighted fundamental lemma for non-split groups, or the articles referred to in [Art13] as [A25], [A26] or [A27]). All these results are strictly stronger than the hypotheses necessary for Theorem 1.1.5, so that we can deduce Conjecture 1.1.4, and hence consequently also Conjecture 1.1.3, for quasi-split symplectic, special orthogonal, unitary, general symplectic and odd general spin groups, and a weaker result involving an outer automorphism for general special orthogonal groups; see Corollary 7.4.1. However, due to some technical reasons, we do not treat the case of even general spin groups (essentially because we do not yet know if certain transfer factors relevant to it are invariant under the appropriate outer automorphism).
Having discussed the relation between the subrings $\mathcal{Z}_1(\mathrm{G})$ and $\mathcal{Z}_2(\mathrm{G})$ of $\mathbb{C}[\Omega(\mathrm{G})] = \mathcal{Z}(\mathrm{G})$, we now discuss their relation with $\mathbb{C}[\Omega({}^L\mathrm{G})]$, again following the treatment of Haines in [Hai14]. For this,

let us first recall the identification $\mathcal{Z}(\mathrm{G}) = \mathbb{C}[\Omega(\mathrm{G})]$. Note that each $z \in \mathcal{Z}(\mathrm{G})$ acts by a scalar $\hat{z}(\pi)$ on each irreducible smooth representation π of $\mathrm{G}(F)$, and hence determines a function $\hat{z} : \mathrm{Irr}(\mathrm{G}) \to \mathbb{C}$, $\mathrm{Irr}(\mathrm{G})$ denoting the set of isomorphism classes of irreducible admissible representations of $\mathrm{G}(F)$. On the other hand, the Bernstein variety $\Omega(\mathrm{G})$ is the set of $\mathrm{G}(F)$-conjugacy classes of cuspidal pairs (M, σ) consisting of a Levi subgroup $\mathrm{M} \subset \mathrm{G}$ and a supercuspidal representation $\sigma \in \mathrm{Irr}(\mathrm{M})$. One can show that for each $z \in \mathcal{Z}(\mathrm{G})$, the associated function $\hat{z} : \mathrm{Irr}(\mathrm{G}) \to \mathbb{C}$ factors through the cuspidal support map $\mathrm{Irr}(\mathrm{G}) \to \Omega(\mathrm{G})$, identifying $\mathcal{Z}(\mathrm{G})$ with the ring $\mathbb{C}[\Omega(\mathrm{G})]$ of regular functions in $\Omega(\mathrm{G})$.

On the other hand, assuming the existence of a local Langlands correspondence for G and its Levi subgroups satisfying some reasonable extra compatibilities that Haines calls LLC+ for G (see [Hai14, Definition 5.2.1]), one can show that the map sending $\pi \in \mathrm{Irr}(\mathrm{G})$ to its infinitesimal character $\lambda(\varphi_\pi) \in \Omega({}^L\mathrm{G})$, where φ_π is the Langlands parameter of π and $\lambda(\varphi_\pi)$ is the well-defined $\hat{\mathrm{G}}$-conjugacy class of homomorphisms $W_F \to {}^L\mathrm{G}$ given by

$$\lambda(\varphi_\pi)(w) = \left(\varphi_\pi\left(w, \begin{pmatrix} \|w\|^{1/2} & \\ & \|w\|^{-1/2} \end{pmatrix}\right)\right), \tag{1}$$

descends to a regular morphism (i.e., morphism of algebraic varieties) $p_1 : \Omega(\mathrm{G}) \to \Omega({}^L\mathrm{G})$, that is surjective (see [Hai14, Proposition 5.5.1] or Proposition 4.3.2 of this paper, recalling that G is quasi-split for this introduction). Dually, we get the required embedding $p_1^* : \mathbb{C}[\Omega({}^L\mathrm{G})] \hookrightarrow \mathbb{C}[\Omega(\mathrm{G})] = \mathcal{Z}(\mathrm{G})$ of coordinate rings.

Then a conjecture of Scholze and Shin, at least in its strong form as interpreted in [Hai14, Remark 5.5.4], says:

Conjecture 1.1.6. The image $p_1^*(\mathbb{C}[\Omega({}^L\mathrm{G})]) \subset \mathbb{C}[\Omega(\mathrm{G})] = \mathcal{Z}(\mathrm{G})$ of p_1^* equals $\mathcal{Z}_2(\mathrm{G})$.

See [Hai14, Remark 5.5.4] for an explanation of why expected properties relating the local Langlands correspondence to stable characters imply the inclusion "$\subset$" in Conjecture 1.1.6. To deal with this, we need at least three hypotheses: the existence of tempered L-packets (i.e., with appropriate stability properties, as in Hypothesis 2.7.1), LLC+ for G (Hypothesis 2.10.3), and the compatibility between LLC and tempered L-packets (Hypothesis 2.10.12).

However, the inclusion "$\supset$" does not seem clear from the discussion so far, so we make use of a hypothesis that the Langlands parameters of L-packets that contain only supercuspidal representations are trivial on the $\mathrm{SL}_2(\mathbb{C})$-factor of the Weil-Deligne group $W_F' = W_F \times \mathrm{SL}_2(\mathbb{C})$ (see Hypothesis 2.11.1); here again we recall that G is quasi-split. With these, we prove in Corollary 5.5.2:

Proposition 1.1.7. *Assume the existence of tempered L-packets, LLC+, the compatibility between LLC and tempered L-packets, and that the Langlands parameters of supercuspidal packets take the expected form (see Hypotheses 2.7.1) 2.10.3, 2.10.12 and 2.11.1). Then Conjecture 1.1.6 is valid, i.e., the image $p_1^*(\mathbb{C}[\Omega({}^L\mathrm{G})])$ of p_1^* equals $\mathcal{Z}_2(\mathrm{G}) = \mathcal{Z}_1(\mathrm{G})$.*

Since the hypotheses assumed in the proposition are known for classical groups by the work of Arthur, Mœglin and Mok (see Proposition 7.2.4), up to an outer automorphism in the even special orthogonal case, we conclude in Proposition 7.4.4:

Proposition 1.1.8. *If* G *is a quasi-split symplectic, odd special orthogonal or unitary group, then Conjecture 1.1.6 is valid, and the same holds up to an outer automorphism in the even special orthogonal case.*

To use in the proof of Proposition 1.1.7, assuming the existence of tempered L-packets, we introduce a new hypothesis called the existence of a 'stable cuspidal support' (Hypothesis 2.11.4), inspired by the notion of an extended cuspidal support due to Mœglin (see, e.g., [Mg14, Section 4]) but slightly different in its precise mechanics. Namely, if one starts with a pair (M, Σ) consisting of a Levi subgroup $\mathrm{M} \subset \mathrm{G}$ and an essentially square integrable packet Σ on $\mathrm{M}(F)$, replaces it by another such pair (M', Σ') such that $\mathrm{M}' \subset \mathrm{M}$ and such that some $\sigma \in \Sigma$ is an irreducible subquotient of a representation parabolically induced from some $\sigma' \in \Sigma'$, and iterates this process, the hypothesis says that the set of possibilities for the final pair (L, Υ) should constitute a single orbit under $\mathrm{M}(F)$-conjugation (one can also consider more general packets instead of just essentially square integrable ones). One can show that this hypothesis follows from Hypotheses 2.7.1, 2.10.3, 2.10.12 and 2.11.1. Assuming the existence of tempered L-packets as well as of stable cuspidal supports, we define a map $p_2 : \Omega(\mathrm{G}) \to \Omega^{\mathrm{st}}(\mathrm{G})$ from $\Omega(\mathrm{G})$ to a 'harmonic analytic' variant $\Omega^{\mathrm{st}}(\mathrm{G})$ of $\Omega({}^L\mathrm{G})$ consisting of conjugacy classes of pairs (M, Σ) where $\mathrm{M} \subset \mathrm{G}$ is a Levi subgroup and Σ is a supercuspidal packet for $\mathrm{M}(F)$, and prove that the hypothesis on the existence of a stable cuspidal support implies that $p_2^*(\mathbb{C}[\Omega^{\mathrm{st}}(\mathrm{G})]) \subset \mathbb{C}[\Omega(\mathrm{G})] = \mathcal{Z}(\mathrm{G})$ equals $\mathcal{Z}_2(\mathrm{G})$ (see Proposition 5.5.1).

One might ask how far one could proceed in a direction converse to that of Theorem 1.1.5: how much can we expect $\mathcal{Z}_2(\mathrm{G})$ to tell us about tempered L-packets? For this, fix a $\mathbb{C}$-algebra homomorphism $\varrho : \mathcal{Z}_2(\mathrm{G}) \to \mathbb{C}$ (one might refer to it as a 'stable infinitesimal character'), and assume that there exists at least one $\sigma \in \mathrm{Irr}_2(\mathrm{G})$ on which $\mathcal{Z}_2(\mathrm{G})$ acts via ϱ. Then the set

$$\Sigma_\varrho := \{\sigma \in \mathrm{Irr}_2(\mathrm{G}) \mid \hat{z}(\sigma) = \varrho(z) \, \forall \, z \in \mathcal{Z}_2(\mathrm{G})\}$$

may be a priori infinite, but since its elements all have the same central character (because of an obvious embedding of the center $\mathrm{Z_G}(F)$ of $\mathrm{G}(F)$ in $\mathcal{Z}_2(\mathrm{G})$),

$$\Theta_{\Sigma_\varrho} := \sum_{\sigma \in \Sigma_\varrho} d(\sigma)\Theta_\sigma$$

is well-defined (use Remark 2.2.5), where $d(\sigma)$ denotes the formal degree of σ. Using the Plancherel formula and the fact that $f \mapsto f(1)$ is a stable distribution, it is easy to see that Θ_{Σ_ϱ} is stable. The point is that, if we know all the hypotheses we have discussed so far (Hypotheses 2.7.1, 2.10.3, 2.10.12 and 2.11.1), so that $\mathcal{Z}_2(\mathrm{G}) = p_1^*(\mathbb{C}[\Omega({}^L\mathrm{G})])$, then it is easy to see that the discrete series L-packets on G are precisely the Θ_{Σ_ϱ} as above: in other words, a discrete series L-packet or more generally a tempered L-packet is determined by its infinitesimal character, i.e., the set of tempered Langlands parameters inject via the map $\varphi \mapsto \lambda(\varphi)$ of (1) into the set $\Omega({}^L\mathrm{G})$, just as the set of Arthur parameters injects into that of Langlands parameters.

A variant of this last assertion in the context of classical groups is [Mg14, Remark 4.1], where (up to accounting for a possible outer automorphism for even special orthogonal and general spin groups) one has, in place of the infinitesimal character, an equivalent notion in the form of the extended cuspidal support: [Mg14, Theorem 4.14] uses precisely the extended cuspidal support to determine discrete series packets on classical groups. Looking at discrete series packets this way also seems analogous to the fact that Harish-Chandra's classification of discrete series packets for real groups was by means of their infinitesimal character.

From this perspective, one could also ask if one can define a local Langlands correspondence by defining homomorphisms $\varrho_\varphi : \mathcal{Z}_2(\mathrm{G}) \to \mathbb{C}$ dictated by infinitesimal characters of Langlands parameters φ, and realize the L-packet Σ_φ corresponding to a discrete series parameter φ as simply Σ_{ϱ_φ}. But this is already a well-known idea: at least if one replaces $\mathcal{Z}_2(\mathrm{G})$ by a possibly different ring, this is the pattern seen in the local Langlands correspondence as defined by Scholze for GL_n ([Sch13]), by Genestier and Lafforgue over local function fields ([GL17]), and by Fargues and Scholze over local fields including those over characteristic zero, with one key difference: the elements of the Bernstein center constructed in [GL17] and [FS21], using excursion operators, are not known to be stable (though this may be known by the work of Bertoloni Meli, Hamann and Nguyen, for odd unitary groups associated to an unramified extension of $\mathbb{Q}_p$ as well as for GSp_4; see Remark 7.4.5).

Going back to the Σ_ϱ, it may be fair to say that the reason we cannot define a useful notion of discrete series L-packets using $\mathcal{Z}_2(\mathrm{G})$ yet, is that we do not know $\mathcal{Z}_2(\mathrm{G})$ to be large enough. On the other hand, it is probably easier to produce elements in $\mathcal{Z}_1(\mathrm{G})$ than in $\mathcal{Z}_2(\mathrm{G})$, though to use $\mathcal{Z}_1(\mathrm{G})$ to define Σ_ϱ as above we would also need to know that $\mathcal{Z}_1(\mathrm{G})$ is an algebra, and not just a subspace, of $\mathcal{Z}(\mathrm{G})$, as asserted by the weaker form of the stable center conjecture, namely, Conjecture 1.1.3. In [BKV16], R. Bezrukavnikov, D. Kazhdan and Y. Varshavsky show that if p is 'very good' for G, then for all $r \geq 0$, what they call the depth r projector (and denote by E_r) belongs to $\mathcal{Z}_1(\mathrm{G})$. Let us remark as an aside that, in future work with Li and Oi, we hope to prove that when $p \gg 0$, the depth r projector of [BKV16] belongs to $\mathcal{Z}_2(\mathrm{G})$, which will at least imply that each Σ_ϱ defined as above, with $\varrho : \mathcal{Z}_2(\mathrm{G}) \to \mathbb{C}$ a ring homomorphism, is finite.

1.2. **The $\mathcal{Z}$-transfer conjecture.** Haines formulated forms of the $\mathcal{Z}$-transfer conjecture (see [Hai14, Conjectures 6.2.2 and 6.2.3]) to study how the Bernstein center behaves with respect to endoscopy. Let H be a connected reductive group over F that is endoscopic to G. One knows that there is a 'transfer' of functions from $C_c^\infty(\mathrm{G}(F))$ to $C_c^\infty(\mathrm{H}(F))$ dictated by a certain matching of orbital integrals, under simplifying hypotheses for this introduction to let us avoid having to involve a z-extension. Dually, distributions on $\mathrm{H}(F)$ that are stable transfer to distributions on $\mathrm{G}(F)$. It is a nontrivial consequence of [Art96], generalized in [MW16, Chapter XI] to the twisted setting, that under this map, stable virtual tempered characters on $\mathrm{H}(F)$ transfer to virtual tempered characters on $\mathrm{G}(F)$. In contrast, the elements of the stable Bernstein center, when looked at as elements of the Bernstein center rather than as distributions, work differently: at least when G is quasi-split, one can expect a ring homomorphism $\mathcal{Z}_2(\mathrm{G}) \to \mathcal{Z}_2(\mathrm{H})$ (i.e., in the opposite direction), which has the following property: if $z \mapsto z_\mathrm{H}$ according to this homomorphism, and if $f \in C_c^\infty(\mathrm{G}(F))$ and $f^\mathrm{H} \in C_c^\infty(\mathrm{H}(F))$ have matching orbital integrals, then so do $z * f$ and $z_\mathrm{H} * f^\mathrm{H}$. As Haines explains in [Hai14, Section 6.2], the $\mathcal{Z}$-transfer conjecture contains, formally, the "the fundamental lemma implies spherical transfer" result, and hopefully, more generally and using slightly different terminology, results of the "the fundamental lemma for the unit elements implies the fundamental lemma for the entire Hecke algebra" type results. Thus, it should also give some 'philosophical explanation' for why a result of Lemaire and Mishra ([LM20]), with the Weyl-averaging in its formulation, looks the way it does. In this context, Haines remarks that this conjecture should help study the transfer of the test functions that lend their name to the test function conjecture (see [Hai14, Conjecture 6.1.1]). However, without the luxury of a local Langlands correspondence, we are unable to see a description for such a transfer $z \mapsto z_\mathrm{H}$ that is at least as transparent as that of matching of orbital integrals.

It is also worth mentioning that a sub-case of the $\mathcal{Z}$-transfer conjecture concerns the endsocopic 'Jacquet-Langlands-type' or 'Deligne-Kazhdan-Vigneras-type' transfer from a quasi-split form G^* of G to G. In this case, one expects a ring homomorphism $\mathcal{Z}_2(\mathrm{G}^*) \to \mathcal{Z}_2(\mathrm{G})$ (thus, this time the source is on the side of the endoscopic group) with the following property: if z^* has image z and if $f \in C_c^\infty(\mathrm{G}(F))$ and $f^* \in C_c^\infty(\mathrm{G}^*(F))$ have matching orbital integrals, then so do $z * f$ and $z^* * f^*$. This reversal of direction is not surprising, because it is for quasi-split groups that the condition of relevance for Langlands parameters is automatic, and hence for a possibly non-quasi-split group G, it is $\mathcal{Z}_2(\mathrm{G}^*)$ and not $\mathcal{Z}_2(\mathrm{G})$ that $\mathbb{C}[\Omega({}^L\mathrm{G})]$ can be expected to equal. We are not even sure if $\mathcal{Z}_2(\mathrm{G}^*)$ has a greater claim than $\mathcal{Z}_2(\mathrm{G})$ does for being called the stable Bernstein center of G.

Moreover, in this case, the description of the map $z^* \mapsto z$ should be no less transparent than the transfer of stable distributions from G^* to G: if $e(\mathrm{G})$ denotes the Kottwitz sign of G, and the measures on $\mathrm{G}^*(F)$ and $\mathrm{G}(F)$ are chosen compatibly (see [Kot88, page 631]), then z, as a distribution, should simply be the product of $e(\mathrm{G})$ and the endoscopic transfer of the stable distribution z^* from G^* to G, though in practice it is not at all obvious that this transfer belongs to $\mathcal{Z}(\mathrm{G})$.

There are at least two situations in which the $\mathcal{Z}$-transfer conjecture is known completely: the case of base-change for GL_n, proved in [Hai14, Proposition 6.2.4], and the Jacquet-Langlands or Deligne-Kazhdan-Vigneras transfer between GL_n and an inner form of it (see [Coh18]).

In [Hai14], the $\mathcal{Z}$-transfer conjecture is stated only in two situations: in the situation of standard ('untwisted') endoscopy and for the subring of $\mathcal{Z}(\mathrm{G})$ that Haines calls the 'geometric Bernstein center' (see [Hai14, Definition 6.2.1]), and also in the situation of base-change, but for all elements of the stable Bernstein center in its $\mathbb{C}[\Omega({}^L\mathrm{G})]$ realization. In [Hai14, Section 6.2.1], Haines remarks that one could formulate a general $\mathcal{Z}$-transfer conjecture involving a map $\mathbb{C}[\Omega({}^L\mathrm{G})] \to \mathbb{C}[\Omega({}^L\mathrm{H})]$ dual to the obvious map $\Omega({}^L\mathrm{H}) \to \Omega({}^L\mathrm{G})$, provided one can show that the latter map is algebraic.

We study the $\mathcal{Z}$-transfer conjecture in Section 6 in a general twisted endoscopic setting, considering an endoscopic datum $(\mathrm{H}, \mathcal{H}, \tilde{s})$ for a pair $(\mathrm{G}, \tilde{\mathrm{G}}, \mathbf{a})$, with $\mathbf{a}$ a cocycle giving to a unitary character $\omega : \mathrm{G}(F) \to \mathbb{C}^\times$, where $\mathcal{H}$ may not be isomorphic to ${}^L\mathrm{H}$. Choosing auxiliary data that includes a z-extension $\mathrm{H}_1 \to \mathrm{H}$ of H with kernel C_1 and a unitary character μ of $\mathrm{C}_1(F)$, we show that the embedding $\mathcal{H} \hookrightarrow {}^L\mathrm{G}$ yields a map $\Omega({}^L\mathrm{H}_1)_\mu \to \Omega({}^L\mathrm{G})$ of varieties, where $\Omega({}^L\mathrm{H}_1)_\mu \subset \Omega({}^L\mathrm{H}_1)$ is the closed subvariety defined by a condition reflecting the stipulation that the central character should restrict to μ on $\mathrm{C}_1(F)$. Thus, assuming various conjectures, the corresponding map $\mathbb{C}[\Omega({}^L\mathrm{G})] \to \mathbb{C}[\Omega({}^L\mathrm{H}_1)_\mu]$ of coordinate rings should identify with a map $\mathcal{Z}_2(\mathrm{G}) \to \mathcal{Z}_2(\mathrm{H}_1)_\mu$, where $\mathcal{Z}_2(\mathrm{H}_1)_\mu$ is a quotient of $\mathcal{Z}_2(\mathrm{H}_1)$ with which the latter ring acts on those smooth representations of $\mathrm{H}_1(F)$ on which $\mathrm{C}_1(F)$ acts via μ.

In any case, assuming various conjectures, let us make $\mathbb{C}[\Omega({}^L\mathrm{G})]$ (resp., $\mathbb{C}[\Omega({}^L\mathrm{H}_1)_\mu]$) act on functions, distributions and representations of $\mathrm{G}(F)$ (resp., functions, distributions and representations of $\mathrm{H}_1(F)$ on which $\mathrm{C}_1(F)$ acts via μ) through its map to $\mathcal{Z}_2(\mathrm{G})$ (resp., to $\mathcal{Z}_2(\mathrm{H}_1)_\mu$). We then prove the following theorem, which we state very informally, referring to Theorem 6.2.3 and Proposition 6.3.1 for more precise statements and the body of the paper for the notation.

Theorem 1.2.1. *(i) If we assume that* G *and* H_1 *satisfy the existence of tempered* L*-packets, an LLC+ that is compatible with these tempered* L*-packets as well as with the endoscopic transfer* $SD_\mu(\tilde{\mathrm{H}}_1) \to D(\tilde{\mathrm{G}}, \omega)$ *of tempered characters, and* $z \mapsto z_1$ *denotes the map* $\mathbb{C}[\Omega({}^L\mathrm{G})] \to \mathbb{C}[\Omega({}^L\mathrm{H}_1)_\mu]$*, then whenever* $f \in C_c^\infty(\tilde{\mathrm{G}}(F))$ *and* $f_1 \in C_\mu^\infty(\tilde{\mathrm{H}}_1(F))$ *have matching orbital integrals, so do* $z * f$ *and* $z_1 * f_1$*.*

(ii) Assume that G *and a quasi-split inner form* G^* *satisfy the existence of tempered* L*-packets, and that these packets transfer in the expected way between (Levi subgroups of)* G^* *and* G*. Give* $\mathrm{G}(F)$ *and* $\mathrm{G}^*(F)$ *compatible Haar measures. Then the map* $D^* \mapsto e(\mathrm{G})D$*, where the distribution* D *on* $\mathrm{G}(F)$ *is the endoscopic transfer of the stable distribution* D^* *on* $\mathrm{G}^*(F)$*, and* $e(\mathrm{G})$ *is the Kottwitz sign of* G*, restricts to a map* $\mathcal{Z}_2(\mathrm{G}^*) \to \mathcal{Z}_2(\mathrm{G})$ *that is a ring homomorphism, and satisfies the property that whenever* $f \in C_c^\infty(\mathrm{G}(F))$ *and* $f^* \in C_c^\infty(\mathrm{G}^*(F))$ *have matching orbital integrals, so do* $z * f$ *and* $z^* * f^*$*.*

(i) of the theorem is very soft and formal modulo deep results of [Art96] and [MW16], while (ii) can be viewed as a generalization of a result in [Coh18].

Let us make some comments on the case where F is replaced by the field $\mathbb{R}$. In this case, one might be interested in the center of the category of $(\mathfrak{g}, K)$-modules with a given central character. This ring accepts a homomorphism from the center $\mathcal{Z}(\mathfrak{g})$ of the universal enveloping algebra of G. Therefore, at least a partial analogue of the $\mathcal{Z}$-transfer conjecture in this situation could be [MW16, I.2.8, Corollary], which shows that endoscopic transfer respects an appropriate homomorphism $\mathcal{Z}(\mathfrak{g}) \to \mathcal{Z}(\mathfrak{h})$ of universal enveloping algebras. However, D. Prasad pointed us to the paper [MS08] by Muic and Savin, which shows that if G is quasi-split but not split, then the center of the category of $(\mathfrak{g}, K)$-modules with central character χ is much larger than the center of the enveloping algebra. We do not know if, accordingly, a more general archimedean analogue of the $\mathcal{Z}$-transfer conjecture exists.

Another interesting and unconditional "$\mathcal{Z}$-transfer" result is that of Waldspurger in [Wal21], that the depth zero projector behaves well with respect to endoscopy: if H is endoscopic to G, p is large enough, and $f \in C_c^\infty(\mathrm{G}(F))$ and $f^{\mathrm{H}} \in C_c^\infty(\mathrm{H}(F))$ have matching orbital integrals, then so do $E_0 * f$ and $E_{0,\mathrm{H}} * f^{\mathrm{H}}$, where E_0 and $E_{0,\mathrm{H}}$ are respectively the depth zero projectors on $\mathrm{G}(F)$ and $\mathrm{H}(F)$ (this follows from [Wal21, Theorem 1 and Theorem 2]). In other words, maps $\mathcal{Z}_2(\mathrm{G}) \to \mathcal{Z}_2(\mathrm{H})$ as discussed earlier should take the depth zero projector to the depth zero projector. In future work with Li and Oi, we hope to prove similar results for depth r projectors with $r > 0$, and for other Bernstein projectors.

1.3. **Other results: atomically stable packets, Plancherel measures, inner forms of classical groups, and depth preservation.** Let us describe a few more results in the paper, that were arrived at while making the above considerations. The proof of Theorem 1.1.5 follows Shahidi's proof of the constancy of the Plancherel μ-function on discrete series L-packets of Levi subgroups, under an assumption almost equivalent to (perhaps slightly weaker than) our assumption on the existence of tempered L-packets (see [Sha90, Proposition 9.3]). Indeed, Shahidi studied the Plancherel expansion of $f \mapsto f(1)$ to show that if Σ is a discrete series packet on a Levi subgroup $\mathrm{M} \subset \mathrm{G}$, then $\mu(\sigma_1) = \mu(\sigma_2)$ for all $\sigma_1, \sigma_2 \in \Sigma$, where μ is the Plancherel μ-function associated to the parabolic induction from M to G (see [Wal03, Section V]). Similarly, given $z \in \mathcal{Z}_1(\mathrm{G})$, the proof that $z \in \mathcal{Z}_2(\mathrm{G})$ (under the assumption that tempered L-packets exist) goes through first showing that if Σ is a discrete series packet on a Levi subgroup $\mathrm{M} \subset \mathrm{G}$, then $\hat{z}(\sigma_1) = \hat{z}(\sigma_2)$ for all $\sigma_1, \sigma_2 \in \Sigma$, where $\hat{z}(\sigma_i)$ refers to the value of $\hat{z}$ on any irreducible subquotient of a representation of $\mathrm{G}(F)$ parabolically induced from σ_i. The reason this helps is that one can then combine this with Arthur's deep result that if $f \in C_c^\infty(\mathrm{G}(F))$ satisfies that $\Theta(f) = 0$ for all stable tempered virtual characters Θ on $\mathrm{G}(F)$, then f is unstable, i.e., all its stable orbital integrals vanish (this follows from [Art96, Theorems 6.1 and 6.2], but it may be more convenient to see this from the statement of the twisted version given in [MW16, Corollary XI.5.2]), and the fact that the existence of tempered L-packets gives a simple description for the set of stable tempered virtual characters. While Shahidi considers the Plancherel expansion of $f \mapsto f(1)$, here one considers the expansion of $f \mapsto z(f^\vee) = z * f(1)$, where $f^\vee$ is given by $f^\vee(g) = f(g^{-1})$, to get the constancy of $\sigma \mapsto \hat{z}(\sigma)\mu(\sigma)$ on Σ, from which the constancy of $\sigma \mapsto \hat{z}(\sigma)$ on Σ follows (using Shahidi's result mentioned above, which can be recovered by taking z to be the Dirac measure at the identity element).

We find it convenient to use the above Plancherel expansion argument to prove the following unconditional result on the way, one that does not depend on a strong assumption like the existence of tempered L-packets (see Corollary 5.2.11(i) for more details):

Proposition 1.3.1. *Let $z \in \mathcal{Z}_1(\mathrm{G})$. Let $\zeta : \mathrm{A}_\mathrm{M}(F) \to \mathbb{C}^\times$ be a smooth unitary character, where A_M is the maximal split torus contained in the center of M. Write $\mathrm{Irr}_{\zeta,2}(\mathrm{M})$ for the subset of $\mathrm{Irr}_2(\mathrm{M})$ consisting of representations whose central character restricts to ζ on $\mathrm{A}_\mathrm{M}(F)$. Then*

$$\sum_{\sigma \in \mathrm{Irr}_{\zeta,2}(\mathrm{M})} d(\sigma)\hat{z}(\sigma)\mu(\sigma)\Theta_\sigma, \tag{2}$$

which makes sense as a distribution on $\mathrm{G}(F)$ by Remark 2.2.5, is stable.

This suggests that, to conclude the equality $\hat{z}(\sigma_1)\mu(\sigma_1) = \hat{z}(\sigma_2)\mu(\sigma_2)$ for two given representations σ_1, σ_2 that belong to a candidate discrete series L-packet (such as a Kaletha packet), we might not need the full strength of the existence of tempered L-packets: it will suffice if we know that the function $d(\sigma_1)^{-1}f_{\sigma_1} - d(\sigma_2)^{-1}f_{\sigma_2}$ is unstable, where f_{σ_i} is a pseudocoefficient for σ_i among those representations of $\mathrm{M}(F)$ whose central character restricts to ζ on $\mathrm{A}_\mathrm{M}(F)$.

Thus, we consider what we call 'atomically stable' discrete series L-packets, adapting terminology from [MY20]: a finite subset $\Sigma \subset \mathrm{Irr}_2(\mathrm{M})$ is atomically stable if it supports a nonzero stable virtual character Θ_Σ with the property that every stable elliptic virtual character Θ on $\mathrm{M}(F)$ can be uniquely written as $c\Theta_\Sigma + \Theta'$, where c is a scalar and Θ' is supported outside Σ. This notion is a natural one and hence is almost certainly well-known to experts, but we could not find a reference in literature. We also warn the reader that it is different from the notion with the same name in [Kal22, Conjecture 2.2] (we apologize for the clash of terminology). One then shows that the Plancherel μ-function as well as $\hat{z}$, for any $z \in \mathcal{Z}_1(\mathrm{G})$, are constant on any atomically stable discrete series packet on a Levi subgroup of G. While pursuing these considerations, it is not hard to see that for any atomically stable discrete series packet Σ, Θ_Σ is a scalar multiple of $\sum d(\sigma)\Theta_\sigma$, with σ running over Σ and $d(\sigma)$ denoting the formal degree of σ.

If tempered L-packets are known to exist, then it is easy to see that every discrete series packet on a Levi subgroup is atomically stable. It turns out that examples can be given even when tempered L-packets are not known to exist. We describe two ways to check that a given finite set of discrete series representations constitutes an atomically stable discrete series packet. Again, this should be known to experts, since these two results are very simple consequences of [Art96], but we were unaware of them and could not find them in literature. It was a remark of Mœglin in [Mg14, Section 4.8] that suggested the first to us, and it was [LMW18, Section 4.6, Lemma 3] that suggested the second.

To describe the first, let M be a connected reductive group over F. Given a virtual discrete series character Θ on $\mathrm{M}(F)$, recalling that it is completely determined by the values it takes as a locally constant function on the set $\mathrm{M}(F)_{\mathrm{ell}}$ of strongly regular *elliptic* semisimple elements of $\mathrm{M}(F)$, let us denote by $\Theta^{\mathrm{st}} : \mathrm{M}(F)_{\mathrm{ell}} \to \mathbb{C}$ the function that takes γ to the average of the $\Theta(\gamma')$ as γ' varies over representatives for the conjugacy classes in the stable conjugacy class of γ. It is not obvious that $\Theta^{\rm st}$ is the set of values taken by any virtual character on $\mathrm{M}(F)_{\rm ell}$, but one knows from a deep result of [Art96] that it is so. The first way to detect atomic stability is as follows (see Proposition 3.4.2):

Proposition 1.3.2. *A finite subset $\Sigma \subset \mathrm{Irr}_2(\mathrm{M})$ is an atomically stable discrete series packet if and only if the following two conditions are satisfied:*

- *The $\Theta^{\rm st}_\sigma$, as σ varies over Σ, are all proportional to each other; and*
- *Some linear combination of the Θ_σ with all coefficients nonzero, as σ varies over Σ, is a stable distribution.*

As remarked earlier, the proof is not hard: if we assume for simplicity that M is semisimple, the result follows easily once one computes that $\Theta^{\rm st}_\sigma$ is the image of Θ_σ under the projection map $D_{\rm ell}(\mathrm{M}) \to SD_{\rm ell}(\mathrm{M})$, where $D_{\rm ell}(\mathrm{M})$ is the space of elliptic virtual characters on $\mathrm{M}(F)$, $SD_{\rm ell}(\mathrm{M}) \subset D_{\rm ell}(\mathrm{M})$ is the subspace of stable elliptic virtual characters on $\mathrm{M}(F)$, and the projection is with respect to the elliptic inner product.

The second way to detect atomic stability is only a sufficient condition, which we state slightly informally and imprecisely as follows; see Proposition 3.4.11 for the more precise statement:

Proposition 1.3.3. *If a finite subset $\Sigma \subset \mathrm{Irr}_2(\mathrm{M})$ has a crude 'endoscopic decomposition', in the sense that we can write*

$$\sum_{\sigma\in\Sigma} \mathbb{C}\Theta_\sigma = \sum_{\underline{\mathrm{H}}} \mathbb{C}\Theta^{\mathrm{M}}_{\underline{\mathrm{H}}},$$

where $\underline{\mathrm{H}}$ runs over a set of distinct relevant elliptic endoscopic data for M and $\Theta^{\mathrm{M}}_{\underline{\mathrm{H}}}$ is the transfer to $\mathrm{M}(F)$ of some stable elliptic virtual character on (a z-extension of) $\mathrm{H}(F)$ via $\underline{\mathrm{H}}$, then Σ is an atomically stable discrete series packet.

This proposition follows easily from the result in [Art96] (though we follow the exposition in [LMW18]) that endoscopic transfer from stable elliptic virtual characters on relevant elliptic endoscopic groups gives us a decomposition of $D_{\rm ell}(\mathrm{M})$ that is orthogonal for the elliptic inner product. This way of detecting atomic stability is harder to implement, but has the advantage that the necessary work has already been done by Kaletha in the case of regular supercuspidal packets when $p \gg 0$.

Thus, we conclude that when $p \gg 0$, Kaletha's regular supercuspidal packets are atomically stable. This implies (see Remark 3.4.13 for a few more details) a weak compatibility result, comparing Kaletha's local Langlands correspondence with those of Arthur, Mœglin and Mok: when $p \gg 0$, Kaletha's regular supercuspidal packets on quasi-split special orthogonal and symplectic (resp., unitary) groups are also packets in the sense of [Art13] (resp., [Mok15]); an analogous comment applies with [Mg14] in place of [Art13] and [Mok15], provided one accounts for an outer automorphism in the case of even special orthogonal groups. However, we do not have any result on the compatibility between the relevant Langlands parametrizations.

Recall that already in [Sha90], Shahidi had proposed that his proof of the constancy of the μ-function on discrete series packets on Levi subgroups should generalize to transfer to inner forms, and should thus make local Langlands-Shahidi L-functions available for inner forms, modulo the generic packet conjecture. Such transfers have been known by the work of Choiy and Heiermann (see [Cho14] and [Hei16]).

Inspired by the transfer of μ-functions across inner forms by Choiy and Heiermann, one can ask if the stability of (2) given by Proposition 1.3.1 can be enhanced to a transfer between inner forms. This leads to the following proposition, which we state informally and refer to Corollary 5.2.11(ii) for more details:

Proposition 1.3.4. *Given an inner twist between* G *and its quasi-split inner form* G^* *that transfers a Levi subgroup* $\mathrm{M} \subset \mathrm{G}$ *to a Levi subgroup* $\mathrm{M}^* \subset \mathrm{G}^*$, (2) *above generalizes to:*

$$\sum_{\sigma^*\in\mathrm{Irr}_{\zeta,2}(\mathrm{M}^*)} d(\sigma^*)\hat{z}(\sigma^*)\mu(\sigma^*)\Theta_{\sigma^*} \text{ transfers to (scalar)} \cdot \sum_{\sigma\in\mathrm{Irr}_{\zeta,2}(\mathrm{M})} d(\sigma)\hat{z}(\sigma)\mu(\sigma)\Theta_\sigma. \tag{3}$$

The scalar in the above proposition is given sort of explicitly in Corollary 5.2.11(ii). Among other things, it involves the Kottwitz sign $e(\mathrm{G}) = e(\mathrm{M})$ of G.

While Proposition 1.3.1 used that $f \mapsto f(1)$ is stable (by [Kot88, Proposition 1]), Proposition (1.3.4) uses that the distribution $f^* \mapsto f^*(1)$ on $\mathrm{G}^*(F)$ transfers to the product of $e(\mathrm{G})$ and the distribution $f \mapsto f(1)$ on $\mathrm{G}(F)$ (by [Kot88, Proposition 2]). Instead of using the result of [Art96] that the instability of a function can be checked on stable characters, one uses that whether or not two functions have matching orbital integrals can be checked by seeing that various (non-explicit) character identities are satisfied; this follows from [Art96, Lemma 6.3], as explained in [LM20]: see the equivalence of the conditions (A) and (B) in page 587 of that reference.

Given Propositions 1.3.1 and 1.3.4, the following informally stated proposition, whose first (resp., second) assertion generalizes Shahidi's constancy of the μ-function on discrete series L-packets (resp., the transfer of μ-functions as in the works of Choiy and Heiermann), is not hard to see; we refer to Corollary 5.2.12 for more details:

Proposition 1.3.5. *(i) If Σ is an atomically stable discrete series packet on a Levi subgroup $\mathrm{M} \subset \mathrm{G}$, and $z \in \mathcal{Z}_1(\mathrm{G})$, then for all $\sigma_1, \sigma_2 \in \Sigma$ we have $\mu(\sigma_1) = \mu(\sigma_2)$ and $\hat{z}(\sigma_1) = \hat{z}(\sigma_2)$.*

(ii) In the setting of Proposition 1.3.4, if an atomically stable discrete series packet Σ^* on M^* transfers to an atomically stable discrete series packet Σ on M in a sense that is not hard to formulate, then for all $\sigma \in \Sigma$ and $\sigma^* \in \Sigma^*$, $\mu^*(\sigma^*)$ is the product of $\mu(\sigma)$ and an explicit constant that does not depend on Σ^* or Σ. If moreover $z \in \mathcal{Z}(\mathrm{G})$ is a transfer of $z^* \in \mathcal{Z}_1(\mathrm{G}^*)$ in the sense that as a distribution on $\mathrm{G}(F)$, z is the product of $e(\mathrm{G})$ and the endoscopic transfer of z^* viewed as a distribution on $\mathrm{G}^*(F)$, then for all $\sigma^* \in \Sigma^*$ and $\sigma \in \Sigma$, we have

$$\hat{z}^*(\sigma^*) = \hat{z}(\sigma). \tag{4}$$

In particular, when $p \gg 0$, the Plancherel μ-function associated to regular supercuspidal packets on Levi subgroups transfers well across inner forms.

We discuss two applications for Proposition 1.3.5. First, the arguments of Shahidi following [Sha90, Conjecture 9.4] should now make available the normalization of intertwining operators using Langlands-Shahidi L-functions, for those atomically stable discrete series packets on $\mathrm{G}(F)$ that transfer to atomically stable discrete series packets on the quasi-split form $\mathrm{G}^*(F)$ that can be shown to be generic, and in particular for regular supercuspidal packets when $p \gg 0$. For some more explanation, see Subsubsection 5.3.1.

Let us also remark that there is a much more delicate and subtle strengthening of the aforementioned transfer of Plancherel μ-functions, called the local intertwining relation (and which is due to Arthur), addressing which is beyond the scope of this paper. One can hope that forthcoming work of Kaletha will shed light on it. Let us also take this opportunity to mention that a 'relatively local' approach towards proving some special cases of the local intertwining relation when the induced representation is irreducible, is given by the Goldberg-Shahidi method of residues of intertwining operators: see [Sha92] and [Var]. We hope that, at least in some very special situations, and assuming $p \gg 0$, it could yield, by 'relatively local' methods, an answer to a question that the above considerations bring to the fore: whether the Langlands-Shahidi L-functions and ϵ-factors associated to regular supercuspidal packets on Levi subgroups agree with the corresponding Artin L-functions and ϵ-factors associated to the Langlands parameters assigned to them by Kaletha.

To state the second application of Proposition 1.3.5, fix $r \geq 0$ and let $E_r \in \mathcal{Z}(\mathrm{G})$ be the depth r projector in the sense of [BKV16]; thus, for an irreducible admissible representation σ of $\mathrm{G}(F)$, $\hat{E}_r(\sigma)$ equals 1 or 0 depending on whether or not the depth of σ is at most r. Assuming $p \gg 0$, the second application of Proposition 1.3.5 gives the constancy of depth (see [MP96]) on atomically stable discrete series L-packets, and the fact that transfer of atomically stable discrete series L-packets across inner forms respects depth; we refer to Corollary 5.3.4 and Proposition 5.3.5 for more details.

Proposition 1.3.6. *(i) If p is very good for G in the sense of [BKV16], and Σ is an atomically stable discrete series packet on a Levi subgroup $\mathrm{M} \subset \mathrm{G}$, then the elements of Σ have the same depth.*

(ii) *Let* G^* *be a quasi-split inner form of* G*, and let* E_r^* *be the depth* r *projector on it. Assume* p *to be very good for* G*, and that* $\mathfrak{g}$ *has a bilinear form that behaves well with respect to its Moy-Prasad filtrations (as in* [AR00, Proposition 4.1]*). Then:*

  (a) E_r^* *belongs to* $\mathcal{Z}_1(\mathrm{G}^*)$ *and transfers as a stable Bernstein center element to* E_r*, in the sense that when viewed as distributions, and with* $\mathrm{G}(F)$ *and* $\mathrm{G}^*(F)$ *given compatible Haar measures,* E_r^* *transfers to* $e(\mathrm{G})E_r$*.*

  (b) *Moreover, if we are in the setting of Proposition 1.3.4, and if an atomically stable discrete series packet* Σ^* *on* M^* *transfers to an atomically stable discrete series packet* Σ *on* M*, then for all* $\sigma \in \Sigma$ *and* $\sigma^* \in \Sigma^*$*, we have* $\mathrm{depth}(\sigma) = \mathrm{depth}(\sigma^*)$*.*

(i) of the above proposition is an immediate consequence of the stability of E_r (as given by [BKV16], since p is 'very good' for G) and Proposition 1.3.5(i). As for (ii) of the above proposition, the assertion (a) is Proposition 5.3.5, while the assertion (b) follows from the assertion (a) and Proposition 1.3.5(ii). In future joint work with Li and Oi alluded to above, we hope to generalize (ii)(a) of the above proposition to an assertion about the behavior of the depth r projector with respect to endoscopic transfer.

Now we come to the question of extracting more mileage from (2) and (3). The former can be interpreted as saying that, if M is a Levi subgroup of G and ζ is a unitary smooth character of $\mathrm{A}_{\mathrm{M}}(F)$, then Θ and $T(\Theta)$ are stable, where

$$\Theta = \sum_{\sigma\in \mathrm{Irr}_{\zeta,2}(\mathrm{M})} d(\sigma)\Theta_\sigma, \qquad \text{and} \qquad T(\sum_{\sigma\in \mathrm{Irr}_{\zeta,2}(\mathrm{M})} a_\sigma\Theta_\sigma) = \sum_{\sigma\in \mathrm{Irr}_{\zeta,2}(\mathrm{M})} a_\sigma \mu(\sigma)\Theta_\sigma,$$

where Θ is interepreted as an element and T as a linear transformation, of the vector space consisting of 'possibly infinite linear combinations' of characters of elements of $\mathrm{Irr}_{\zeta,2}(\mathrm{M})$.

Thus, we can try to fix M and vary G, or rather fix G and realize it as a factor of Levi subgroups of varying larger groups G^+. This is not possible in all situations, but classical and general spin groups and their inner forms are especially suited to such considerations: if G is such a group, it can be written as some G_n for a series $\{\mathrm{G}_m\}_m$ of groups of the same type as G, and for appropriate $m > n$, G_m has Levi subgroups that are a product of G with either a product of inner forms of general linear groups or a product of restrictions of scalars of general linear groups. Combining this with the multiplicativity of the Plancherel measure ([Wal03, Lemma V.2.1]), one can get a ring of operators on a vector space containing Θ as above. Taking simultaneous eigendecomposition then yields 'packets' of discrete series representations of $\mathrm{G}(F)$ such that σ_1 and σ_2 are in the same packet if and only if a condition of the following form is satisfied: $\mu(\sigma_1 \otimes \tau) = \mu(\sigma_2 \otimes \tau)$ for all τ belonging to a certain collection of representations of inner forms of groups of GL-type.

This brings us to the consideration of characterizing Langlands parameters using Plancherel measures in the spirit of Gan and Takeda (see [GT11]). More precisely, the idea is to characterize discrete series L-packets on the quasi-split inner form G^* of G, as defined by Arthur, Mœglin or Mok, up to an outer automorphism in the even special orthogonal case, by those expressions of the form $\mu(\sigma^* \otimes \tau^*)$ that transfer in an appropriate sense to G, and appeal to (3) to transfer these packets to G. For a technical reason mentioned before, we do not consider even general spin groups. These considerations give us the following very informally stated theorem, where we refer to Theorems 7.3.3 and 7.3.10 for the more precise statements:

Theorem 1.3.7. *If* G *is an inner form of a quasi-split classical or odd general spin group* G^* *over* F*, and if* G^* *is not even special orthogonal, there is a crude local Langlands correspondence for discrete series representations of* $\mathrm{G}(F)$*, which is characterized using Plancherel measures, and whose fibers are atomically stable discrete series L-packets. When* G^* *is even special orthogonal, a weaker analogue involving an outer automorphism applies.*

In the above theorem, even for a precise statement of the latter assertion, special care is needed for those inner forms of even special orthogonal groups that Arthur refers to in [Art13, Chapter 9] as not symmetric. We emphasize that much finer results on the local Langlands correspondence than Theorem 1.3.7 are already known in all these cases except possibly for inner forms of odd general spin groups, by [Art13, Chapter 9], the work of Mœglin and Renard ([MR18]), the work of Kaletha, Minguez, Shin and White ([KMSW14]), and the work of Ishimoto ([Ish23]). For non-quasi-split inner forms of unitary groups, there is also a characterization using the theta correspondence, in the work of Chen and Zou ([CZ20]). We also remark that if $p \gg 0$, and G is an inner form of a quasi-split symplectic or odd special orthogonal group G^* over F, regular supercuspidal L-packets on $G(F)$ as defined by Kaletha are also L-packets in the sense of Theorem 1.3.7: see Remark 7.3.17 for more details.

A result of Oi in [Oi22] allows us to deduce from Proposition 1.3.6, but with some extra effort in some cases related to inner forms of even special orthogonal groups, the following, where we refer to Corollary 7.4.3 for more details:

Proposition 1.3.8. *Let* G *be an inner form of a special orthogonal, symplectic or unitary group, and assume that p is larger than a certain explicit constant depending only on the absolute root datum of* G*. If* G *is quasi-split, the local Langlands correspondence for* G *as defined by Arthur, Mœglin or Mok (defined up to an outer automorphism in the even special orthogonal case) preserves depth. In general, the same applies to the crude local Langlands correspondence of Theorem 1.3.7.*

In the case of non-quasi-split unitary groups, the above proposition is already known by the work of Oi ([Oi21]).

Now let us discuss the organization of this paper. After setting some notation and reviewing some results (Subsections 2.1, 2.2, 2.3, 2.4, 2.5), we fix in Subsection 2.6 a system $\{\mathcal{O}_{\mathrm{M}}\}_{\mathrm{M}}$ indexed by Levi subgroups M of a fixed arbitrary connected reductive group G over our p-adic field F, where each $\mathcal{O}_{\mathrm{M}}$ is a subgroup of the group Aut(M) of (F-rational) automorphisms of M, imposing some hypotheses on the collection $\{\mathcal{O}_{\mathrm{M}}\}_{\mathrm{M}}$. In Subsection 2.7, we spell out our hypothesis on the existence of tempered L-packets with appropriate stability properties (Hypothesis 2.7.1), draw some obvious consequences and set related notation. In Subsections 2.8 and 2.9 we set notation and recall basic facts concerning Langlands parameters and infinitesimal characters. In Subsection 2.10, we state our LLC+ hypothesis (Hypothesis 2.10.3), which involves discrete series representations, and explain its extension to more general representations using Langlands classification. In Subsection 2.11, we discuss the remaining main hypotheses — the ones on the compatibility between stability and LLC, Langlands parameters for supercuspidal packets and the existence of stable cuspidal support (Hypotheses 2.10.12, 2.11.1 and 2.11.4) — and show that the third of these follow from the earlier hypotheses.

Section 3 deals with results about stable virtual elliptic characters and their endoscopic transfer, as well as with atomically stable discrete series L-packets. We set some notation and review some basic facts in Subsection 3.1. In Subsection 3.2, we discuss basic facts about stable distributions on a non-quasi-split group, such as why for any $f \in C_c^\infty(\mathrm{G}(F))$, the vanishing of $\Theta(f)$ for every stable tempered virtual character Θ implies that f is unstable. While these results are contained in [Art96], only in the quasi-split case do these results seem to be stated in [Art96] in the exact manner in which we want to use it, so what we do is to explain the process of deduction. We then introduce atomically stable discrete series L-packets in Subsection 3.3, and explain in Subsection 3.4 two ways of detecting atomic stability that follow from Arthur's formalism in [Art96].

Section 4 studies the three kinds of Bernstein varieties of inerest to this paper: the usual one $\Omega(\mathrm{G})$, the variety $\Omega({}^L\mathrm{G})$ of Vogan where we follow the approach of Haines, and the harmonic analytic variant $\underline{\Omega}^{\mathrm{st}}(\mathrm{G})$ of the latter (under some hypotheses). The definitions are recalled (or given in the case of $\underline{\Omega}^{\mathrm{st}}(\mathrm{G})$) in Subsection 4.1. Subsection 4.2 is an aside relating connected components of $\Omega({}^L\mathrm{G})$ to inertial Langlands parameters. In Subsection 4.3, we study maps $p_1 : \Omega(\mathrm{G}) \to \Omega({}^L\mathrm{G})$, $p_2 : \Omega(\mathrm{G}) \to \underline{\Omega}^{\mathrm{st}}(\mathrm{G})$ and $p_{12} : \underline{\Omega}^{\mathrm{st}}(\mathrm{G}) \to \Omega({}^L\mathrm{G})$, as well as their properties and relationship between them, under varying combinations of hypotheses. p_2 and p_{12} are only defined when G is quasi-split.

Section 5 is where we study the relationship between $\mathcal{Z}_1(\mathrm{G})$ and $\mathcal{Z}_2(\mathrm{G})$. In Subsection 5.1, we review basic facts concerning the Bernstein center $\mathcal{Z}(\mathrm{G})$ and review $\mathcal{Z}_1(\mathrm{G})$ and $\mathcal{Z}_2(\mathrm{G})$, and introduce the variant $\mathcal{Z}_{2,\mathcal{O}}(\mathrm{G})$ that takes the presence of $\{\mathcal{O}_{\mathrm{M}}\}_{\mathrm{M}}$ in our hypotheses to account. In Subsection 5.2, adapting Shahidi's work, we prove (2) and (3), and deduce Shahidi-style constancy on atomically stable discrete series packets for μ and $\hat{z}$ with $z \in \mathcal{Z}_1(\mathrm{G})^{\mathcal{O}}$. In Subsection 5.3, we give two applications to atomically stable discrete series packets on Levi subgroups: to the normalization of intertwining operators using Langlands-Shahidi L-functions, and the behavior of Moy-Prasad depth; the latter involves an easy proof showing that the of depth r projector of [BKV16] transfers well across inner forms. In Subsection 5.4, we deduce consequences such as Theorem 1.1.5. In Subsection 5.5, we relate these to the third candidate of our interest for the stable Bernstein center, namely, we study combinations of our hypotheses that show these to be the pull-back of $\mathbb{C}[\Omega({}^L\mathrm{G})]$ under p_1^* (assuming in particular that G is quasi-split).

In Section 6, we study the $\mathcal{Z}$-transfer conjecture. In Subsection 6.1, we discuss the aforementioned map $\Omega({}^L\mathrm{H}_1)_\mu \to \Omega({}^L\mathrm{G})$ and show that it is a map of algebraic varieties; this involves showing that $\Omega({}^L\mathrm{H}_1)_\mu$ is isomorphic to a variety $\Omega(\mathcal{H})$ that is defined analogously to $\Omega({}^L\mathrm{G})$, but with ${}^L\mathrm{G}$ replaced by the component $\mathcal{H}$ of the endoscopic datum under consideration. Using this map, and under a large number of hypotheses, we give a proof of the $\mathcal{Z}$-transfer conjecture in Subsection 6.2. In Subsection 6.3, we give a weak generalization of some of the work of Cohen in [Coh18] for general linear groups; namely, we show that the existence of tempered L-packets with appropriate stability properties for G and G^*, together with appropriate endoscopic transfers between them, implies that the product of $e(\mathrm{G})$ and endoscopic transfer at the level of distributions (where the measures on $\mathrm{G}(F)$ and $\mathrm{G}^*(F)$ are compatibly chosen), gives us the expected ring homomorphism $\mathcal{Z}_2(\mathrm{G}^*) \to \mathcal{Z}_2(\mathrm{G})$ that behaves well with respect to endoscopic transfer of functions.

Section 7 contains the results mentioned earlier concerning classical groups and their inner forms. Subsection 7.1 consists of general remarks about Hypothesis 2.7.1, not specific to classical groups. In Subsection 7.2, we explain how the work of Arthur, Mœglin and Mok give various hypotheses of interest (Hypotheses 2.7.1, 2.10.3, 2.10.12 and 2.11.1, and hence also Hypothesis 2.11.4) for quasi-split classical and odd general spin groups, up to an outer automorphism for even special orthogonal groups (some additional cases are covered for Hypothesis 2.7.1). One also deduces Hypothesis 2.7.1 for quasi-split general symplectic and even general special orthogonal groups from the work of Xu ([Xu18]), the latter only up to an outer automorphism. In Subsection 7.3, we prove Hypotheses 2.7.1, 2.10.3 and 2.10.12 for inner forms of classical and general spin groups, except that for inner forms of split even special orthogonal groups that are associated to a division algebra (i.e., for those inner forms of classical groups that are not symmetric in the sense of [Art13, Chapter 9]), one proves a slightly differently formulated version of these hypotheses, because the nontrivial outer automorphism does not have an F-rational lift in these cases. In Subsection 7.4, we summarize the consequences for classical groups and their inner forms in the form of the identification of the three forms of stable Bernstein center considered here, as well as depth preservation (and transfer of depth across inner forms under milder hypotheses). We also explain how, using the $\mathcal{Z}$-transfer conjecture, one can recover Mœglin's result, for quasi-split classical groups, that Langlands parameters of representations inside an Arthur packet all have the same infinitesimal character.

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2. Some notation, preliminaries, preparation, and the main hypotheses

Throughout this paper, notation that we define for a group will be applied with obvious modification to other groups. For instance, once we define the object $D(\mathrm{M})$ or $\Omega(\mathrm{G})$ associated to the connected reductive group M or G, then for any connected reductive group G′, we will use $D(\mathrm{G}')$ or $\Omega(\mathrm{G}')$ to denote the analogous object associated to G′.

2.1. Some notation.

2.1.1. *Miscellaneous notation.* For an abstract group $\mathscr{G}$ that acts on a mathematical object X, we will denote by $X^{\mathscr{G}}$ (resp., $X_{\mathscr{G}}$) the invariants (resp., the coinvariants) for the action of $\mathscr{G}$ on X, provided such a thing makes sense. For any mathematical object X, we will write $\mathrm{Aut}(X)$ for the group of automorphisms of X, when the meaning of 'automorphisms' is clear from the context. If $\mathscr{G}$ is a topological group, $\mathrm{Hom}_{\mathrm{cts}}(\mathscr{G}, \mathbb{C}^\times)$ will denote the group of (quasi-)characters of $\mathscr{G}$, i.e., of continuous homomorphisms $\mathscr{G} \to \mathbb{C}^\times$.
For a ring R, an R-algebra R', a module M over R and a scheme X over R, we will write $M_{R'}$ for $M \otimes_R R'$ and $X_{R'}$ or $X \times_R R'$ for the base-change of X from R to R'.
M^0 will denote the identity component of an algebraic group M defined over a field. The Lie algebra of an algebraic group denoted by a roman letter (e.g., G) will be denoted by the corresponding fraktur letter (e.g., $\mathfrak{g}$). If X is a variety (resp., algebraic group) over a valued field F, $\mathrm{X}(F)$ will be viewed as a topologial space (resp., topological group) with the "Hausdorff topology" associated to the valuation. Whenever X is a complex variety, we may abbreviate $\mathrm{X}(\mathbb{C})$ to X.

2.1.2. *Tori.* If T is a torus defined over a field F, we will denote by $X^*(\mathrm{T})$ (resp., $X_*(\mathrm{T})$) the character lattice (resp., the cocharacter lattice) of the base-change T_{F^s} of T to F^s, and view it together with the $\mathrm{Gal}(F^s/F)$-action on it, where F^s will be a separable closure of F that will be clear from the context. Moreover, given such a T, $\mathrm{A_T} \subset \mathrm{T}$ will denote the maximal split subtorus and $\mathrm{T} \to \mathrm{S_T}$ the maximal split quotient torus.

2.1.3. *Derived group, outer automorphisms etc.* If M is a connected reductive group over a field F, we will write $\mathrm{M_{der}}, \mathrm{M_{ad}}$ and $\mathrm{M_{sc}}$ respectively for the derived group of M, the adjoint group of M, and the simply connected cover of $\mathrm{M_{der}}$, respectively. Moreover, $\mathrm{Out(M)}$ will denote the group of outer automorphisms of the base-change $\mathrm{M}_{\bar F}$ of M to a suitable algebraic closure $\bar F$ of F (i.e., the group of all algebraic automorphisms of $\mathrm{M}_{\bar F}$ quotiented by the normal subgroup of the inner automorphisms $\mathrm{Int}\, m$, with m ranging over $\mathrm{M}(\bar F)$).

2.1.4. *Twisted spaces, center and related notation.* For any algebraic group M over a field F, $\tilde{\mathrm{M}}$ will usually denote a twisted space for M — recall that this means that $\tilde{\mathrm{M}}$ is an algebraic variety over F that is given commuting left and right M-actions, which we will write as $(m, \delta) \mapsto m\delta$ and $(\delta, m) \mapsto \delta m$, that are both simply transitive, and satisfying $\tilde{\mathrm{M}}(F) \neq \emptyset$. Note that $m_1\delta m_2$ has an unambiguous meaning for $m_1, m_2 \in \mathrm{M}(F)$ and $\delta \in \tilde{\mathrm{M}}(F)$, as either of the terms in the equality $(m_1\delta)m_2 = m_1(\delta m_2)$.
If δ is an element of a twisted space $\tilde{\mathrm{M}}$ over an algebraic group M over a field F, we will denote by $\mathrm{Int}\,\delta$ the unique automorphism of M such that $\delta \cdot m = \mathrm{Int}\,\delta(m) \cdot \delta$. $\mathrm{Ad}\,\delta$ will then denote the derivative of $\mathrm{Int}\,\delta$. $\mathrm{Z}_{\tilde{\mathrm{M}}}$ will denote the intersection of the kernels of the $\mathrm{Int}\,\delta$ as δ ranges over $\tilde{\mathrm{M}}$, and $\mathrm{A}_{\tilde{\mathrm{M}}}$ the maximal split torus contained in $\mathrm{Z}_{\tilde{\mathrm{M}}}$.
Often, an algebraic group M over a field F will be implicitly considered as a twisted space over itself using its left and right multiplication. The group $\mathrm{Z_M}$ thus defined coincides with the center of M, and, for each $\delta \in \mathrm{M}$, the automorphisms $\mathrm{Int}\,\delta$ and $\mathrm{Ad}\,\delta$ thus defined coincide with conjugation by δ and the adjoint action of δ, respectively.

2.1.5. *The p-adic field F and related notation.* Henceforth we fix a finite extension F of $\mathbb{Q}_p$ for some prime p, an algebraic closure $\bar F$ of F, and a uniformizer ϖ for the ring of integers of F. Let $\mathfrak{O} = \mathfrak{O}_F \subset F$ be the ring of integers of F, and q the cardinality of the residue field of F. Let $|\cdot| : \bar F \to \mathbb{R}$ denote the usual extension to $\bar F$ of the normalized absolute value on F. We will denote by $\Gamma := \mathrm{Gal}(\bar F/F)$ and by $W_F \subset \Gamma$ the absolute Galois group and the Weil group of F, by $I_F \subset W_F$ the inertia subgroup, and by $W'_F := W_F \times \mathrm{SL}_2(\mathbb{C})$ the Weil-Deligne group of F. Let $\mathrm{Fr} \in W_F/I_F$ stand for the element that induces the Frobenius automorphism of the residue field. We denote by $\|\cdot\| : W_F \to \mathbb{R}_{>0}$ the composite of the normalized absolute value on $F^\times$ and the abelianization homomorphism $W_F \to F^\times$ that is normalized to send (any representative for) Fr to a uniformizer in the ring of integers of F.

2.1.6. *Discrete series etc.* For a connected reductive group M over F, a 'discrete series representation of $\mathrm{M}(F)$' will refer to a unitary irreducible smooth representation of $\mathrm{M}(F)$ whose matrix coefficients are square-integrable modulo the center, while an 'essentially square-integrable representation of $\mathrm{M}(F)$' will refer to a twist of a discrete series representation of $\mathrm{M}(F)$ by a (not necessarily unitary) continuous (quasi-)character $\chi \in \mathrm{Hom}_{\mathrm{cts}}(\mathrm{M}(F), \mathbb{C}^\times)$. If M is a connected reductive group over F, we denote by $\mathrm{Irr}(\mathrm{M})$(resp., $\mathrm{Irr}_{\mathrm{temp}}(\mathrm{M})$; resp., $\mathrm{Irr}_2(\mathrm{M})$; resp., $\mathrm{Irr}_2^+(\mathrm{M})$) the set of isomorphism classes of irreducible representations of $\mathrm{M}(F)$ that are admissible (resp., tempered; resp., discrete series; resp., essentially square-integrable).
If $\mathcal{Z} \subset \mathrm{M}(F)$ is a central subgroup that is understood from the context, and $\zeta : \mathcal{Z} \to \mathbb{C}^\times$ is a smooth character, then we denote by $\mathrm{Irr}_2(\mathrm{M})_\zeta \subset \mathrm{Irr}_2(\mathrm{M}), \mathrm{Irr}_2^+(\mathrm{M})_\zeta \subset \mathrm{Irr}_2^+(\mathrm{M}), \mathrm{Irr}_{\mathrm{temp}}(\mathrm{M})_\zeta \subset \mathrm{Irr}_{\mathrm{temp}}(\mathrm{M})$ and $\mathrm{Irr}(\mathrm{M})_\zeta \subset \mathrm{Irr}(\mathrm{M})$ the subsets consisting of (isomorphism classes) of representations whose central character restricts to ζ on $\mathcal{Z}$.

2.1.7. *Levi subgroups and parabolic induction.* Let M be a connected reductive group over F. For each Levi subgroup $\mathrm{L} \subset \mathrm{M}$, we denote by $W_{\mathrm{M}}(\mathrm{L})$, and by $W(\mathrm{L})$ when M is understood from the context, the group of F-rational points of the quotient, of the normalizer of L in M, by L. Then every element of $W(\mathrm{L})$ can be represented by an element of $\mathrm{M}(F)$, letting us identify $W(\mathrm{L})$ with the quotient of the normalizer of $\mathrm{L}(F)$ in $\mathrm{M}(F)$, by $\mathrm{L}(F)$ (here is a quick argument: choosing a parabolic subgroup Q of M with L as a Levi subgroup and having unipotent radical N, $W(\mathrm{L})$ acts freely on the set of parabolic subgroups of M that are $\mathrm{M}(\bar{F})$-conjugate to Q and have L as a Levi subgroup; but the normalizer of $\mathrm{L}(F)$ in $\mathrm{M}(F)$ acts transitively on this set — use [Bor91, Proposition 20.5], which gives both the surjectivity of $\mathrm{M}(F) \to (\mathrm{Q}\backslash\mathrm{M})(F)$ and the fact that the set of Levi subgroups of Q is a torsor under $\mathrm{N}(F)$-conjugation).
Let $\mathrm{M}_1 \subset \mathrm{M}$ be a Levi subgroup (of a parabolic subgroup of M). Then for any parabolic subgroup $\mathrm{P}_1 \subset \mathrm{M}$ with Levi subgroup M_1, we will write $\mathrm{Ind}_{\mathrm{P}_1}^{\mathrm{M}}$ for the associated (normalized) parabolic induction functor, taking smooth representations of $\mathrm{M}_1(F)$ to smooth representations of $\mathrm{M}(F)$. The map induced by the functor $\mathrm{Ind}_{\mathrm{P}_1}^{\mathrm{M}}$ at the level of virtual characters is independent of the choice of P_1, and hence will be written $\mathrm{Ind}_{\mathrm{M}_1}^{\mathrm{M}}$. Sometimes we will refer to a subquotient of $\mathrm{Ind}_{\mathrm{M}_1}^{\mathrm{M}} \sigma$, by which we will mean a subquotient of $\mathrm{Ind}_{\mathrm{P}_1}^{\mathrm{P}} \sigma$ — this notion is independent of the choice of P_1, though $\mathrm{Ind}_{\mathrm{P}_1}^{\mathrm{M}} \sigma$ itself (and what its subrepresentations and quotient representations are) depends on P_1. We have not defined the twisted harmonic analytic version of the notation just introduced, as we will not need it.

2.1.8. *Twisted representations and virtual characters.* In this subsubsection, let M be an arbitrary reductive group over F and $\tilde{\mathrm{M}}$ a twisted space associated to M, with the property that for some $\delta \in \tilde{\mathrm{M}}(F)$, $\mathrm{Int}\,\delta$ is a semisimple automorphism of M whose restriction to $\mathrm{A}_{\tilde{\mathrm{M}}}$ is of finite order. Let $\omega : \mathrm{M}(F) \to \mathbb{C}^\times$ be a continuous character attached to a cocycle $\mathbf{a} \in H^1(W_F, \mathrm{Z}_{\hat{\mathrm{M}}})$ (in the sense referred to in Subsubsection 2.5.2 below). Assume that ω is unitary.
An ω-invariant distribution on $\tilde{\mathrm{M}}(F)$ refers to a $\mathbb{C}$-linear map $C_c^\infty(\tilde{\mathrm{M}}(F)) \to \mathbb{C}$ such that $D(\tilde{f} \circ \mathrm{Int}\, m) = \omega(m) D(\tilde{f})$ for all $m \in \mathrm{M}(F)$.
Recall that a representation of $(\tilde{\mathrm{M}}(F), \omega)$, or an ω-representation of $\tilde{\mathrm{M}}(F)$, is a representation (σ, V) of $\mathrm{M}(F)$ together with a map $\tilde{\sigma} : \tilde{\mathrm{M}}(F) \to \mathrm{Aut}_{\mathbb{C}}(V)$, such that $\tilde{\sigma}(m_1 \delta m_2) = \omega(m_2)\sigma(m_1)\tilde{\sigma}(\delta)\sigma(m_2)$ for all $m_1, m_2 \in \mathrm{M}(F)$ and $\delta \in \tilde{\mathrm{M}}(F)$. We will refer to σ as the representation of $\mathrm{M}(F)$ underlying $\tilde{\sigma}$. We will say that $\tilde{\sigma}$ is smooth or admissible or $\mathrm{M}(F)$-irreducible or of finite length if the representation σ of $\mathrm{M}(F)$ that underlies it, is (see [MgW18, Section 2.5]). Note that $\mathrm{M}(F)$-irreducibility is stronger than the 'obvious' notion of irreducibility. We will refer to $\tilde{\sigma}$ as tempered if $\tilde{\sigma}$ is unitary and σ is tempered.
Given an admissible representation $\tilde{\sigma}$ of $\tilde{\mathrm{M}}(F)$ (the underlying representation σ of $\mathrm{M}(F)$ being suppressed from the notation), we will denote by $\Theta_{\tilde{\sigma}}$ the (easily checked to be ω-invariant) distribution

$C_c^\infty(\tilde{\mathrm{M}}(F)) \to \mathbb{C}$ that takes $\tilde{f} \in C_c^\infty(\tilde{\mathrm{M}}(F))$ to $\operatorname{tr}\tilde{\sigma}(\tilde{f})$, where:

$$\tilde{\sigma}(\tilde{f}) = \left(v \mapsto \int_{\tilde{\mathrm{M}}(F)} \tilde{f}(\delta) \cdot \tilde{\sigma}(\delta) v \, d\delta\right),$$

for some fixed choice of a measure on $\tilde{\mathrm{M}}(F)$ obtained by transferring a Haar measure on $\mathrm{M}(F)$ via any isomorphism $\mathrm{M}(F) \to \tilde{\mathrm{M}}(F)$ obtained as $m \mapsto \delta \cdot m$ or $m \mapsto m \cdot \delta$ for some $\delta \in \tilde{\mathrm{M}}(F)$. All these isomorphisms indeed yield the same measure on $\tilde{\mathrm{M}}(F)$ that is independent of δ. One knows that any such $\Theta_{\tilde{\sigma}}$ can be realized by integration against a locally integrable function on $\tilde{\mathrm{M}}(F)$ that is locally constant on the set of regular semisimple elements of $\tilde{\mathrm{M}}(F)$ (see [LH17, Corollary 5.8.3] and use, as discussed in [LMW18, Section 3.1], that F has characteristic zero). By abuse of notation, we will use $\Theta_{\tilde{\sigma}}$ to also denote this function, called the Harish-Chandra character of $\tilde{\sigma}$. We can talk of formal complex linear combinations $\sum c_i\tilde{\sigma}_i$ of finite-length admissible ω-representations $\tilde{\sigma}_i$ of $\tilde{\mathrm{M}}(F)$, and thus make sense of character distributions or Harish-Chandra characters associated to such formal linear combinations as well: $\Theta_{\sum c_i\tilde{\sigma}_i} = \sum c_i\Theta_{\tilde{\sigma}_i}$. Such distributions and functions will be referred to as virtual characters associated to the ω-representation theory of $\tilde{\mathrm{M}}(F)$. Let Θ be such a virtual character. Θ is said to be supported on a set Σ of isomorphism classes of M-irreducible admissible ω-representations of $\tilde{\mathrm{M}}(F)$, if we can write $\Theta = \sum_i c_i\Theta_{\tilde{\sigma}_i}$ with $\tilde{\sigma}_i \in \Sigma$ for each i. Θ is said to be supported outside another such set Σ', if Σ can be chosen so that no $\mathrm{M}(F)$-representation underlying an element of Σ underlies an element of Σ'.

2.1.9. *Some spaces of distributions.*

Notation 2.1.1. Let $\mathrm{M}, \tilde{\mathrm{M}}, \omega$ be as in Subsubsection 2.1.8.

(i) Following [LMW18, Section 3.1], or the definition of "$D_{temp}(\tilde{G}(F), \omega)$" in [MgW18, Section 2.9], let $D(\tilde{\mathrm{M}}, \omega)$ denote the complex vector space of ω-invariant distributions on $\mathrm{M}(F)$ spanned by the characters of tempered $\mathrm{M}(F)$-irreducible ω-representations of $\tilde{\mathrm{M}}(F)$; it is spanned by characters of representations $\tilde{\sigma}_{\boldsymbol{\tau}}$ associated to certain triplets τ as in [MgW18, Section 2.9].

(ii) Following [MgW18, Section 2.12], we consider the subspace $D_{\mathrm{ell}}(\tilde{\mathrm{M}}, \omega) \subset D(\tilde{\mathrm{M}}, \omega)$ spanned by the characters of those $\tilde{\sigma}_{\boldsymbol{\tau}}$ such that the triplet τ is elliptic as defined in [MgW18, Section 2.11]; it is the twisted version of the analogous notion considered by Arthur.

(iii) We refer to [MW16] for the notion of orbital integrals $O(\gamma, \omega, \cdot)$, and their special cases $O(\gamma, \cdot) = O(\gamma, \mathbb{1}, \cdot)$, defined on appropriate function spaces (like suitable $C_\mu^\infty(\tilde{\mathrm{M}}(F))$ as below).

(iv) Let $\mathscr{Z} \subset \mathrm{M}(F)$ be a central subgroup, and $\mu : \mathscr{Z} \to \mathbb{C}^\times$ a continuous character. In such a situation we will use the following notation, often suppressing from the notation the dependence on $\mathscr{Z}$ when it is understood in the context.
- We will let $C_\mu^\infty(\tilde{\mathrm{M}}(F))$ be the space of smooth functions $f_1 : \tilde{\mathrm{M}}(F) \to \mathbb{C}$, compactly supported modulo $\mathscr{Z}$, such that $f_1(z_1\gamma_1) = \mu(z_1)^{-1}f_1(\gamma_1)$ for all $z_1 \in \mathscr{Z}$ and $\gamma_1 \in \tilde{\mathrm{M}}(F)$. If $\mathscr{Z}$ is not clear from the context, or if $\mathscr{Z} = \mathrm{C}_1(F)$ with $\mathrm{C}_1 \subset \mathrm{M}$ a central subgroup the dependence on which we do not wish to suppress, we will write $C_{\mathscr{Z},\mu}^\infty(\tilde{\mathrm{M}}(F))$ or $C_{\mathrm{C}_1,\mu}^\infty(\tilde{\mathrm{M}}(F))$ or $C_{\mathrm{C}_1(F),\mu}^\infty(\tilde{\mathrm{M}}(F))$ in place of $C_\mu^\infty(\tilde{\mathrm{M}}(F))$.
- We will let $D_{\mathscr{Z},\mu}(\tilde{\mathrm{M}}, \omega) = D_\mu(\tilde{\mathrm{M}}, \omega)$ (resp., $D_{\mathcal{Z},\mu,\mathrm{ell}}(\tilde{\mathrm{M}}, \omega) = D_{\mu,\mathrm{ell}}(\tilde{\mathrm{M}}, \omega)$) denote the subspace of $D(\tilde{\mathrm{M}}, \omega)$ (resp., $D_{\mathrm{ell}}(\tilde{\mathrm{M}}, \omega)$) generated by characters of ω-representations $(\pi, \tilde{\pi})$ of $\tilde{\mathrm{M}}(F)$ with the property that π has a central character that restricts to μ on $\mathscr{Z}$; this agrees with the notation in [LMW18, Sections 4.3 and 4.4].

(v) The above notation will be adapted, without further comment, to deal with usual invariant harmonic analysis — $D_{\mathrm{ell}}(\tilde{\mathrm{M}}), D_{\mathscr{Z},\mu}(\tilde{\mathrm{M}}), D_{\mu,\mathrm{ell}}(\tilde{\mathrm{M}})$ etc. will denote $D_{\mathrm{ell}}(\tilde{\mathrm{M}}, \mathbb{1})$, $D_{\mathscr{Z},\mu}(\tilde{\mathrm{M}}, \mathbb{1})$, $D_{\mu,\mathrm{ell}}(\tilde{\mathrm{M}}, \mathbb{1})$ etc., where $\mathbb{1}$ denotes the trivial character of $\mathrm{M}(F)$. Further, $D_{\mathrm{ell}}(\mathrm{M}), D_{\mathscr{Z},\mu}(\mathrm{M}), D_{\mu,\mathrm{ell}}(\mathrm{M})$ etc. will denote $D_{\mathrm{ell}}(\tilde{\mathrm{M}}'), D_{\mathscr{Z},\mu}(\tilde{\mathrm{M}}'), D_{\mu,\mathrm{ell}}(\tilde{\mathrm{M}}')$ etc., where $\tilde{\mathrm{M}}'$ equals M thought of as a twisted space over itself using left and right multiplication.

(vi) Now suppose further that we are in the case where $\tilde{\mathrm{M}}$ has the property that, for all $\delta \in \tilde{\mathrm{M}}(\bar{F})$, the automorphism $\operatorname{Int}\delta$ of M is inner (i.e., equal to $\operatorname{Int} m$ for some $m \in \mathrm{M}_{\mathrm{ad}}(\bar{F})$).

The latter property is what [MW16] refers to as $\tilde{\mathrm{M}}$ being 'á torsion intérieure'. Further, assume that ω is trivial, and that either M is quasi-split, or that we are in the case where the twisted space $\tilde{\mathrm{M}}$ is isomorphic to M acting on itself by left and right multiplication. [1] In this case:

- We refer to [KS99] or [MW16] for the notion of the stable orbital integrals $SO(\gamma, \cdot)$.
- In the setting of twisted endoscopy, a function belonging to $C_c^\infty(\tilde{\mathrm{M}}(F))$ or some suitable $C_\mu^\infty(\tilde{\mathrm{M}}(F))$, whose stable orbital integrals all vanish, will be called unstable. The condition that the stable orbitals vanish only needs to be checked at semisimple elements that are strongly regular in the sense of having an abelian centralizer.
- A stable distribution is one that vanishes on unstable functions (in the context of an appropriate space of distributions).
- Therefore, various spaces of distributions defined above have their stable variants, which are their subspaces consisting of those distributions that are stable: $SD(\tilde{\mathrm{M}}) \subset D(\tilde{\mathrm{M}}), SD_{\mathrm{ell}}(\tilde{\mathrm{M}}) \subset D_{\mathrm{ell}}(\tilde{\mathrm{M}}), SD_{\mathscr{Z},\mu,\mathrm{ell}}(\tilde{\mathrm{M}}) \subset D_{\mathscr{Z},\mu,\mathrm{ell}}(\tilde{\mathrm{M}})$ etc. Again, this makes sense of $SD_{\mathrm{ell}}(\mathrm{M}), SD(\mathrm{M})$ etc., thinking of M as a twisted space over itself under left and right multiplication.

(vii) Let $\mathscr{Z} \subset \mathrm{M}(F)$ be a central subgroup. Any choice of a Haar measure on $\mathscr{Z}$ gives us an obvious map $C_c^\infty(\tilde{\mathrm{M}}(F)) \to C_\mu^\infty(\tilde{\mathrm{M}}(F))$, through which the elements of $D_\mu(\tilde{\mathrm{M}}, \omega)$ and $D_{\mu,\mathrm{ell}}(\tilde{\mathrm{M}}, \omega)$ factor, letting us view $D_\mu(\tilde{\mathrm{M}}, \omega)$ and $D_{\mu,\mathrm{ell}}(\tilde{\mathrm{M}}, \omega)$ as linear forms on $C_\mu^\infty(\tilde{\mathrm{M}}(F))$. We will similarly make sense of $SD_\mu(\tilde{\mathrm{M}})$ and $SD_{\mu,\mathrm{ell}}(\tilde{\mathrm{M}})$ as linear forms on $C_\mu^\infty(\tilde{\mathrm{M}}(F))$, in those contexts in which we have defined SD (see (vi) above).

We will use the above notation only when μ is unitary.

2.2. **Review of miscellaneous results.** In this subsection, let M be a connected reductive group over F.

Definition 2.2.1. Let $\mathrm{Q} \subset \mathrm{M}$ be a parabolic subgroup, and $\chi : \mathrm{Q}(F) \to \mathbb{C}^\times$ an unramified character (this notion is recalled in Notation 2.5.1 below). Then χ is said to be Q-dominant if for some (or equivalently, any) maximal split torus A_0 of M contained in Q, and any coroot $\lambda : \mathbb{G}_m \to \mathrm{A}_0$ associated to a root of A_0 in the unipotent radical of Q (one knows that the coroots λ belong to $X_*(\mathrm{A}_0)$ and not just to $X_*(\mathrm{A}_0) \otimes \mathbb{Q}$), the character $\chi \circ \lambda : F^\times \to \mathbb{C}^\times$ is of the form $|\cdot|^s$, where the complex number s, well-defined modulo $2\pi i(\log q)^{-1}\mathbb{Z} \subset \mathbb{C}$, has a nonnegative real part.

The following notation will be used only in this section.

Notation 2.2.2. Let L be a Levi subgroup of a parabolic subgroup Q of M, and let $\upsilon \in \mathrm{Irr}_2^+(\mathrm{L})$. One knows that one can write $\upsilon = \upsilon' \otimes \chi'$, where $\upsilon' \in \mathrm{Irr}_2(\mathrm{L})$ and $\chi' : \mathrm{L}(F) \to \mathbb{C}^\times$ is an unramified character. We say that υ is Q-dominant if χ', viewed as a character $\mathrm{Q}(F) \to \mathbb{C}^\times$ by inflation, is. This notion is independent of the decomposition $\upsilon = \upsilon' \otimes \chi'$, since given two such decompositions $\upsilon' \otimes \chi'$ and $\upsilon'' \otimes \chi''$ of υ, $\chi'(\chi'')^{-1}$ is unitary.

We now recall the version of the Langlands classification involving essentially square-integrable representations:

Proposition 2.2.3. *(i) Given $\sigma \in \mathrm{Irr}(\mathrm{M})$, there exists a pair (L, υ) consisting of a Levi subgroup* L *of* M *and a representation $\upsilon \in \mathrm{Irr}_2^+(\mathrm{L})$, uniquely determined up to $\mathrm{M}(F)$-conjugacy, such that σ is an irreducible quotient (not just subquotient) of $\mathrm{Ind}_{\mathrm{Q}}^{\mathrm{M}} \upsilon$, where* Q *is a choice of a parabolic subgroup of* M *such that* Q *has* L *as a Levi subgroup and υ is* Q*-dominant (it is standard that such a* Q *exists).*

(ii) Sending σ to the $\mathrm{M}(F)$-conjugacy class of (L, υ) as in (i) gives a finite-to-one map from $\mathrm{Irr}(\mathrm{M})$ *to the set of $\mathrm{M}(F)$-conjugacy classes of pairs (L, υ) with $\mathrm{L} \subset \mathrm{M}$ a Levi subgroup and $\upsilon \in \mathrm{Irr}_2^+(\mathrm{L})$. Thus, we get a finite-to-one surjective map*

$$\mathrm{Irr}(\mathrm{M}) \to \bigsqcup_{\mathrm{L}} \mathrm{Irr}_2^+(\mathrm{L})/W_{\mathrm{M}}(\mathrm{L}), \tag{5}$$

[1] This combination of assumptions may not be very natural, but we stick to it for simplicity.

where L *runs over any set of representatives for the* $M(F)$*-conjugacy classes of Levi subgroups of* M *(and* $W_M(L)$ *is as in Subsubsection 2.1.7).*

(iii) For a pair (L, υ)*, with* L *occurring in* (5) *and* $\upsilon \in \mathrm{Irr}_2^+(L)$*, the fiber of* (5) *over the image of* υ *in* $\mathrm{Irr}_2^+(L)/W_L$ *consists of all the irreducible* quotients *of* $\mathrm{Ind}_Q^M \upsilon$*, where* Q *is any choice of a parabolic subgroup of* M *such that* L *is a Levi subgroup of* Q *and* υ *is* Q*-dominant.*

Proof. We omit the proof, since it is well-known, and can be found in [ABPS14, Theorem 1.2]. Let us remark that the proof combines two ingredients, the first being [Wal03, Proposition III.4.1], which asserts the existence of a finite-to-one surjective map defined similarly as in (5):

$$\mathrm{Irr}_{\mathrm{temp}}(M) \to \bigsqcup_{L} \mathrm{Irr}_2(L)/W_M(L). \tag{6}$$

(6) is the restriction of (5) to $\mathrm{Irr}_{\mathrm{temp}}(M)$, and its fibers have a description similar to the one given for (5) in Proposition 2.2.3(iii). The second ingredient is the usual statement of Langlands classification (e.g., [SZ18, Theorem 1.4]). □

Remark 2.2.4. We now recall some easy facts about stable virtual characters that we will use. Let $\sum_{\sigma\in\Sigma} c_\sigma \Theta_\sigma$ be a stable virtual character on $M(F)$, for some $\Sigma \subset \mathrm{Irr}(M)$.

(i) For any central subgroup $Z \subset M(F)$ and any smooth character $\chi : Z \to \mathbb{C}^\times$, $\sum_{\sigma\in\Sigma_\chi} c_\sigma\Theta_\sigma$ is also a stable virtual character, where $\Sigma_\chi \subset \Sigma$ is the subset consisting of representations whose central character restricts to χ on Z.

(ii) For any isomorphism $M \to M'$ of reductive groups over F, and any smooth character $\chi' : M'(F) \to \mathbb{C}^\times$ that is trivial on $M'_{\mathrm{der}}(F)$, $\sum_{\sigma\in\Sigma} c_\sigma \Theta_{(\sigma\circ\beta^{-1})\otimes\chi'} = ((\sum_{\sigma\in\Sigma} c_\sigma\Theta_\sigma)\circ\beta^{-1})\chi'$ is a stable virtual character on $M'(F)$.

Remark 2.2.5. Later, we will have use for distributions on $M(F)$ of the form:

$$\sum_{\sigma\in\mathrm{Irr}_2(M)_\zeta} c_\sigma\Theta_\sigma,$$

where $\zeta : \mathcal{Z} \to \mathbb{C}^\times$ is a smooth character of a central subgroup $\mathscr{Z}$ of $M(F)$ containing $A_M(F)$, and $c_\sigma \in \mathbb{C}$ for each $\sigma \in \mathrm{Irr}_2(M)_\zeta$. We claim that such infinite sums makes sense, i.e., for each $f \in C_c^\infty(M(F))$, or equivalently for each $f \in C_\zeta^\infty(M(F))$, $\Theta_\sigma(f) = 0$ for all but finitely many $\sigma \in \mathrm{Irr}_2(M)_\zeta$. This is easy to deduce from [Wal03, Theorem VIII.1.2] using standard facts, as observed in [MW16, Corollary XI.4.1].

2.3. **The root datum, the dual group and the L-group.** For this subsection too, let M be an arbitrary connected reductive group over F (which will vary in sentences where we discuss the functoriality of constructions involving it). We refer to [MW16, Section I.1.2] for a review of the notion of 'the pinned Borel pair' or 'the pinning' $\mathcal{E}_M^* := (B_M^*, T_M^*, (E_\alpha^*)_{\alpha\in\Delta})$ attached to M — briefly, the pinnings in $M \times_F \bar{F}$ form an inverse system in a suitable way, and one thinks of the inverse limit of this inverse system as 'the' pinning attached to M.

Notation 2.3.1. (i) We will denote by $\Psi(M)$ the absolute based root datum of M obtained from the pinning of M (independently of any choice of a Borel pair), which is a four tuple $(X_M, \Delta_M, X_M^\vee, \Delta_M^\vee)$ together with a Γ-action on it (i.e., a compatible collection of Γ-actions on its entries). [2] We may often write $\Psi(M) = (X_M, \Delta_M, X_M^\vee, \Delta_M^\vee)$, with the Γ-action understood.

(ii) If $\Psi(M) = (X_M, \Delta_M, X_M^\vee, \Delta_M^\vee)$, we let $\Psi(M)^\vee$ be the dual based root datum $(X_M^\vee, \Delta_M^\vee, X_M, \Delta_M)$, which gets a Γ-action from the Γ-action on $\Psi(M)$.

(iii) Similarly, to each complex connected reductive group $\hat{M}$, we can associate 'the' pinning of that group, and a based root datum $\Psi(\hat{M})$.

[2]This notation is justified by the fact that the set $\Delta_M \subset X_M$ of simple roots together with the set $\Delta_M^\vee \subset X_M^\vee$ of the corresponding coroots completely determines all roots and coroots, as well as the bijection $\alpha \mapsto \alpha^\vee$.

(iv) Following [Bor79, Section 1.4], the assignment $\mathrm{M} \mapsto \Psi(\mathrm{M})$ extends to a functor from the category whose objects are connected reductive groups over F and whose morphisms are normal homomorphisms (i.e., homomorphisms $f : \mathrm{M} \to \mathrm{M}'$ of connected reductive groups such that $f(\mathrm{M})$ is a normal subgroup of M': since f has characteristic zero, the separability condition of [Kot84, Section 1.8] is automatic), to a category whose objects are based root data with a Γ-action and which has a suitable notion of morphisms, simplified in our case because F has characteristic zero. We do not write out the relevant definition of morphisms of root data precisely, but note that for isomorphisms $f : \mathrm{M} \to \mathrm{M}'$ of reductive groups it is clear how to make sense of the associated maps $\Psi(f)$ of root data and their functoriality, and this is essentially the only case that we will need.

Remark 2.3.2. To go ahead, it will help us to know that, given any exact sequence

$$1 \to \mathscr{G}' \to \mathscr{G} \to \mathscr{G}'' \to 1$$

of abstract groups (i.e., $\mathscr{G} \to \mathscr{G}''$ is a surjection whose kernel is the image of $\mathscr{G}'$), we have a homomorphism:

$$\mathscr{G}'' \to \mathrm{Out}(\mathscr{G}') \tag{7}$$

from $\mathscr{G}''$ to the group $\mathrm{Out}(\mathscr{G}')$ of outer automorphisms of $\mathscr{G}'$, obtained by factoring the composite of the conjugation action $\mathscr{G} \to \mathrm{Aut}(\mathscr{G}')$ with the map $\mathrm{Aut}(\mathscr{G}') \to \mathrm{Out}(\mathscr{G}')$. For later use we also note that, since we have a well-defined restriction map $\mathrm{Out}(\mathscr{G}') \to \mathrm{Aut}(Z_{\mathscr{G}'})$, where $Z_{\mathscr{G}'}$ is the center of $\mathscr{G}'$, we also get, by composing with (7), an action:

$$\mathscr{G}'' \to \mathrm{Aut}(Z_{\mathscr{G}'}) \tag{8}$$

of $\mathscr{G}''$ on $Z_{\mathscr{G}'}$.
In what follows, we will encounter several situations of this kind, in each of which $\mathscr{G}''$ will be isomorphic to W_F and $\mathscr{G}'$ will be a complex connected reductive group; the right-hand sides of (7) and (8) will be interpreted in terms of algebraic morphisms. Often, but not always, $\mathscr{G}$ will be a split L-group in the sense we now proceed to discuss.

We will take one of the well-known perspectives on dual groups and L-groups along the lines found in Weissman's work, and of which we learnt from D. Prasad.

Definition 2.3.3. (i) By a dual group of M, we mean a connected reductive group $\hat{\mathrm{M}}$ over $\mathbb{C}$, together with an isomorphism $\Psi(\hat{\mathrm{M}}) \to \Psi(\mathrm{M})^\vee$, which we will think of as the identity so as to write $\Psi(\hat{\mathrm{M}}) = (X_{\mathrm{M}}^\vee, \Delta_{\mathrm{M}}^\vee, X_{\mathrm{M}}, \Delta_{\mathrm{M}})$, and using which we will view $\Psi(\hat{\mathrm{M}})$ as acted on by Γ. Note that any two dual groups for M admit an isomorphism between them that is determined up to composition with an inner conjugation, and that we have fixed bijections:

$$\mathrm{Out}(\hat{\mathrm{M}}) \cong \mathrm{Aut}(\Psi(\hat{\mathrm{M}})) \to \mathrm{Aut}(\Psi(\mathrm{M})) \cong \mathrm{Out}(\mathrm{M}). \tag{9}$$

(ii) By an L-group equipped with preferred sections, or a split L-group in short, we mean the following data:
- A topological group $\mathcal{M}$ together with a surjection $\mathcal{M} \to W_F$ of topological groups;
- a structure of a complex connected reductive group on the kernel of $\mathcal{M} \to W_F$, which we will denote by $\mathcal{M}^0$, think of as a complex connected reductive group and refer to as the identity component of $\mathcal{M}$, such that $\mathrm{Int}\, m$ restricts to an algebraic automorphism of $\mathcal{M}^0$ for all $m \in \mathcal{M}$; and
- a choice of an $\mathcal{M}^0$-conjugacy class of sections $s : W_F \to \mathcal{M}$ to the map $\mathcal{M} \to W_F$, which we will call preferred sections, satisfying the following property: for some (or equivalently, any) preferred section $s : W_F \to \mathcal{M}$, the action $w \mapsto \mathrm{Int}\, s(w)$ of W_F on $\mathcal{M}^0$ is an L-action, i.e., one that preserves some pinning of $\mathcal{M}^0$ and factors through the Weil group $W_E \subset W_F$ of some finite extension E/F in $\bar{F}$, and hence extends to an action of Γ on $\mathcal{M}^0$.

(iii) By an L-group of M, we mean a split L-group ${}^L\mathrm{M}$, together with an isomorphism $\Psi(\hat{\mathrm{M}}) \to \Psi(\mathrm{M})^\vee$ where we write $\hat{\mathrm{M}} := ({}^L\mathrm{M})^0$, with the property that this isomorphism transports the W_F-action on $\Psi(\hat{\mathrm{M}})$ induced by $w \mapsto \mathrm{Int}\, s(w)$, where $s : W_F \to {}^L\mathrm{M}$ is some or equivalently any preferred section, to the obvious W_F-action on $\Psi(\mathrm{M})^\vee$. In particular, $(\hat{\mathrm{M}}, \Psi(\hat{\mathrm{M}}) \to \Psi(\mathrm{M})^\vee)$ is a dual group for M. It is well-known that an L-group for M exists: usually one chooses a dual group $\hat{\mathrm{M}}$, lifts the action of W_F on $\Psi(\hat{\mathrm{M}})$ to one on $\hat{\mathrm{M}}$ by forcing it to fix a pinning, and takes ${}^L\mathrm{M}$ to be $\hat{\mathrm{M}} \rtimes W_F$. Conversely, any choice of a preferred section gives us an action of W_F on $\hat{\mathrm{M}} := ({}^L\mathrm{M})^0$, and then a realization ${}^L\mathrm{M} \cong \hat{\mathrm{M}} \rtimes W_F$. See Notation 2.3.4(iv) below for the sense in which an L-group of M is unique.

(iv) If $\mathcal{M}$ is a split L-group, we will view Γ as acting on $\mathrm{Z}_{\mathcal{M}^0}$ by algebraic automorphisms as follows. First, we get an action of $W_F \subset \Gamma$ on $\mathrm{Z}_{\mathcal{M}^0}$ through a chain of the following form, obtained from the considerations of Remark 2.3.2 (i.e., from 'algebrized' versions of (7) and (8) of that remark):

$$W_F = \mathcal{M}/\mathcal{M}^0 \to \mathrm{Out}(\mathcal{M}^0) \to \mathrm{Z}_{\mathcal{M}^0}.$$

This action can also be described as given by $\mathrm{Int} \circ s$, where $s : W_F \to \mathcal{M}$ is a preferred section, and hence factors through the quotient W_F/W_E for some finite extension E/F, and hence extends to an action of Γ. Similarly, we also get an action of $W_F = \mathcal{M}/\mathcal{M}^0$ on $\Psi(\mathcal{M}^0)$, via $W_F = \mathcal{M}/\mathcal{M}^0 \to \mathrm{Out}(\mathcal{M}^0) = \mathrm{Aut}(\Psi(\mathcal{M}^0))$.

We will often resort to standard abuse of notation, referring to a dual group of M as just $\hat{\mathrm{M}}$, or to a split L-group as just $\mathcal{M}$, or to an L-group of M as just ${}^L\mathrm{M}$, with the remaining data involved considered as understood. We will typically work with fixed choices of $\hat{\mathrm{M}}$ and ${}^L\mathrm{M}$, and identify $({}^L\mathrm{M})^0 = \hat{\mathrm{M}}$, but will not work with any fixed preferred section of ${}^L\mathrm{M}$ or identify ${}^L\mathrm{M}$ with $\hat{\mathrm{M}} \rtimes W_F$ unless otherwise stated.

Notation 2.3.4. (i) Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be split L-groups. An isomorphism $\mathcal{M}_1 \to \mathcal{M}_2$ of topological groups over W_F is said to be an isomorphism of split L-groups if it sends a preferred section to a preferred section, and restricts to an isomorphism of algebraic groups $\mathcal{M}_1^0 \to \mathcal{M}_2^0$. Using [Kot84, Corollary 1.7], it is easy to see that, up to post-composition with $\mathrm{Int}\,\mathcal{M}_2^0$ or equivalently pre-composition with $\mathrm{Int}\,\mathcal{M}_1^0$, such an isomorphism is determined by the map of root data $\Psi(\mathcal{M}_1^0) \to \Psi(\mathcal{M}_2^0)$ induced by the restriction $\mathcal{M}_1^0 \to \mathcal{M}_2^0$.

(ii) Suppose $\mathrm{M}_1 \to \mathrm{M}_2$ is an isomorphism of connected reductive groups over F. We will refer to an isomorphism ${}^L\mathrm{M}_2 \to {}^L\mathrm{M}_1$ of split L-groups as dual to $f : \mathrm{M}_1 \to \mathrm{M}_2$, if the induced map $\Psi(\hat{\mathrm{M}}_2) \to \Psi(\hat{\mathrm{M}}_1)$ is dual to $\Psi(f) : \Psi(\mathrm{M}_1) \to \Psi(\mathrm{M}_2)$. By the discussion in (i) above, the collection of isomorphisms of split L-groups ${}^L\mathrm{M}_2 \to {}^L\mathrm{M}_1$ that are dual to a given isomorphism $\mathrm{M}_1 \to \mathrm{M}_2$ is a single orbit under post-composition by $\mathrm{Int}\,\hat{\mathrm{M}}_1$, or equivalently under pre-composition by $\mathrm{Int}\,\hat{\mathrm{M}}_2$. Clearly, this definition generalizes to define what it means for ${}^L\mathrm{M}_2 \to {}^L\mathrm{M}_1$ to be dual to a Γ-equivariant isomorphism $\Psi(\mathrm{M}_1) \to \Psi(\mathrm{M}_2)$.

(iii) Often, if $\beta : \mathrm{M}_1 \to \mathrm{M}_2$ is an isomorphism of reductive groups, ${}^L\beta : {}^L\mathrm{M}_2 \to {}^L\mathrm{M}_1$ will denote a *choice* of an element in the $\hat{\mathrm{M}}_1$-conjugacy class of homomorphisms that are dual to β in the sense of (ii) above.

(iv) It follows from the discussion above that, given any reductive group M, ${}^L\mathrm{M}$ is unique up to an isomorphism of split L-groups which itself is uniquely determined up to conjugation by an element of its identity component $\hat{\mathrm{M}}$. Henceforth, we will implicitly fix a choice of ${}^L\mathrm{M}$ and hence of $\hat{\mathrm{M}} = ({}^L\mathrm{M})^0$ for every reductive group M over F that we will encounter in what follows.

(v) We saw in (ii) above that to any isomorphism $\mathrm{M}_1 \to \mathrm{M}_2$ of connected reductive groups over F, we can functorially define a dual map ${}^L\mathrm{M}_2 \to {}^L\mathrm{M}_1$, albeit considered up to composition with $\mathrm{Int}\,\hat{\mathrm{M}}_1$. A similar but easier proof shows the same to be the case for when $\mathrm{M}_1 \to \mathrm{M}_2$ is either injective with image a central torus in M_2, or surjective with the image M_2 a torus. In fact, one can do the same when $\mathrm{M}_1 \to \mathrm{M}_2$ is any normal homomorphism of connected reductive groups, but we will not need it.

Langlands parameters and related constructions involving these groups will usually be 'invariant under conjugation', so that the ambiguity of inner automorphisms mentioned in the above remark will not matter.

Remark 2.3.5. Unlike [Bor79], we are using the Weil form of the L-group, which is what suits the theory of endoscopy. But this difference is harmless for our purposes, and we will therefore use results from [Bor79] without further comment, skipping the minor checks needed. Nevertheless, this difference will occasionally reflect in our treatment, e.g., in that we will use preferred sections to talk of temperedness for a Langlands parameter; in fact, it has already been reflected in our requiring the preferred sections, in Definition 2.3.3(ii), to factor through W_E for a finite extension E/F in $\bar{F}$.

2.4. **Parabolic subgroups of** M **and of** ${}^L\mathrm{M}$**.** In this subsection, we let M be a connected reductive group over F.

2.4.1. *Conjugacy classes of parabolic subgroups of* M *and* ${}^L\mathrm{M}$ *in terms of* $\Psi(\mathrm{M})$*.* Recall from [Bor79, Section 3.3] (keeping in mind Remark 2.3.5) that a subgroup $\mathcal{Q}$ of ${}^L\mathrm{M}$ is called parabolic if $\mathcal{Q}^0 := \mathcal{Q}\cap\hat{\mathrm{M}}$ is a parabolic subgroup of $\hat{\mathrm{M}}$, and the projection ${}^L\mathrm{M} \to W_F$ restricts to a surjection $\mathcal{Q} \to W_F$ (in which case, $\mathcal{Q}$ is the normalizer of $\mathcal{Q}^0$ in ${}^L\mathrm{M}$). Recall that, in such a situation, a Levi subgroup of $\mathcal{Q}$ refers to the normalizer $\mathcal{L}$ in $\mathcal{Q}$ of a Levi subgroup $\mathcal{L}^0$ of $\mathcal{Q}^0$ (and automatically satisfies that $\mathcal{Q}$ is the semidirect product of $\mathcal{L}$ and the unipotent radical of $\mathcal{Q}^0$; in particular, $\mathcal{L} \to W_F$ is surjective). Of course, by abuse of notation, a Levi subgroup of a parabolic subgroup of ${}^L\mathrm{M}$ will be called a Levi subgroup of ${}^L\mathrm{M}$.

Remark 2.4.1. If (Q,L) is a parabolic-Levi pair in $\mathrm{M}_{\bar{F}}$, it will help to note that the pinnings $\mathcal{E} = (\mathrm{B}_\mathrm{M}, \mathrm{T}_\mathrm{M}, \{E_\alpha\}_{\alpha\in\Delta_\mathrm{M}})$ of $\mathrm{M}_{\bar{F}}$ with $\mathrm{B}_\mathrm{M} \subset \mathrm{Q}_{\bar{F}}$ are all $\mathrm{Q}(\bar{F})$-conjugate, and that the set of such pinnings maps $\operatorname{Int}\mathrm{L}(\bar{F})$-equivariantly onto the set of pinnings of $\mathrm{L}_{\bar{F}}$.

Notation 2.4.2.

(i) For each parabolic-Levi pair (Q,L) in $\mathrm{M}_{\bar{F}}$ or M, Remark 2.4.1 gives subsets $\Delta_\mathrm{Q} = \Delta_{\mathrm{Q}_{\bar{F}}} \subset \Delta_\mathrm{M}$ and $\Delta^\vee_\mathrm{Q} = \Delta^\vee_{\mathrm{Q}_{\bar{F}}} \subset \Delta^\vee_\mathrm{M}$, matching each other under the bijection $\Delta_\mathrm{M} \to \Delta^\vee_\mathrm{M}$ and depending only on Q, as well as an identification:

$$\Psi(\mathrm{L}) \overset{\text{using }(\mathrm{Q},\mathrm{L})}{=} (X_\mathrm{M}, \Delta_\mathrm{Q}, X^\vee_\mathrm{M}, \Delta^\vee_\mathrm{Q}). \tag{10}$$

For fixed L, this identification of $\Psi(\mathrm{L})$ will be referred to as the embedding $\Psi(\mathrm{L}) \hookrightarrow \Psi(\mathrm{M})$ determined by the choice of Q (by abuse of notation: this is not an 'embedding', and goes in the 'wrong direction'). Note that if pairs (L,Q) and $(\mathrm{L}',\mathrm{Q}')$ are $\mathrm{M}(\bar{F})$-conjugate, then the resulting identification $\Psi(\mathrm{L}) \to \Psi(\mathrm{L}')$, which is independent of any conjugating element, is compatible with the embeddings of $\Psi(\mathrm{L})$ and $\Psi(\mathrm{L}')$ in $\Psi(\mathrm{M})$ determined by Q and Q'. Note also that (10) is Γ-equivariant if we started with $\mathrm{Q},\mathrm{L} \subset \mathrm{M}$ (i.e., defined over F).

(ii) For each parabolic-Levi pair $(\mathcal{Q}^0,\mathcal{L}^0)$ in $\hat{\mathrm{M}}$, an analogue of Remark 2.4.1 gives subsets $\Delta_{\mathcal{Q}^0} \subset \Delta_{\hat{\mathrm{M}}} = \Delta^\vee_\mathrm{M}$ and $\Delta^\vee_{\mathcal{Q}^0} \subset \Delta^\vee_{\hat{\mathrm{M}}} = \Delta_\mathrm{M}$, matching each other under the bijection $\Delta_{\hat{\mathrm{M}}} = \Delta^\vee_\mathrm{M} \to \Delta_\mathrm{M} = \Delta^\vee_{\hat{\mathrm{M}}}$ and depending only on $\mathcal{Q}^0$, as well as an identification:

$$\Psi(\mathcal{L}^0) \overset{\text{using }(\mathcal{Q}^0,\mathcal{L}^0)}{=} (X^\vee_\mathrm{M}, \Delta_{\mathcal{Q}^0}, X_\mathrm{M}, \Delta^\vee_{\mathcal{Q}^0}). \tag{11}$$

For fixed $\mathcal{L}^0$, this identification will be referred to as the embedding $\Psi(\mathcal{L}^0) \hookrightarrow \Psi(\hat{\mathrm{M}})$ determined by the choice of $\mathcal{Q}^0$. It enjoys a behavior with respect to $\hat{\mathrm{M}}$-conjugation, that follows the lines of the discussion in (i) above. For a parabolic-Levi pair $(\mathcal{Q},\mathcal{L})$ in ${}^L\mathrm{M}$, we will write $\Delta_\mathcal{Q} = \Delta_{\mathcal{Q}^0}$ and $\Delta^\vee_\mathcal{Q} = \Delta^\vee_{\mathcal{Q}^0}$, and refer to the identification $\Psi(\mathcal{L}^0) \hookrightarrow \Psi(\hat{\mathrm{M}})$ determined by $\mathcal{Q}^0$ also as the one determined by $\mathcal{Q}$; note that in this case, (11) is Γ-equivariant in an appropriate sense.

(iii) Thus, if (Q,L) and $(\mathcal{Q}^0,\mathcal{L}^0)$ are parabolic-Levi pairs in $\mathrm{M}_{\bar{F}}$ and $\hat{\mathrm{M}}$, respectively, such that $\Delta_\mathrm{Q} = \Delta^\vee_{\mathcal{Q}^0}$, then (i) and (ii) give us an identification $\Psi(\hat{\mathrm{L}}) = \Psi(\mathrm{L})^\vee = \Psi(\mathcal{L}^0)$, which we will write as $\Psi(\hat{\mathrm{L}}) \overset{\mathrm{Q},\mathcal{Q}^0}{=} \Psi(\mathcal{L}^0)$. If further (Q,L) is a parabolic-Levi pair in M and $(\mathcal{Q},\mathcal{L})$ is a parabolic-Levi pair in ${}^L\mathrm{M}$, then this identification $\Psi(\hat{\mathrm{L}}) \overset{\mathrm{Q},\mathcal{Q}^0}{=} \Psi(\mathcal{L}^0)$ is Γ-equivariant.

Recall from [Bor79, Sections 3.2 and 3.3] the bijection between the following collections of objects:

(i) Γ-stable $\mathrm{M}(\bar{F})$-conjugacy classes of parabolic subgroups of $\mathrm{M}_{\bar{F}}$ (recall that such a conjugacy class may not contain a parabolic subgroup that is Γ-stable, though it does if M is quasi-split);
(ii) Γ-invariant subsets $\Delta_1 \subset \Delta_{\mathrm{M}}$;
(iii) Γ-invariant subsets $\Delta_1^\vee \subset \Delta_{\mathrm{M}}^\vee$;
(iv) $\hat{\mathrm{M}}$-conjugacy classes of parabolic subgroups $\mathcal{Q} \subset {}^L\mathrm{M}$.

Here, the bijections (i) ↔ (ii) and (iii) ↔ (iv) are given respectively by $\mathrm{Q} \leftrightarrow \Delta_{\mathrm{Q}}$ and $\Delta_{\mathcal{Q}} \leftrightarrow \mathcal{Q}$, while (ii) ↔ (iii) is given by $\Delta_1 \leftrightarrow \Delta_1^\vee := \{\alpha^\vee \mid \alpha \in \Delta_1\}$.

Notation 2.4.3. Recall that a parabolic subgroup $\mathcal{Q} \subset {}^L\mathrm{M}$ is said to be relevant if the associated Γ-stable $\mathrm{M}(\bar{F})$-conjugacy class of parabolic subgroups of $\mathrm{M}_{\bar{F}}$ (as per the bijection between the collections (iv) and (i) above) has an element that is Γ-stable, i.e., obtained by base-change from a parabolic subgroup of M (defined over F). A Levi subgroup $\mathcal{L} \subset {}^L\mathrm{M}$ is said to be relevant if it is a Levi subgroup of a relevant parabolic subgroup of ${}^L\mathrm{M}$.

Remark 2.4.4. If M is quasi-split, then all parabolic subgroups of ${}^L\mathrm{M}$ are relevant, so that each of the above collections is in bijection with the set of $\mathrm{M}(F)$-conjugacy classes of parabolic subgroups Q of M.

2.4.2. *Levi subgroups of ${}^L\mathrm{M}$ as split L-groups.*

Proposition 2.4.5. *Let $\mathcal{L}$ be a Levi subgroup of ${}^L\mathrm{M}$. Then the preferred sections of ${}^L\mathrm{M}$ whose images are contained in $\mathcal{L}$ form a single $\mathcal{L}^0$-conjugacy class. Moreover, given any such section $s : W_F \to \mathcal{L}$, the action* $\mathrm{Int} \circ s$ *of W_F on $\mathcal{L}^0$ is an L-action (see Definition 2.3.3(ii)).*

Notation 2.4.6. (i) Once we prove Proposition 2.4.5, we may and shall view an arbitrary Levi subgroup $\mathcal{L}$ of ${}^L\mathrm{M}$ as canonically a split L-group in the sense of Definition 2.3.3(ii), whose preferred sections are those preferred sections of ${}^L\mathrm{M}$ whose images are contained in $\mathcal{L}$.

(ii) Given parabolic-Levi pairs (Q, L) in M and $(\mathcal{Q}, \mathcal{L})$ in ${}^L\mathrm{M}$, such that $\Delta_{\mathcal{Q}} = \Delta_{\mathrm{Q}}^\vee$, the Γ-equivariant identification $\Psi(\hat{\mathrm{L}}) \overset{\mathrm{Q},\mathcal{Q}^0}{=} \Psi(\mathcal{L}^0)$ from Notation 2.4.2(iii), together with the considerations of Notation 2.3.4(i), give us an identification

$$\iota_{\mathrm{M},\mathrm{L}} : {}^L\mathrm{L} \overset{\cong}{\to} \mathcal{L} \subset {}^L\mathrm{M} \tag{12}$$

that is uniquely determined up to post-composition by an element of $\mathrm{Int}\,\mathcal{L}^0$ or equivalently pre-composition by an element of $\mathrm{Int}\,\hat{\mathrm{L}}$. Here, we are suppressing $\mathrm{Q}, \mathcal{L}$ and $\mathcal{Q}$ from the notation $\iota_{\mathrm{M},\mathrm{L}}$; this is because we will only be interested in the $\hat{\mathrm{M}}$-conjugacy class of $\iota_{\mathrm{M},\mathrm{L}}$, which we will see in Corollary 2.4.10 below to depend only on L.

Proof of Proposition 2.4.5. We will adapt much of the proof of [Kot84, Lemma 1.6]. We use a preferred section $s : W_F \to {}^L\mathrm{M}$ to get an isomorphism ${}^L\mathrm{M} = \hat{\mathrm{M}} \rtimes W_F$, and recall that the resulting action of W_F on $\hat{\mathrm{M}}$, being an L-action, extends to an action of Γ which preserves a pinning of $\hat{\mathrm{M}}$. We choose such a pinning, so that we can talk of standard parabolic and Levi subgroups of ${}^L\mathrm{M}$. Without loss of generality, we may assume that $\mathcal{L}$ is a standard Levi subgroup of a standard parabolic subgroup $\mathcal{Q}$ of ${}^L\mathrm{M}$, so that we can write $\mathcal{L} = \mathcal{L}^0 \rtimes W_F$, with $W_F \hookrightarrow \mathcal{L}$ agreeing with s. Hence $s : W_F = \{1\} \rtimes W_F \hookrightarrow \mathcal{L}^0 \rtimes W_F = \mathcal{L}$ satisfies that $\mathrm{Int} \circ s$ preseres a pinning of $\mathcal{L}^0$, and it is now easy to see that the second statement of the proposition follows if we prove the first.

Thus, it is enough to assume given $x \in \hat{\mathrm{M}}$ such that $x^{-1}(1 \rtimes W_F)x \subset \mathcal{L}^0 \rtimes W_F$, and to show that $x^{-1}(1 \rtimes W_F)x = y^{-1}(1 \rtimes W_F)y$ for some $y \in \mathcal{L}^0$. In other words, we are given $x_\sigma \in \mathcal{L}^0$ for all $\sigma \in W_F$ so that the identity $x^{-1}\sigma(x) = x_\sigma$ holds, and we need to show that there exists $y \in \mathcal{L}^0$ such that $y^{-1}\sigma(y) = x_\sigma$ for all $\sigma \in W_F$ (though we will not use it, note that this amounts to showing that, for the map $H^1(W_F, \mathcal{L}^0) \to H^1(W_F, \hat{\mathrm{M}})$ of pointed sets, the preimage of the distinguished point is singleton).

Let $\mathcal{B} \subset \mathcal{Q}$ be the standard Borel subgroup underlying our pinning of $\hat{\mathrm{M}}$. By the Bruhat decomposition, we can write $\hat{\mathrm{M}}$ as the disjoint union of the $\mathcal{B}^0 w \mathcal{Q}^0$, as w ranges over minimal length representatives for the right $W_{\mathcal{L}^0}$-cosets in $W_{\hat{\mathrm{M}}}$, where we write $W_{\mathcal{L}^0}$ and $W_{\hat{\mathrm{M}}}$ for the Weyl groups of $\mathcal{L}^0$ and $\hat{\mathrm{M}}$ with respect to the maximal torus underlying our pinning, and where we use $\mathcal{B}^0$ to specify the simple roots and hence the length function. Since $x^{-1}\sigma(x) = x_\sigma \in \mathcal{L}^0$ for all $\sigma \in W_F$, it follows that the coset $\mathcal{B}^0 x \mathcal{Q}^0$ is σ-invariant, and hence so is the minimal length representative determining it. This representative is the image of a Γ-fixed element w of $\hat{\mathrm{M}}$, by [Bor79, Lemma 6.2] (which applies though M is not split over a cyclic extension of F; e.g., the set of Weyl group representatives determined by our standard pinning is Γ-invariant, since Γ preserves the pinning). We use this representative w to write x as $uwvy$, where u, v respectively belong to the unipotent radicals of $\mathcal{B}^0$ and $\mathcal{Q}^0$, and $y \in \mathcal{L}^0$. Such a decomposition is not unique, but it is unique once we also require u, as we may, to lie in the Γ-invariant subgroup $w\mathcal{U}^- w^{-1}$, where $\mathcal{U}^-$ is the unipotent radical of the parabolic subgroup of $\hat{\mathrm{M}}$ that contains $\mathcal{L}^0$ and is opposite to $\mathcal{Q}^0$. Thus, applying the uniqueness of this decomposition to the equality $\sigma(x) = x x_\sigma$, we conclude that $\sigma(y) = y x_\sigma$, as desired. □

Corollary 2.4.7. *Suppose $\mathcal{L}_1, \mathcal{L}_2 \subset {}^L\mathrm{M}$ are Levi subgroups, and suppose $\mathrm{Int}\, x(\mathcal{L}_1) = \mathcal{L}_2$ for some $x \in \hat{\mathrm{M}}$. Then $\mathrm{Int}\, x : \mathcal{L}_1 \to \mathcal{L}_2$ is an isomorphism of split L-groups.*

Proof. This is obvious from Proposition 2.4.5 and the definition (see Notation 2.3.4(i)). □

The following lemma, at least in the special case where $\mathrm{L}_1 = \mathrm{L}_2, \mathrm{Q}_1 = \mathrm{Q}_2$ and $\mathcal{Q}_1 = \mathcal{Q}_2$, is part of the set up in the theory of R-groups:

Lemma 2.4.8. *Assume that $\mathrm{L}_1, \mathrm{L}_2 \subset \mathrm{M}$ and $\mathcal{L}_1, \mathcal{L}_2 \subset {}^L\mathrm{M}$ are Levi subgroups. For $i = 1, 2$, let Q_i be a parabolic subgroup of M with L_i as a Levi subgroup and $\mathcal{Q}_i$ a parabolic subgroup of ${}^L\mathrm{M}$ with $\mathcal{L}_i$ as a Levi subgroup, such that $\Delta_{\mathcal{Q}_i} = \Delta^\vee_{\mathrm{Q}_i}$, yielding by Notation 2.4.2(iii) an identification $\Psi(\hat{\mathrm{L}}_i) \overset{\mathrm{Q}_i, \mathcal{Q}_i^0}{=} \Psi(\mathcal{L}_i^0)$. Define*

$$W(\mathrm{L}_1, \mathrm{L}_2) := \{m \in \mathrm{M}(F) \mid \mathrm{Int}\, m(\mathrm{L}_1) = \mathrm{L}_2\}/\mathrm{L}_1(F), \quad \textit{and} \quad W(\mathcal{L}_2, \mathcal{L}_1) := \{\hat{m} \in \hat{\mathrm{M}}(F) \mid \mathrm{Int}\, \hat{m}(\mathcal{L}_2) = \mathcal{L}_1\}/\mathcal{L}_2^0,$$

where the quotients have an obvious interpretation. Then:

(i) The obvious map from $W(\mathrm{L}_1, \mathrm{L}_2)$ to the group $\mathrm{Isom}(\Psi(\mathrm{L}_1), \Psi(\mathrm{L}_2))$ of isomorphisms from $\Psi(\mathrm{L}_1)$ to $\Psi(\mathrm{L}_2)$ is injective, as is the map from $W(\mathcal{L}_2, \mathcal{L}_1)$ to the analogously defined group $\mathrm{Isom}(\Psi(\mathcal{L}_2^0), \Psi(\mathcal{L}_1^0))$.

(ii) There is a unique bijection $W(\mathrm{L}_1, \mathrm{L}_2) \to W(\mathcal{L}_2, \mathcal{L}_1)$ under which $w \in W(\mathrm{L}_1, \mathrm{L}_2)$ and $\hat{w} \in W(\mathcal{L}_2, \mathcal{L}_1)$ correspond if and only if their images in $\mathrm{Isom}(\Psi(\mathrm{L}_1), \Psi(\mathrm{L}_2))$ and $\mathrm{Isom}(\Psi(\mathcal{L}_2^0), \Psi(\mathcal{L}_1^0))$ are dual to each other, as made sense of by the above identifications $\Psi(\hat{\mathrm{L}}_1) = \Psi(\mathcal{L}_1^0)$ and $\Psi(\hat{\mathrm{L}}_2) = \Psi(\mathcal{L}_2^0)$.

Before proving the lemma, let us restate it as a corollary:

Corollary 2.4.9. *Assume the scenario of Lemma 2.4.8. Define $\iota_{\mathrm{M}, \mathrm{L}_i} : {}^L\mathrm{L}_i \to {}^L\mathrm{M}$ as in Notation 2.4.6(ii) for $i = 1, 2$, using $(\mathrm{Q}_i, \mathrm{L}_i)$ and $(\mathcal{Q}_i, \mathcal{L}_i)$. Let $\tilde{W}({}^L\mathrm{L}_2, {}^L\mathrm{L}_1)$ be the set of all isomorphisms ${}^L\mathrm{L}_2 \to {}^L\mathrm{L}_1$ that are dual to $\mathrm{Int}\, m|_{\mathrm{L}_1}$, for some $m \in \mathrm{M}(F)$ such that $\mathrm{Int}\, m(\mathrm{L}_1) = \mathrm{L}_2$. Let $\tilde{W}(\mathcal{L}_2, \mathcal{L}_1)$ be the set of all isomorphisms of split L-groups $\mathcal{L}_2 \to \mathcal{L}_1$ of the form $\mathrm{Int}\, \hat{m}$ for some $\hat{m} \in \hat{\mathrm{M}}$. Then:*

(i) Under the identifications $\iota_{\mathrm{M}, \mathrm{L}_1} : {}^L\mathrm{L}_1 \to \mathcal{L}_1$ and $\iota_{\mathrm{M}, \mathrm{L}_2} : {}^L\mathrm{L}_2 \to \mathcal{L}_2$, $\tilde{W}({}^L\mathrm{L}_2, {}^L\mathrm{L}_1)$ is transported to $\tilde{W}(\mathcal{L}_2, \mathcal{L}_1)$.

(ii) L_1 and L_2 are $\mathrm{M}(F)$-conjugate if and only if $\mathcal{L}_1$ and $\mathcal{L}_2$ are $\hat{\mathrm{M}}$-conjugate (note that this statement is independent of $\mathrm{Q}_1, \mathrm{Q}_2, \mathcal{Q}_1$ and $\mathcal{Q}_2$).

Proof. (i) follows from Lemma 2.4.8 (both (i) and (ii); combine these with the considerations of Notation 2.3.4(i)). (ii) is immediate from (i). □

Proof of Lemma 2.4.8. The injectivity of $W(\mathrm{L}_1, \mathrm{L}_2) \to \mathrm{Isom}(\Psi(\mathrm{L}_1), \Psi(\mathrm{L}_2))$ follows from the fact that if $m \in \mathrm{M}(F)$ normalizes $\mathrm{L}_1(F)$, but does not belong to $\mathrm{L}_1(F)$, then it acts nontrivially on $\mathrm{Z}_{\mathrm{L}_1}(F)$, and hence is not an inner automorphism. A similar argument involving the Levi subgroup

$\mathcal{L}_2^0 \subset \hat{\mathrm{M}}$ gives the injectivity of $W(\mathcal{L}_2, \mathcal{L}_1) \to \mathrm{Isom}(\Psi(\mathcal{L}_2), \Psi(\mathcal{L}_1))$, and hence finishes the proof of (i).

[MW16, Section I.3.1, (8)] constructs a bijection $W(\mathrm{L}_1, \mathrm{L}_2) \to W(\mathcal{L}_2, \mathcal{L}_1)$; one can check that it satisfies the prescription of (ii). While this suffices, for the convenience of the reader we will sketch a version of this verification here that has the disadvantage of being lengthier and more informal, but whose outline the author finds it easier to relate to.

Let W denote the Weyl group of $\Psi(\mathrm{M})$ — or equivalently of M, defined as an inverse limit over pinnings. The Weyl groups $W(\mathrm{L}_1)$ of L_1 and $W(\mathrm{L}_2)$ of L_2 identify, using the choices of Q_1 and Q_2, with the subgroups $W(\Delta_{\mathrm{Q}_1})$ and $W(\mathrm{L}_2) = W(\Delta_{\mathrm{Q}_2})$ of W generated respectively by reflections about the elements of Δ_{Q_1} and Δ_{Q_2}. We similarly have the Weyl group $\hat{W}$ of $\hat{\mathrm{M}}$, and $W(\mathcal{L}_1) = W(\Delta_{\mathcal{Q}_1}), W(\mathcal{L}_2) = W(\Delta_{\mathcal{Q}_2}) \subset \hat{W}$. All of these carry compatible Γ-actions, and we have a 'root by root' Γ-equivariant identification $W = \hat{W}$ taking $W(\mathrm{L}_i) = W(\Delta_{\mathrm{Q}_i})$ to $W(\mathcal{L}_i) = W(\Delta_{\mathcal{Q}_i})$ for $i = 1, 2$. Let Φ_{Q_i} (resp., $\Phi_{\mathcal{Q}_i}$) stand for the set of roots in the $\mathbb{Z}$-span of Δ_{Q_i} (resp., $\Delta_{\mathcal{Q}_i}$).

We may $\mathrm{M}(F)$-conjugate $(\mathrm{Q}_1, \mathrm{L}_1)$ and separately also $(\mathrm{Q}_2, \mathrm{L}_2)$, to assume that both of these contain a fixed minimal parabolic-Levi pair $(\mathrm{P}_0, \mathrm{M}_0)$. We choose a Borel pair $(\mathrm{B}, \mathrm{T}) \subset (\mathrm{P}_{0,\bar{F}}, \mathrm{M}_{0,\bar{F}})$. For any $\sigma \in \mathrm{Gal}(\bar{F}/F)$, it is easy to see that there exists $m_{0,\sigma} \in \mathrm{M}_0(\bar{F})$ such that ${}^{\sigma}(\mathrm{B}, \mathrm{T}) = (\mathrm{Int}\, m_{0,\sigma}(\mathrm{B}), \mathrm{Int}\, m_{0,\sigma}(\mathrm{T}))$.

It is now easy to see using conjugacy of maximal tori that there exists an injection:

$$\begin{aligned} W(\mathrm{L}_1, \mathrm{L}_2) &\hookrightarrow W(\mathrm{L}_{2,\bar{F}}, \mathrm{T}) \backslash \{w \in W(\mathrm{M}_{\bar{F}}, \mathrm{T}) \mid w(\mathrm{L}_{1,\bar{F}}) = \mathrm{L}_{2,\bar{F}}\} / W(\mathrm{L}_{1,\bar{F}}, \mathrm{T}) \\ &\cong W(\Delta_{\mathrm{Q}_2}) \backslash \{w \in W \mid w(\Phi_{\mathrm{Q}_1}) = \Phi_{\mathrm{Q}_2}\} / W(\Delta_{\mathrm{Q}_1}) \cong \{w \in W \mid w(\Delta_{\mathrm{Q}_1}) = \Delta_{\mathrm{Q}_2}\}, \end{aligned} \tag{13}$$

where the first map sends the image of $m \in \mathrm{M}(F)$ in $W(\mathrm{L}_1, \mathrm{L}_2)$ to the image of $m_2 m$, for any $m_2 \in \mathrm{L}_2(\bar{F})$ such that $m_2 m \mathrm{T} m^{-1} m_2^{-1} = \mathrm{T}$. Here, the last identification in (13) involves minimal length representatives, as made sense of using (B, T). It is easy to see that (13) is Γ-equivariant, i.e., each element of its image is Γ-fixed: use the computation that for each $\sigma \in \mathrm{Gal}(\bar{F}/F)$, $\mathrm{Int}\, m_{0,\sigma}^{-1}(\sigma(m_2 m)) \in \mathrm{L}_2(\bar{F}) m$. Further, we claim that the image of (13) is simply the set of Γ-fixed points of the target: this is easily verified using the fact that $\mathrm{M}(F) \to (\mathrm{L}_2 \backslash \mathrm{M})(F)$ is a surjection (which follows since $\mathrm{M}(F) \to (\mathrm{Q}_2 \backslash \mathrm{M})(F)$ is a surjection, and since the unipotent radical of Q_2 is split unipotent).

Now we study the dual variant of (13). Using a preferred section $s : W_F \to {}^L\mathrm{M}$, identify ${}^L\mathrm{M} = \hat{\mathrm{M}} \rtimes W_F$. We choose a Borel pair $(\mathcal{B}, \mathcal{T})$ in ${}^L\mathrm{M}$ such that $(\mathcal{B}^0, \mathcal{T}^0)$ underlies a W_F-invariant pinning of $\hat{\mathrm{M}}$. Again, we may $\hat{\mathrm{M}}$-conjugate $(\mathcal{Q}_1, \mathcal{L}_1)$ and $(\mathcal{Q}_2, \mathcal{L}_2)$ separately, and assume without loss of generality that these contain $(\mathcal{B}, \mathcal{T})$. We get a map analogous to (13) involving a set $W(\mathcal{L}_1, \mathcal{L}_2)$, defined just like $W(\mathcal{L}_2, \mathcal{L}_1)$ but with $\mathcal{L}_1$ and $\mathcal{L}_2$ swapped:

$$\begin{aligned} W(\mathcal{L}_1, \mathcal{L}_2) &\hookrightarrow W(\mathcal{L}_2^0, \mathcal{T}^0) \backslash \{w \in W(\hat{\mathrm{M}}, \mathcal{T}^0) \mid w(\mathcal{L}_1) = \mathcal{L}_2\} / W(\mathcal{L}_2^0, \mathcal{T}^0) \\ &\cong W(\Delta_{\mathcal{Q}_2}) \backslash \{w \in \hat{W} \mid w(\Phi_{\mathcal{Q}_1}) = \Phi_{\mathcal{Q}_2}\} / W(\Delta_{\mathcal{Q}_1}) \cong \{w \in \hat{W} \mid w(\Delta_{\mathcal{Q}_1}) = \Delta_{\mathcal{Q}_2}\}, \end{aligned} \tag{14}$$

again using minimal length representatives as made sense of using $(\mathcal{B}^0, \mathcal{T}^0)$. Since we assumed that $(\mathcal{Q}_1, \mathcal{L}_1)$ and $(\mathcal{Q}_2, \mathcal{L}_2)$ are standard with respect to $(\mathcal{B}, \mathcal{T})$, it is easy to verify that the image is contained in $(\hat{W})^{\Gamma}$, and then, using that elements of $(\hat{W})^{\Gamma}$ have Γ-fixed representatives (as we saw in the proof of Proposition 2.4.5), that the image of (14) is just the set of Γ-fixed elements of the target.

The Γ-equivariant identification $W = \hat{W}$, which is a 'transpose inverse' at the level of automorphisms of character lattices, clearly takes the target of (13) to the target of (14). It is immediately verified that the composition of this identification with $(w \mapsto w^{-1})$, which is a 'transpose' at the level of automorphisms of character lattices, is exactly as in (ii): (13) and (14) respect maps of based root data in an obvious sense, which at the level of the targets of these maps have naive descriptions thanks to the conditions $w(\Delta_{\mathrm{Q}_1}) = \Delta_{\mathrm{Q}_2}$ and $w(\Delta_{\mathcal{Q}_1}) = \Delta_{\mathcal{Q}_2}$. ☐

The following corollary is well-known (e.g., it is mentioned just below [Hai14, Definition 5.2.1]), though not obvious:

Corollary 2.4.10. *Let* L *be a Levi subgroup of* M*. The collection of embeddings* $\iota_{\mathrm{M},\mathrm{L}} : {}^L\mathrm{L} \to {}^L\mathrm{M}$ *obtained as in* (12)*, for varying pairs* (Q,L) *and* $(\mathcal{Q},\mathcal{L})$ *as in Notation 2.4.6(ii) but involving our fixed Levi subgroup* L*, forms a single conjugacy class for* $\hat{\mathrm{M}}$*-conjugation.*

Proof. Given a Levi subgroup $\mathrm{L} \subset \mathrm{M}$, it is immediate that Q and $(\mathcal{Q},\mathcal{L})$ as in Notation 2.4.6(ii) exist, so the collection of the $\iota_{\mathrm{M},\mathrm{L}}$ is nonempty. Given an embedding $\iota_1 : {}^L\mathrm{L} \hookrightarrow {}^L\mathrm{M}$ defined using pairs $(\mathrm{L},\mathrm{Q}_1)$ and $(\mathcal{L}_1,\mathcal{Q}_1)$, and another embedding ι_2 defined using pairs $(\mathrm{L},\mathrm{Q}_2)$ and $(\mathcal{L}_2,\mathcal{Q}_2)$, set $\mathcal{L}_i := \iota_i({}^L\mathrm{L}_i)$, and apply Corollary 2.4.9(i) with $\mathrm{L}_1 = \mathrm{L}_2 = \mathrm{L}$, considering the identity map $({}^L\mathrm{L}_2 = {}^L\mathrm{L} \to {}^L\mathrm{L} = {}^L\mathrm{L}_1) \in W({}^L\mathrm{L}_2, {}^L\mathrm{L}_1)$. What this gives us is precisely that there exists $\hat{m} \in \hat{\mathrm{M}}$ such that $\mathrm{Int}\,\hat{m} \circ \iota_2 = \iota_1$, as desired. □

Notation 2.4.11. Given any Levi subgroup $\mathrm{L} \subset \mathrm{M}$, $\iota_{\mathrm{M},\mathrm{L}} : {}^L\mathrm{L} \to {}^L\mathrm{M}$ will denote a (typically implicitly fixed) choice of a member of the $\hat{\mathrm{M}}$-conjugacy class of embeddings ${}^L\mathrm{L} \to {}^L\mathrm{M}$ as in Corollary 2.4.10, or equivalently, as in (12). We emphasize that each $\mathcal{Q}$ and $\mathcal{L}$ as in Notation 2.4.6(ii) are automatically relevant, so $\iota_{\mathrm{M},\mathrm{L}}({}^L\mathrm{L}) \subset {}^L\mathrm{M}$ is always a relevant Levi subgroup. Each time we make a statement or an argument involving such a map $\iota_{\mathrm{M},\mathrm{L}}$, the reader will be implicitly expected to note it to be independent, in an appropriate sense, of the choice of $\iota_{\mathrm{M},\mathrm{L}}$.

The following corollary is well-known (e.g., see [SZ18, 2.3(6) and the remark following it]):

Corollary 2.4.12. *Sending a Levi subgroup* $\mathrm{L} \subset \mathrm{M}$ *to the conjugacy class of* $\iota_{\mathrm{M},\mathrm{L}}({}^L\mathrm{L})$*, induces a bijection from the set of* $\mathrm{M}(F)$*-conjugacy classes of Levi subgroups of* M *to the set of* $\hat{\mathrm{M}}$*-conjugacy classes of relevant Levi subgroups of* ${}^L\mathrm{M}$*.*

Proof. It follows from Corollary 2.4.9(ii) that sending L to $\iota_{\mathrm{M},\mathrm{L}}({}^L\mathrm{L})$ indeed gives a well-defined injection from the set of $\mathrm{M}(F)$-conjugacy classes of Levi subgroups of M to the set of $\hat{\mathrm{M}}$-conjugacy classes of Levi subgroups of ${}^L\mathrm{M}$. It is easy to see that the image consists precisely of conjugacy classes of relevant Levi subgroups: for instance, if $(\mathcal{Q},\mathcal{L})$ is a parabolic-Levi pair in ${}^L\mathrm{M}$ with $\mathcal{Q}$ relevant, so that $\Delta_{\mathcal{Q}} = \Delta^\vee_{\mathrm{Q}}$ for some parabolic subgroup $\mathrm{Q} \subset \mathrm{M}$, it follows that for any Levi subgroup $\mathrm{L} \subset \mathrm{Q}$, we have a choice of $\iota_{\mathrm{M},\mathrm{L}}$ with image equal to $\mathcal{L}$. □

Corollary 2.4.13. *Let* $\mathrm{L} \subset \mathrm{M}$ *be a Levi subgroup. Then* $\mathrm{Z}_{\hat{\mathrm{M}}} \subset \mathcal{L} := \iota_{\mathrm{M},\mathrm{L}}(\hat{\mathrm{L}})$*, and* $\iota^{-1}_{\mathrm{M},\mathrm{L}} : \mathcal{L} \to {}^L\mathrm{L}$ *restricts to an embedding* $\mathrm{Z}_{\hat{\mathrm{M}}} \hookrightarrow \mathrm{Z}_{\hat{\mathrm{L}}}$ *that is independent of the choice of* $\iota_{\mathrm{M},\mathrm{L}}$ *within its* $\hat{\mathrm{M}}$*-conjugacy class.*

Proof. Since $\mathcal{L}^0 = \iota_{\mathrm{M},\mathrm{L}}(\hat{\mathrm{L}}) \subset \hat{\mathrm{M}}$ is a Levi subgroup, it contains $\mathrm{Z}_{\hat{\mathrm{M}}}$, and gives an embedding $\mathrm{Z}_{\hat{\mathrm{M}}} \hookrightarrow \mathrm{Z}_{\mathcal{L}^0} \xrightarrow{\iota^{-1}_{\mathrm{M},\mathrm{L}}} \mathrm{Z}_{\hat{\mathrm{L}}}$. Since $\mathrm{Z}_{\hat{\mathrm{M}}}$ is centralized by $\hat{\mathrm{M}}$, it is easy to see that this embedding does not change when $\iota_{\mathrm{M},\mathrm{L}}$ is replaced by an $\hat{\mathrm{M}}$-conjugate. □

2.4.3. *Functoriality of the* $\iota_{\mathrm{M},\mathrm{L}}$ *and some further consequences.*

Construction 2.4.14. In the situation of Notation 2.4.11, it is sometimes convenient to have an explicit choice of an embedding $\iota_{\mathrm{M},\mathrm{L}}$, which we now describe, following [Bor79, Section 3.4]. Let $\mathrm{Q} \subset \mathrm{M}$ be a choice of a parabolic subgroup with L as a Levi subgroup, giving by Notation 2.4.2 a Γ-equivariant identification $\Psi(\hat{\mathrm{L}}) \cong (X^\vee_{\mathrm{M}}, \Delta^\vee_{\mathrm{Q}}, X_{\mathrm{M}}, \Delta_{\mathrm{Q}})$. We realize ${}^L\mathrm{M}$ as $\hat{\mathrm{M}} \rtimes W_F$ using a preferred section $s : W_F \to {}^L\mathrm{M}$, and choose an $\mathrm{Int}\,s(W_F)$-invariant pinning in $\hat{\mathrm{M}}$. We do the same with ${}^L\mathrm{L}$. Let $(\mathcal{Q},\mathcal{L})$ be the unique standard (for the chosen pinning) parabolic-Levi pair in ${}^L\mathrm{M}$, with the property that $\Delta_{\mathcal{Q}} = \Delta^\vee_{\mathrm{Q}}$. This gives us a unique Γ-equivariant identification $\Psi(\hat{\mathrm{L}}) \cong \Psi(\mathcal{L}^0)$, and hence a unique Γ-invariant isomorphism $\hat{\mathrm{L}} \to \mathcal{L}^0$ respecting the standard pinnings of the two groups. Thus, we also get the embedding ${}^L\mathrm{L} = \hat{\mathrm{L}} \rtimes W_F \cong \mathcal{L}^0 \rtimes W_F = \mathcal{L} \hookrightarrow {}^L\mathrm{M}$, which is a choice for $\iota_{\mathrm{M},\mathrm{L}}$.

Proposition 2.4.15. *Suppose* $\mathrm{L}_1, \mathrm{L}_2$ *are Levi subgroups of* M *with* $\mathrm{L}_1 \subset \mathrm{L}_2$*. Then* $\iota_{\mathrm{M},\mathrm{L}_2} \circ \iota_{\mathrm{L}_2,\mathrm{L}_1} : {}^L\mathrm{L}_1 \hookrightarrow {}^L\mathrm{M}$ *necessarily belongs to the* $\hat{\mathrm{M}}$*-conjugacy class of* $\iota_{\mathrm{M},\mathrm{L}_1}$*.*

Proof. The result being well-known, we will be slightly informal. By Corollary 2.4.10, it is enough to find a single choice of each of $\iota_{\mathrm{M},\mathrm{L}_2}, \iota_{\mathrm{M},\mathrm{L}_1}$ and $\iota_{\mathrm{L}_2,\mathrm{L}_1}$, such that $\iota_{\mathrm{M},\mathrm{L}_1} = \iota_{\mathrm{M},\mathrm{L}_2} \circ \iota_{\mathrm{L}_2,\mathrm{L}_1}$. Using

Construction 2.4.14, it is easy to see that this follows if we can choose parabolic subgroups $\mathrm{Q}_1, \mathrm{Q}_2 \subset \mathrm{M}$, with Levi subgroups $\mathrm{L}_1, \mathrm{L}_2$, respectively, such that the resulting "embeddings" $\Psi(\mathrm{L}_1) \hookrightarrow \Psi(\mathrm{L}_2), \Psi(\mathrm{L}_2) \hookrightarrow \Psi(\mathrm{M})$ and $\Psi(\mathrm{L}_1) \hookrightarrow \Psi(\mathrm{M})$ in the sense mentioned in Notation 2.4.2 satisfy that the third of these is the 'composite' of the first two in an appropriate sense. It is easy to see that this compatibility of "embeddings" follows automatically if $\mathrm{Q}_1 \subset \mathrm{Q}_2$. For such a choice, e.g., first choose Q_2, and let Q_1 be generated by the unipotent radical of Q_2 together with any parabolic subgroup of L_2 having L_1 as a Levi subgroup. □

Lemma 2.4.16. *Suppose* $\mathrm{L}_1, \mathrm{L}_2$ *are Levi subgroups of reductive groups* $\mathrm{M}_1, \mathrm{M}_2$ *over* F, *and suppose* $\beta : \mathrm{M}_1 \to \mathrm{M}_2$ *is an isomorphism of reductive groups such that* $\mathrm{L}_2 = \beta(\mathrm{L}_1)$. *If* ${}^L\beta : {}^L\mathrm{M}_2 \to {}^L\mathrm{M}_1$ *and* ${}^L(\beta|_{\mathrm{L}_1}) : {}^L\mathrm{L}_2 \to {}^L\mathrm{L}_1$ *are dual to* $\beta : \mathrm{M}_1 \to \mathrm{M}_2$ *and* $\beta|_{\mathrm{L}_1} : \mathrm{L}_1 \to \mathrm{L}_2$ *(in the sense of Notation 2.3.4(ii)), then:*

(i) If $\iota_{\mathrm{M}_1,\mathrm{L}_1}$ *is defined using pairs* $(\mathrm{Q}_1, \mathrm{L}_1)$ *and* $(\mathcal{Q}_1, \mathcal{L}_1)$, *and* $\iota_{\mathrm{M}_2,\mathrm{L}_2}$ *using pairs* $(\beta(\mathrm{Q}_1), \mathrm{L}_2)$ *and* $({}^L\beta^{-1}(\mathcal{Q}_1), {}^L\beta^{-1}(\mathcal{L}_1))$, *then the identifications induced by* $(\iota_{\mathrm{M}_1,\mathrm{L}_1}, \iota_{\mathrm{M}_2,\mathrm{L}_2})$ *transport* ${}^L(\beta|_{\mathrm{L}_1})$ *to* ${}^L\beta|_{\mathcal{L}_2}$ *up to* $\mathrm{Int}\, \mathcal{L}_2^0$ *or equivalently* $\mathrm{Int}\, \hat{\mathrm{L}}_2$, *i.e.,*

$$ {}^L\beta \circ \iota_{\mathrm{M}_2,\mathrm{L}_2} \in \iota_{\mathrm{M}_1,\mathrm{L}_1} \circ {}^L(\beta|_{\mathrm{L}_1}) \circ \mathrm{Int}\, \hat{\mathrm{L}}_2. $$

(ii) More generally, ${}^L\beta \circ \iota_{\mathrm{M}_2,\mathrm{L}_2}$ *is* $\hat{\mathrm{M}}_1$*-conjugate to* $\iota_{\mathrm{M}_1,\mathrm{L}_1} \circ {}^L(\beta|_{\mathrm{L}_1})$.

Proof. (i) is straightforward diagram-chasing, while (ii) follows from (i) and Corollary 2.4.10. □

Corollary 2.4.17. *The bijection between the set of* $\mathrm{M}(F)$*-conjugacy classes of Levi subgroups of* M *and the set of* $\hat{\mathrm{M}}$*-conjugacy classes of relevant Levi subgroups of* ${}^L\mathrm{M}$, *induced by* $\mathrm{L} \mapsto \iota_{\mathrm{M},\mathrm{L}}({}^L\mathrm{L})$ *(see Corollary 2.4.12), respects the action of* $\mathrm{Aut}(\mathrm{M})$ *on either set (where* $\beta \in \mathrm{Aut}(\mathrm{M})$ *takes the* $\hat{\mathrm{M}}$*-conjugacy class of a Levi subgroup* $\mathcal{L} \subset {}^L\mathrm{M}$ *to that of* $({}^L\beta)^{-1}(\mathcal{L})$ *for any* ${}^L\beta \in \mathrm{Aut}({}^L\mathrm{M})$ *dual to* β*).*

Proof. This is immediate from Lemma 2.4.16. □

The following proposition uses Lemma 2.4.8 or equivalently Corollary 2.4.9 to slightly generalize it to a statement involving automorphisms.

Proposition 2.4.18. *Let* $\mathcal{O}_{\mathrm{M}} \subset \mathrm{Aut}(\mathrm{M})$ *be a subset, and let* $\mathcal{O}_{\mathrm{M}}^+ = \{\beta \circ \mathrm{Int}\, m \mid \beta \in \mathcal{O}_{\mathrm{M}}, m \in \mathrm{M}(F)\}$. *Let* $\mathrm{L}_1, \mathrm{L}_2 \subset \mathrm{M}$ *be Levi subgroups. For* $i = 1, 2$, *fix choices of* $\iota_{\mathrm{M},\mathrm{L}_i} : {}^L\mathrm{L}_i \to {}^L\mathrm{M}$, *and let* $\mathcal{L}_i = \iota_{\mathrm{M},\mathrm{L}_i}({}^L\mathrm{L}_i) \subset {}^L\mathrm{M}$. *Then, via the isomorphisms* $\iota_{\mathrm{M},\mathrm{L}_1} : {}^L\mathrm{L}_1 \to \mathcal{L}_1$ *and* $\iota_{\mathrm{M},\mathrm{L}_2} : {}^L\mathrm{L}_2 \to \mathcal{L}_2$, *the first of the two sets below is transported to the second:*

- *The set* $\mathcal{O}_{\mathrm{M}}({}^L\mathrm{L}_1, {}^L\mathrm{L}_2)$ *of isomorphisms* ${}^L\mathrm{L}_2 \to {}^L\mathrm{L}_1$ *that are dual to* $\beta|_{\mathrm{L}_1} : \mathrm{L}_1 \to \mathrm{L}_2$ *for some* $\beta \in \mathcal{O}_{\mathrm{M}}^+$ *such that* $\beta(\mathrm{L}_1) = \mathrm{L}_2$;
- *The set* $\mathcal{O}_{\mathrm{M}}(\mathcal{L}_1, \mathcal{L}_2)$ *of isomorphisms* $\mathcal{L}_2 \to \mathcal{L}_1$ *obtained by restricting automorphisms* ${}^L\mathrm{M} \to {}^L\mathrm{M}$ *that are dual to some element of* $\mathcal{O}_{\mathrm{M}}$ *(or equivalently,* $\mathcal{O}_{\mathrm{M}}^+$*).*

Proof. Without loss of generality, we may and do assume that $\mathcal{O}_{\mathrm{M}} = \{\beta\}$ is singleton. Note that $\mathcal{O}_{\mathrm{M}}({}^L\mathrm{L}_1, {}^L\mathrm{L}_2) \neq \emptyset$ if and only if $\beta(\mathrm{L}_1)$ and L_2 are $\mathrm{M}(F)$-conjugate, which by Corollary 2.4.12 is equivalent to $\iota_{\mathrm{M},\beta(\mathrm{L}_1)}({}^L(\beta(\mathrm{L}_1)))$ and $\iota_{\mathrm{M},\mathrm{L}_2}({}^L\mathrm{L}_2) = \mathcal{L}_2$ being $\hat{\mathrm{M}}$-conjugate, which in turn, by Lemma 2.4.16, is equivalent to $({}^L\beta)^{-1} \circ \iota_{\mathrm{M},\mathrm{L}_1}({}^L\mathrm{L}_1) = ({}^L\beta)^{-1}(\mathcal{L}_1)$ being $\hat{\mathrm{M}}$-conjugate to $\iota_{\mathrm{M},\mathrm{L}_2}({}^L\mathrm{L}_2) = \mathcal{L}_2$ for some (or any) ${}^L\beta : {}^L\mathrm{M} \to {}^L\mathrm{M}$ dual to β, which in turn is equivalent to the condition that $\mathcal{O}(\mathcal{L}_1, \mathcal{L}_2) \neq \emptyset$.

Thus, we now assume that $\mathcal{O}(\mathrm{L}_1, \mathrm{L}_2) \neq \emptyset$ and $\mathcal{O}(\mathcal{L}_1, \mathcal{L}_2) \neq \emptyset$, in which case we may replace β with some $\mathrm{Int}\, m \circ \beta$ and assume that $\beta(\mathrm{L}_1) = \mathrm{L}_2$. We can also fix ${}^L\beta : {}^L\mathrm{M} \to {}^L\mathrm{M}$ that is dual to β, and such that ${}^L\beta(\mathcal{L}_2) = \mathcal{L}_1$. Now the task is to show that the set of isomorphisms ${}^L\mathrm{L}_2 \to {}^L\mathrm{L}_1$ dual to some $(\mathrm{Int}\, m \circ \beta)|_{\mathrm{L}_1}$ with $m \in \mathrm{M}(F)$ normalizing L_2, is transported by $(\iota_{\mathrm{M},\mathrm{L}_1}, \iota_{\mathrm{M},\mathrm{L}_2})$ to the set of isomorphisms $\mathcal{L}_2 \to \mathcal{L}_1$ of the form ${}^L\beta \circ \mathrm{Int}\, \hat{m}$ for some $\hat{m} \in \hat{\mathrm{M}}$ normalizing $\mathcal{L}_2$.

Note that the statement of the proposition does not change if we replace $\iota_{\mathrm{M},\mathrm{L}_2}$ by a different choice with image $\mathcal{L}_2$; this is because such choices differ by $\hat{\mathrm{M}}_2$-conjugacy, which can be absorbed into the possibilities for $\mathrm{Int}\, \hat{m}$. Thus, we assume without loss of generality that, if $\iota_{\mathrm{M},\mathrm{L}_1}$ is defined using the datum $(\mathrm{L}_1, \mathrm{Q}_1, \mathcal{L}_1, \mathcal{Q}_1)$, then $\iota_{\mathrm{M},\mathrm{L}_2}$ is defined using $(\beta(\mathrm{L}_1) = \mathrm{L}_2, \beta(\mathrm{Q}_1), {}^L\beta^{-1}(\mathcal{L}_1) =$

$\mathcal{L}_2$, ${}^L\beta^{-1}(\mathcal{Q}_1)$). This choice has the advantage that, by Lemma 2.4.16(i), ${}^L\beta \circ \iota_{\mathrm{M}_2,\mathrm{L}_2} \in \iota_{\mathrm{M}_1,\mathrm{L}_1} \circ {}^L(\beta|_{\mathrm{L}_1}) \circ \mathrm{Int}\,\hat{\mathrm{L}}_2$, i.e., $(\iota_{\mathrm{M},\mathrm{L}_1}, \iota_{\mathrm{M},\mathrm{L}_2})$ transports ${}^L(\beta|_{\mathrm{L}_1})$ to ${}^L\beta|_{\mathcal{L}_2}$, up to $\mathrm{Int}\,\mathcal{L}_2^0$.
With this, our task becomes to show that $(\iota_{\mathrm{M},\mathrm{L}_2}, \iota_{\mathrm{M},\mathrm{L}_2})$ transports the set of isomorphisms ${}^L\mathrm{L}_2 \to {}^L\mathrm{L}_2$ that are dual to $\mathrm{Int}\, m$ for some $m \in \mathrm{M}(F)$ normalizing L_2, to the set of isomorphisms $\mathcal{L}_2 \to \mathcal{L}_2$ of the form $\mathrm{Int}\,\hat{m}$ for some $\hat{m} \in \hat{\mathrm{M}}(F)$ normalizing $\mathcal{L}_2$. But this is Corollary 2.4.9(i). □

2.5. **Complex characters of p-adic reductive groups and Langlands duality.**

2.5.1. *Unramified characters.*

Notation 2.5.1. Suppose P is a linear algebraic group over F, which is not necessarily reductive.

(i) We denote by S_P the maximal split torus quotient of P.
(ii) Recall that a character $\chi : \mathrm{P}(F) \to \mathbb{C}^\times$ is said to be unramified if $\chi(x) = 1$ for all $x \in \mathrm{P}(F)$ such that $|\mu(x)| = 1$ for all algebraic characters $\mu : \mathrm{P} \to \mathbb{G}_m$. We denote by $X^{\mathrm{unr}}(\mathrm{P})$ the group of unramified characters $\mathrm{P}(F) \to \mathbb{C}^\times$. Then $X^{\mathrm{unr}}(\mathrm{P}) \subset \mathrm{Hom}_{\mathrm{cts}}(\mathrm{P}(F), \mathbb{C}^\times)$.
(iii) Let $X^{\mathrm{unr-uni}}(\mathrm{P}) \subset X^{\mathrm{unr}}(\mathrm{P})$ be the subgroup of unitary characters, and $X^{\mathrm{unr}}(\mathrm{P})_{>0} \subset X^{\mathrm{unr}}(\mathrm{P})$ the subgroup consisting of characters taking values in the multiplicative group $\mathbb{R}_{>0}$ of positive real numbers.
(iv) $X^{\mathrm{unr}}(\mathrm{P})$ will be viewed as a complex torus in the usual way (see Remark 2.5.2(ii) below). The product map
$$X^{\mathrm{unr}}(\mathrm{P})_{>0} \times X^{\mathrm{unr-uni}}(\mathrm{P}) \to X^{\mathrm{unr}}(\mathrm{P})$$
is easily seen to be an isomorphism, identifying $X^{\mathrm{unr}}(\mathrm{P})_{>0}$ with the set of hyperbolic elements of $X^{\mathrm{unr}}(\mathrm{P})$ in the sense of [SZ18, Section 5.1], and $X^{\mathrm{unr-uni}}(\mathrm{P})$ with the maximal compact subgroup of $X^{\mathrm{unr}}(\mathrm{P})$.

Remark 2.5.2. (i) Pull-back gives an isomorphism $X^{\mathrm{unr}}(\mathrm{M}) \cong X^{\mathrm{unr}}(\mathrm{P})$, where M is a Levi quotient of P.
(ii) Recall that $X^{\mathrm{unr}}(\mathrm{S}_\mathrm{P}) \to X^{\mathrm{unr}}(\mathrm{P})$ is surjective with finite kernel. Indeed, for surjectivity, combine the injectivity of the abelian group $\mathbb{C}^\times$ with the fact that for any $\chi \in X^{\mathrm{unr}}(\mathrm{P})$ and $p \in \mathrm{P}(F)$, $\chi(p)$ depends only on the image of p under $\mathrm{P}(F) \to \mathrm{S}_\mathrm{P}(F) \to \mathrm{S}_\mathrm{P}(F)/\mathrm{S}_\mathrm{P}(\mathfrak{O})$ (use that any algebraic character $\mu : \mathrm{P} \to \mathbb{G}_m$ factors through $\mathrm{P} \to \mathrm{S}_\mathrm{P}$). For the finiteness of the kernel, one immediately reduces to the case where $\mathrm{P} = \mathrm{M}$ is reductive, and notes that we have a chain of restriction maps $X^{\mathrm{unr}}(\mathrm{S}_\mathrm{M}) \to X^{\mathrm{unr}}(\mathrm{M}) \to X^{\mathrm{unr}}(\mathrm{A}_\mathrm{M})$, whose composite has finite kernel since $\mathrm{A}_\mathrm{M} \to \mathrm{S}_\mathrm{M}$ is an isogeny.
Since $X^{\mathrm{unr}}(\mathrm{S}_\mathrm{P}) \cong \mathrm{Hom}(X_*(\mathrm{S}_\mathrm{P}), \mathbb{C}^\times)$ is a complex torus, it follows that so is $X^{\mathrm{unr}}(\mathrm{P})$. Clearly, $X^{\mathrm{unr}}(\mathrm{S}_\mathrm{P}) \to X^{\mathrm{unr}}(\mathrm{P})$ restricts to an isomorphism $X^{\mathrm{unr}}(\mathrm{S}_\mathrm{P})_{>0} \to X^{\mathrm{unr}}(\mathrm{P})_{>0}$.
(iii) If $\chi \in \mathrm{Hom}_{\mathrm{cts}}(\mathrm{P}(F), \mathbb{C}^\times)$ is valued in $\mathbb{R}_{>0}$, then $\chi \in X^{\mathrm{unr}}(\mathrm{P})$ — assuming without loss of generality that P is reductive, and using that $\mathbb{R}_{>0}$ is torsion free, this follows from the well-known fact that the subgroup of $p \in \mathrm{P}(F)$ such that $|\mu(p)| = 1$ for all algebraic characters $\mu : \mathrm{P} \to \mathbb{G}_m$, is generated by the compact subgroups of $\mathrm{P}(F)$ (e.g., [FP21, Lemma 4.8]).

2.5.2. *Two constructions of Langlands from* [Bor79, Section 10.1 and Section 10.2]. Recall that, if M is a connected reductive group over F, Γ acts on $\mathrm{Z}_{\hat{\mathrm{M}}}$ through algebraic automorphisms (see Definition 2.3.3(iv)).

Notation 2.5.3. Let M be a connected reductive group over F. We will occasionally refer to, but not use substantially, the set $\mathscr{S}(W_F, {}^L\mathrm{M})$ of continuous sections to ${}^L\mathrm{M} \to W_F$. If we use a preferred section $s : W_F \to {}^L\mathrm{M}$ to make W_F act on $\hat{\mathrm{M}}$ via $\mathrm{Int} \circ s$, then $\mathscr{S}(W_F, {}^L\mathrm{M})$ identifies with $H^1(W_F, \hat{\mathrm{M}})$, where an element of $H^1(W_F, \hat{\mathrm{M}})$ represented by a cocycle α identifies with the section $w \mapsto \alpha(w)s(w)$ inside $\mathscr{S}(W_F, {}^L\mathrm{M})$.

Notation 2.5.4. For any reductive group M over F, we have a homomorphism of groups:

$$\mathscr{S}(W_F, {}^L\mathrm{M})/(\mathrm{Int}\,\hat{\mathrm{M}}) \to \mathrm{Hom}_{\mathrm{cts}}(\mathrm{Z}_\mathrm{M}(F), \mathbb{C}^\times) \tag{15}$$

and an isomorphism of groups (which we normalize using the "Fr goes to a uniformizer" convention, analogous to [Hai14, footnote 2]) involving the group $\mathscr{S}(W_F, Z_{\hat{M}} \rtimes W_F)$ of continuous sections to $Z_{\hat{M}} \rtimes W_F \to W_F$:

(16)
$$\mathscr{S}(W_F, Z_{\hat{M}} \rtimes W_F)/(\operatorname{Int} Z_{\hat{M}}) = H^1(W_F, Z_{\hat{M}}) \xrightarrow{\cong} \operatorname{Hom}_{\mathrm{cts}}(\operatorname{coker}(M_{\mathrm{sc}}(F) \to M(F)), \mathbb{C}^\times) \subset \operatorname{Hom}_{\mathrm{cts}}(M(F), \mathbb{C}^\times),$$

which are constructions of Langlands (see [Bor79, Section 10.1 and Section 10.2] and also [Kal15, Section 5.5]). We will denote these by $\varphi \mapsto \zeta_\varphi$ and $\alpha \mapsto \chi_\alpha$, respectively.

Remark 2.5.5. We are not describing $\varphi \mapsto \zeta_\varphi$ or $\alpha \mapsto \chi_\alpha$ explicitly, because we will have very little use for the former, and only need an easy case of the latter (to be explicated in Subsubsection 2.5.3 below). But it is easy to give parts/cases of these definitions, which we now do, in terms of local Langlands duality for tori:

- χ_α is easy to define if α is the image of some α_0 under $H^1(W_F, Z^0_{\hat{M}}) \to H^1(W_F, Z_{\hat{M}})$: $Z^0_{\hat{M}}$ with its Γ-action can be checked to be Langlands dual to the maximal torus quotient M/M_{der} of M, and χ_α is obtained by pulling back the character of $(M/M_{\mathrm{der}})(F)$ that is Langlands dual to α_0.
- $\zeta_\varphi|_{Z^0_M(F)}$ is easy to define: pushing $\varphi \in \mathscr{S}(W_F, {}^L M)/(\operatorname{Int}\hat{M})$ forward under any morphism ${}^L M \to {}^L(Z^0_M)$ dual to $Z^0_M \hookrightarrow M$ (in the sense of Notation 2.3.4(v)) gives one an element of $\mathscr{S}(W_F, {}^L(Z^0_M))/(\operatorname{Int}\widehat{Z^0_M}) \cong H^1(W_F, \widehat{Z^0_M})$, to which $\zeta_\varphi|_{Z^0_M(F)}$ is Langlands dual.

Remark 2.5.6. We will use the following two standard properties of these constructions.

(i) They are functorial for normal homomorphisms of reductive groups.
(ii) Suppose $L \subset M$ is a Levi subgroup, and $\varphi_L \in \mathscr{S}(W_F, {}^L L)/(\operatorname{Int}\hat{L})$. Making a choice of $\iota = \iota_{M,L} : {}^L L \to {}^L M$, we have a map $\iota \circ - =: \iota_* : \mathscr{S}(W_F, {}^L L)/(\operatorname{Int}\hat{L}) \to \mathscr{S}(W_F, {}^L M)/(\operatorname{Int}\hat{M})$. Then $\zeta_{\iota_*(\varphi_L)} = \zeta_{\varphi_L}|_{Z_M(F)}$. Moreover, for all $\alpha_M \in H^1(W_F, Z_{\hat{M}})$, denoting by α_L its image in $H^1(W_F, Z_{\hat{L}})$ under the embedding $Z_{\hat{M}} \hookrightarrow Z_{\hat{L}}$ obtained from $\iota_{M,L}$ (see Corollary 2.4.13), we have $\chi_{\alpha_M}|_{L(F)} = \chi_{\alpha_L}$.

Remark 2.5.7. Let M be a connected reductive group over F. If $\varphi \in \mathscr{S}(W_F, {}^L M)/(\operatorname{Int}\hat{M})$ and $\alpha \in H^1(W_F, Z_{\hat{M}})$, then we have a well-defined element $\alpha \cdot \varphi \in \mathscr{S}(W_F, \hat{M})/(\operatorname{Int}\hat{M})$, defined using pointwise product at the level of cocycles. One can show, along the lines of [Bor79, Equation (4), Section 10.2], that $\zeta_{\alpha\cdot\varphi} = \zeta_\varphi \cdot (\chi_\alpha|_{Z_M(F)})$.

2.5.3. *Langlands duality for characters and the Kottwitz homomorphism.* The following lemma is given a more insightful derivation, based on standard results in literature, in [Hai14, Section 3.3.1]. We nevertheless give a proof (based on some standard results), because it helps illustrate the role that S_M will play.

Lemma 2.5.8. *Let* M *be a connected reductive group over F. Then, via the chain of inclusions*

$$(Z^{I_F}_{\hat{M}})^0_{\mathrm{Fr}} := ((Z^{I_F}_{\hat{M}})_{\mathrm{Fr}})^0 \hookrightarrow (Z^{I_F}_{\hat{M}})_{\mathrm{Fr}} \xrightarrow{\cong} H^1(W_F/I_F, Z^{I_F}_{\hat{M}}) \hookrightarrow H^1(W_F, Z_{\hat{M}}),$$

where the last map is given by inflation and the map before that is inverse to evaluating at the Frobenius element Fr*, and the inclusion*

$$X^{\mathrm{unr}}(M) \hookrightarrow \operatorname{Hom}_{\mathrm{cts}}(M(F), \mathbb{C}^\times),$$

(16) *restricts to an isomorphism (a restriction of the Kottwitz isomorphism):*

$$(Z^{I_F}_{\hat{M}})^0_{\mathrm{Fr}} \cong X^{\mathrm{unr}}(M). \tag{17}$$

Moreover, (17) *is an isomorphism of complex tori, and is functorial in* M.

Remark 2.5.9. In the above lemma, we have stipulated, for relative lightness of notation, that the ambiguous $(Z^{I_F}_{\hat{M}})^0_{\mathrm{Fr}}$ be read as $((Z^{I_F}_{\hat{M}})_{\mathrm{Fr}})^0$, rather than as $((Z^{I_F}_{\hat{M}})^0)_{\mathrm{Fr}}$. As we learnt from a discussion with D. Prasad, the quotient map $((Z^{I_F}_{\hat{M}})^0)_{\mathrm{Fr}} \to (Z^{I_F}_{\hat{M}})^0_{\mathrm{Fr}} = ((Z^{I_F}_{\hat{M}})_{\mathrm{Fr}})^0$ is not an isomorphism when M is an unramified but non-split form of GSO_{2n}. This convention will be applicable henceforth: in case of ambiguity as to whether to take Fr-coinvariants or the identity component first, one should take Fr-coinvariants first.

Proof of Lemma 2.5.8. The functoriality in M is a special case of the well-known functoriality of (16), so let us prove that (17) is indeed a restriction of (16) as well as an isomorphism. If M is a split torus, this is easy to see explicitly (use Remark 2.5.5). In this case, (17) is the isomorphism

$$X^{\mathrm{unr}}(\mathrm{M}) = \mathrm{Hom}(X_*(\mathrm{M}), \mathbb{C}^\times) = \mathrm{Hom}(X^*(\hat{\mathrm{M}}), \mathbb{C}^\times) = \hat{\mathrm{M}} = \mathrm{Z}_{\hat{\mathrm{M}}}^{\Gamma,0} = (\mathrm{Z}_{\hat{\mathrm{M}}}^{I_F})^0_{\mathrm{Fr}}$$

of complex tori, where the first map is given by $\chi \mapsto (\mu \mapsto \chi(\mu(\varpi)))$ in accordance with our normalization of (16). In particular, (17) holds with M replaced by $\mathrm{S_M}$. Since $X^{\mathrm{unr}}(\mathrm{S_M}) \to X^{\mathrm{unr}}(\mathrm{M})$ is surjective with finite kernel, as we mentioned in Remark 2.5.2(ii), we will be done by the functoriality of (16) if we show that the image of

$$\hat{\mathrm{S}}_{\mathrm{M}} = \mathrm{Z}_{\hat{\mathrm{M}}}^{\Gamma,0} \to H^1(W_F, \mathrm{Z}_{\hat{\mathrm{M}}}) \tag{18}$$

equals $(\mathrm{Z}_{\hat{\mathrm{M}}}^{I_F})^0_{\mathrm{Fr}} \subset H^1(W_F/I_F, \mathrm{Z}_{\hat{\mathrm{M}}}^{I_F})$, where we used Lemma 2.5.10 below to identify $\hat{\mathrm{S}}_{\mathrm{M}}$ with $\mathrm{Z}_{\hat{\mathrm{M}}}^{\Gamma,0}$. It is immediate, from the cocycle-level, that this image is contained in $H^1(W_F/I_F, (\mathrm{Z}_{\hat{\mathrm{M}}})^{I_F}) = (\mathrm{Z}_{\hat{\mathrm{M}}}^{I_F})_{\mathrm{Fr}}$. The image of (18) is therefore that of $\mathrm{Z}_{\hat{\mathrm{M}}}^{\Gamma,0} = (\mathrm{Z}_{\hat{\mathrm{M}}}^{I_F})^{\mathrm{Fr},0} \to (\mathrm{Z}_{\hat{\mathrm{M}}}^{I_F})_{\mathrm{Fr}}$, which by the finiteness of the kernel and comparison of dimension is easily seen to be $(\mathrm{Z}_{\hat{\mathrm{M}}}^{I_F})^0_{\mathrm{Fr}}$: the group of Fr-invariants and the group of Fr-coinvariants of the complex torus $(\mathrm{Z}_{\hat{\mathrm{M}}}^{I_F})^0$ have the same dimension, as Fr induces an automorphism of $X^*((\mathrm{Z}_{\hat{\mathrm{M}}}^{I_F})^0) \otimes \mathbb{Q}$ that has finite order and is hence semisimple. □

Lemma 2.5.10. *The (central and hence unique) map* $\hat{\mathrm{S}}_{\mathrm{M}} \to \hat{\mathrm{M}}$ *dual to* $\mathrm{M} \to \mathrm{S_M}$ *identifies* $\hat{\mathrm{S}}_{\mathrm{M}}$ *with* $\mathrm{Z}_{\hat{\mathrm{M}}}^{\Gamma,0}$.

Proof. Note that $\mathrm{S_M}$ is also the maximal split torus quotient of $\mathrm{M}/\mathrm{M_{der}}$, and use the identification $X^*(\mathrm{M}) = X_*(\mathrm{Z}_{\hat{\mathrm{M}}})$ from [Kot84, (1.8.2)], to get:

$$X^*(\mathrm{S_M}) = X^*(\mathrm{M}/\mathrm{M_{der}})^\Gamma = X^*(\mathrm{M})^\Gamma = X_*(\mathrm{Z}_{\hat{\mathrm{M}}})^\Gamma = X_*(\mathrm{Z}_{\hat{\mathrm{M}}}^{\Gamma,0}).$$

□

As a summary of sorts, we have a commutative diagram:
(19)

$$\begin{array}{ccccccc}
H^1(W_F/I_F, \mathrm{Z}_{\hat{\mathrm{M}}}^{\Gamma,0}) \cong \mathrm{Z}_{\hat{\mathrm{M}}}^{\Gamma,0} & \longrightarrow & (\mathrm{Z}_{\hat{\mathrm{M}}}^{I_F})^0_{\mathrm{Fr}} & \hookrightarrow & (\mathrm{Z}_{\hat{\mathrm{M}}}^{I_F})_{\mathrm{Fr}} \cong H^1(W_F/I_F, \mathrm{Z}_{\hat{\mathrm{M}}}^{I_F}) & \longrightarrow & H^1(W_F, \mathrm{Z}_{\hat{\mathrm{M}}}) \\
\downarrow \cong & & \downarrow \cong & & & & \downarrow \alpha \mapsto \chi_\alpha \\
X^{\mathrm{unr}}(\mathrm{S_M}) & \longrightarrow & X^{\mathrm{unr}}(\mathrm{M}) & & \longrightarrow & & \mathrm{Hom_{cts}}(\mathrm{M}(F), \mathbb{C}^\times),
\end{array}$$

for a discussion of whose maps we refer to the preceding discussion.

Notation 2.5.11. If $\alpha \in \mathrm{Z}_{\hat{\mathrm{M}}}^{\Gamma,0} = H^1(W_F/I_F, \mathrm{Z}_{\hat{\mathrm{M}}}^{\Gamma,0})$, we will use χ_α to denote two different objects: the image of α in $X^{\mathrm{unr}}(\mathrm{S_M})$ under the Kottwitz homomorphism, or the image of this image under $X^{\mathrm{unr}}(\mathrm{S_M}) \to X^{\mathrm{unr}}(\mathrm{M})$, namely $\chi_{\alpha'}$, where α' is the image of α in $(\mathrm{Z}_{\hat{\mathrm{M}}}^{I_F})^0_{\mathrm{Fr}} \subset H^1(W_F, \mathrm{Z}_{\hat{\mathrm{M}}})$. The context will make it clear as to which of these two objects we are referring to.

2.6. **Groups of automorphisms 'up to which' we will work.** We will now fix our connected reductive group G over F, as well as a collection $\{\mathcal{O}_{\mathrm{M}}\}_{\mathrm{M}}$ indexed by Levi subgroups $\mathrm{M} \subset \mathrm{G}$ (subject the to some conditions), where each $\mathcal{O}_{\mathrm{M}}$ is a subgroup of $\mathrm{Aut(M)}$.

Notation 2.6.1. (i) For the rest of this paper, let G be a fixed connected reductive group over F.

(ii) We fix a collection $\{\mathcal{O}_{\mathrm{M}}\}_{\mathrm{M}}$ indexed by Levi subgroups $\mathrm{M} \subset \mathrm{G}$, where $\mathcal{O}_{\mathrm{M}} \subset \mathrm{Aut(M)}$ is a group of (F-rational algebraic) automorphisms of M, subject to the conditions of (iv) below; we will abbreviate $\mathcal{O}_{\mathrm{G}}$ to $\mathcal{O}$.

(iii) Given Levi subgroups $\mathrm{L}, \mathrm{L}_1, \mathrm{L}_2 \subset \mathrm{M} \subset \mathrm{G}$ and Levi subgroups $\mathcal{L}, \mathcal{L}_1, \mathcal{L}_2 \subset {}^L\mathrm{M}$, we define groups $\mathcal{O}^+_{\mathrm{M}}(\mathrm{L}_2, \mathrm{L}_1), \mathcal{O}^+_{\mathrm{M}}({}^L\mathrm{L}_1, {}^L\mathrm{L}_2), \mathcal{O}^+_{\mathrm{M}}(\mathcal{L}_1, \mathcal{L}_2), \mathcal{O}^+_{\mathrm{M,L}}, \mathcal{O}^+_{\mathrm{M}}, \mathcal{O}^+_{\mathrm{M},{}^L\mathrm{L}}$ and $\mathcal{O}^+_{\mathrm{M},\mathcal{L}}$ as follows:

- We let $\mathcal{O}^+_{\mathrm{M}}(\mathrm{L}_2, \mathrm{L}_1)$ be the set of all $\beta|_{\mathrm{L}_1} : \mathrm{L}_1 \to \mathrm{L}_2$, as β runs over the elements of $\mathcal{O}_{\mathrm{M}} \circ \mathrm{Int}\,\mathrm{M}(F)$ (which we will soon abbreviate to $\mathcal{O}^+_{\mathrm{M}}$) such that $\beta(\mathrm{L}_1) = \mathrm{L}_2$. We abbreviate $\mathcal{O}^+_{\mathrm{M}}(\mathrm{L}, \mathrm{L})$ to $\mathcal{O}^+_{\mathrm{M,L}}$ and $\mathcal{O}^+_{\mathrm{M,M}} = \mathcal{O}_{\mathrm{M}} \circ \mathrm{Int}\,\mathrm{M}(F)$ to $\mathcal{O}^+_{\mathrm{M}}$;

- Consistently with Proposition 2.4.18, we let $\mathcal{O}^+_{\mathrm{M}}({}^L\mathrm{L}_1, {}^L\mathrm{L}_2)$ be the set of all maps ${}^L\mathrm{L}_2 \to {}^L\mathrm{L}_1$ that are dual to some element of $\mathcal{O}_{\mathrm{M}}(\mathrm{L}_2, \mathrm{L}_1)$, and we abbreviate $\mathcal{O}^+_{\mathrm{M}}({}^L\mathrm{L}, {}^L\mathrm{L})$ to $\mathcal{O}^+_{\mathrm{M}, {}^L\mathrm{L}}$;
- Consistently with Proposition 2.4.18, we let $\mathcal{O}^+_{\mathrm{M}}(\mathcal{L}_1, \mathcal{L}_2)$ be the set of all ${}^L\beta|_{\mathcal{L}_2} : \mathcal{L}_2 \to \mathcal{L}_1$ as ${}^L\beta : {}^L\mathrm{M} \to {}^L\mathrm{M}$ varies over maps that are dual to some element of $\mathcal{O}_{\mathrm{M}}$ and such that ${}^L\beta(\mathcal{L}_2) = \mathcal{L}_1$, and we abbreviate $\mathcal{O}^+_{\mathrm{M}}(\mathcal{L}, \mathcal{L})$ to $\mathcal{O}^+_{\mathrm{M},\mathcal{L}}$.

(iv) We subject the collection $\{\mathcal{O}_{\mathrm{M}}\}_{\mathrm{M}}$ to the following conditions:
  (a) For each M, each element of $\mathcal{O}_{\mathrm{M}}$ acts as the identity on A_{M};
  (b) If $\mathrm{L}, \mathrm{M} \subset \mathrm{G}$ are Levi subgroups, and $\beta \in \mathcal{O}^+_{\mathrm{G}} = \mathcal{O}_{\mathrm{G}} \circ \operatorname{Int} \mathrm{G}(F)$ satisfies that $\beta(\mathrm{L}) \subset \mathrm{M}$, then under the map $\mathrm{Aut}(\mathrm{L}) \to \mathrm{Aut}(\beta(\mathrm{L}))$ given by transport by β, the image of $\mathcal{O}^+_{\mathrm{L}} = \mathcal{O}_{\mathrm{L}} \circ \operatorname{Int} \mathrm{L}(F)$ is contained in $\mathcal{O}^+_{\mathrm{M},\beta(\mathrm{L})}$ (so, as an important special case, $\mathcal{O}_{\mathrm{L}} \subset \mathcal{O}^+_{\mathrm{L}} \subset \mathcal{O}^+_{\mathrm{M},\mathrm{L}}$);
  (c) The image of $\mathcal{O}_{\mathrm{G}}$ in $\mathrm{Out}(\mathrm{G})$ is finite.

(v) Since the dual of an automorphism β of a connected reductive group M over F is a well-defined $\hat{\mathrm{M}}$-conjugacy class of automorphisms ${}^L\beta : {}^L\mathrm{M} \to {}^L\mathrm{M}$, any group of automorphisms of M acts on the set of $\hat{\mathrm{M}}$-conjugacy classes of maps $X \to {}^L\mathrm{M}$, where X is any set, topological space etc.: $\beta \in \mathrm{Aut}(\mathrm{M})$ takes the $\hat{\mathrm{M}}$-conjugacy class of $\dot\varphi : X \to {}^L\mathrm{M}$ to $({}^L\beta)^{-1} \circ \dot\varphi$. We will apply these considerations to let groups such as $\mathcal{O}_{\mathrm{M}}$ and $\mathcal{O}^+_{\mathrm{M},\mathrm{L}}$ etc. defined above act on appropriate spaces of Langlands parameters.

Remark 2.6.2. (i) An important example of a collection $\{\mathcal{O}_{\mathrm{M}}\}_{\mathrm{M}}$ as above is the one where each $\mathcal{O}_{\mathrm{M}}$ is trivial. We get another example by fixing any group $\mathcal{O}$ of automorphisms of G with finite image in $\mathrm{Out}(\mathrm{G})$, and taking $\mathcal{O}_{\mathrm{M}}$ to be the group of automorphisms of M induced by those elements of $\mathcal{O}^+_{\mathrm{G}} = \mathcal{O} \circ \operatorname{Int} \mathrm{G}(F)$ that act as the identity on Z_{M} (and hence preserve the centralizer M of Z_{M}).

(ii) The typical example we have in mind for a nontrivial collection $\{\mathcal{O}_{\mathrm{M}}\}$ is in the case where G is a quasi-split form of SO_{2n}, GSO_{2n} or GSpin_{2n}, with $\mathcal{O} = \mathcal{O}_{\mathrm{G}}$ a two element group of automorphisms of G, one of which is outer, and the $\mathcal{O}_{\mathrm{M}}$ as in the latter example of (i) above (i.e., consisting of those automorphisms of M induced by elements of $\mathcal{O}^+_{\mathrm{G}}$ that act as the identity on Z_{M}).

(iii) As (ii) suggests, the reason for introducing the collection $\{\mathcal{O}_{\mathrm{M}}\}_{\mathrm{M}}$ is to be able to make a weaker statement in cases where we don't have a 'canonical' local Langlands correspondence for the M as such, but only one up to the action of the $\mathcal{O}_{\mathrm{M}}$; this applies to the study of quasi-split forms of SO_{2n} in [Art13], of GSO_{2n} in [Xu16] and [Xu18], and of GSpin_{2n} in [Mg14]. A reader who is not particular about cases of this sort may assume each $\mathcal{O}_{\mathrm{M}}$ to be trivial or to be simply the group of all inner conjugations $\operatorname{Int} m$ with $m \in \mathrm{M}(F)$, in which case a lot of the definitions and results below simplify.

(iv) For each Levi subgroup of $\mathrm{M} \subset \mathrm{G}$, $\mathcal{O}^+_{\mathrm{G},\mathrm{M}}$ contains the group of conjugations of M by the elements of the normalizer $\mathrm{N}_{\mathrm{G}}(\mathrm{M})(F)$ of $\mathrm{M}(F)$ in $\mathrm{G}(F)$. If $\mathcal{O}_{\mathrm{G}}$ is trivial, then this containment is easily checked to be an equality.

(v) If one wishes to impose the 'twisting by a character desideratum' and the 'central character desideratum' (see Remark 2.10.4 below) on a 'local Langlands correspondence for M up to the $\mathcal{O}_{\mathrm{M}}$', one would need to additionally impose on the collection $\{\mathcal{O}_{\mathrm{M}}\}_{\mathrm{M}}$ the following condition:
Stronger condition: For each $\beta \in \mathcal{O}_{\mathrm{M}}$, β acts trivially on Z_{M}, and further, each ${}^L\beta : {}^L\mathrm{M} \to {}^L\mathrm{M}$ dual to β is the identity on $\mathrm{Z}_{\hat{\mathrm{M}}}$.
We will impose only the desideratum of twisting by unramified characters, which explains our condition in Notation 2.6.1(iv) that $\mathcal{O}_{\mathrm{M}}$ acts as the identity on A_{M}.

(vi) We will only consider the action of $\mathcal{O}_{\mathrm{M}}$ on objects related to invariant harmonic analysis on M, so replacing $\mathcal{O}_{\mathrm{M}}$ by $\mathcal{O}^+_{\mathrm{M}}$ will not change any of the analysis that follows. The only reason we write $\mathcal{O}_{\mathrm{M}}$ instead of $\mathcal{O}^+_{\mathrm{M}}$ in what follows, is that it can be convenient to think of a finite group of automorphisms (which $\mathcal{O}^+_{\mathrm{M}}$ almost never is, while $\mathcal{O}_{\mathrm{M}}$ is allowed, though not required, to be trivial).

(vii) A lot of the time we will consider only the action of $\mathcal{O}_{\mathrm{M}}$ on objects such as a set of L-packets or a set of (non-enhanced) Langlands parameters associated to M, so our dependence on $\mathcal{O}_{\mathrm{M}}$ will often, though not always, be only through its image in Out(M).

Lemma 2.6.3. *Let* $\mathrm{L} \subset \mathrm{M} \subset \mathrm{G}$ *be Levi subgroups, and let* $\beta \in \mathcal{O}_{\mathrm{G}}^{+} = \mathcal{O} \circ \operatorname{Int} \mathrm{G}(F)$.

(i) β *transports* $\mathcal{O}_{\mathrm{M},\mathrm{L}}^{+}$ *isomorphically onto* $\mathcal{O}_{\beta(\mathrm{M}),\beta(\mathrm{L})}^{+}$.

(ii) $\operatorname{Int} \mathrm{M}(F) \subset \mathcal{O}_{\mathrm{G},\mathrm{M}}^{+}$ *is of finite index. Equivalently, the image of* $\mathcal{O}_{\mathrm{G},\mathrm{M}}^{+}$ *in* Out(M) *is finite.*

(iii) For any Levi subgroup $\mathrm{M}' \subset \mathrm{G}$*, the collection* $\{\mathcal{O}_{\mathrm{L}'}\}_{\mathrm{L}'}$*, as* L' *varies over the Levi subgroups of* M'*, satisfies the analogues, for* M' *in place of* G*, of the hypotheses imposed on the collection* $\{\mathcal{O}_{\mathrm{M}}\}_{\mathrm{M}}$ *in Notation 2.6.1(iv).*

Proof. By (iv)b of Notation 2.6.1, β transports $\mathcal{O}_{\mathrm{M},\mathrm{M}}^{+} = \mathcal{O}_{\mathrm{M}}^{+}$ into $\mathcal{O}_{\beta(\mathrm{M}),\beta(\mathrm{M})}^{+} = \mathcal{O}_{\beta(\mathrm{M})}^{+}$. In doing so, it clearly transports the set $\mathcal{O}_{\mathrm{M},\mathrm{L}}^{+}$ of automorphisms in $\mathcal{O}_{\mathrm{M}}^{+}$ that preserve L to the set $\mathcal{O}_{\beta(\mathrm{M}),\beta(\mathrm{L})}^{+}$ of automorphisms in $\mathcal{O}_{\beta(\mathrm{M})}^{+}$ that preserve $\beta(\mathrm{L})$. By making a similar argument with β^{-1}, (i) follows. Since $\mathcal{O}_{\mathrm{G}}^{+}$ has finite image in Out(G), and $\operatorname{Int} \mathrm{G}(F)$ has finite index in $\operatorname{Int} \mathrm{G}_{\mathrm{ad}}(F)$, $\operatorname{Int} \mathrm{G}(F)$ has finite index in $\mathcal{O}_{\mathrm{G}}^{+}$. Thus, some finite-index subgroup of $\mathcal{O}_{\mathrm{G},\mathrm{M}}^{+}$ acts on M by restrictions of elements of $\operatorname{Int} \mathrm{G}(F)$. Since the normalizer of $\mathrm{M}(F)$ in $\mathrm{G}(F)$ has finite image in Out(M), some smaller finite-index subgroup of $\mathcal{O}_{\mathrm{G},\mathrm{M}}^{+}$ acts on M by elements of $\operatorname{Int} \mathrm{M}(F)$. From this, (ii) follows. Now (iii) is easy to verify (using (ii)). ☐

Lemma 2.6.4. *Let* M *be a Levi subgroup of* G*. Suppose* $\beta \in \operatorname{Aut}(\mathrm{G})$ *preserves* M *and restricts to an element of* $\mathcal{O}_{\mathrm{M}}^{+}$ *on it. Then for each parabolic subgroup* $\mathrm{P} \subset \mathrm{G}$ *with* M *as a Levi subgroup,* $\beta(\mathrm{P}) = \mathrm{P}$.

Proof. If $\mathrm{P} \subset \mathrm{G}$ is a parabolic subgroup with M as a Levi subgroup, there exists a homomorphism $\mu : \mathbb{G}_m \to \mathrm{A}_{\mathrm{M}}$ such that Lie P is the subspace of Lie G on which $\mathrm{Ad} \circ \mu$ acts by nonnegative weights. Hence the lemma follows from the fact that $\mathcal{O}_{\mathrm{M}}^{+}$ acts trivially on A_{M}. ☐

Lemma 2.6.5. *Suppose* M *is a Levi subgroup of* G*. Then the obvious actions of* $\mathcal{O}_{\mathrm{M}}^{+}$ *on* $\mathrm{Z}_{\hat{\mathrm{M}}}^{\Gamma,0}$, $(\mathrm{Z}_{\hat{\mathrm{M}}}^{I_F})_{\mathrm{Fr}}^{0}$, $X^{\mathrm{unr}}(\mathrm{M})$ *and* $X^{\mathrm{unr}}(\mathrm{S}_{\mathrm{M}})$ *are trivial.*

Proof. It suffices to prove the assertions involving $X^{\mathrm{unr}}(\mathrm{S}_{\mathrm{M}})$ and $X^{\mathrm{unr}}(\mathrm{M})$, for then one can conclude those involving $\mathrm{Z}_{\hat{\mathrm{M}}}^{\Gamma,0}$ and $(\mathrm{Z}_{\hat{\mathrm{M}}}^{I_F})_{\mathrm{Fr}}^{0}$ by the functoriality of the Kottwitz isomorphism (see Lemmas 2.5.8 and 2.5.10). The assertion for $X^{\mathrm{unr}}(\mathrm{S}_{\mathrm{M}})$ follows from the fact that $\mathrm{A}_{\mathrm{M}} \to \mathrm{S}_{\mathrm{M}}$ is an isogeny, so that the elements of $\mathcal{O}_{\mathrm{M}}^{+}$ induce the identity automorphism of S_{M}. The assertion for $X^{\mathrm{unr}}(\mathrm{M})$ follows from the assertion for $X^{\mathrm{unr}}(\mathrm{S}_{\mathrm{M}})$, since the restriction map $X^{\mathrm{unr}}(\mathrm{S}_{\mathrm{M}}) \to X^{\mathrm{unr}}(\mathrm{M})$ is surjective and respects the action of $\mathcal{O}_{\mathrm{M}}^{+}$. ☐

2.7. L-packets from the point of view of stability of distributions.

2.7.1. *The main hypothesis for L-packets to be defined from the perspective of stability.* In much of what follows, Remark 2.6.2(vi) can be helpful to keep in mind. Informally, the following hypothesis says that '$\mathcal{O}$-coarsened' tempered L-packets can be defined based on the notion of stability of distributions (see also [Sha90, Section 9], and the notion of atomic stability in [MY20, Section 4]).

Hypothesis 2.7.1 (Existence of tempered L-packets)**.** For each Levi subgroup M of G, there exists a collection $\Phi_2(\mathrm{M})$ of finite subsets of $\mathrm{Irr}_2(\mathrm{M})$ partitioning it, and a virtual character Θ_Σ for each $\Sigma \in \Phi_2(\mathrm{M})$, such that the following two properties are satisfied:

(i) For each $\Sigma \in \Phi_2(\mathrm{M})$, Θ_Σ is a nonzero stable $\mathcal{O}_{\mathrm{M}}$-invariant (or equivalently, $\mathcal{O}_{\mathrm{M}}^{+}$-invariant) virtual character on $\mathrm{M}(F)$ of the form $\sum_{\sigma \in \Sigma} c_\sigma \Theta_\sigma$ (thus, Σ is $\mathcal{O}_{\mathrm{M}}^{+}$-invariant as well).

(ii) $\{\Theta_\Sigma \mid \Sigma \in \Phi_2(\mathrm{M})\}$ is a complex vector space basis for the subspace $SD_{\mathrm{ell}}(\mathrm{M})^{\mathcal{O}_{\mathrm{M}}} = SD_{\mathrm{ell}}(\mathrm{M})^{\mathcal{O}_{\mathrm{M}}^{+}}$ of $SD_{\mathrm{ell}}(\mathrm{M})$ fixed by $\mathcal{O}_{\mathrm{M}}$ or equivalently by $\mathcal{O}_{\mathrm{M}}^{+}$.

Proposition 2.7.2. *Suppose Hypothesis 2.7.1 is satisfied. Then* Θ_Σ *is a multiple of* $\sum_{\sigma \in \Sigma} d(\sigma) \Theta_\sigma$*, where* $d(\sigma)$ *stands for the formal degree of* σ*. In particular, if* $\Theta_\Sigma = \sum_{\sigma \in \Sigma} c_\sigma \Theta_\sigma$*, then* $c_\sigma \neq 0$ *for each* $\sigma \in \Sigma$*.*

Proof. This is proved exactly as in Proposition 3.3.6(ii) below, and in fact follows from it. Note that, while we do not prove Proposition 3.3.6(ii) either, its proof is an easier variant of the proof of Proposition 3.3.7(ii) below it. □

Lemma 2.7.3. *Assume Hypothesis 2.7.1. Let* $\mathrm{M} \subset \mathrm{G}$ *be a Levi subgroup.*

(i) $\Phi_2(\mathrm{M})$ *can be described as the set of* $\Sigma \subset \mathrm{Irr}_2(\mathrm{M})$ *satisfying the following condition: there exists a nonzero stable* $\mathcal{O}_{\mathrm{M}}$*-invariant virtual character* Θ'_Σ *supported on* Σ*, with the property that every stable* $\mathcal{O}_{\mathrm{M}}$*-invariant virtual character* $\Theta \in SD_{\mathrm{ell}}(\mathrm{M})^{\mathcal{O}_{\mathrm{M}}}$ *can be uniquely written in the form* $c_1\Theta'_\Sigma + c_2\Theta'$ *for a (automatically stable and* $\mathcal{O}_{\mathrm{M}}$*-invariant) virtual character* Θ' *supported outside* Σ*, and complex numbers* c_1, c_2*.*

(ii) If $\beta \in \mathcal{O}_{\mathrm{G}}^+$ *and* $\mathrm{M}' = \beta(\mathrm{M})$*, and* $\chi' : \mathrm{M}'(F) \to \mathbb{C}^\times$ *is a smooth unitary character on which* $\mathcal{O}_{\mathrm{M}'}$ *acts trivially (this is automatic if* χ *is unramified, by Lemma 2.6.5), then we have a bijection* $\Phi_2(\mathrm{M}) \to \Phi_2(\mathrm{M}')$ *sending each* $\Sigma \in \Phi_2(\mathrm{M})$ *to* $\Sigma' := \{(\sigma \circ \beta^{-1}) \otimes \chi' \mid \sigma \in \Sigma\}$*. Moreover,* $\Theta_{\Sigma'}$ *is a scalar multiple of* $(\Theta_\Sigma \circ \beta^{-1})\chi'$*.*

Remark 2.7.4. (i) of the lemma implies that, when Hypothesis 2.7.1 is satisfied, $\Phi_2(\mathrm{M})$ is uniquely determined, and not a choice made along with assuming the hypothesis. On the other hand, each Θ_Σ is uniquely determined up to a nonzero scalar.

Proof of Lemma 2.7.3. (i) is immediate, but it needs that $c_\sigma \neq 0$ whenever $\Sigma \in \Phi_2(\mathrm{M})$ and $\sigma \in \Sigma$ (to prevent proper subsets of such a Σ from satisfying the condition of (i)), a consequence of Proposition 2.7.2. (ii) follows from (i) and the fact that β transports $\mathcal{O}_{\mathrm{M}}^+$ to $(\mathcal{O}_{\mathrm{M}'})^+$ (see Lemma 2.6.3(i)) and $SD_{\mathrm{ell}}(\mathrm{M})$ to $SD_{\mathrm{ell}}(\mathrm{M}')$, etc. □

We will now define analogous sets $\Phi(\mathrm{M}), \Phi_{\mathrm{temp}}(\mathrm{M})$ and $\Phi_2^+(\mathrm{M})$, using the Langlands classification of Proposition 2.2.3, or rather its corollary in the form of the following $\mathcal{O}_{\mathrm{M}}$-invariant version:

Corollary 2.7.5. *Let* $\mathrm{M} \subset \mathrm{G}$ *be a Levi subgroup.*

(i) The map (5) *induces a finite-to-one surjective map*

$$\mathrm{Irr}(\mathrm{M})/\mathcal{O}_{\mathrm{M}} = \mathrm{Irr}(\mathrm{M})/\mathcal{O}_{\mathrm{M}}^+ \to \bigsqcup_{\mathrm{L}} \mathrm{Irr}_2^+(\mathrm{L})/\mathcal{O}_{\mathrm{M},\mathrm{L}}^+, \tag{20}$$

where the L *runs over a set of representatives for the* $\mathcal{O}_{\mathrm{M}}^+$*-orbits of Levi subgroups of* M*. It restricts to an analogously defined map:*

$$\mathrm{Irr}_{\mathrm{temp}}(\mathrm{M})/\mathcal{O}_{\mathrm{M}} = \mathrm{Irr}_{\mathrm{temp}}(\mathrm{M})/\mathcal{O}_{\mathrm{M}}^+ \to \bigsqcup_{\mathrm{L}} \mathrm{Irr}_2(\mathrm{L})/\mathcal{O}_{\mathrm{M},\mathrm{L}}^+. \tag{21}$$

(ii) For a pair (L, υ)*, with* L *occurring in* (20) *and* $\upsilon \in \mathrm{Irr}_2^+(\mathrm{L})$*, the fiber of* (20) *over the image of* υ *in* $\mathrm{Irr}_2^+(\mathrm{L})/\mathcal{O}_{\mathrm{M},\mathrm{L}}^+$ *is the union of the* $\mathcal{O}_{\mathrm{M}}^+$*-orbits (or equivalently the* $\mathcal{O}_{\mathrm{M}}$*-orbits) of the irreducible* quotients *of* $\mathrm{Ind}_{\mathrm{Q}}^{\mathrm{M}} \upsilon$*, where* Q *is a choice as in Proposition 2.2.3(iii). An analogous description applies to the fibers of* (21)*.*

Proof. One gets (i) from Proposition 2.2.3(ii) simply by quotienting with $\mathcal{O}_{\mathrm{M}}^+ \supset \mathrm{Int}\,\mathrm{M}(F)$. The finite-to-one-ness follows from the fact that $\mathrm{Int}\,\mathrm{M}(F) \subset \mathcal{O}_{\mathrm{M}}^+$ is of finite index (see Lemma 2.6.3(ii)). (ii) is an immediate consequence of Proposition 2.2.3(iii). □

Notation 2.7.6. Henceforth, whenever Hypothesis 2.7.1 is satisfied, in addition to fixing the Θ_Σ as in it (the $\Phi_2(\mathrm{M})$ being automatically fixed — see Remark 2.7.4), we also define the following objects:

(i) For each Levi subgroup $\mathrm{M} \subset \mathrm{G}$, we define $\Phi_2^+(\mathrm{M}) = \{\Sigma \otimes \chi \mid \Sigma \in \Phi_2(\mathrm{M}), \chi \in X^{\mathrm{unr}}(\mathrm{M})\}$.

(ii) If $\mathrm{L} \subset \mathrm{M} \subset \mathrm{G}$ are Levi subgroups and $\Upsilon \in \Phi_2^+(\mathrm{L})$ (as defined in (i)), then we let Υ^{M} be the preimage, under (20), of the image of Υ in $\mathrm{Irr}_2^+(\mathrm{L})/\mathcal{O}_{\mathrm{M},\mathrm{L}}^+$: here we assume without loss of generality that L occurs on the right-hand side of (20). In other words, Υ^{M} is the collection of the $\mathcal{O}_{\mathrm{M}}$-conjugates of the irreducible quotients of $\mathrm{Ind}_{\mathrm{Q}}^{\mathrm{M}} \upsilon$, where υ runs over Υ and Q is as in Proposition 2.2.3(iii): note that the same Q works for all $\upsilon \in \Upsilon$. If $\Upsilon \in \Phi_2(\mathrm{L})$, then by unitarity, we can replace the word 'quotients' by 'subquotients' in the previous sentence.

(iii) For $\mathrm{M} \subset \mathrm{G}$ a Levi subgroup, we let $\varPhi(\mathrm{M})$ (resp., $\varPhi_{\mathrm{temp}}(\mathrm{M})$) be the set of all Υ^{M} as (L, Υ) ranges over pairs consisting of a Levi subgroup $\mathrm{L} \subset \mathrm{M}$ and $\Upsilon \in \varPhi_2^+(\mathrm{L})$ (resp., $\Upsilon \in \varPhi_2(\mathrm{L})$).

(iv) If $\Sigma \in \varPhi_{\mathrm{temp}}(\mathrm{M})$, choosing (L, Υ) such that $\Sigma = \Upsilon^{\mathrm{M}}$, we let $\Theta_\Sigma = \mathrm{Avg}_{\mathcal{O}_{\mathrm{M}}}(\mathrm{Ind}_{\mathrm{L}}^{\mathrm{M}} \Theta_\Upsilon)$, where $\mathrm{Avg}_{\mathcal{O}_{\mathrm{M}}}$ refers to averaging with respect to the action of $\mathcal{O}_{\mathrm{M}}$ (which acts through the finite quotient $\mathcal{O}_{\mathrm{M}}^+ / \operatorname{Int} \mathrm{M}(F)$). Note that Θ_Σ, which is a virtual character supported on Σ, is well-defined, since (L, Υ) is well-defined up to $\mathcal{O}_{\mathrm{M}}^+$-conjugation by (20).

Lemma 2.7.7. *Let $\mathrm{M} \subset \mathrm{G}$ be a Levi subgroup.*

(i) $\varPhi(\mathrm{M}), \varPhi_2^+(\mathrm{M})$ and $\varPhi_{\mathrm{temp}}(\mathrm{M})$ consist of $\mathcal{O}_{\mathrm{M}}^+$-invariant sets, and they are partitions of $\mathrm{Irr}(\mathrm{M}), \mathrm{Irr}_2^+(\mathrm{M})$ and $\mathrm{Irr}_{\mathrm{temp}}(\mathrm{M})$, respectively.

(ii) Θ_Σ is well-defined for each $\Sigma \in \Phi_{\mathrm{temp}}(\mathrm{M})$, and the collection of the Θ_Σ forms a basis for $SD(\mathrm{M})^{\mathcal{O}_{\mathrm{M}}}$.

Proof. Since every element of $\mathrm{Irr}_2^+(\mathrm{M})$ can be written as $\sigma \otimes \chi$ with $\sigma \in \mathrm{Irr}_2(\mathrm{M})$ and $\chi \in X^{\mathrm{unr}}(\mathrm{M})$, it is immediate that the union of the $\varPhi_2^+(\mathrm{M})$ equals $\mathrm{Irr}_2^+(\mathrm{M})$. By Lemma 2.6.5, each element of $\varPhi_2^+(\mathrm{M})$ is also $\mathcal{O}_{\mathrm{M}}^+$-invariant. If $\Sigma_1, \Sigma_2 \in \varPhi_2^+(\mathrm{M})$, $\chi_1, \chi_2 \in X^{\mathrm{unr}}(\mathrm{M})$ and $\Sigma_1 \otimes \chi_1$ intersects $\Sigma_2 \otimes \chi_2$, then $\chi_1 \chi_2^{-1} \in X^{\mathrm{unr}}(\mathrm{M})$ restricts to a unitary character on $\mathrm{Z}_{\mathrm{M}}(F)$ and is hence unitary, so that $\Sigma_1 \otimes \chi_1 = \Sigma_2 \otimes \chi_2$ by Lemma 2.7.3(ii).

Thus, we have proved the assertion of (i) for $\varPhi_2^+(\mathrm{M})$. Applying this with M replaced by various Levi subgroups $\mathrm{L} \subset \mathrm{M}$, the assertion of (i) for $\varPhi(\mathrm{M})$ (resp., $\varPhi_{\mathrm{temp}}(\mathrm{M})$) then follows from (20) (resp., (21)) and the fact that the elements of $\mathcal{O}_{\mathrm{M,L}}^+$ permute $\varPhi_2^+(\mathrm{L})$ (resp., $\varPhi_2(\mathrm{L})$), by Lemma 2.7.3(ii).

(ii) follows from Proposition 3.2.8 later below, applied with M in place of G, and the corresponding restriction of the collection $\{\mathcal{O}_{\mathrm{M}}\}_{\mathrm{M}}$ (as justified by Lemma 2.6.3(iii)). ☐

Remark 2.7.8. Thus, Hypothesis 2.7.1 also has the consequence that each $SD(\mathrm{M})^{\mathcal{O}_{\mathrm{M}}}$ and in particular $SD(\mathrm{G})^{\mathcal{O}_{\mathrm{G}}} = SD(\mathrm{G})^{\mathcal{O}}$, has a basis consisting of virtual characters whose supports are pairwise disjoint and together exhaustive. By the same argument as in Lemma 2.7.3(i), the elements of such a basis are uniquely determined up to scaling, and hence $\varPhi_{\mathrm{temp}}(\mathrm{M})$ has an alternate characterization as in Lemma 2.7.3(i).

Notation 2.7.9. Henceforth, for any Levi subgroup $\mathrm{M} \subset \mathrm{G}$, the elements of $\varPhi_2(\mathrm{M})$ (resp., $\varPhi_{\mathrm{temp}}(\mathrm{M})$) will be referred to as the discrete series L-packets (resp., tempered L-packets) on $\mathrm{M}(F)$ up to the action of $\mathcal{O}_{\mathrm{M}}$ in the sense of Hypothesis 2.7.1.

Remark 2.7.10. Suppose that Hypothesis 2.7.1 is satisfied with the collection $\{\mathcal{O}_{\mathrm{M}}\}_{\mathrm{M}}$ replaced by a collection $\{\mathcal{O}'_{\mathrm{M}}\}_{\mathrm{M}}$ satisfying analogous conditions, where $\mathcal{O}'_{\mathrm{M}}$ is a normal subgroup of $\mathcal{O}_{\mathrm{M}}$ for each Levi subgroup $\mathrm{M} \subset \mathrm{G}$. Since we may assume that $c_\sigma > 0$ for each $\sigma \in \Sigma$ by Proposition 2.7.2, it is easy to see by averaging and using the idea of the proof of Lemma 2.7.3, that Hypothesis 2.7.1 is satisfied (without replacing $\{\mathcal{O}_{\mathrm{M}}\}_{\mathrm{M}}$ by $\{\mathcal{O}'_{\mathrm{M}}\}_{\mathrm{M}}$).

2.8. **Langlands parameters.** In this subsection, let M be a connected reductive group over F.

Definition 2.8.1. (i) An element $x \in {}^L\mathrm{M}$ is said to be semisimple if $\operatorname{Int} x$ preserves a maximal torus of $\hat{\mathrm{M}}$.

(ii) By a Langlands parameter for M, we will mean the $\hat{\mathrm{M}}$-conjugacy class of a homomorphism $\dot{\varphi} : W'_F := W_F \times \mathrm{SL}_2(\mathbb{C}) \to {}^L\mathrm{M}$ that is admissible relevant. Here, $\dot{\varphi}$ being 'admissible' means that $\dot{\varphi}$ satisfies the following three properties: $\dot{\varphi}$ is continuous; $\dot{\varphi}|_{\mathrm{SL}_2(\mathbb{C})}$ is an algebraic map $\mathrm{SL}_2(\mathbb{C}) \to \hat{\mathrm{M}}$; and for all $w \in W_F$ we have that $\dot{\varphi}(w) \in {}^L\mathrm{M}$ is semisimple and maps to w under ${}^L\mathrm{M} \to W_F$. $\dot{\varphi}$ being 'relevant' means that $\dot{\varphi}(W_F) \subsetneq \mathcal{M}$ if $\mathcal{M}$ is a Levi subgroup of ${}^L\mathrm{M}$ which is not relevant in the sense of Notation 2.4.3.

(iii) Let $\Phi(\mathrm{M})$ denote the set of Langlands parameters for M. Let $\Phi_{\mathrm{temp}}(\mathrm{M})$ denote the subset of those $\varphi \in \Phi(\mathrm{M})$ with the property that some (or equivalently, any) $\dot{\varphi} : W'_F \to {}^L\mathrm{M}$ representing φ is a bounded homomorphism, i.e., for some (or equivalently, any) preferred section $s : W_F \to {}^L\mathrm{M}$, there exists a bounded subset $C \subset \hat{\mathrm{M}}$ such that $\varphi(W_F) \subset Cs(W_F)$. Let $\Phi_2(\mathrm{M})$ (resp., $\Phi_2^+(\mathrm{M})$) denote the subset of $\Phi_{\mathrm{temp}}(\mathrm{M})$ (resp., $\Phi(\mathrm{M})$) consisting of those Langlands parameters φ which (have representatives that) do not factor through any proper Levi subgroup of ${}^L\mathrm{M}$. Elements of $\Phi_{\mathrm{temp}}(\mathrm{M})$ (resp., $\Phi_2(\mathrm{M})$) or their representatives will be referred to as tempered or bounded parameters (resp., discrete series parameters).

(iv) Let $x \in {}^L\mathrm{M}$ be semisimple in the sense of (i) above. Then x is called elliptic if for some (or equivalently, any) preferred section $s : W_F \to {}^L\mathrm{M}$, there exists a compact subset $C \subset \hat{\mathrm{M}}$ such that the subgroup of ${}^L\mathrm{M}$ generated by x is contained in $Cs(W_F)$. Further, x is called hyperbolic if it belongs to $\hat{\mathrm{M}}$ and is hyperbolic as an element of $\hat{\mathrm{M}}$ (in the sense recalled in [SZ18, Section 5.1]).

The definitions above agree with the usual ones, by Lemma 2.8.5 below.

Remark 2.8.2. In the spirit of the parenthetical 'have representatives that' in (iii) above, we will often describe properties of Langlands parameters in terms of their representatives, the independence of the choice of which will be implicitly left to the reader to check.

Remark 2.8.3. If we fix a preferred section $s : W_F \to {}^L\mathrm{M}$, yielding an identification ${}^L\mathrm{M} \cong \hat{\mathrm{M}} \rtimes W_F$ and in particular an action of W_F on $\hat{\mathrm{M}}$, and hence also an action of W_F' via $W_F' \to W_F$, we can use the identification from [Bor79, Section 8.2] (analogous to that in Notation 2.5.3) to view the set of Langlands parameters for M as a subset of $H^1(W_F', \hat{\mathrm{M}})$: a Langlands parameter represented by $w' \mapsto a(w')s(w)$, where w is the image of w' in W_F, is represented by the image in $H^1(W_F', \hat{\mathrm{M}})$ of $a : W_F' \to \hat{\mathrm{M}}$, which is verified to be a cocycle of W_F' in $\hat{\mathrm{M}}$.

Notation 2.8.4. (i) Through $W_F' \to W_F$, the prescription of Definition 2.3.3(iv) also defines an action of W_F' on $\mathrm{Z}_{\hat{\mathrm{M}}}$, independently of any choice of a preferred section.

(ii) Now that we can talk of $H^1(W_F, \mathrm{Z}_{\hat{\mathrm{M}}})$ (or $H^1(W_F', \mathrm{Z}_{\hat{\mathrm{M}}})$), we can also make it act on $\Phi(\mathrm{M})$, as in [Bor79, Section 8.5]: if φ is a Langlands parameter for M represented by a homomorphism $\dot\varphi : W_F' \to {}^L\mathrm{M}$, and if (the inflation to $H^1(W_F', \mathrm{Z}_{\hat{\mathrm{M}}})$ of) α is represented by a cocycle $\dot\alpha$, then $\alpha\cdot\varphi \in \Phi(\mathrm{M})$ is represented by the admissible relevant homomorphism $W_F' \to {}^L\mathrm{M}$ given by $w \mapsto \dot\alpha(w)\dot\varphi(w) \in \mathrm{Z}_{\hat{\mathrm{M}}} \cdot {}^L\mathrm{M} = {}^L\mathrm{M}$. Note that there is a similarly defined action of $\mathrm{Z}_{\hat{\mathrm{M}}}^{\Gamma,0} = H^1(W_F/I_F, \mathrm{Z}_{\hat{\mathrm{M}}}^{\Gamma,0})$ on $\Phi(\mathrm{M})$, which factors through the above action via the map $\mathrm{Z}_{\hat{\mathrm{M}}}^{\Gamma,0} \to (\mathrm{Z}_{\hat{\mathrm{M}}}^{I_F})_{\mathrm{Fr}}^0 \subset H^1(W_F, \mathrm{Z}_{\hat{\mathrm{M}}})$.

(iii) Restriction along the naive inclusion $W_F \to W_F'$ (given by $w \mapsto (w, 1)$) gives a map $\Phi(\mathrm{M}) \to \mathscr{S}(W_F, {}^L\mathrm{M})/(\mathrm{Int}\,\hat{\mathrm{M}})$, composing which with (15) gives us a map $\Phi(\mathrm{M}) \to \mathrm{Hom}_{\mathrm{cts}}(\mathrm{Z}_\mathrm{M}(F), \mathbb{C}^\times)$, which we denote by $\varphi \mapsto \zeta_\varphi$ as well. It is easy to check that this construction agrees with that in [Bor79, Section 10.1]. It is easy to see that Remark 2.5.6 and Remark 2.5.7 also admit obvious variants with the $\mathscr{S}(W_F, -)$ replaced by the $\Phi(-)$; in what follows, when we invoke either of these remarks, it is to be understood that this variant is also being referenced.

Lemma 2.8.5. *Realize ${}^L\mathrm{M}$ as $\hat{\mathrm{M}} \rtimes W_F$ using a preferred section $s : W_F \to {}^L\mathrm{M}$. Let $x \in {}^L\mathrm{M}$.*

(i) x is semisimple in the sense of Definition 2.8.1(i) if and only if it is semisimple in the usual sense (as explained in [Kot84, Section 10] *or* [SZ18, Section 5.1]*, and recalled in the proof).*

(ii) Assume that x is semisimple. Then x is elliptic (resp., hyperbolic) in the sense of Definition 2.8.1(iv) if and only if it is elliptic (resp., hyperbolic) in the usual sense (explained in [SZ18, Section 5.1]*).*

(iii) Assume that x is semisimple. Then there exist unique $x_e \in {}^L\mathrm{M}$ and $x_h \in \hat{\mathrm{M}}$ such that x_e is elliptic, x_h is hyperbolic, and $x_e x_h = x = x_h x_e$.

(iv) An admissible relevant homomorphism $\dot\varphi : W_F' \to {}^L\mathrm{M}$ represents an element $\varphi \in \Phi_{\mathrm{temp}}(\mathrm{M})$ if and only if $\dot\varphi(W_F) \subset {}^L\mathrm{M}$ consists entirely of elliptic elements.

Proof. First we prove (i). Recall that if K/F is an extension in $\bar F$ splitting M, the semisimplicity of x in the usual sense is equivalent to that of its image, call it $\bar x$, in the quotient $\hat{\mathrm{M}} \rtimes \mathrm{Gal}(K/F)$ of $\hat{\mathrm{M}} \rtimes W_F = {}^L\mathrm{M}$. Let us prove the implication "$\Rightarrow$" first, so assume that $\mathrm{Int}\,x$ and hence $\mathrm{Int}\,\bar x$ preserves a maximal torus of $\hat{\mathrm{M}}$. Then some power of $\bar x$ belongs to $\hat{\mathrm{M}}$ and centralizes a

maximal torus of $\hat{\mathrm{M}}$, and hence belongs to this torus. This forces $\bar{x}$ to be semisimple, since $\mathbb{C}$ has characteristic zero. To prove the sufficiency, note that if x is semisimple in the usual sense, so is the automorphism $\operatorname{Int}\bar{x}$ of $\hat{\mathrm{M}}$ (by the definition of a semisimple automorphism, as given shortly before [Ste68, Theorem 7.5]). Therefore, by [Ste68, Theorem 7.5], $\operatorname{Int}\bar{x}$ and hence $\operatorname{Int} x$ preserves a Borel pair in $\hat{\mathrm{M}}$.
We now sketch a proof of (ii). The assertion dealing with hyperbolic elements is tautological (the relevant definitions in [SZ18, Section 5.1] and Definition 2.8.1(iv) agree). It is easy to see that x is elliptic in the sense of [SZ18, Section 5.1] if and only if its image in each factor $\hat{\mathrm{M}} \rtimes \mathrm{Gal}(K/F)$ as in the proof of (i) is contained in a compact subgroup of $\hat{\mathrm{M}} \rtimes \mathrm{Gal}(K/F)$, which is easily seen to be equivalent to x being elliptic in the sense of Definition 2.8.1(iv). This proves (ii), given which (iii) follows from the discussion in [SZ18, Section 5.1]. The implication "⇒" of (iv) is immediate. On the other hand, if $w \in W_F$ is a lift of Fr, then it is easy to see from the compactness of I_F, and the fact that $\dot{\varphi}|_{W_F}$ is continuous and hence smooth, that the ellipticity of $\dot{\varphi}(w)$ is already sufficient for the temperedness of φ. □

2.9. **Infinitesimal characters.** Now let us recall the notion of an infinitesimal character from [Hai14, Section 5.1]. For this subsection too, M is an arbitrary reductive group over F.

Definition 2.9.1. By an infinitesimal character for M we refer to the $\hat{\mathrm{M}}$-conjugacy class of a homomorphism $\lambda : W_F \to {}^L\mathrm{M}$, which is admissible (see Definition 2.8.1), though we do not impose any condition of relevance. Let $\Omega({}^L\mathrm{M})$ denote the set of infinitesimal characters of M.

Notation 2.9.2. If a homomorphism $\dot{\varphi} : W_F' \to {}^L\mathrm{M}$ represents a Langlands parameter φ for M, we will denote by $\lambda(\varphi)$ the well-defined $\operatorname{Int}\hat{\mathrm{M}}$-orbit of the homomorphism $W_F \to {}^L\mathrm{M}$ given by:

$$w \mapsto \dot{\varphi}\left(w, \begin{pmatrix} \|w\|^{1/2} & 0 \\ 0 & \|w\|^{-1/2} \end{pmatrix}\right).$$

It is easy to see that $\lambda(\varphi) \in \Omega({}^L\mathrm{M})$.

Remark 2.9.3. Using the discussion of Notation 2.6.1(v), we get an action of Aut(M) on $\Phi(\mathrm{M})$ and $\Omega({}^L\mathrm{M})$. The action of Aut(M) on $\Phi(\mathrm{M})$ preserves $\Phi_2(\mathrm{M}), \Phi_2^+(\mathrm{M})$ and $\Phi_{\mathrm{temp}}(\mathrm{M})$. Further, the map $\Phi(\mathrm{M}) \to \Omega({}^L\mathrm{M})$ given by $\varphi \mapsto \lambda(\varphi)$ is Aut(M)-equivariant. Note that if $\mathrm{L} \subset \mathrm{M} \subset \mathrm{G}$ are Levi subgroups, then since $\mathcal{O}_{\mathrm{L}}^+ \subset \mathcal{O}_{\mathrm{M,L}}^+$ (see (iv)b in Notation 2.6.1), we have a surjection $\Phi(\mathrm{L})/\mathcal{O}_{\mathrm{L}} = \Phi(\mathrm{L})/\mathcal{O}_{\mathrm{L}}^+ \twoheadrightarrow \Phi(\mathrm{L})/\mathcal{O}_{\mathrm{M,L}}^+$, and similarly with $\Omega(\mathrm{L})$ in place of $\Phi(\mathrm{L})$.

Remark 2.9.4. The following lemma is well-known to experts and reflects in results found in the literature, e.g., in the first remark of [Mg14, Section 4], with the consequence that the L-packet of a tempered representation of a quasi-split classical group is determined (up to an outer automorphism) by the cuspidal support of its transfer to the appropriate general linear group.

Lemma 2.9.5. *Any tempered Langlands parameter for* M *is determined by its infinitesimal character, i.e., if* $\varphi_1, \varphi_2 \in \Phi_{\mathrm{temp}}(\mathrm{M})$ *satisfy that* $\lambda(\varphi_1) = \lambda(\varphi_2) \in \Omega({}^L\mathrm{M})$*, then* $\varphi_1 = \varphi_2 \in \Phi(\mathrm{M})$.

Proof. The proof is the same as for the assertion that the "$\psi \mapsto \phi_\psi$" map defines an injection from the set of Arthur parameters for M into the set of Langlands parameters for M (see [Art84, Proposition 1.3], or alternatively [Sha11, Section 3]), but we give the details for the convenience of the reader.
We confuse φ_1, φ_2 with homomorphisms $W_F' \to {}^L\mathrm{M}$ that represent them, and similarly with $\lambda(\varphi_1)$ and $\lambda(\varphi_2)$. Conjugating one of these representatives if necessary, we may and do assume that $\lambda(\varphi_1) = \lambda(\varphi_2) : W_F \to {}^L\mathrm{M}$, so that for all $w \in W_F$:

$$\varphi_1\left(w, \begin{pmatrix} \|w\|^{1/2} & \\ & \|w\|^{-1/2} \end{pmatrix}\right) = \varphi_2\left(w, \begin{pmatrix} \|w\|^{1/2} & \\ & \|w\|^{-1/2} \end{pmatrix}\right). \tag{22}$$

We claim that $\varphi_1|_{W_F} = \varphi_2|_{W_F}$: this is a very special case of the Langlands classification for L-parameters, as in [SZ18, Lemma 5.2], as we now explain.

For each $w \in W_F$ and $i \in \{1,2\}$, $\varphi_i(w)$ is elliptic by Lemma 2.8.5(iv), while $\varphi_i\left(1, \begin{pmatrix} \|w\|^{1/2} & \\ & \|w\|^{-1/2} \end{pmatrix}\right) \in \hat{\mathrm{M}} \subset {}^L\mathrm{M}$ is hyperbolic (because it lies in the image of the identity component of the group of real points of the diagonal maximal torus of $\mathrm{SL}_2(\mathbb{C})$). Therefore, by the uniqueness of the polar decomposition (Lemma 2.8.5(iii)), both $\varphi_i(w)$ and $\varphi_i\left(1, \begin{pmatrix} \|w\|^{1/2} & \\ & \|w\|^{-1/2} \end{pmatrix}\right)$ are determined as the elliptic and hyperbolic parts, respectively, of their product. Applying this to (22), it follows that $\varphi_1|_{W_F} = \varphi_2|_{W_F}$.

For the rest of this proof alone, write T for the diagonal maximal torus of the $\mathrm{SL}_2(\mathbb{C})$-factor of W'_F. Thus, φ_1, φ_2 agree on W_F as well as on T, and it suffices to show that the homomorphisms $\varphi_1|_{\mathrm{SL}_2(\mathbb{C})}$ and $\varphi_2|_{\mathrm{SL}_2(\mathbb{C})}$, valued in the connected centralizer C of $\varphi_1(W_F) = \varphi_2(W_F)$ in $\hat{\mathrm{M}}$, are C-conjugate. A result of Mal'cev (see [CM93, Theorem 3.4.12]) says that if C′ is a connected reductive group over $\mathbb{C}$ and if $\varphi'_1, \varphi'_2 : \mathrm{SL}_2(\mathbb{C}) \to \mathrm{C}'$ are algebraic homomorphisms such that $\varphi'_1|_{\mathrm{T}}$ and $\varphi'_2|_{\mathrm{T}}$ are C′-conjugate, then so are φ'_1 and φ'_2. Applying this with $\mathrm{C}' = \mathrm{C}$ and $\varphi'_i = \varphi_i|_{\mathrm{SL}_2(\mathbb{C})}$ for $i = 1, 2$, and using that $\varphi_1|_{\mathrm{T}} = \varphi_2|_{\mathrm{T}}$, it follows that $\varphi_1|_{\mathrm{SL}_2(\mathbb{C})}$ and $\varphi_2|_{\mathrm{SL}_2(\mathbb{C})}$ are C-conjugate, as needed. □

2.10. **Hypothesis on the local Langlands correspondence.** Before stating our hypothesis on the local Langlands correspondence, we need to make sure to account for the fact that certain constructions are well-defined modulo the action of the groups $\mathcal{O}_{\mathrm{M}}$.

Remark 2.10.1. Let $\mathrm{M} \subset \mathrm{G}$ be a Levi subgroup.

(i) We claim that for all $\alpha \in (\mathrm{Z}_{\hat{\mathrm{M}}}^{I_F})^0_{\mathrm{Fr}} \subset H^1(W_F/I_F, \mathrm{Z}_{\hat{\mathrm{M}}}^{I_F}) \subset H^1(W_F, \mathrm{Z}_{\hat{\mathrm{M}}})$, and hence also for $\alpha \in \mathrm{Z}_{\hat{\mathrm{M}}}^{\Gamma,0} = H^1(W_F/I_F, \mathrm{Z}_{\hat{\mathrm{M}}}^{\Gamma,0})$, $\varphi \mapsto \alpha \cdot \varphi$ (see Notation 2.8.4(ii)) descends to a well-defined map $\Phi(\mathrm{M})/\mathcal{O}_{\mathrm{M}} \to \Phi(\mathrm{M})/\mathcal{O}_{\mathrm{M}}$. This follows from Lemma 2.6.5 and the fact that the map $(\alpha, \varphi) \mapsto \alpha \cdot \varphi$ is clearly equivariant for $\mathcal{O}_{\mathrm{M}}$ or equivalently for $\mathcal{O}_{\mathrm{M}}^+$. If the stronger condition on the $\mathcal{O}_{\mathrm{M}}$ discussed in Remark 2.6.2(v) holds, then the same applies to all $\alpha \in H^1(W_F, \mathrm{Z}_{\hat{\mathrm{M}}})$.

(ii) Recall the assignment $\varphi \mapsto \zeta_\varphi$ ($\varphi \in H^1(W'_F, \hat{\mathrm{M}})$) from Notation 2.8.4(iii). By the functoriality of this assignment, replacing φ by an element in its $\mathcal{O}_{\mathrm{M}}^+$-orbit replaces ζ_φ by an element in its $\mathcal{O}_{\mathrm{M}}^+$-orbit, and hence leaves ζ_φ unchanged under the stronger condition on the $\mathcal{O}_{\mathrm{M}}$ discussed in Remark 2.6.2(v). Thus, under this stronger condition we may talk of ζ_φ for $\varphi \in \Phi(\mathrm{M})/\mathcal{O}_{\mathrm{M}}$.

Remark 2.10.2. (i) We will use the following without further remark: if $\mathrm{L} \subset \mathrm{M} \subset \mathrm{G}$ are Levi subgroups, then since $\mathcal{O}_{\mathrm{L}} \subset \mathcal{O}_{\mathrm{M,L}}^+$ (see (iv)b in Notation 2.6.1), it follows from Lemma 2.4.16(ii) that $\iota_{\mathrm{M,L}} \circ -$ descends to well-defined maps $\Phi(\mathrm{L})/\mathcal{O}_{\mathrm{L}} \to \Phi(\mathrm{M})/\mathcal{O}_{\mathrm{M}}^+ = \Phi(\mathrm{M})/\mathcal{O}_{\mathrm{M}}$ and $\Omega({}^L\mathrm{L})/\mathcal{O}_{\mathrm{L}} \to \Omega({}^L\mathrm{M})/\mathcal{O}_{\mathrm{M}}$, and then further to well-defined maps $\Phi(\mathrm{L})/\mathcal{O}_{\mathrm{M,L}}^+ \to \Phi(\mathrm{M})/\mathcal{O}_{\mathrm{M}}$ and $\Omega({}^L\mathrm{L})/\mathcal{O}_{\mathrm{M,L}}^+ \to \Omega({}^L\mathrm{M})/\mathcal{O}_{\mathrm{M}}$. These maps are clearly independent of the choice of $\iota_{\mathrm{M,L}}$ in its $\hat{\mathrm{M}}$-conjugacy class.

(ii) Let $\beta \in \mathcal{O}_{\mathrm{G}}^+$, and let $\mathrm{M} \subset \mathrm{G}$ be a Levi subgroup. Then β transports $\mathcal{O}_{\mathrm{M}}^+$ to $\mathcal{O}_{\beta(\mathrm{M})}^+$ (see Lemma 2.6.3(i)). Compatibly with this transport, any choice of ${}^L(\beta|_{\mathrm{M}}) : {}^L\beta(\mathrm{M}) \to {}^L\mathrm{M}$ (see Notation 2.3.4(iii)) transports the $\mathcal{O}_{\beta(\mathrm{M})}^+$-action on $\Phi(\beta(\mathrm{M}))$ to the $\mathcal{O}_{\mathrm{M}}^+$-action on $\Phi(\mathrm{M})$. Hence ${}^L(\beta|_{\mathrm{M}}) \circ -$ induces a well-defined map $\Phi(\beta(\mathrm{M}))/\mathcal{O}_{\beta(\mathrm{M})} = \Phi(\beta(\mathrm{M}))/\mathcal{O}_{\beta(\mathrm{M})}^+ \to \Phi(\mathrm{M})/\mathcal{O}_{\mathrm{M}}^+ = \Phi(\mathrm{M})/\mathcal{O}_{\mathrm{M}}$.

(iii) Suppose $\mathrm{L} \subset \mathrm{M} \subset \mathrm{G}$ are Levi subgroups, and let $\beta \in \mathcal{O}_{\mathrm{M}}^+$, so that we have ${}^L(\beta|_{\mathrm{L}}) \circ - : \Phi(\beta(\mathrm{L})) \to \Phi(\mathrm{L})$. By Lemma 2.4.16(ii), we have an equality of two maps $\Phi(\beta(\mathrm{L})) \to \Phi(\mathrm{M})$:

$$\iota_{\mathrm{M,L}} \circ {}^L(\beta|_{\mathrm{L}}) \circ - = {}^L\beta \circ \iota_{\mathrm{M},\beta(\mathrm{L})} \circ - : \Phi(\beta(\mathrm{L})) \to \Phi(\mathrm{M}).$$

This can also be viewed as an equality of two maps $\Phi(\beta(\mathrm{L}))/\mathcal{O}_{\beta(\mathrm{L})} \to \Phi(\mathrm{M})/\mathcal{O}_{\mathrm{M}}$, thanks to (i) and (ii) above.

The following conjecture is a variant of what is called LLC+ for G in [Hai14, Section 5.2]; unlike in [Hai14, Section 5.1] we state it only for discrete series representations, and use Langlands classification to extend it to its 'admissible dual' version (which we do in Theorem 2.10.10 below):

Hypothesis 2.10.3. (LLC+)

(i) For each Levi subgroup $\mathrm{M} \subset \mathrm{G}$, there is a finite-to-one surjective map $\mathrm{Irr}_2(\mathrm{M})/\mathcal{O}_\mathrm{M} \to \Phi_2(\mathrm{M})/\mathcal{O}_\mathrm{M}$, which we will denote by $\sigma \mapsto \varphi_\sigma$ (suppressing M from the notation), satisfying the desideratum of compatibility with twisting by unramified unitary characters: for all α belonging to $H^1(W_F/I_F, \mathrm{Z}_{\hat{\mathrm{M}}}^{\Gamma,0}) = \mathrm{Z}_{\hat{\mathrm{M}}}^{\Gamma,0}$ such that $\chi_\alpha \in X^{\mathrm{unr}}(\mathrm{M})$ is unitary, we have $\varphi_{\sigma\otimes\chi_\alpha} = \alpha \cdot \varphi_\sigma \in \Phi_2(\mathrm{M})/\mathcal{O}_\mathrm{M}$ (see Notation 2.5.11 for χ_α, Remark 2.10.1(i) for $\alpha \cdot \varphi_\sigma$). Thus, unlike in [Hai14], we do not impose a lot of the desiderata from [Bor79].

(ii) (A variant of [Hai14, Conjecture 5.2.2]) "Infinitesimal character is preserved by parabolic induction": Let L, M be Levi subgroups of G with $\mathrm{L} \subset \mathrm{M}$. Suppose $\sigma \in \mathrm{Irr}_2(\mathrm{M})$ is a subquotient of the representation $\mathrm{Ind}_\mathrm{Q}^\mathrm{M}(\upsilon \otimes \chi_\alpha)$ of $\mathrm{M}(F)$, for some unitary supercuspidal representation υ of $\mathrm{L}(F)$, some parabolic subgroup Q of M with L as a Levi subgroup, and the character χ_α associated to some $\alpha \in \mathrm{Z}_{\hat{\mathrm{L}}}^{\Gamma,0} = H^1(W_F/I_F, \mathrm{Z}_{\hat{\mathrm{L}}}^{\Gamma,0})$ (note that $\chi_\alpha \in X^{\mathrm{unr}}(\mathrm{L})$). Then $\iota_{\mathrm{M},\mathrm{L}} \circ \lambda(\alpha \cdot \varphi_\upsilon) = \lambda(\varphi_\sigma)$, in the sense that the map $\iota_{\mathrm{M},\mathrm{L}} : \Omega({}^L\mathrm{L})/\mathcal{O}_\mathrm{L} \to \Omega({}^L\mathrm{M})/\mathcal{O}_\mathrm{M}$ from Remark 2.10.2(i) takes $\lambda(\alpha \cdot \varphi_\upsilon)$ to $\lambda(\varphi_\sigma)$. Here, $\alpha \cdot \varphi_\upsilon$ is as in Remark 2.10.1(i).

(iii) (Compare with [Hai14, Conjecture 5.2.7]) If $\mathrm{M} \subset \mathrm{G}$ is a Levi subgroup and $\beta \in \mathcal{O}_\mathrm{G}^+$, and ${}^L(\beta|_\mathrm{M}) : {}^L\beta(\mathrm{M}) \to {}^L\mathrm{M}$ is dual to $\beta|_\mathrm{M} : \mathrm{M} \to \beta(\mathrm{M})$, then for all $\sigma \in \mathrm{Irr}_2(\mathrm{M})$, we have ${}^L(\beta|_\mathrm{M}) \circ \varphi_{\sigma\circ\beta^{-1}} = \varphi_\sigma$ (see Remark 2.10.2(ii) for the meaning of ${}^L(\beta|_\mathrm{M}) \circ -$).

Remark 2.10.4. The conditions imposed by the desiderata (1)-(4) of [Bor79, Section 10.3] on the maps $\sigma \mapsto \varphi_\sigma$ in (i) of the above hypothesis are as follows. Let $\mathrm{M} \subset \mathrm{G}$ be a Levi subgroup and σ a discrete series representation of $\mathrm{M}(F)$. Then:

(i) *Central character.* The central character of σ equals the character ζ_{φ_σ} (see Remark 2.10.1(ii)). We haven't imposed this.

(ii) *Twisting by a character.* For all $\alpha \in H^1(W_F, \mathrm{Z}_{\hat{\mathrm{M}}})$ such that $\varphi_{\sigma\otimes\chi_\alpha}$ makes sense and $\mathcal{O}_\mathrm{M}$ fixes χ_α, $\varphi_{\sigma\otimes\chi_\alpha} = \alpha \cdot \varphi_\sigma$ (here χ_α is as in Subsubsection 2.5.2, and $\alpha \cdot \varphi_\sigma$ is as in Remark 2.10.1(i)). In the restricted context of Hypothesis 2.10.3, $\varphi_{\sigma\otimes\chi_\alpha}$ makes sense only when $\sigma \otimes \chi_\alpha$ is a discrete series representation, or equivalently only when χ_α is unitary; we have imposed this condition only when $\chi_\alpha \in X^{\mathrm{unr-uni}}(\mathrm{M})$.

(iii) *L-packets of essentially square-integrable representations.* σ is essentially square-integrable if and only if $\varphi_\sigma \in \Phi_2^+(\mathrm{M})/\mathcal{O}_\mathrm{M}$. This is automatic in the restricted setting of Hypothesis 2.10.3, since σ is a discrete series representation and φ_σ is hypothesized to lie in $\Phi_2(\mathrm{M})/\mathcal{O}_\mathrm{M}$.

(iv) *Tempered L-packets.* σ is tempered if and only if $\varphi \in \Phi_{\mathrm{temp}}(\mathrm{M})$. This too is automatic in the our restricted setting of Hypothesis 2.10.3.

2.10.1. *Extending the local Langlands correspondence to the admissible dual using Langlands classification.*

Notation 2.10.5. Let $\mathrm{M} \subset \mathrm{G}$ be a Levi sugbgroup. If $\mathcal{L} \subset {}^L\mathrm{M}$ is a relevant Levi subgroup, write $\Phi_2^+(\mathcal{L})$ for the set of $\mathcal{L}^0$-conjugacy classes of admissible homomorphisms $\dot{\varphi} : W_F' \to {}^L\mathrm{M}$, such that $\mathcal{L}$ is minimal among the Levi subgroup of ${}^L\mathrm{M}$ containing $\dot{\varphi}(W_F')$. $\Phi_2^+(\mathcal{L})$ is acted on by the group $\mathcal{O}_{\mathrm{M},\mathcal{L}}^+$ of Notation 2.6.1(iii).

Lemma 2.10.6. *Let $\mathrm{M} \subset \mathrm{G}$ be a Levi subgroup.*

(i) Let $\varphi_i \in \Phi_2^+(\mathcal{L}_i)$ for $i = 1, 2$, where $\mathcal{L}_1, \mathcal{L}_2 \subset {}^L\mathrm{M}$ are Levi subgroups. Then the images of φ_1 and φ_2 in $\Phi(\mathrm{M})/\mathcal{O}_\mathrm{M}$ are equal if and only if $\varphi_1 = {}^L\beta \circ \varphi_2$ for some ${}^L\beta \in \mathcal{O}_\mathrm{M}^+(\mathcal{L}_1, \mathcal{L}_2)$ (see Notation 2.6.1(iii)).

(ii) The obvious maps $\Phi_2^+(\mathcal{L}) \to \Phi(\mathrm{M})$ give us decompositions:

$$\Phi(\mathrm{M}) = \bigsqcup_{\mathcal{L}} \Phi_2^+(\mathcal{L})/W_{\hat{\mathrm{M}}}(\mathcal{L}), \tag{23}$$

where $\mathcal{L}$ runs over a set of representatives for the $\hat{\mathrm{M}}$-conjugacy classes of relevant Levi subgroups of ${}^L\mathrm{M}$, and

$$\Phi(\mathrm{M})/\mathcal{O}_{\mathrm{M}} = \bigsqcup_{\mathcal{L}} \Phi_2^+(\mathcal{L})/\mathcal{O}^+_{\mathrm{M},\mathcal{L}}, \tag{24}$$

where $\mathcal{L}$ runs over a set of representatives for the orbits of relevant Levi subgroups of ${}^L\mathrm{M}$ under the group $\mathcal{O}^+_{\mathrm{M},\,{}^L\mathrm{M}} = \mathcal{O}^+_{\mathrm{M}}({}^L\mathrm{M},\,{}^L\mathrm{M})$ (i.e., for the $\mathcal{O}_{\mathrm{M}}$-orbits of $\hat{\mathrm{M}}$-conjugacy classes of relevant Levi subgroups of ${}^L\mathrm{M}$).

Proof. (24) is immediate once we prove (i), while (23) is the special case of (24) where each $\mathcal{O}_{\mathrm{L}}$ is trivial. Therefore, it suffices to prove (i).

The "⇒" implication of (i) is immediate, so let us prove "⇐". $\Phi(\mathrm{M})/\mathcal{O}_{\mathrm{M}}$ is also the quotient, by $\mathcal{O}^+_{\mathrm{M},\,{}^L\mathrm{M}}$, of the set of all admissible relevant homomorphisms $W'_F \to {}^L\mathrm{M}$. Thus, replacing φ_2 and $\mathcal{L}_2$ by ${}^L\beta' \circ \varphi_2$ and ${}^L\beta'(\mathcal{L}_2)$ for some $\beta' \in \mathcal{O}_{\mathrm{M}}$, we may and do assume without loss of generality that $\varphi_1 = \varphi_2$, and that $\dot{\varphi} : W'_F \to {}^L\mathrm{M}$ represents $\varphi_1 = \varphi_2$.

Now use [Bor79, Proposition 3.6], which says that the Levi subgroups of ${}^L\mathrm{M}$ which are minimal among those that $\dot{\varphi}(W'_F)$ form a single conjugacy class under the centralizer of $\dot{\varphi}(W'_F)$ in $\hat{\mathrm{M}}$; this gives us an $\hat{\mathrm{M}}$-conjugation between φ_1 and φ_2 taking $\mathcal{L}_2$ to $\mathcal{L}_1$, finishing the proof of the lemma. □

Corollary 2.10.7. *Let* $\mathrm{M} \subset \mathrm{G}$ *be a Levi subgroup.*

(i) Let $\varphi_i \in \Phi_2^+(\mathrm{L}_i)$ for $i = 1, 2$, where $\mathrm{L}_1, \mathrm{L}_2 \subset \mathrm{M}$ are Levi subgroups. Then $\iota_{\mathrm{M},\mathrm{L}_1} \circ \varphi_1, \iota_{\mathrm{M},\mathrm{L}_2} \circ \varphi_2 \in \Phi(\mathrm{M})/\mathcal{O}_{\mathrm{M}}$ (made sense of using Remark 2.10.2(i)) are equal if and only if $\varphi_1 = {}^L\beta \circ \varphi_2$ for some ${}^L\beta \in \mathcal{O}^+_{\mathrm{M}}({}^L\mathrm{L}_1,\,{}^L\mathrm{L}_2)$ (see Notation 2.6.1(iii)).

(ii) The maps $\Phi(\mathrm{L}) \to \Phi(\mathrm{M})$ given by $\iota_{\mathrm{M},\mathrm{L}} \circ -$ (see Remark 2.10.2(i)) induce decompositions:

$$\Phi(\mathrm{M}) = \bigsqcup_{\mathrm{L}} \Phi_2^+(\mathrm{L})/W(\mathrm{L}), \tag{25}$$

where L *runs over a set of representatives for the* $\mathrm{M}(F)$*-conjugacy classes of Levi subgroups of* M*, and*

$$\Phi(\mathrm{M})/\mathcal{O}_{\mathrm{M}} = \bigsqcup_{\mathrm{L}} \Phi_2^+(\mathrm{L})/\mathcal{O}^+_{\mathrm{M},\mathrm{L}}, \tag{26}$$

where L *runs over a set of representatives for the* $\mathcal{O}^+_{\mathrm{M}}$*-orbits of Levi subgroups of* M.

Proof. Given any Levi subgroup $\mathrm{L} \subset \mathrm{M}$ and a choice of $\iota_{\mathrm{M},\mathrm{L}}$ with $\iota_{\mathrm{M},\mathrm{L}}({}^L\mathrm{L}) = \mathcal{L}$, $\iota_{\mathrm{M},\mathrm{L}} \circ -$ induces a bijection $\Phi_2^+(\mathrm{L}) \cong \Phi_2^+(\mathcal{L})$. Now (i) (resp., (ii)) of the corollary easily follows from combining the corresponding assertion of Lemma 2.10.6 with Proposition 2.4.18 (resp., with Corollary 2.4.17 and Proposition 2.4.18). □

Remark 2.10.8. Let $\beta \in \mathcal{O}^+_{\mathrm{G}}$. Let $\mathrm{M} \subset \mathrm{G}$ be a Levi subgroup, and recall that ${}^L(\beta|_{\mathrm{M}}) : {}^L\beta(\mathrm{M}) := {}^L(\beta(\mathrm{M})) \to {}^L\mathrm{M}$ is a choice of a dual to $\beta|_{\mathrm{M}}$.

(i) Since β transports $\mathcal{O}^+_{\mathrm{M}}$ to $\mathcal{O}^+_{\beta(\mathrm{M})}$ and each $\mathcal{O}^+_{\mathrm{M},\mathrm{L}}$ to $\mathcal{O}^+_{\beta(\mathrm{M}),\beta(\mathrm{L})}$ (Lemma 2.6.3(i)), it follows that applying β to (20) gives an analogue of (20) with M replaced by $\beta(\mathrm{M})$.

(ii) We can get a decomposition as in (24) by applying ${}^L(\beta|_{\mathrm{M}})$ to an analogous decomposition for ${}^L\beta(\mathrm{M})$: this uses that, since β takes $\mathcal{O}^+_{\mathrm{M}}$ to $\mathcal{O}^+_{\beta(\mathrm{M})}$, for each Levi subgroup $\mathcal{L}' \subset {}^L\beta(\mathrm{M})$, ${}^L(\beta|_{\mathrm{M}})$ transports $\mathcal{O}^+_{\beta(\mathrm{M}),\mathcal{L}'}$ isomorphically to $\mathcal{O}^+_{\mathrm{M},\,{}^L(\beta|_{\mathrm{M}})(\mathcal{L}')}$.

(iii) For each Levi subgroup $\mathrm{L} \subset \mathrm{M}$, since $\iota_{\mathrm{M},\mathrm{L}} \circ {}^L(\beta|_{\mathrm{L}})$ and ${}^L(\beta|_{\mathrm{M}}) \circ \iota_{\beta(\mathrm{M}),\beta(\mathrm{L})}$ are $\hat{\mathrm{M}}$-conjugate (Lemma 2.4.16(ii)), it is easy to see that a decomposition as in (26) for M can be obtained from that for $\beta(\mathrm{M})$, by mapping $\Phi(\beta(\mathrm{M}))$ to $\Phi(\mathrm{M})$ using ${}^L(\beta|_{\mathrm{M}})$ on the one hand, and mapping, for each Levi subgroup $\mathrm{L} \subset \mathrm{M}$, $\Phi_2^+(\beta(\mathrm{L}))$ to $\Phi_2^+(\mathrm{L})$ by applying ${}^L(\beta|_{\mathrm{L}})$, on the other.

Notation 2.10.9. Whenever we assume Hypothesis 2.10.3, we will assume that the maps $\mathrm{Irr}_2(\mathrm{M})/\mathcal{O}_{\mathrm{M}} \to \Phi_2(\mathrm{M})/\mathcal{O}_{\mathrm{M}}$ as in that hypothesis have been chosen, and that they are extended to maps

$$\mathrm{Irr}_2^+(\mathrm{M})/\mathcal{O}_{\mathrm{M}} \to \Phi_2^+(\mathrm{M})/\mathcal{O}_{\mathrm{M}} \tag{27}$$

and

$$\mathrm{Irr}(\mathrm{M})/\mathcal{O}_{\mathrm{M}} \to \Phi(\mathrm{M})/\mathcal{O}_{\mathrm{M}}, \tag{28}$$

still denoted $\sigma \mapsto \varphi_\sigma$, as follows:

(i) (27) is defined by requiring that $\varphi_{\sigma\otimes\chi_\alpha} = \alpha \cdot \varphi_\sigma$ for each $\alpha \in H^1(W_F/I_F, \mathrm{Z}_{\hat{\mathrm{M}}}^{\Gamma,0})$ or equivalently each $\alpha \in (\mathrm{Z}_{\hat{\mathrm{M}}}^{I_F})^0_{\mathrm{Fr}} \subset H^1(W_F/I_F, \mathrm{Z}_{\hat{\mathrm{M}}}^{I_F})$: to see that this is well-defined, use (a)-(d) that follow: (a) every element of $\mathrm{Irr}_2^+(\mathrm{M})$ is of the form $\sigma \otimes \chi$ for a unique pair $(\sigma, \chi) \in \mathrm{Irr}_2(\mathrm{M}) \times X^{\mathrm{unr}}(\mathrm{M})_{>0}$; (b) $X^{\mathrm{unr}}(\mathrm{S_M})_{>0} \cong X^{\mathrm{unr}}(\mathrm{M})_{>0} \cong X^{\mathrm{unr}}(\mathrm{A_M})_{>0}$ acts freely on $\mathrm{Irr}(\mathrm{M})$ (look at central characters); (c) the map $\alpha \mapsto \chi_\alpha$ is an $\mathcal{O}_{\mathrm{M}}$-equivariant surjective map $\mathrm{Z}_{\hat{\mathrm{M}}}^{\Gamma,0} \to X^{\mathrm{unr}}(\mathrm{M}) \cong X^{\mathrm{unr}}(\mathrm{M})_{>0} \times X^{\mathrm{unr-uni}}(\mathrm{M})$ with all the actions of $\mathcal{O}_{\mathrm{M}}$ being trivial (see Lemma 2.6.5); (d) the compatibility $\varphi_{\sigma\otimes\chi_\alpha} = \alpha \cdot \varphi_\sigma$ has already been imposed whenever $\sigma \in \mathrm{Irr}_2(\mathrm{M})$ and $\chi_\alpha \in X^{\mathrm{unr-uni}}(\mathrm{M})$.

(ii) We define (28) by:

$$\mathrm{Irr}(\mathrm{M})/\mathcal{O}_{\mathrm{M}} \to \bigsqcup_{\mathrm{L}} \mathrm{Irr}_2^+(\mathrm{L})/\mathcal{O}_{\mathrm{M,L}}^+ \to \bigsqcup_{\mathrm{L}} \Phi_2^+(\mathrm{L})/\mathcal{O}_{\mathrm{M,L}}^+ \to \Phi(\mathrm{M})/\mathcal{O}_{\mathrm{M}}, \tag{29}$$

where the first map is given by (20), the third map is from (26), and the middle map is obtained by factoring (27) with M replaced by L: note that (27) with M replaced by L is equivariant for $\mathcal{O}_{\mathrm{M,L}}^+ \supset \mathcal{O}_{\mathrm{L}}$, by (iii) of Hypothesis 2.10.3 and the fact that the surjection $H^1(W_F/I_F, \mathrm{Z}_{\hat{\mathrm{L}}}^{\Gamma,0}) \to X^{\mathrm{unr}}(\mathrm{L})$ given by $\alpha \mapsto \chi_\alpha$ is equivariant for $\mathrm{Aut}(\mathrm{L}) \supset \mathcal{O}_{\mathrm{M,L}}^+$ (though the $\mathcal{O}_{\mathrm{M,L}}^+$-actions involved are not trivial). A similar argument shows that (28) is independent of the choices of the representatives L involved.

In what follows, it may be helpful to keep in mind notation from Remark 2.10.2. One form of the well-known Langlands classification amounts to most assertions of the following theorem, and compensates for our having stated Hypothesis 2.10.3 only for discrete series representations, unlike in [Hai14].

Theorem 2.10.10. *Assume Hypothesis 2.10.3, so that we have the maps* $\mathrm{Irr}_2^+(\mathrm{M})/\mathcal{O}_{\mathrm{M}} \to \Phi_2^+(\mathrm{M})/\mathcal{O}_{\mathrm{M}}$ *and* $\mathrm{Irr}(\mathrm{M})/\mathcal{O}_{\mathrm{M}} \to \Phi(\mathrm{M})/\mathcal{O}_{\mathrm{M}}$ *of* (27) *and* (28)*, denoted* $\sigma \mapsto \varphi_\sigma$*, as in Notation 2.10.9 (as* M *ranges over Levi subgroups of* G*). These maps have the following properties, parallel to those in Hypothesis 2.10.3:*

(i) The maps $\mathrm{Irr}_2^+(\mathrm{M})/\mathcal{O}_{\mathrm{M}} \to \Phi_2^+(\mathrm{M})/\mathcal{O}_{\mathrm{M}}$ *and* $\mathrm{Irr}(\mathrm{M})/\mathcal{O}_{\mathrm{M}} \to \Phi(\mathrm{M})/\mathcal{O}_{\mathrm{M}}$ *are finite-to-one and surjective, and satisfy that* $\varphi_{\sigma\otimes\chi_\alpha} = \alpha \cdot \varphi_\sigma$ *(for* σ *in* $\mathrm{Irr}_2^+(\mathrm{M})$ *or* $\mathrm{Irr}(\mathrm{M})$*, and* $\alpha \in \mathrm{Z}^0_{{}^L\mathrm{M}} = \mathrm{Z}_{\hat{\mathrm{M}}}^{\Gamma,0} = H^1(W_F/I_F, \mathrm{Z}_{\hat{\mathrm{M}}}^{\Gamma,0})$*; see Notation 2.5.11). Moreover, these maps satisfy the desiderata (iii) and (iv) of Remark 2.10.4, i.e.,* $\mathrm{Irr}_2^+(\mathrm{M})/\mathcal{O}_{\mathrm{M}}$ *and* $\mathrm{Irr}_{\mathrm{temp}}(\mathrm{M})/\mathcal{O}_{\mathrm{M}}$ *are the full preimages of* $\Phi_2^+(\mathrm{M})$ *and* $\Phi_{\mathrm{temp}}(\mathrm{M})$*, respectively, under* $\mathrm{Irr}(\mathrm{M})/\mathcal{O}_{\mathrm{M}} \to \Phi(\mathrm{M})/\mathcal{O}_{\mathrm{M}}$*.*

(ii) (A generalization of Hypothesis 2.10.3(ii); compare with [Hai14, Conjecture 5.2.2]*). Let* M', M *be Levi subgroups of* G *with* $\mathrm{M}' \subset \mathrm{M}$*, and* σ' *and* σ *irreducible admissible representations of* $\mathrm{M}'(F)$ *and* $\mathrm{M}(F)$ *with* σ *an irreducible subquotient of* $\mathrm{Ind}_{\mathrm{M}'}^{\mathrm{M}} \sigma'$*. Then* $\iota_{\mathrm{M,M}'}(\lambda(\varphi_{\sigma'})) = \lambda(\varphi_\sigma) \in \Omega({}^L\mathrm{M})/\mathcal{O}_{\mathrm{M}}$*.*

(iii) (A generalization of Hypothesis 2.10.3(iii); compare with [Hai14, Conjecture 5.2.7]*). If* $\mathrm{M} \subset \mathrm{G}$ *is a Levi subgroup,* $\beta \in \mathcal{O}_{\mathrm{G}}^+$*, and* ${}^L(\beta|_{\mathrm{M}}) : {}^L\beta(\mathrm{M}) \to {}^L\mathrm{M}$ *is dual to* $\beta|_{\mathrm{M}} : \mathrm{M} \to \beta(\mathrm{M})$*, then for all irreducible admissible representations* σ *of* $\mathrm{M}(F)$*,* ${}^L(\beta|_{\mathrm{M}}) \circ \varphi_{\sigma\circ\beta^{-1}} = \varphi_\sigma \in \Phi(\mathrm{M})/\mathcal{O}_{\mathrm{M}}$*.*

Proof. Since the arguments involved are standard, parts of the proof will be only sketched. Let us prove (i), fixing a Levi subgroup $\mathrm{M} \subset \mathrm{G}$.

The surjectivity of (27) and that of (28) are immediate from the surjectivity assumption in Hypothesis 2.10.3(i), once we show that for each Levi subgroup $\mathrm{L} \subset \mathrm{M}$, every element $\varphi' \in \Phi_2^+(\mathrm{L})$

is of the form $\alpha \cdot \varphi$ for some $\alpha \in Z_{\hat{M}}^{\Gamma,0}$ and $\varphi \in \Phi_2(L)$. To see this, apply [SZ18, Proposition 5.3] (essentially, Langlands classification for paramemters) with L in place of the group G there, and note that the group denoted "$Z(C_{\hat{G}}(Im(\phi)))^0$" there is simply $Z_{\hat{L}}^{\Gamma,0}$ in our situation: if not, then by [Bor79, Lemma 3.5], $\varphi'(W_F)$ would be contained in a proper Levi subgroup of ${}^L\mathrm{L}$, contradicting that $\varphi' \in \Phi_2^+(L)$. The maps $\mathrm{Irr}_2^+(M)/\mathcal{O}_M \to \Phi_2^+(M)/\mathcal{O}_M$ are finite-to-one, since the maps $\mathrm{Irr}_2(M)/\mathcal{O}_M \to \Phi_2(M)/\mathcal{O}_M$ are, and since the action of $X^{\mathrm{unr}}(M)_{>0}$ on $\mathrm{Irr}_2^+(M)/\mathcal{O}_M$ and that of the group $(Z_{\hat{M}}^{\Gamma,0})_h := \{\alpha \in Z_{\hat{M}}^{\Gamma,0} \mid \chi_\alpha \in X^{\mathrm{unr}}(M)_{>0}\}$ of hyperbolic elements in $Z_{\hat{M}}^{\Gamma,0}$ on $\Phi_2^+(M)/\mathcal{O}_M$ are free; for the assertion involving $X^{\mathrm{unr}}(M)_{>0}$, use that $X^{\mathrm{unr}}(S_M)_{>0} \to X^{\mathrm{unr}}(M)_{>0} \to X^{\mathrm{unr}}(A_M)_{>0}$ are bijections, and for the assertion involving $(Z_{\hat{M}}^{\Gamma,0})_h$, use that $(Z_{\hat{M}}^{\Gamma,0})_h$ injects into the group $(\hat{M}/(\hat{M})_{\mathrm{der}})_\Gamma$ of Γ-coinvariants of $\hat{M}/\hat{M}_{\mathrm{der}}$, since $Z_{\hat{M}}^{\Gamma,0} \to (\hat{M}/(\hat{M})_{\mathrm{der}})_\Gamma$ is an isogeny, on whose target $\mathcal{O}_M$ acts trivially by Lemma 2.6.5. Since the maps (20) and (26) are also finite-to-one, it follows that each of the three maps that are used to define (29) is finite-to-one, so the map $\mathrm{Irr}(M) \to \Phi(M)$ is finite-to-one.

The desideratum (iii) of Remark 2.10.4 is immediate from the construction. For the desideratum (iv) of Remark 2.10.4: combine (21) from Corollary 2.7.5 (Langlands classification for tempered representations) with the analogous equality that (26) clearly restricts to:

$$\Phi_{\mathrm{temp}}(M)/\mathcal{O}_M = \bigsqcup_{L} \Phi_2(L)/\mathcal{O}_{M,L}^+.$$

This proves (i) of the theorem.

Now let us prove (iii) of the theorem. The equality ${}^L(\beta|_M) \circ \varphi_{\sigma\circ\beta^{-1}} = \varphi_\sigma$ is automatic if $\sigma \in \mathrm{Irr}_2(M)$, by Hypothesis 2.10.3(iii). If $\sigma \in \mathrm{Irr}_2^+(M)$, the same follows by the definition of (27) and the equality $\chi_\alpha \circ \beta^{-1} = \chi_{{}^L(\beta|_M)^{-1}(\alpha)} \in X^{\mathrm{unr}}(S_{\beta(M)})$ for $\alpha \in Z_{\hat{M}}^{\Gamma,0}$, obtained by applying the functoriality of the local Langlands correspondence for tori to the map $S_{\beta(M)} \to S_M$ induced by β^{-1}, whose dual is the map $Z_{\hat{M}}^{\Gamma,0} \to Z_{\widehat{\beta(M)}}^{\Gamma,0}$ obtained by restricting ${}^L(\beta|_M)^{-1}$ (use Lemma 2.5.10). Since this applies with M replaced by any Levi subgroup $L \subset M$, (iii) of the theorem follows from (i) and (iii) of Remark 2.10.8, together with the definition of the map $\mathrm{Irr}(M)/\mathcal{O}_M \to \Phi(M)/\mathcal{O}_M$ in (29).

Now let us verify (ii) of the theorem, which is a strengthening of Hypothesis 2.10.3(ii).

For this, let us first reduce to the case where the inducing representation σ' is supercuspidal. Let (L, υ) be a cuspidal support for σ'; it is then also a cuspidal support for σ. Use Proposition 2.4.15 to assume without loss of generality that $\iota_{M,L} = \iota_{M,M'} \circ \iota_{M',L}$. Therefore, once we prove the case where the inducing representation is supercuspidal, we can apply it to σ being a subquotient of $\mathrm{Ind}_L^M \upsilon$ as well as to σ' being a subquotient of $\mathrm{Ind}_L^{M'} \upsilon$, to get:

$$\lambda(\varphi_\sigma) = \iota_{M,L}(\lambda(\varphi_\upsilon)) = \iota_{M,M'} \circ \iota_{M',L}(\lambda(\varphi_\upsilon)) = \iota_{M,M'}(\lambda(\varphi_{\sigma'})).$$

Thus, we now assume that σ' is supercuspidal, i.e., (M', σ') is a cuspidal support of σ. In what follows, we will use the following observation:

Observation: If $\iota_{M,M'}(\lambda(\varphi_{\sigma'})) = \lambda(\varphi_\sigma)$, then for all $\chi \in X^{\mathrm{unr}}(M)$, $\iota_{M,M'}(\lambda(\varphi_{\sigma'\otimes\chi|_{M'(F)}})) = \lambda(\varphi_{\sigma\otimes\chi})$.

To see this observation, assume that $\chi = \chi_\alpha$ for some $\alpha \in Z_{\hat{M}}^{\Gamma,0}$, and denote by α' its image in $Z_{\hat{M}'}^{\Gamma,0}$ under the embedding $Z_{\hat{M}} \hookrightarrow Z_{\hat{M}'}$ obtained from $\iota_{M,M'}$ (see Corollary 2.4.13), i.e., $\iota_{M,M'}(\alpha') = \alpha$. From Remark 2.5.6(ii), it follows that $\chi_{\alpha'} = \chi_\alpha|_{M'(F)}$. Using that $\varphi_{\sigma\otimes\chi_\alpha} = \alpha \cdot \varphi_\sigma$ and $\varphi_{\sigma'\otimes\chi_{\alpha'}} = \alpha' \cdot \varphi_{\sigma'}$, as given by (i) of the theorem, we get:

$$\lambda(\varphi_{\sigma\otimes\chi}) = \lambda(\alpha\cdot\varphi_\sigma) = \iota_{M,M'}(\alpha')\cdot\iota_{M,M'}(\lambda(\varphi_{\sigma'})) = \iota_{M,M'}(\lambda(\alpha'\cdot\varphi_{\sigma'})) = \iota_{M,M'}(\lambda(\varphi_{\sigma'\otimes\chi_{\alpha'}})) = \iota_{M,M'}(\lambda(\varphi_{\sigma'\otimes\chi|_{M'(F)}})),$$

proving the observation.

In the case where $\sigma \in \mathrm{Irr}_2(M)$ and σ' is supercuspidal, the claim of (ii) is almost the same as Hypothesis 2.10.3(ii): writing $\sigma' = \sigma'' \otimes \chi_{\alpha''}$, where σ'' is unitary supercuspidal and $\alpha'' \in Z_{\hat{M}'}^{\Gamma,0}$, so that $\varphi_{\sigma'} = \alpha'' \cdot \varphi_{\sigma''}$, we get from Hypothesis 2.10.3(ii) that:

$$\lambda(\varphi_\sigma) = \iota_{M,M'} \circ \lambda(\alpha'' \cdot \varphi_{\sigma''}) = \iota_{M,M'} \circ \lambda(\varphi_{\sigma'}).$$

The case where $\sigma \in \mathrm{Irr}_2^+(\mathrm{M})$ and σ' is supercuspidal then follows by the observation above.
Recall that φ_σ is defined using (29). Let L, Q and $\upsilon \in \mathrm{Irr}_2^+(\mathrm{L})$ be associated to σ as in Langlands classification, i.e., as in Proposition 2.2.3(i). Thus, σ is an irreducible quotient of $\mathrm{Ind}_\mathrm{Q}^\mathrm{M} \upsilon$, and the $\mathcal{O}_\mathrm{M}^+$-orbit of σ corresponds to the $\mathcal{O}_{\mathrm{M},\mathrm{L}}^+$-orbit of (L, υ) under (20) (which we may assume features L on its right-hand side). A cuspidal support of υ is then also a cuspidal support of σ, and hence $\mathrm{M}(F)$-conjugate to (M', σ'). Since (iii) has been proved and since $\mathrm{Int}\,\mathrm{M}(F) \subset \mathcal{O}_\mathrm{G}^+$, we $\mathrm{M}(F)$-conjugate (L, υ) without loss of generality to assume that (M', σ') is a cuspidal support for υ.
Thus, we have:

$$\lambda(\varphi_\sigma) = \iota_{\mathrm{M},\mathrm{L}} \circ \lambda(\varphi_\upsilon) = \iota_{\mathrm{M},\mathrm{L}} \circ \iota_{\mathrm{L},\mathrm{M}'} \circ \lambda(\varphi_{\sigma'}) = \iota_{\mathrm{M},\mathrm{M}'} \circ \lambda(\varphi_{\sigma'}),$$

where the first step uses the definition of φ_σ as $\varphi_\sigma = \iota_{\mathrm{M},\mathrm{L}} \circ \varphi_\upsilon \in \Phi(\mathrm{M})$, the second step uses the known case where σ' is supercuspidal and σ is essentially square-integrable (but applied with (L, υ) in place of (M, σ)), and the third step uses Proposition 2.4.15. ☐

2.10.2. *The local Langlands correspondence and L-packets.*

Notation 2.10.11. Let $\mathrm{M} \subset \mathrm{G}$ be a Levi subgroup.

(i) For any $\varphi \in \Phi(\mathrm{M})/\mathcal{O}_\mathrm{M}$, we will denote by $\Sigma(\varphi)$ the set of all (isomorphism classes of) irreducible admissible representations σ of $\mathrm{M}(F)$ with the property that $\varphi = \varphi_\sigma$; it is finite by Theorem 2.10.10.
(ii) We will also refer to each of the finite sets $\Sigma(\varphi)$ obtained in this way, where $\varphi \in \Phi(\mathrm{M})/\mathcal{O}_\mathrm{M}$, as an L-packet on $\mathrm{M}(F)$ in the sense of Hypothesis 2.10.3, up to the action of $\mathcal{O}_\mathrm{M}$ (although this involves the implicit choice of the maps $\sigma \mapsto \varphi_\sigma$, which we fixed in Notation 2.10.9).
(iii) Given an L-packet $\Sigma(\varphi)$ on $\mathrm{M}(F)$ up to the action of $\mathcal{O}_\mathrm{M}$ in the sense of Hypothesis 2.10.3, with $\varphi \in \Phi(\mathrm{M})$, we will call Σ a discrete series L-packet (resp., an essentially square-integrable L-packet; resp., a tempered L-packet) if $\varphi \in \Phi_2(\mathrm{M})$ (resp., $\varphi \in \Phi_2^+(\mathrm{M})$; resp., $\varphi \in \Phi_{\mathrm{temp}}(\mathrm{M})$). Note that, by the desiderata (iii) and (iv) of Remark 2.10.4 (proved in Theorem 2.10.10), $\Sigma(\varphi)$ is a discrete series L-packet (resp., an essentially square-integrable L-packet; resp., a tempered L-packet) if and only if some or equivalently any representation in it belongs to the discrete series (resp., is essentially square-integrable; resp., is tempered).

The following hypothesis, which is also expected to be satisfied by the local Langlands correspondence, addresses the potential conflict between the notion of discrete series L-packets according to Notation 2.10.11 and the one according to Notation 2.7.9:

Hypothesis 2.10.12. (LLC+ and stability) Assume that Hypotheses 2.7.1 and 2.10.3 and are satisfied with the following compatibility between them: for each Levi subgroup $\mathrm{M} \subset \mathrm{G}$, the map $\varphi \mapsto \Sigma(\varphi)$ defines a bijection $\Phi_2(\mathrm{M})/\mathcal{O}_\mathrm{M} \to \mathit{\Phi}_2(\mathrm{M})$.

Lemma 2.10.13. *Assume Hypothesis 2.10.12 (and in particular Hypotheses 2.7.1 and 2.10.3), and let* $\mathrm{M} \subset \mathrm{G}$ *be a Levi subgroup. Then the map* $\varphi \mapsto \Sigma(\varphi)$ *defines bijections* $\Phi_2^+(\mathrm{M})/\mathcal{O}_\mathrm{M} \to \mathit{\Phi}_2^+(\mathrm{M})$ *and* $\Phi(\mathrm{M})/\mathcal{O}_\mathrm{M} \to \mathit{\Phi}(\mathrm{M})$.

Proof. Since $\mathit{\Phi}_2^+(\mathrm{M})$ is a partition of $\mathrm{Irr}_2^+(\mathrm{M})$ (see Lemma 2.7.7(i)), and so is the set of all $\Sigma(\varphi)$ with φ ranging over $\Phi_2^+(\mathrm{M})/\mathcal{O}_\mathrm{M}$ (by the surjectivity of $\mathrm{Irr}(\mathrm{M}) \to \Phi(\mathrm{M})/\mathcal{O}_\mathrm{M}$ and the desideratum (iii) of Remark 2.10.4, proved in Theorem 2.10.10(i)), the claim regarding the bijection $\Phi_2^+(\mathrm{M})/\mathcal{O}_\mathrm{M} \to \mathit{\Phi}_2^+(\mathrm{M})$ follows if we show that each $\Sigma(\varphi)$ with $\varphi \in \Phi_2^+(\mathrm{M})/\mathcal{O}_\mathrm{M}$ belongs to $\mathit{\Phi}_2^+(\mathrm{M})$. But each such φ can be written as $\alpha' \cdot \varphi'$ with $\alpha' \in \mathrm{Z}_{\hat{\mathrm{M}}}^{\Gamma,0}$ and $\varphi' \in \Phi_2(\mathrm{M})/\mathcal{O}_\mathrm{M}$, as observed in the proof of Theorem 2.10.10(i), so that $\Sigma(\varphi) = \Sigma(\varphi') \otimes \chi_\alpha$, which belongs to $\mathit{\Phi}_2^+(\mathrm{M})$ by Hypothesis 2.10.12 and the definition of $\mathit{\Phi}_2^+(\mathrm{M})$ (Notation 2.7.6(i)). This proves the assertion involving the bijection $\Phi_2^+(\mathrm{M})/\mathcal{O}_\mathrm{M} \to \mathit{\Phi}_2^+(\mathrm{M})$. The assertion involving the bijection $\Phi(\mathrm{M})/\mathcal{O}_\mathrm{M} \to \mathit{\Phi}(\mathrm{M})$ follows from this, together with the definition of $\mathit{\Phi}(\mathrm{M})$ (Notation 2.7.6(iii)), and the definition of the map $\mathrm{Irr}(\mathrm{M}) \to \Phi(\mathrm{M})/\mathcal{O}_\mathrm{M}$ in (29). ☐

Remark 2.10.14. We emphasize that even when Hypothesis 2.10.3 is satisfied, the choice of the maps $\sigma \mapsto \varphi_\sigma$ will in general be too random to be compatible with 'the' local Langlands correspondence for the relevant Levi subgroups, though 'the' local Langlands correspondence is expected to fit the description of that hypothesis. Even when Hypotheses 2.7.1 and 2.10.12 are satisfied in addition, the maps $\sigma \mapsto \varphi_\sigma$ will be determined up to barely more than a permutation of the various $\Phi(\mathrm{M})/\mathcal{O}_\mathrm{M}$ (which they will be thanks to Lemma 2.7.3). It is in cases where 'the correct' local Langlands correspondence (up to some $\{\mathcal{O}_\mathrm{M}\}_\mathrm{M}$) is established, and the maps $\sigma \mapsto \varphi_\sigma$ are chosen to be compatible with it, that the theorems we prove will be 'really' meaningful, rather than just technically valid.

2.11. **Additional hypotheses on the local Langlands correspondence.** To ensure that various proposed descriptions of the stable Bernstein center agree with each other, certain statements about supercuspidal supports of representations in L-packets need to be proved, as pointed out by Haines in [Hai14, Remark 5.5.4]. In this subsection, assuming G to be quasi-split, we will instead make two such statements into hypotheses.

2.11.1. *Langlands parameters of L-packets consisting entirely of supercuspidal representations.*

Hypothesis 2.11.1. (Supercuspidal packets) (Applicable only when G is quasi-split). Assume the LLC+ hypothesis (Hypothesis 2.10.3). For each Levi subgroup $\mathrm{M} \subset \mathrm{G}$ and each $\varphi \in \Phi_2^+(\mathrm{M})/\mathcal{O}_\mathrm{M}$, the following are equivalent:

(a) $\Sigma(\varphi)$ consists entirely of (not necessarily unitary) supercuspidal representations; and
(b) (Any representative $\dot{\varphi} : W_F' \to {}^L\mathrm{M}$ for) φ factors through the projection from $W_F' = W_F \times \mathrm{SL}_2(\mathbb{C})$ to its first factor.

As Bertoloni-Meli pointed out to us, this hypothesis is related to [Hai14, Proposition 5.6.1]. Nevertheless, it does not seem to follow from that proposition.

Remark 2.11.2. In fact, the implication (b) $\Rightarrow$ (a) in the above hypothesis follows from the LLC+ hypothesis (Hypothesis 2.10.3), as in the easy implication "$\Leftarrow$" of [Hai14, Proposition 5.6.1]; let us recall this argument. Suppose that φ factors through the projection to the W_F-factor, but that $\Sigma(\varphi)$ contains a nonsupercuspidal representation σ. Consider a cuspidal support (M', σ') for σ. Then $\varphi = \varphi_\sigma$ equals $\lambda(\varphi) = \lambda(\varphi_\sigma)$, which equals $\iota_{\mathrm{M},\mathrm{M}'}(\lambda(\varphi_{\sigma'}))$ by Theorem 2.10.10(ii), and hence factors through the proper Levi subgroup $\iota_{\mathrm{M},\mathrm{M}'}({}^L\mathrm{M}')$ of ${}^L\mathrm{M}$, contradicting the desideratum (iii) of Remark 2.10.4.

2.11.2. *Stable cuspidal support.*

Notation 2.11.3. Assume that G is quasi-split, and assume Hypothesis 2.7.1.

(i) Given pairs (M, Σ) and (M', Σ') with M, M' Levi subgroups of G, $\Sigma \in \mathit{\Phi}(\mathrm{M})$ and $\Sigma' \in \mathit{\Phi}(\mathrm{M}')$, we write $(\mathrm{M}, \Sigma) \succeq (\mathrm{M}', \Sigma')$ if $\mathrm{M}' \subset \mathrm{M}$, and there exist $\sigma \in \Sigma$ and $\sigma' \in \Sigma'$ such that σ is an irreducible subquotient of $\mathrm{Ind}_{\mathrm{M}'}^{\mathrm{M}} \sigma'$. We write $(\mathrm{M}, \Sigma) \succ (\mathrm{M}', \Sigma')$ if $(\mathrm{M}, \Sigma) \succeq (\mathrm{M}', \Sigma')$ and $(\mathrm{M}, \Sigma) \neq (\mathrm{M}', \Sigma')$.
(ii) Let $\mathrm{L}, \mathrm{M} \subset \mathrm{G}$ be Levi subgroups, and let $\Upsilon \in \mathit{\Phi}(\mathrm{L})$ and $\Sigma \in \mathit{\Phi}(\mathrm{M})$. We say that (L, Υ) is a potential stable cuspidal support for (M, Σ) if there exists a maximal chain

$$(\mathrm{M}, \Sigma) = (\mathrm{M}_0, \Sigma_0) \succ (\mathrm{M}_1, \Sigma_1) \succ \cdots \succ (\mathrm{M}_n, \Sigma_n) = (\mathrm{L}, \Upsilon). \tag{30}$$

(iii) Given (M, Σ) as in (ii), we say that it has a stable cuspidal support if its potential stable cuspidal supports belong to a single orbit under $\mathcal{O}_\mathrm{M}^+$, in which case each potential stable cuspidal support of (M, Σ) will also be referred to as a stable cuspidal support of (M, Σ).

Hypothesis 2.11.4. (Existence of stable cuspidal support; applicable only when G is quasi-split). Assume that G is quasi-split, and assume Hypothesis 2.7.1. We assume that every pair (M, Σ), where $\mathrm{M} \subset \mathrm{G}$ is a Levi subgroup and $\Sigma \in \mathit{\Phi}(\mathrm{M})$, has a stable cuspidal support.

Remark 2.11.5. (i) Note that every (M, Σ) has a potential stable cuspidal support, and that for any such potential stable cuspidal support (L, Υ), Υ consists entirely of supercuspidal representations. Thus, the nontrivial assumption in Hypothesis 2.11.4 is the uniqueness of the potential stable cuspidal supports up to the action of $\mathcal{O}_\mathrm{M}^+$.

(ii) It is easy to see from the definition of $\Phi(\mathrm{M})$ (involving Langlands classification) that Hypothesis 2.11.4 needs to only be checked for pairs (M, Σ) with $\Sigma \in \Phi_2(\mathrm{M})$.

Proposition 2.11.6. *Assume that* G *is quasi-split, and assume the hypotheses on the existence of tempered L-packets, LLC+, LLC+ and stability, and supercuspidal packets (Hypotheses 2.7.1, 2.10.3, 2.10.12 and 2.11.1). Then the hypothesis on the existence of stable cuspidal support (Hypothesis 2.11.4) holds.*

Proof. Let $(\mathrm{L}_1, \Upsilon_1)$ and $(\mathrm{L}_2, \Upsilon_2)$ be potential stable cuspidal supports for (M, Σ), and let $v_i \in \Upsilon_i$ for $i = 1, 2$. Thus, for $i = 1, 2$, Υ_i consists entirely of supercuspidal representations, so that by Hypothesis 2.11.1, we can identify φ_{v_i} with $\lambda(\varphi_{v_i})$. Hence it follows from Lemma 2.10.13 and repeated applications of Theorem 2.10.10(ii) that the image of φ_{v_1} in $\Phi_2^+(\mathrm{L}_1)/\mathcal{O}^+_{\mathrm{M},\mathrm{L}_1}$ and that of φ_{v_2} in $\Phi_2^+(\mathrm{L}_2)/\mathcal{O}^+_{\mathrm{M},\mathrm{L}_2}$ define the same element of $\Phi(\mathrm{M})/\mathcal{O}_{\mathrm{M}}$ as per (26), namely, $\lambda(\varphi_\sigma)$, where σ is any element of Σ (note that $\lambda(\varphi_\sigma)$ automatically satisfies the relevance condition as G is quasi-split). Therefore, by Corollary 2.10.7(i), there exists an element of $\mathcal{O}^+_{\mathrm{M}}$ that transports $(\mathrm{L}_1, \varphi_{v_1})$ to $(\mathrm{L}_2, \varphi_{v_2})$. By the property in (iii) of Theorem 2.10.10 (together with the inclusion $\mathcal{O}^+_{\mathrm{M}} \subset \mathcal{O}^+_{\mathrm{G},\mathrm{M}}$) and by Lemma 2.10.13, this element transports $(\mathrm{L}_1, \Upsilon_1)$ to $(\mathrm{L}_2, \Upsilon_2)$. □

3. Some results on stable virtual characters and atomically stable packets

3.1. **Elliptic characters and endoscopic transfer.** We will typically assume the three hypotheses stated in [MW16, Section I.1.5]:

Notation 3.1.1. Let $(\mathrm{M}, \tilde{\mathrm{M}}, \mathbf{a})$ be a triple where $(\mathrm{M}, \tilde{\mathrm{M}})$ is a twisted space (see Subsubsection 2.1.1), and $\mathbf{a}$ is a cocycle representing an element of $H^1(W_F, \mathrm{Z}_{\hat{\mathrm{M}}})$. Let $\omega : \mathrm{M}(F) \to \mathbb{C}^\times$ be the quasi-character associated to $\mathbf{a}$ (see Subsubsection 2.5.2), which we assume to be unitary. The purpose of this notation is to record the following hypothesis (to be imposed later):

- (i) $\tilde{\mathrm{M}}(F) \neq \emptyset$;
- (ii) θ^* has finite order, where θ^* is the object constructed towards the end of [MW16, Section I.1.2], as an automorphism of 'the pinned Borel pair' attached to M (see Subsection 2.3).
- (iii) ω is trivial on $\mathrm{Z}_{\tilde{\mathrm{M}}}(F)$ (else the theory is empty).

Notation 3.1.2. For a triple $(\mathrm{M}, \tilde{\mathrm{M}}, \mathbf{a})$ and the associated character $\omega : \mathrm{M}(F) \to \mathbb{C}^\times$ as in Notation 3.1.1, satisfying the hypotheses of that notation, we will often use the following notation:

- (i) As in [MW16, I.4.11], $\mathcal{E}(\tilde{\mathrm{M}}, \mathbf{a})$ will denote the set of isomorphism classes of relevant elliptic endoscopic data for $(\tilde{\mathrm{M}}, \mathbf{a})$. If we simply write $\mathcal{E}(\mathrm{M})$, it will stand for the set $\mathcal{E}(\mathrm{M}, \mathbb{1})$, where M is thought of as a twisted space over itself with respect to left and right multiplication, and $\mathbb{1}$ stands for the zero element of $H^1(W_F, \mathrm{Z}_{\hat{\mathrm{M}}})$ (thus, $\mathcal{E}(\mathrm{M})$ consists of endoscopic data for standard, untwisted, endoscopy).
- (ii) We will write a typical element of $\mathcal{E}(\tilde{\mathrm{M}}, \mathbf{a})$ or $\mathcal{E}(\mathrm{M})$ as $\underline{\mathrm{H}}$, and given such an $\underline{\mathrm{H}}$, write H for its underlying endoscopic group. This is an abuse of notation, since H does not determine $\underline{\mathrm{H}}$.
- (iii) For each endoscopic datum $\underline{\mathrm{H}} = (\mathrm{H}, \mathcal{H}, \tilde{s}) \in \mathcal{E}(\tilde{\mathrm{M}}, \mathbf{a})$ (the notation is chosen as in [MW16, Section I.1.5] — we will recall more of it in a later section when it becomes necessary), we will denote by $(\mathrm{H}, \tilde{\mathrm{H}})$ the associated twisted space as in [MW16, Section I.1.7]; it has the property that for each $\gamma \in \tilde{\mathrm{H}}(\bar{F})$, $\operatorname{Int}\gamma$ is of the form $\operatorname{Int} h$ for some $h \in \mathrm{H}_{\mathrm{ad}}(\bar{F})$ (this is the meaning of 'est á torsion intérieure' in (3) of [MW16, Section I.1.7]). For each such $\underline{\mathrm{H}}$, we will also often choose some auxiliary data as in [MW16, Section I.2.1], but also satisfying the extra condition of [MW16, Section I.7.1, (3)] (which may be imposed as ω is unitary); these yield for us a 5-tuple $(\mathrm{H}_1 \to \mathrm{H}, \hat{\xi}_1, \tilde{\mathrm{H}}_1 \to \tilde{\mathrm{H}}, \mathrm{C}_1, \mu)$, where:
  - $\mathrm{H}_1 \to \mathrm{H}$ is a z-extension, i.e., its kernel is an induced torus and the derived group of H_1 is simply connected;
  - $\hat{\xi}_1$ will be recalled later when it becomes necessary;
  - C_1 is the kernel of $\mathrm{H}_1 \to \mathrm{H}$ (and is hence an induced torus);

- $\mu : \mathrm{C}_1(F) \to \mathbb{C}^\times$ is a character (this is the λ_1 of [MW16, Section I.2.1]), which is unitary since we have imposed [MW16, Section I.7.1, (3)] (see towards the end of [MW16, Section I.7.1]);
- $\tilde{\mathrm{H}}_1$ is a twisted space with underlying group H_1, satisfying $\tilde{\mathrm{H}}_1(F) \neq \emptyset$, and the map $\tilde{\mathrm{H}}_1 \to \tilde{\mathrm{H}}$ is compatible in the obvious way with the homomorphism $\mathrm{H}_1 \to \mathrm{H}$.

Typically, when we make these choices, we will suppress the dependence of these objects on $\underline{\mathrm{H}}$ for lightness of notation.

(iv) There is a notion of endoscopic transfer of functions, which is a linear map from $C_c^\infty(\tilde{\mathrm{M}}(F))$ to the quotient of $C_\mu^\infty(\tilde{\mathrm{H}}_1(F))$ by the subspace consisting of the unstable functions in it, i.e., functions whose stable orbital integrals all vanish (see, e.g., [MW16, Section 1.2.4]). By [MW16, Corollary XI.5.1] (keeping in mind the convention from [MW16, Section XI.1] of calling an ω-representation just a representation), dual to this map is a map $\mathbf{T}_{\underline{\mathrm{H}}} : SD_\mu(\tilde{\mathrm{H}}_1) \to D(\tilde{\mathrm{M}}, \omega)$, restricting to a map $\mathbf{T}_{\underline{\mathrm{H}},\mathrm{ell}} : SD_{\mu,\mathrm{ell}}(\tilde{\mathrm{H}}_1) \to D_{\mathrm{ell}}(\tilde{\mathrm{M}}, \omega)$ (thus, one can show that pulling back under endoscopic transfer of functions takes $SD_\mu(\tilde{\mathrm{H}}_1)$ to $D(\tilde{\mathrm{M}}, \omega)$ and $SD_{\mu,\mathrm{ell}}(\tilde{\mathrm{H}}_1)$ to $D_{\mathrm{ell}}(\tilde{\mathrm{M}}, \omega)$). As explained around [LMW18, Section 4.4, (4)], the latter factors through the projection $SD_{\mu,\mathrm{ell}}(\tilde{\mathrm{H}}_1) \to SD_{\mu,\mathrm{ell}}(\tilde{\mathrm{H}}_1)_{\mathrm{Aut}(\underline{\mathrm{H}})}$ from $SD_{\mu,\mathrm{ell}}(\tilde{\mathrm{H}}_1)$ to its space of coinvariants for an action of a certain outer automorphism group $\mathrm{Aut}(\underline{\mathrm{H}})$ of H determined by the endoscopic datum $\underline{\mathrm{H}} \in \mathcal{E}(\tilde{\mathrm{M}}, \omega)$ (this group $\mathrm{Aut}(\underline{\mathrm{H}})$ is recalled in [MW16, I.1.5]), and these add together to give us an isomorphism of complex vector spaces :

$$\bigoplus_{\underline{\mathrm{H}} \in \mathcal{E}(\tilde{\mathrm{M}}, \mathbf{a})} \mathbf{T}_{\underline{\mathrm{H}}} = \bigoplus_{\underline{\mathrm{H}} \in \mathcal{E}(\tilde{\mathrm{M}}, \mathbf{a})} \mathbf{T}_{\underline{\mathrm{H}},\mathrm{ell}} : \bigoplus_{\underline{\mathrm{H}} \in \mathcal{E}(\tilde{\mathrm{M}}, \mathbf{a})} SD_{\mu,\mathrm{ell}}(\tilde{\mathrm{H}}_1)_{\mathrm{Aut}(\underline{\mathrm{H}})} \to D_{\mathrm{ell}}(\tilde{\mathrm{M}}, \omega). \tag{31}$$

For $\Theta \in D_{\mathrm{ell}}(\tilde{\mathrm{M}}, \omega)$ and $\underline{\mathrm{H}} \in \mathcal{E}(\tilde{\mathrm{M}}, \mathbf{a})$, we will let $\Theta^{\underline{\mathrm{H}}}$ denote the component of Θ along the subspace of $D_{\mathrm{ell}}(\tilde{\mathrm{M}}, \omega)$ obtained from the contribution of $\underline{\mathrm{H}}$ in the above decomposition of $D_{\mathrm{ell}}(\tilde{\mathrm{M}}, \omega)$.

Remark 3.1.3. We emphasize that, in (31), each $\underline{\mathrm{H}}$ contributes a different 'μ', i.e., the 'μ' of $SD_{\mu,\mathrm{ell}}(\tilde{\mathrm{H}}_1)_{\mathrm{Aut}(\underline{\mathrm{H}})}$ depends on $\underline{\mathrm{H}}$ as well. This dependence is suppressed from notation for lightness.

Remark 3.1.4. Fix $(\mathrm{M}, \tilde{\mathrm{M}}, \mathbf{a}), \omega, \underline{\mathrm{H}}$ and $(\mathrm{H}_1 \to \mathrm{H}, \hat{\xi}_1, \tilde{\mathrm{H}}_1 \to \tilde{\mathrm{H}}, \mathrm{C}_1, \mu)$ as in Notation 3.1.2, except that we do not yet assume that $\underline{\mathrm{H}}$ is elliptic.

(i) Suppose $\underline{\mathrm{H}}$ is elliptic and relevant. Let $(\mathrm{L}_{\mathrm{H}}, \tilde{\mathrm{L}}_{\mathrm{H}})$ be a Levi subspace of $(\mathrm{H}, \tilde{\mathrm{H}})$, and $(\mathrm{L}_1, \tilde{\mathrm{L}}_1)$ its inverse image in $(\mathrm{H}_1, \tilde{\mathrm{H}}_1)$. We now state the compatibility between parabolic induction and endoscopic transfer as follows.

- If $\mathrm{L}_{\mathrm{H}} \subset \mathrm{H}$ is not relevant in the sense described in [MW16, Section I.3.4], then under the endoscopic transfer map $SD_\mu(\tilde{\mathrm{H}}_1) \to D(\tilde{\mathrm{M}}, \omega)$, the image of any virtual character parabolically induced from $\tilde{\mathrm{L}}_1$ is zero.
- Suppose L_{H} is relevant in the sense described in [MW16, Section I.3.4]. Thus, [MW16, Section I.3.4] constructs a Levi subspace $(\mathrm{L}, \tilde{\mathrm{L}}) \subset (\mathrm{M}, \tilde{\mathrm{M}})$ and an elliptic relevant endoscopic datum $\underline{\mathrm{L}}_{\mathrm{H}}$ for $(\tilde{\mathrm{L}}, \mathbf{a}_{\tilde{\mathrm{L}}})$ with underlying group L_{H}, where $\mathbf{a}_{\tilde{\mathrm{L}}}$ is a cocycle representing the image of $\mathbf{a}$ in $H^1(W_F, \mathrm{Z}_{\hat{\mathrm{L}}})$. Then $(\mathrm{L}_1, \tilde{\mathrm{L}}_1, \mu)$ is part of a choice of auxiliary data for $\underline{\mathrm{L}}_{\mathrm{H}}$ obtained from those for $\underline{\mathrm{H}}$, as discussed in [MW16, Section I.3.3 or Section I.3.4]. For any virtual character $\Theta_1 \in SD_\mu(\tilde{\mathrm{L}}_1)$ parabolically inducing to $\mathrm{Ind}_{\tilde{\mathrm{L}}_1}^{\tilde{\mathrm{H}}_1} \Theta_1 =: \Theta_1^{\tilde{\mathrm{H}}_1} \in D_\mu(\tilde{\mathrm{H}}_1)$ and endoscopically transferring via $\underline{\mathrm{L}}_{\mathrm{H}}$ to $\Theta \in D(\tilde{\mathrm{L}}, \omega|_{\mathrm{L}(F)})$, $\Theta_1^{\tilde{\mathrm{H}}_1}$ belongs to $SD_\mu(\tilde{\mathrm{H}}_1)$, and its endoscopic transfer to $D(\tilde{\mathrm{M}}, \omega)$ under $\underline{\mathrm{H}}$ equals the parabolically induced character $\mathrm{Ind}_{\tilde{\mathrm{L}}}^{\tilde{\mathrm{M}}} \Theta =: \Theta^{\tilde{\mathrm{M}}}$.

These assertions are present in [MW16, Section I.4.11]. Slightly more precisely, recalling that parabolic induction is dual to the 'constant term' map, and using the assertion from the discussion below [MW16, Proposition I.4.11] that the image of the map [MW16, I.4.11(4)] is contained in the space denoted $I_+^{\mathcal{E}}(\tilde{G}(F), \omega)$ there, the former (resp., the latter) assertion follows from the condition (3) (resp., the condition (2)) in the definition of $I_+^{\mathcal{E}}(\tilde{G}(F), \omega)$ given at the beginning of [MW16, Section I.4.11].

(ii) For simplicity, we now assume that we are in the situation of standard endoscopy, and suppose that $\underline{\mathrm{H}}$ is not elliptic, i.e., the obvious injection $\mathrm{Z}_{\hat{\mathrm{M}}}^{\Gamma,0} \to \mathrm{Z}_{\hat{\mathrm{H}}}^{\Gamma,0}$ is not bijective. Thus, $\dim \mathrm{A_H} > \dim \mathrm{A_M}$, and it is easy to see that no elliptic strongly regular semisimple element of $\mathrm{M}(F)$ matches any semisimple element of $\mathrm{H}(F)$. Therefore, the image of the endoscopic transfer map $SD_\mu(\mathrm{H}_1) \to D(\mathrm{M})$ consists of virtual characters that vanish on the set $\mathrm{M}(F)_{\mathrm{ell}}$ of elliptic strongly regular elements of $\mathrm{M}(F)$. This implies (using a standard fact, (34) below) that this image is contained in the span of virtual characters that are fully induced from proper Levi subgroups of M.

3.2. **Unstable functions and stable characters on a non-quasi-split group.**

Notation 3.2.1. (i) In this subsection, given a connected reductive group M over F, we will denote by M^* its quasi-split inner form, and implicitly fix an inner twist $\psi_{\mathrm{M}^*} : \mathrm{M}^*_{\bar{F}} \to \mathrm{M}_{\bar{F}}$ from M^* to M unless otherwise specified. Note that ψ_{M^*} fixes an identification ${}^L\mathrm{M}^* = {}^L\mathrm{M}$, helping realize M^* as the endoscopic group underlying some $\underline{\mathrm{M}}^* \in \mathcal{E}(\mathrm{M})$, which is uniquely determined up to isomorphism. For $\underline{\mathrm{H}} = \underline{\mathrm{M}}^*$, we may and shall assume that the associated auxiliary data as in Notation 3.1.2(iii) satisfy $\mu = 1, \tilde{\mathrm{H}} = \mathrm{H}$ and $\tilde{\mathrm{H}}_1 = \mathrm{H}_1$, and identify $C^\infty_\mu(\mathrm{H}_1(F))$ with $C^\infty_c(\mathrm{M}^*(F))$, $SD_{\mu,\mathrm{ell}}(\tilde{\mathrm{H}}_1)$ with $SD_{\mathrm{ell}}(\mathrm{M}^*)$ etc. When we talk of endoscopic transfer between M and M^* (i.e., between $C^\infty_c(\mathrm{M}(F))$ and $C^\infty_c(\mathrm{M}^*(F))$ or the pull-back from $SD(\mathrm{M}^*)$ to $D(\mathrm{M})$), the reference will be to such a fixed endoscopic datum.

(ii) It is easy to see that the inner twist ψ_{M^*} fixed in (i) above identifies $\mathrm{A}_{\mathrm{M}^*}$, $\mathrm{Z}_{\mathrm{M}^*}$, $\mathrm{S}_{\mathrm{M}^*}$, $X^{\mathrm{unr}}(\mathrm{M}^*)$, $X^{\mathrm{unr-uni}}(\mathrm{M}^*)$ etc. with $\mathrm{A_M}, \mathrm{Z_M}, \mathrm{S_M}, X^{\mathrm{unr}}(\mathrm{M}), X^{\mathrm{unr-uni}}(\mathrm{M})$ etc. We will use this to transfer central characters, unramified characters etc. between $\mathrm{M}^*(F)$ and $\mathrm{M}(F)$.

(iii) Sometimes we will consider a 'variant with central character' of these notions: if $\mathrm{Z} \subset \mathrm{M}$ is a central subgroup and $\zeta : \mathrm{Z}(F) \to \mathbb{C}^\times$ is a unitary character, then endoscopic transfer also defines a map from $C^\infty_{\mathrm{Z},\zeta}(\mathrm{M}(F))$ to the quotient of $C^\infty_{\mathrm{Z},\zeta}(\mathrm{M}^*(F))$ by its subspace of unstable functions, where these function spaces are as in Notation 2.1.1(iv), and where Z is also viewed as a central subgroup of M^* as described in (ii) above. The map $SD_{\mathrm{Z},\zeta}(\mathrm{M}^*) \to SD_{\mathrm{Z},\zeta}(\mathrm{M})$ dual to this transfer (between $C^\infty_{\mathrm{Z},\zeta}(\mathrm{M}(F))$ and $C^\infty_{\mathrm{Z},\zeta}(\mathrm{M}^*(F))$) is also obtained by restricting the dual map $SD(\mathrm{M}^*) \to SD(\mathrm{M})$ for the transfer between $C^\infty_c(\mathrm{M}(F))$ and $C^\infty_c(\mathrm{M}^*(F))$.

(iv) Sometimes, we will give $\mathrm{M}(F)$ and $\mathrm{M}^*(F)$ measures that are compatible in the sense explained in [Kot88, page 631]: this means that, for some scalar $c > 0$ and some top-degree differential form ω on M defined over F, these measures are $c|\omega|$ and $c|(\psi_{\mathrm{M}^*})^*(\omega)|$.

(v) There is an injection from the set of $\mathrm{M}(F)$-conjugacy classes of Levi subgroups of M to the set of $\mathrm{M}^*(F)$-conjugacy classes of Levi subgroups of M^*, under which the conjugacy class of $\mathrm{M}_1 \subset \mathrm{M}$ maps to that of $\mathrm{M}_1^* \subset \mathrm{M}^*$ if and only if $\psi_{\mathrm{M}^*}((\mathrm{M}_1^*)_{\bar{F}})$ is $\mathrm{M}(\bar{F})$-conjugate to $(\mathrm{M}_1)_{\bar{F}}$, or equivalently, the bijection of Corollary 2.4.12 takes M_1^* and M_1 to the same conjugacy class of Levi subgroups of ${}^L\mathrm{M}^* = {}^L\mathrm{M}$ (this identification ${}^L\mathrm{M}^* = {}^L\mathrm{M}$ obtained from ψ_{M^*} or equivalently from $\underline{\mathrm{M}}^*$). Here, to make sense of the former description of this injection, we use Solleveld's result that conjugacy of Levi subgroups may be checked after base-changing to $\bar{F}$ (see [Sol20, Theorem A]). A Levi subgroup $\mathrm{M}_1^* \subset \mathrm{M}^*$ is said to be $\underline{\mathrm{M}}^*$-relevant if its conjugacy class lies in the image of this map; this agrees with the notion of relevance from [MW16, Section I.3.4], that we used earlier.

(vi) Now let $\mathrm{M}_1 \subset \mathrm{M}$ be a Levi subgroup, and consider inner twists ψ^* in $\psi_{\mathrm{M}^*} \circ \mathrm{Int}\,\mathrm{M}^*(\bar{F}) = \mathrm{Int}\,\mathrm{M}(\bar{F}) \circ \psi_{\mathrm{M}^*}$ such that $(\psi^*)^{-1}$ takes, for some or equivalently any parabolic subgroup $\mathrm{Q} \subset \mathrm{M}$ with Levi subgroup M_1, $(\mathrm{Q}_{\bar{F}}, (\mathrm{M}_1)_{\bar{F}})$ to $(\mathrm{Q}^*_{\bar{F}}, (\mathrm{M}_1^*)_{\bar{F}})$ for some parabolic-Levi pair $(\mathrm{Q}^*, \mathrm{M}_1^*)$ in M^*: to see that this condition is independent of Q, note that these are precisely the inner twists $\psi^* \in \psi_{\mathrm{M}^*} \circ \mathrm{Int}\,\mathrm{M}^*(\bar{F})$, that satisfy the property that ${}^\sigma\psi^* \circ (\psi^*)^{-1} \in \mathrm{Int}\,\mathrm{M}_1(\bar{F})$ for all $\sigma \in \mathrm{Gal}(\bar{F}/F)$, and hence satisfy the same property with Q replaced by any other parabolic subgroup $\mathrm{Q}' \subset \mathrm{M}$ with M_1 as a Levi subgroup. Given any such inner twist ψ^*, $(\psi^*)^{-1}((\mathrm{M}_1)_{\bar{F}})$ is of the form $(\mathrm{M}_1^*)_{\bar{F}}$ for some Levi subgroup $\mathrm{M}_1^* \subset \mathrm{M}^*$. Thus, any such ψ^* restricts to an inner twist $\psi_{\mathrm{M}_1^*}$ from such an M_1^* to M_1, realizing an endoscopic datum $\underline{\mathrm{M}}_1^*$ for M_1 with M_1^* as the unerlying group. Here is a second way to describe the resulting identification ${}^L\mathrm{M}_1 = {}^L\mathrm{M}_1^*$ up to $\mathrm{Int}\,\hat{\mathrm{M}}_1$-conjugacy. We can choose parabolic subgroups $\mathrm{Q} \subset \mathrm{M}$ and $\mathrm{Q}^* \subset \mathrm{M}^*$ with $\psi^*(\mathrm{Q}^*_{\bar{F}}) = \mathrm{Q}_{\bar{F}}$, so that Q and Q^* correspond to the conjugacy class of a common parabolic subgroup $\mathcal{Q} \subset {}^L\mathrm{M} = {}^L\mathrm{M}^*$. Choosing a Levi subgroup $\mathcal{L} \subset \mathcal{Q}$, we get using the pairs $(\mathrm{Q}, \mathrm{M}_1), (\mathrm{Q}^*, \mathrm{M}_1^*)$ and $(\mathcal{Q}, \mathcal{L})$ embeddings $\iota_{\mathrm{M},\mathrm{M}_1} : {}^L\mathrm{M}_1 \to {}^L\mathrm{M}$ and $\iota_{\mathrm{M}^*,\mathrm{M}_1^*} : {}^L\mathrm{M}_1^* \to {}^L\mathrm{M}^*$ with the same image $\mathcal{L}$, and using these embeddings, a realization of M_1^* as an elliptic endoscopic group of M_1, that can also be seen to agree with $\underline{\mathrm{M}}_1^*$. Henceforth, given a Levi subgroup $\mathrm{M}_1 \subset \mathrm{M}$ as above, we will often choose 'Levi subgroup matching data' consisting of a Levi subgroup $\mathrm{M}_1^* \subset \mathrm{M}^*$ together with an inner twist $\psi^* = \psi_{\mathrm{M}^*} \circ \mathrm{Int}\, m^*$ restricting to $\psi_{\mathrm{M}_1^*}$ as above, and the resulting realization $\underline{\mathrm{M}}_1^*$ of M_1^* as an elliptic endoscopic group of M_1. Sometimes, we will also fix auxiliary choices $\mathrm{Q}, \mathrm{Q}^*, \mathcal{Q}, \psi_{\mathrm{M}_1^*}$ etc. as above. This endoscopic datum and the resulting map $SD(\mathrm{M}_1^*) \to D(\mathrm{M}_1)$, as well as various isomorphisms such as the map $W(\mathrm{M}_1^*) \to W(\mathrm{M}_1)$ considered in (vii) below, depend on these auxiliary choices, but in a harmless way. In what follows, this dependence will be suppressed for lightness of notation.

(vii) Suppose $\mathrm{M}_1 \subset \mathrm{M}$ is a Levi subgroup, and a pair $(\mathrm{M}_1^*, \psi_{\mathrm{M}_1^*})$ is assigned to M_1 as in (vi) above. Let us study the impact of changing the choice of $(\mathrm{M}_1^*, \psi_{\mathrm{M}_1^*})$ to a different one, $((\mathrm{M}_1^*)', \psi_{(\mathrm{M}_1^*)'})$. Choose a parabolic subgroup $\mathrm{Q} \subset \mathrm{M}$ with M_1 as a Levi subgroup, and note that $(\mathrm{M}_1^*, \psi_{\mathrm{M}_1^*}^{-1}(\mathrm{Q}_{\bar{F}}))$ and $((\mathrm{M}_1^*)', \psi_{(\mathrm{M}_1^*)'}^{-1}(\mathrm{Q}_{\bar{F}}))$ are obtained by base-change from conjugate parabolic pairs in M^*. It follows that $\psi_{(\mathrm{M}_1^*)'} = \psi_{\mathrm{M}_1^*} \circ \mathrm{Int}(m_1^* w)$ for some $w \in \mathrm{M}^*(F)$ transporting $(\mathrm{M}_1^*)'$ to M_1^* and some $m_1^* \in \mathrm{M}_1^*(\bar{F})$. It is then easy to see that the identifications ${}^L\mathrm{M}_1 = {}^L\mathrm{M}_1^*$ and ${}^L\mathrm{M}_1 = {}^L(\mathrm{M}_1')^*$ as in (vi) differ from each other by the isomorphism ${}^L\mathrm{M}_1^* = {}^L(\mathrm{M}_1')^*$ that is dual to $\mathrm{Int}\, w : (\mathrm{M}_1')^* \to \mathrm{M}_1^*$.

(viii) It is easy to see that $\psi_{\mathrm{M}_1^*}$, though not defined over F, induces an (F-)isomorphism $W(\mathrm{M}_1) \cong W(\mathrm{M}_1^*)$ between the Weyl groups of M_1 in M and M_1^* in M^*, where $W(\mathrm{M}_1^*)$ and $W(\mathrm{M}_1)$ are described in terms of $\bar{F}$-points using the discussion of Subsubsection 2.1.7.

Remark 3.2.2. Let M be a connected reductive group over F. We collect a few useful facts concerning the endoscopic transfer between M and M^* (see Notation 3.2.1).

(i) The transfer factors between M^* and M can be normalized such that, if the stable conjugacy classes of $\gamma^* \in \mathrm{M}^*(F)$ and $\gamma \in \mathrm{M}(F)$ correspond to each other, then $\Delta(\gamma^*, \gamma) = 1$ (while this surely exists somewhere in the literature, since we have managed to not be able to locate a reference, here is a summary: as per [LS87], the transfer factor Δ_I and the relative transfer factor $\Delta_1 = \Delta_{III,1}$ are trivial because the element 's' in the endoscopic datum is the identity, the transfer factors Δ_{II} and Δ_{IV} are trivial because all roots of M come from M^*, and the transfer factor $\Delta_2 = \Delta_{III,2}$ is trivial because, in the notation of [LS87, (3.5)], we have $\xi \circ \xi_{T_H} = \xi_T$).

(ii) (i), together with the fact that the set of stable conjugacy classes of strongly regular semisimple elements of $\mathrm{M}(F)$ injects into the analogous set for $\mathrm{M}^*(F)$ under the matching of semisimple elements in the theory of endoscopy (see [Kot82, Section 6]), implies that:
- If $\mathrm{Z} \subset \mathrm{M}$ is a central torus and $\zeta : \mathrm{Z}(F) \to \mathbb{C}^\times$ is a unitary character, then a function $f \in C_c^\infty(\mathrm{M}(F))$ (resp., $C_{\mathrm{Z},\zeta}^\infty(\mathrm{M}(F))$) is unstable if and only if some or equivalently any endoscopic transfer $f^* \in C_c^\infty(\mathrm{M}^*(F))$ (resp., $f^* \in C_{\mathrm{Z},\zeta}^\infty(\mathrm{M}^*(F))$) of f to $\mathrm{M}^*(F)$ is unstable.
- At the level of distributions, it follows that endoscopic transfer takes stable distributions on $\mathrm{M}^*(F)$ to stable distributions on $\mathrm{M}(F)$, and $SD(\mathrm{M}^*)$ to $SD(\mathrm{M}) \subset D(\mathrm{M})$. Restricting to $SD_{\mathrm{ell}}(\mathrm{M})$ and using [MW16, Theorem XI.4], it also induces a map $SD_{\mathrm{ell}}(\mathrm{M}) \to SD_{\mathrm{ell}}(\mathrm{M}^*)$.

(iii) The compatibility between parabolic induction and endoscopic transfer (see Remark 3.1.4) simplifies in this situation. Let $\mathrm{M}_1^* \subset \mathrm{M}^*$ be a Levi subgroup, and $\Theta^* \in SD(\mathrm{M}_1^*)$ a stable tempered character on $\mathrm{M}_1^*(F)$. First, $\mathrm{Ind}_{\mathrm{M}_1^*}^{\mathrm{M}^*} \Theta^*$ is then a stable tempered character on $\mathrm{M}^*(F)$, so it transfers to a distribution on $\mathrm{M}(F)$ under $SD(\mathrm{M}^*) \to SD(\mathrm{M})$. There turn out to be two cases, depending on whether or not $\mathrm{M}_1^* \subset \mathrm{M}^*$ is $\underline{\mathrm{M}}^*$-relevant.

- If M_1^* is not $\underline{\mathrm{M}}^*$-relevant, the assertion is that the image of $\mathrm{Ind}_{\mathrm{M}_1^*}^{\mathrm{M}^*}\Theta^*$ under $SD(\mathrm{M}^*) \to SD(\mathrm{M})$ is 0.
- Suppose M_1^* is relevant, and let the Levi subgroup $\mathrm{M}_1 \subset \mathrm{M}$ and various auxiliary choices be as in Notation 3.2.1(vi). The assertion in this case is that Θ^* transfers to some tempered character Θ on $\mathrm{M}_1(F)$ under the resulting transfer $SD(\mathrm{M}_1^*) \to SD(\mathrm{M}_1)$, and moreover, the stable character $\mathrm{Ind}_{\mathrm{M}_1^*}^{\mathrm{M}^*}\Theta^*$ transfers to $\mathrm{Ind}_{\mathrm{M}_1}^{\mathrm{M}}\Theta$ under the transfer $SD(\mathrm{M}^*) \to SD(\mathrm{M})$, independently of the auxiliary choices of parabolic subgroups involved in Notation 3.2.1(vi).

(iv) Let Levi subgroups $\mathrm{M}_1^* \subset \mathrm{M}^*$ and $\mathrm{M}_1 \subset \mathrm{M}$ and Levi subgroup matching data be chosen as in Notation 3.2.1(vi). It is now easy from the definitions in [LS87] that the transfer of stable conjugacy classes from M_1^* to M_1, and hence by (i) also the endoscopic transfer map $SD(\mathrm{M}_1^*) \to SD(\mathrm{M}_1)$, respects conjugacy under $W(\mathrm{M}_1) = W(\mathrm{M}_1^*)$.

Lemma 3.2.3. *Let* M *be a connected reductive group over* F*. Then:*

(i) The map $SD(\mathrm{M}^*) \to SD(\mathrm{M})$ *respects 'central characters', i.e., the eigendecomposition with respect to* $\mathrm{Z}_{\mathrm{M}}(F) = \mathrm{Z}_{\mathrm{M}^*}(F) \supset \mathrm{A}_{\mathrm{M}^*}(F) = \mathrm{A}_{\mathrm{M}}(F)$*, as well as twisting by* $X^{\mathrm{unr-uni}}(\mathrm{M}^*) = X^{\mathrm{unr-uni}}(\mathrm{M})$ *(see Notation 3.2.1(ii) for these identifications).*

(ii) Let $\mathrm{M}_1^* \subset \mathrm{M}^*, \mathrm{M}_1 \subset \mathrm{M}$ *be as in Notation 3.2.1(vi). Let* $\mathcal{O}'_{\mathrm{M}_1} \subset \mathrm{Aut}(\mathrm{M}_1), \mathcal{O}'_{\mathrm{M}_1^*} \subset \mathrm{Aut}(\mathrm{M}_1^*)$ *be subgroups with the same image* $\bar{\mathcal{O}}'_{\mathrm{M}_1} = \bar{\mathcal{O}}'_{\mathrm{M}_1^*}$ *in* $\mathrm{Out}(\mathrm{M}_1) = \mathrm{Out}(\mathrm{M}_1^*)$ *(e.g., we could have* $\bar{\mathcal{O}}'_{\mathrm{M}_1} = W(\mathrm{M}_1)$ *and* $\bar{\mathcal{O}}'_{\mathrm{M}_1^*} = W(\mathrm{M}_1^*)$*, by the discussion in Notation 3.2.1(viii)). Then the transfer of stable conjugacy classes from* $\mathrm{M}_1^*(F)$ *to* $\mathrm{M}_1(F)$*, as well as the endoscopic transfer map* $SD(\mathrm{M}_1^*) \to SD(\mathrm{M}_1)$*, are equivariant under* $\bar{\mathcal{O}}'_{\mathrm{M}_1} = \bar{\mathcal{O}}'_{\mathrm{M}_1^*}$ *(through which the actions of* $\mathcal{O}'_{\mathrm{M}_1}$ *and* $\mathcal{O}'_{\mathrm{M}_1^*}$ *clearly factor).*

Remark 3.2.4. Of course, one can prove a more general version of (i) of the above lemma, involving twisting by a group of characters that is larger than $X^{\mathrm{unr-uni}}(\mathrm{M})$, but we will not need it.

Proof of Lemma 3.2.3. These assertions being well-known (part of (i) was used in Notation 3.2.1(iii)), we will only sketch the proof. In what follows, we will use that the 'correspondence' of semisimple conjugacy classes between $\mathrm{M}(F)$ and $\mathrm{M}^*(F)$ has the following easy description: the conjugacy classes of semisimple elements $m \in \mathrm{M}(F)$ and $m^* \in \mathrm{M}^*(F)$ correspond if and only if m is $\mathrm{M}(\bar{F})$-conjugate to $\psi(m^*)$.

The assertion in (i) concerning central characters follows from Remark 3.2.2(i) together with the easy observation that the left-regular action of $\mathrm{Z}_{\mathrm{M}}(F) = \mathrm{Z}_{\mathrm{M}^*}(F)$ respects the transfer of stable conjugacy classes from $\mathrm{M}^*(F)$ to $\mathrm{M}(F)$. For the assertion in (i) concerning twisting by $X^{\mathrm{unr-uni}}(\mathrm{M}^*) = X^{\mathrm{unr-uni}}(\mathrm{M})$, combine Remark 3.2.2(i) with the easy observation that the inner twist ψ_{M^*} gives us an identification of the map $\mathrm{M}^* \to \mathrm{S}_{\mathrm{M}^*}$ with the map $\mathrm{M} \to \mathrm{S}_{\mathrm{M}}$ that is manifestly compatible with the transfer of stable conjugacy classes. For (ii), note that if $\beta \in \mathcal{O}'_{\mathrm{M}_1}$ and $\beta^* \in \mathcal{O}'_{\mathrm{M}_1^*}$ have the same image in $\bar{\mathcal{O}}'_{\mathrm{M}_1} = \bar{\mathcal{O}}'_{\mathrm{M}_1^*}$, then strongly regular semisimple elements $m \in \mathrm{M}_1(F)$ and $m^* \in \mathrm{M}_1^*(F)$ match if and only if $\beta(m)$ and $\beta^*(m^*)$ do; now use Remark 3.2.2(i). □

Remark 3.2.5. Let $\mathrm{M}_1^*, (\mathrm{M}_1')^*$ etc. and $w \in \mathrm{M}(F)$ be as in the setting of Notation 3.2.1(vii), from where we recall that the identifications ${}^L\mathrm{M}_1 = {}^L\mathrm{M}_1^*$ and ${}^L\mathrm{M}_1 = {}^L(\mathrm{M}_1')^*$ differ by an isomorphism ${}^L\mathrm{M}_1^* \to {}^L(\mathrm{M}_1')^*$ dual to $\mathrm{Int}\, w$. On the other hand, it is easy to see from Remark 3.2.2(i), as in the proof of the assertions of Lemma 3.2.3, that the maps $SD(\mathrm{M}_1^*) \to SD(\mathrm{M}_1)$ and $SD((\mathrm{M}_1')^*) \to SD(\mathrm{M}_1)$, and hence their restrictions $SD_{\mathrm{ell}}(\mathrm{M}_1^*) \to SD_{\mathrm{ell}}(\mathrm{M}_1)$ and $SD_{\mathrm{ell}}((\mathrm{M}_1')^*) \to SD_{\mathrm{ell}}(\mathrm{M}_1)$, differ by $\mathrm{Int}\, w$.

Proposition 3.2.6. *Let* M *be a connected reductive group over* F*, and let* $D \in D_{\mathrm{ell}}(\mathrm{M})$ *have the property that its restriction to the set* $\mathrm{M}(F)_{\mathrm{ell,srss}} \subset \mathrm{M}(F)$ *consisting of elliptic strongly regular semisimple elements is stable — in other words, recalling that* D *can be viewed as a function* $\mathrm{M}(F) \to \mathbb{C}$ *that is locally constant on the set* $\mathrm{M}(F)_{\mathrm{srss}}$ *of strongly regular semisimple elements of*

$\mathrm{M}(F)$ *and locally integrable on* $\mathrm{M}(F)$, $D(\gamma) = D(\gamma')$ *whenever* $\gamma, \gamma' \in \mathrm{M}(F)_{\mathrm{ell,srss}}$ *are such that* γ' *is* $\mathrm{M}(\bar{F})$*-conjugate to* γ. *Then:*

(i) D *is stable, i.e.,* $D \in SD_{\mathrm{ell}}(\mathrm{M})$.

(ii) D *is the transfer of a stable distribution* $D^* \in SD_{\mathrm{ell}}(\mathrm{M}^*)$ *(in the sense of Notation 3.2.1).*

Proof. In the case where M is quasi-split, (i) (and hence trivially also (ii)) is a well-known result of Arthur; see [MW16, Theorem XI.3]. The general case can be easily deduced from this and some standard facts, as we will see now.

By the discussion in Remark 3.2.2(ii), (ii) implies (i), so it suffices to prove (ii).

As before, we will write $\underline{\mathrm{H}}$ for a typical element of $\mathcal{E}(\mathrm{M})$ and, given $\underline{\mathrm{H}}$, H for the corresponding endoscopic group. (31) specializes to an isomorphism:

$$\bigoplus_{\underline{\mathrm{H}} \in \mathcal{E}(\mathrm{M})} \mathbf{T}_{\underline{\mathrm{H}}} : \bigoplus_{\underline{\mathrm{H}} \in \mathcal{E}(\mathrm{M})} SD_{\mu,\mathrm{ell}}(\mathrm{H}_1)_{\mathrm{Aut}\underline{\mathrm{H}}} \to D_{\mathrm{ell}}(\mathrm{M}). \tag{32}$$

Recall that, when $\underline{\mathrm{H}}$ equals the endoscopic datum $\underline{\mathrm{M}}^* \in \mathcal{E}(\mathrm{M})$ as in Notation 3.2.1, we identify the factor $SD_{\mu,\mathrm{ell}}(\mathrm{H}_1)_{\mathrm{Aut}\underline{\mathrm{H}}}$ with $SD(\mathrm{H})_{\mathrm{Aut}\underline{\mathrm{H}}} = SD_{\mathrm{ell}}(\mathrm{M}^*)$. It suffices to show that D belongs to $\mathbf{T}_{\underline{\mathrm{M}}^*}(SD_{\mathrm{ell}}(\underline{\mathrm{M}}^*))$ under (32).

For this, we recall more specific details on the realization of (32). In this proof, write $C^\infty_{c,\mathrm{cusp}}(\mathrm{M}(F)) \subset C^\infty_c(\mathrm{M}(F))$ for the subspace consisting of cuspidal functions in the sense of [MgW18, Section 7.1], i.e., whose nonelliptic strongly regular semisimple orbital integrals all vanish. Let $\mathcal{I}_{\mathrm{cusp}}(\mathrm{M})$ be the quotient of $C^\infty_{c,\mathrm{cusp}}(\mathrm{M}(F))$ by the subspace consisting of those functions all of whose strongly regular semisimple orbital integrals vanish; this agrees with the notation in [MW16, towards the end of I.3.1]. Similarly, by [MW16, towards the end of Section I.3.1 and towards the end of Section I.2.5], for each $\underline{\mathrm{H}} \in \mathcal{E}(\mathrm{M})$, we have a space $\mathcal{SI}_{\mathrm{cusp}}(\underline{\mathrm{H}})$, a space of stable orbital integrals for functions, not on $\mathrm{H}(F)$, but lying in a space $C^\infty_\mu(\mathrm{H}_1(F))$ associated to a fixed choice of auxiliary data as in Notation 3.1.2(iii), which we now make.

By [Art96, Proposition 3.5], or by [MW16, Proposition I.4.11], as invoked in [LMW18, Section 4.4, (3)], endoscopic transfer from M along the $\underline{\mathrm{H}}$, as $\underline{\mathrm{H}}$ varies over $\mathcal{E}(\mathrm{M})$, descends to an isomorphism of vector spaces:

$$\mathcal{I}_{\mathrm{cusp}}(\mathrm{M}) \xrightarrow{\cong} \bigoplus_{\underline{\mathrm{H}} \in \mathcal{E}(\mathrm{M})} \mathcal{SI}_{\mathrm{cusp}}(\underline{\mathrm{H}})^{\mathrm{Aut}(\underline{\mathrm{H}})}, \tag{33}$$

where $\mathrm{Aut}(\underline{\mathrm{H}})$ is as in (31) (implicit in this isomorphism is the assertion that, if the orbital integrals of $f \in C^\infty_c(\mathrm{M}(F))$ at strongly regular nonelliptic semisimple elements of $\mathrm{M}(F)$ all vanish, then for any $\underline{\mathrm{H}} \in \mathcal{E}(\mathrm{M})$, the stable orbital integrals of any transfer $f^{\underline{\mathrm{H}}} \in C^\infty_\mu(\mathrm{H}_1(F))$ of f satisfy a similar property).

We have a map $D_{\mathrm{ell}}(\mathrm{M}) \to \mathrm{Hom}_{\mathbb{C}}(\mathcal{I}_{\mathrm{cusp}}(\mathrm{M}), \mathbb{C})$, obtained by restricting an element of $D_{\mathrm{ell}}(\mathrm{M})$ to the space $C^\infty_{c,\mathrm{cusp}}(\mathrm{M}(F)) \subset C^\infty_c(\mathrm{M}(F))$. As recalled in [LMW18, Section 4.3, a bit below (5)], this map lets us identify $D_{\mathrm{ell}}(\mathrm{M})$ as a linear subspace of $\mathrm{Hom}_{\mathbb{C}}(\mathcal{I}_{\mathrm{cusp}}(\mathrm{M}), \mathbb{C})$. A similar prescription identifies $SD_{\mu,\mathrm{ell}}(\mathrm{H}_1)$ with a linear subspace of $\mathrm{Hom}_{\mathbb{C}}(\mathcal{SI}_{\mathrm{cusp}}(\underline{\mathrm{H}}), \mathbb{C})$, for each $\underline{\mathrm{H}} \in \mathcal{E}(\mathrm{M})$. Moreover, with these identifications, as explained in [LMW18, Section 4.4, (4)], (32) is obtained by restricting the isomorphism

$$\mathrm{Hom}_{\mathbb{C}}(\mathcal{I}_{\mathrm{cusp}}(\mathrm{M}), \mathbb{C}) \xrightarrow{\cong} \bigoplus_{\underline{\mathrm{H}} \in \mathcal{E}(\mathrm{M})} \mathrm{Hom}_{\mathbb{C}}(\mathcal{SI}_{\mathrm{cusp}}(\mathrm{H})^{\mathrm{Aut}(\underline{\mathrm{H}})}, \mathbb{C})$$

obtained by applying $\mathrm{Hom}_{\mathbb{C}}(-, \mathbb{C})$ to (33) (because $\mathbf{T}_{\underline{\mathrm{H}}}$ is dual to endoscopic transfer).

Using this and the fact that (33) is an isomorphism, it now suffices to show that D, viewed inside $\mathrm{Hom}_{\mathbb{C}}(\mathcal{I}_{\mathrm{cusp}}(\mathrm{M}), \mathbb{C})$, factors as the composite of the projection $\mathcal{I}_{\mathrm{cusp}}(\mathrm{M}) \to \mathcal{SI}_{\mathrm{cusp}}(\underline{\mathrm{M}}^*)^{\mathrm{Aut}(\underline{\mathrm{M}}^*)}$ and some element of $\mathrm{Hom}_{\mathbb{C}}(\mathcal{SI}_{\mathrm{cusp}}(\underline{\mathrm{M}}^*)^{\mathrm{Aut}(\underline{\mathrm{M}}^*)}, \mathbb{C})$ (a priori not necessarily the one obtained from D using (32)). Thus, by Remark 3.2.2(i), it suffices to show that if $f \in C^\infty_{c,\mathrm{cusp}}(\mathrm{M}(F))$, then $D(f)$ depends only on the set of stable orbital integrals of f at strongly regular semisimple elements of $\mathrm{M}(F)$. This follows from the hypothesis on D (that its restriction to the set of elliptic strongly regular semisimple elements is stable), the fact that f belongs to $C^\infty_{c,\mathrm{cusp}}(\mathrm{M}(F))$, and the Weyl integration formula. □

Corollary 3.2.7. *Let* M *be a connected reductive group over F. The transfer $SD(\mathrm{M}^*) \to D(\mathrm{M})$, in the sense of Notation 3.2.1, takes $SD_{\mathrm{ell}}(\mathrm{M}^*)$ isomorphically onto $SD_{\mathrm{ell}}(\mathrm{M})$.*

Proof. The map $SD_{\mathrm{ell}}(\mathrm{M}^*) \to SD_{\mathrm{ell}}(\mathrm{M})$ is injective, since (32) isi an isomorphism, and since $\mathrm{Aut}(\underline{\mathrm{M}}^*)$ is trivial (use the identification ${}^L\mathrm{M}^* = {}^L\mathrm{M}$). The surjectivity of this map $SD_{\mathrm{ell}}(\mathrm{M}^*) \to SD_{\mathrm{ell}}(\mathrm{M})$ follows from Proposition 3.2.6. □

The above proof has the following corollary, in which $W(\mathrm{M})$ is as in Subsubsection 2.1.7:

Proposition 3.2.8. *Let $\tilde{\mathcal{L}}$ denote the set of all Levi subgroups of* G*. Then, inside the space:*

$$D(\mathrm{G}) = \bigoplus_{\mathrm{M} \in \tilde{\mathcal{L}}/\mathrm{G}(F)} \mathrm{Ind}_{\mathrm{M}}^{\mathrm{G}} D_{\mathrm{ell}}(\mathrm{M})^{W(\mathrm{M})} \tag{34}$$

(this identification is defined by parabolic induction; for a proof, see [MgW18, Proposition 2.12]*), we have compatibly an equality*

$$SD(\mathrm{G}) = \bigoplus_{\mathrm{M} \in \tilde{\mathcal{L}}/\mathrm{G}(F)} \mathrm{Ind}_{\mathrm{M}}^{\mathrm{G}} SD_{\mathrm{ell}}(\mathrm{M})^{W(\mathrm{M})}. \tag{35}$$

Moreover we also have the following compatible equality, where we recall our abbreviation $\mathcal{O} = \mathcal{O}_{\mathrm{G}}$:

$$SD(\mathrm{G})^{\mathcal{O}} = \bigoplus_{\mathrm{M} \in \tilde{\mathcal{L}}/\mathcal{O}_{\mathrm{G}}^+} \mathrm{Avg}_{\mathcal{O}} \left(\mathrm{Ind}_{\mathrm{M}}^{\mathrm{G}} SD_{\mathrm{ell}}(\mathrm{M})^{\mathcal{O}_{\mathrm{M}}} \right),$$

where $\mathrm{Avg}_{\mathcal{O}}$ refers to averaging with respect to the action of $\mathcal{O}$ (which makes sense as $\mathcal{O}$ acts via a finite quotient; see the proof of Lemma 2.6.3(ii)).

Proof. It is easy to deduce the latter assertion from the former, noting that

$$o \cdot \mathrm{Ind}_{\mathrm{M}}^{\mathrm{G}} SD_{\mathrm{ell}}(\mathrm{M})^{W(\mathrm{M})} = o \cdot \mathrm{Ind}_{\mathrm{M}}^{\mathrm{G}} SD_{\mathrm{ell}}(\mathrm{M}) = \mathrm{Ind}_{o\cdot\mathrm{M}}^{\mathrm{G}} SD_{\mathrm{ell}}(o \cdot \mathrm{M}) = \mathrm{Ind}_{o\cdot\mathrm{M}}^{\mathrm{G}} SD_{\mathrm{ell}}(o \cdot \mathrm{M})^{W(o\cdot\mathrm{M})},$$

for each $o \in \mathcal{O}_{\mathrm{G}}^+$ and each $\mathrm{M} \in \tilde{\mathcal{L}}$. Therefore let us prove the former.
As observed in [MW16, VIII.2.4] and [LMW18, Remark 3.4, around (2)], when M is quasi-split, this assertion (and even a twisted version of it) follows from [MW16, Corollary XI.3.1]. What we describe will be essentially the proof in [MgW18, Corollary XI.3.1], with only a slight variance, so we will be brief.
The inclusion '$\supset$' is immediate, since parabolic induction preserves the stability of virtual characters (a convenient reference for which is [KV16, Corollary 6.13]). To prove the inclusion '$\subset$', fix $\Theta \in SD(G)$, and, using fixed representatives for $\tilde{\mathcal{L}}/\mathrm{G}(F)$, chosen so as to contain a common minimal Levi subgroup, write $\Theta = \sum_{\mathrm{M}} \mathrm{Ind}_{\mathrm{M}}^{\mathrm{G}} \Theta_{\mathrm{M}}$ according to the decomposition in (34). It is enough to show that the element Θ_{M} of $D_{\mathrm{ell}}(\mathrm{M})$ is stable for each M. By an easy induction argument involving $\tilde{\mathcal{L}}$, partially ordered under reverse inclusion up to conjugacy (see [MgW18, the proof of Corollary XI.3.1]), we may assume that for some fixed $L \in \tilde{\mathcal{L}}$, $\Theta_{\mathrm{M}} = 0$ if M contains a conjugate of L properly, and then prove that Θ_{L} is stable. If $\mathrm{M} \in \tilde{\mathcal{L}}$ is such that $g\gamma g^{-1} \in \mathrm{M}(F)$ for some $g \in \mathrm{G}(F)$ and some γ in the set $\mathrm{L}(F)_{\mathrm{ell,srss}}$ of elliptic strongly regular semisimple elements of $\mathrm{L}(F)$, then $g^{-1}\mathrm{M}g \supset \mathrm{L}$ by hypothesis (because A_{L} equals the maximal split torus in the centralizer of γ by ellipticity, and hence contains $g^{-1}\mathrm{A}_{\mathrm{M}}g$), and hence $\Theta_{\mathrm{M}} = 0$ unless $\mathrm{M} = \mathrm{L}$. Using this, the fact that Θ_{L} was chosen to be fixed under $W(\mathrm{L})$, and van Dijk's formula for induced characters ([vD72, Theorem 3], which takes a particularly simple form at elliptic elements of the Levi subgroup under consideration), it follows that up to a ratio of discriminant factors, which is invariant under $W(\mathrm{L})$ and under stable conjugacy, Θ equals a multiple of Θ_{L} on $\mathrm{L}(F)_{\mathrm{ell,srss}}$. Thus, Θ_{L} is stable when restricted to $\mathrm{L}(F)_{\mathrm{ell,srss}}$, in the sense explained in Proposition 3.2.6. Hence Proposition 3.2.6 implies that Θ_{L} is stable, as desired. □

Corollary 3.2.9. *Let* M *be a connected reductive group over F. The transfer $SD(\mathrm{M}^*) \to D(\mathrm{M})$, in the sense of Notation 3.2.1, takes $SD(\mathrm{M}^*)$ surjectively onto $SD(\mathrm{M})$.*

Proof. In view of the compatibility between endoscopic transfer and parabolic induction (see Remark 3.2.2(iii)), this follows from Proposition 3.2.8 and Corollary 3.2.7. □

Proposition 3.2.10. *Let* M *be a connected reductive group over* F*. Let* $\mathrm{Z} \subset \mathrm{M}$ *be a central induced torus, and* $\zeta : \mathrm{Z}(F) \to \mathbb{C}^\times$ *a smooth unitary character. Recall the space* $C^\infty_{\mathrm{Z},\zeta}(\mathrm{M}(F))$*, and for each Levi subgroup* $\mathrm{L} \subset \mathrm{M}$*, the space* $SD_{\mathrm{Z},\zeta,\mathrm{ell}}(\mathrm{L}) := SD_{\mathrm{Z}(F),\zeta,\mathrm{ell}}(\mathrm{L})$ *(see Notation 2.1.1). Suppose* $f \in C^\infty_{\mathrm{Z},\zeta}(\mathrm{M}(F))$ *has the property that* $\left(\mathrm{Ind}^{\mathrm{M}}_{\mathrm{L}} \Theta\right)(f) = 0$ *for every Levi subgroup* $\mathrm{L} \subset \mathrm{M}$ *and every stable elliptic virtual character* $\Theta \in SD_{\mathrm{Z},\zeta,\mathrm{ell}}(\mathrm{L})$*. Then* f *is unstable.*

Remark 3.2.11. If M is quasi-split, this is immediate from [Art96, Lemma 6.3], as explained in [LM20, page 587]. When Z is trivial, an alternative reference that is more convenient to cross-check (in this quasi-split case) is the combination of [MW16, Corollary XI.5.2(i)] and the description in [MW16, Corollary XI.3.1], invoked earlier, of the space $SD(\mathrm{M})$.

Proof of Proposition 3.2.10. We choose a quasi-split form M^* of M, and fix an endoscopic datum $\underline{\mathrm{M}}^*$ and an inner twist from M^* to M as in Notation 3.2.1(i). By Remark 3.2.2(ii), it is enough to show that any transfer $f^* \in C^\infty_{\mathrm{Z},\zeta}(\mathrm{M}^*(F))$ of f to the quasi-split form M^* of M is unstable (recall from Notation 3.2.1(iii) that Z is viewed, using the fixed inner twist, as a subgroup of M^* as well). The proposition being already known with M replaced by the quasi-split group M^* ([Art96, Lemma 6.3] — here we use that Z is an induced torus), it suffices to show that for any Levi subgroup $\mathrm{L}^* \subset \mathrm{M}^*$ and any $\Theta \in SD_{\mathrm{Z},\zeta,\mathrm{ell}}(\mathrm{L}^*)$, we have $(\mathrm{Ind}^{\mathrm{M}^*}_{\mathrm{L}^*} \Theta)(f^*) = 0$. This follows from the hypothesis of the proposition together with the fact that endoscopic transfer between M and M^* is compatible with parabolic induction (see Remark 3.2.2(iii)), as well as with the central character condition involving (Z, ζ) (see the discussion in Notation 3.2.1(iii)), and takes stable virtual characters to stable virtual characters. □

The following corollary will be useful later:

Corollary 3.2.12. *Assume Hypothesis 2.7.1. Suppose* $f \in C^\infty_c(\mathrm{G}(F))$ *has* $\mathcal{O}$*-invariant image in the space* $\mathcal{I}(\mathrm{G})$ *of coinvariants for the* $\mathrm{G}(F)$*-conjugation action on* $C^\infty_c(\mathrm{G}(F))$*, and suppose that* $D(f) = 0$ *whenever* D *is a virtual character on* $\mathrm{G}(F)$ *obtained by* $\mathcal{O}$*-averaging* $\mathrm{Ind}^{\mathrm{G}}_{\mathrm{M}} \Theta$ *for some Levi subgroup* $\mathrm{M} \subset \mathrm{G}$ *and some* $\Theta \in SD_{\mathrm{ell}}(\mathrm{M})^{\mathcal{O}_{\mathrm{M}}}$*. Then* f *is unstable.*

Proof. Let $\mathrm{M} \subset \mathrm{G}$ be a Levi subgroup, and let $\Theta' \in SD_{\mathrm{ell}}(\mathrm{M})$. By Proposition 3.2.10, it suffices to show that $D'(f) = 0$, where $D' = \mathrm{Ind}^{\mathrm{G}}_{\mathrm{M}} \Theta'$. Let $\Theta \in SD_{\mathrm{ell}}(\mathrm{M})^{\mathcal{O}_{\mathrm{M}}}$ be the $\mathcal{O}_{\mathrm{M}}$-average of Θ', and let $D_0 = \mathrm{Ind}^{\mathrm{G}}_{\mathrm{M}} \Theta$.
Let D be the $\mathcal{O}$-average of D'. Since elements of $\mathcal{O}_{\mathrm{M}}$ are obtained by restricting from $\mathcal{O}^+_{\mathrm{G}}$ (by (iv)b of Notation 2.6.1), and since $\mathcal{O}$ and $\mathcal{O}^+_{\mathrm{G}}$ have the same orbit on the space of invariant distributions on $\mathrm{G}(F)$, D is also the $\mathcal{O}$-average of $D_0 = \mathrm{Ind}^{\mathrm{G}}_{\mathrm{M}} \Theta$, so that $D(f) = 0$. Therefore, using that f has $\mathcal{O}$-invariant image in $\mathcal{I}(\mathrm{G})$, we have $D'(f) = D(f) = 0$, as desired. □

Later, we will need the following variant of Proposition 3.2.10.

Proposition 3.2.13. *Suppose* H_1 *is a quasi-split reductive group over* F*, and* $\tilde{\mathrm{H}}_1$ *is a twisted space over* H_1 *with the property that for all* $\gamma_1 \in \tilde{\mathrm{H}}_1(F)$*, the automorphism* $\mathrm{Int}\,\gamma_1$ *of* H_1 *is inner in the sense of being given by conjugation under an element of* $\mathrm{H}_{1,\mathrm{ad}}(F)$*. Assume further that* $\tilde{\mathrm{H}}_1(F) \neq \emptyset$*. Let* $\mathrm{C}_1 \subset \mathrm{H}_1$ *be a central induced torus, and* $\mu : \mathrm{C}_1(F) \to \mathbb{C}^\times$ *a unitary character. Suppose* $f_1 \in C^\infty_\mu(\tilde{\mathrm{H}}_1(F))$ *has the property that* $\Theta_1(f_1) = 0$ *for all* $\Theta_1 \in SD_\mu(\tilde{\mathrm{H}}_1)$*. Then* f_1 *is unstable.*

Proof. Since $\tilde{\mathrm{H}}_1(F) \neq \emptyset$, by [MW16, Proposition III.2.3], one has an embedding $\mathrm{H}_1 \hookrightarrow \mathrm{H}_2$ of H_1 into a quasi-split reductive group H_2 with the same derived group as H_1, and with the property that, as an H_1-bitorsor, $\tilde{\mathrm{H}}_1$ can be identified with a coset of H_1 in H_2 (and hence with a fiber of $\mathrm{H}_2 \to \mathrm{H}_2/\mathrm{H}_1$). We can find a compact (usually not open) subgroup Z_2 of $\mathrm{Z}_{\mathrm{H}_2}(F)$, containing the identity, with the property that the multiplication map $\tilde{\mathrm{H}}_1(F) \times Z_2 \to \mathrm{H}_2(F)$ identifies the product $\tilde{\mathrm{H}}_1(F) \times Z_2$, as a topological space, with an open subset of $\mathrm{H}_2(F)$. Let f_2 be the pushforward of $f_1 \otimes \mathbb{1}_{Z_2} \in C^\infty_\mu(\tilde{\mathrm{H}}_1(F) \times Z_2)$ to an element of $C^\infty_\mu(\mathrm{H}_2(F))$, where $C^\infty_\mu(\mathrm{H}_2(F))$ is defined just like $C^\infty_\mu(\tilde{\mathrm{H}}_1(F))$, using the same subgroup $\mathrm{C}_1(F)$.
For appropriate choices of measures, it is easy to see that the stable orbital integral of f_2 at any strongly regular semisimple element of $\mathrm{H}_2(F)$ is either zero or equal to the stable orbital integral

of f_1 at some strongly regular semisimple element of $\mathrm{H}_1(F)$. Therefore, it suffices to show that $f_2 \in C^\infty_\mu(\mathrm{H}_2(F))$ is unstable.
Since H_2 is quasi-split and C_1 is an induced torus, this in turn follows from [Art96, Lemma 6.3] if we show that $\Theta_2(f_2) = 0$ for all stable tempered virtual characters $\Theta_2 \in SD_\mu(\mathrm{H}_2)$.
Viewing Θ_2 as a locally integrable function on $\mathrm{H}_2(F)$, it is easy to see that its restriction to $\tilde{\mathrm{H}}_1(F)$, call it Θ_1, belongs to $SD_\mu(\tilde{\mathrm{H}}_1)$: if (π_2, V_2) is an irreducible smooth representation of $\mathrm{H}_2(F)$, the restriction of Θ_{π_2} to $\tilde{\mathrm{H}}_1(F)$ is the character of $\tilde{\mathrm{H}}_1(F)$ acting on the subspace of V_2 that is spanned by those irreducible $\mathrm{H}_1(F)$-subrepresentations that are preserved by $\tilde{\mathrm{H}}_1(F)$. Therefore, $\Theta_1(f_1) = 0$. On the other hand, it is easy to see that $\Theta_2(f_2)$ is some scalar multiple of $\Theta_1(f_1)$, so that $\Theta_2(f_2) = 0$ as well, as desired. □

3.3. **Atomically stable discrete series L-packets.** In many situations where Hypothesis 2.7.1 hasn't been proved, we do have finite sets of representations that deserve to be called discrete series L-packets, in the sense that they satisfy an 'atomic stability' property as in [MY20, Section 4], and hence are necessarily automatically elements of $\Phi_2(\mathrm{M})$ the moment Hypothesis 2.7.1 is true. We will see that this is the case with the notion of L-packets given in Definition 3.3.2 below.

Notation 3.3.1. (i) Henceforth, for a connected reductive group M over F, $e(\mathrm{M})$ denotes its Kottwitz sign (see [Kot83]).
(ii) For this subsection alone, we fix a pair $(\mathrm{M}, \mathcal{O}'_\mathrm{M})$ consisting of reductive group M over F, and a group $\mathcal{O}'_\mathrm{M}$ of automorphisms of M with finite image in $\mathrm{Out}(\mathrm{M})$. For example, M could be a Levi subgroup of G and $\mathcal{O}'_\mathrm{M}$ could equal $\mathcal{O}_\mathrm{M}$.

Definition 3.3.2. Let $\Sigma \subset \mathrm{Irr}_2(\mathrm{M})$ be finite. We say that Σ is an $\mathcal{O}'_\mathrm{M}$-atomically stable L-packet of discrete series representations of $\mathrm{M}(F)$, if there exists a nonzero stable virtual character $\Theta_\Sigma = \sum_{\sigma\in\Sigma} c_\sigma \Theta_\sigma$ on $\mathrm{M}(F)$ supported on Σ, such that Σ and Θ_Σ are preserved under the action of $\mathcal{O}'_\mathrm{M}$, and such that every $\mathcal{O}'_\mathrm{M}$-invariant stable elliptic virtual character $\Theta \in SD_{\mathrm{ell}}(\mathrm{M})^{\mathcal{O}'_\mathrm{M}}$ on $\mathrm{M}(F)$ can be uniquely written in the form $c_1\Theta_\Sigma + c_2\Theta'$ for a (automatically stable and $\mathcal{O}'_\mathrm{M}$-invariant) virtual character Θ' supported outside Σ and complex numbers c_1, c_2. By an atomically stable discrete series L-packet, we refer to an $\mathcal{O}''_\mathrm{M}$-atomically stable discrete series L-packet in the sense just defined, but with $\mathcal{O}''_\mathrm{M} \subset \mathrm{Aut}(\mathrm{M})$ taken to be the trivial group.

Remark 3.3.3. We warn the reader that our use of the term 'atomically stable' clashes with the more appropriate use of the same term in [Kal22, Conjecture 2.2].

Remark 3.3.4. Later, we will see in Proposition 3.4.11, that any finite set Σ of discrete series representations of $\mathrm{M}(F)$ for which one can establish an 'endoscopic decomposition' (in the sense of Definition 3.4.9) satisfies the above property. Thanks to the fact that Kaletha and others have established endoscopic decompositions for various supercuspidal L-packets they have constructed (e.g., see [Kal15]), the scope of the above definition is not subordinate to that of Hypothesis 2.7.1.

Remark 3.3.5. Assume that $\mathcal{O}'_\mathrm{M}$ fixes A_M pointwise, Then the following lemma says that, for any Σ as in Definition 3.3.2, one can take $\Theta_\Sigma = \sum_{\sigma\in\Sigma} d(\sigma)\Theta_\sigma$, where $d(\sigma) \in \mathbb{R}_{>0}$ is the formal degree of σ with respect to any choice of Haar measure on $\mathrm{M}(F)/\mathrm{A}_\mathrm{M}(F)$.

Proposition 3.3.6. *Suppose Σ and Θ_Σ are as in Definition 3.3.2, and assume that $\mathcal{O}'_\mathrm{M}$ fixes A_M pointwise. Fix any Haar measure on $\mathrm{M}(F)/\mathrm{A}_\mathrm{M}(F)$. Then:*

(i) The central characters of the elements of Σ agree on $\mathrm{Z}_\mathrm{M}(F)^{\mathcal{O}'_\mathrm{M}} \supset \mathrm{A}_\mathrm{M}$. In particular, there exists a smooth character $\zeta : \mathrm{A}_\mathrm{M}(F) \to \mathbb{C}^\times$ such that the central character of each $\sigma \in \Sigma$ restricts to ζ on $\mathrm{A}_\mathrm{M}(F)$.
(ii) For some $c \in \mathbb{C} \setminus \{0\}$, $\Theta_\Sigma = c\sum_{\sigma\in\Sigma} d(\sigma)\Theta_\sigma$, where $d(\sigma)$ denotes the formal degree of σ with respect to the chosen Haar measure.
(iii) Suppose we are in the situation of (ii), and let $\zeta : \mathrm{A}_\mathrm{M}(F) \to \mathbb{C}^\times$ be as in (i). Suppose that Θ is an $\mathcal{O}'_\mathrm{M}$-invariant distribution on $\mathrm{M}(F)$ defined by a possibly infinite sum:

$$\Theta = \sum_{\sigma\in\mathrm{Irr}_2(\mathrm{M})_\zeta} c(\sigma)\Theta_\sigma,$$

where $c(\sigma) \in \mathbb{C}$ for each σ (see Remark 2.2.5 for why this infinite sum is well-defined). If further Θ is stable, then for all $\sigma_1, \sigma_2 \in \Sigma$ we have $c(\sigma_1)d(\sigma_1)^{-1} = c(\sigma_2)d(\sigma_2)^{-1}$.

Proof. (i) is an easy consequence of the definitions together with Remark 2.2.4(i).
The proofs of (ii) and (iii) are easier versions of the proofs of (ii) and (iii) of Proposition 3.3.7 that we will prove below, so we will be brief, referring the reader to the proof of Proposition 3.3.7 for more details including of some of the notation. Let us first prove (ii). Write $\Theta_\Sigma = \sum_{\sigma \in \Sigma} c(\sigma)\Theta_\sigma$. Suppose $\sigma_1, \sigma_2 \in \Sigma$; (ii) follows if we show that $c(\sigma_2)d(\sigma_1) = c(\sigma_1)d(\sigma_2)$. For $i = 1, 2$, we let $f_{\sigma_i} \in C^\infty_{\mathrm{A_M}(F),\zeta}(\mathrm{M}(F))$ be a pseudocoefficient for σ_i from among those representations of $\mathrm{M}(F)$ whose central character restricts to ζ on $\mathrm{A_M}(F)$, and let $f_i \in C^\infty_{\mathrm{A_M}(F),\zeta}(\mathrm{M}(F))$ be the average of the pseudocoefficients $f_{\sigma_i} \circ \beta^{-1}$ of the representations $\sigma_i \circ \beta^{-1} \in \Sigma$, as β runs over a set of representatives in $\mathcal{O}'_\mathrm{M}$ for the finite group $\mathcal{O}'_\mathrm{M} \cdot \mathrm{Int}\,\mathrm{M}(F)/\,\mathrm{Int}\,\mathrm{M}(F)$.
We claim that $c(\sigma_2)f_1 - c(\sigma_1)f_2 \in C^\infty_{\mathrm{A_M}(F),\zeta}(\mathrm{M}(F))$ is unstable; this is the analogue of Claim 1 in the proof of Proposition 3.3.7 below. By Proposition 3.2.10, this claim follows if we show that $\Theta(c(\sigma_2)f_1 - c(\sigma_1)f_2) = 0$ for all $\Theta \in SD(\mathrm{M})$. More precisely, the same proposition, together with the fact that $\sigma_1, \sigma_2 \in \mathrm{Irr}_2(\mathrm{M})$, in fact implies that this needs to be checked only for $\Theta \in SD_{\mathrm{ell}}(\mathrm{M})$. Moreover, by the "$\mathcal{O}'_\mathrm{M}$-averaging" process used to define the f_i, we may assume that $\Theta \in SD_{\mathrm{ell}}(\mathrm{M})^{\mathcal{O}'_\mathrm{M}}$, and then using Definition 3.3.2, that either $\Theta = \Theta_\Sigma$, or Θ is supported outside Σ. If $\Theta = \Theta_\Sigma$, then we have $\Theta(c(\sigma_2)f_1 - c(\sigma_1)f_2) = \Theta(c(\sigma_2)f_{\sigma_1} - c(\sigma_1)f_{\sigma_2}) = c(\sigma_2)c(\sigma_1) - c(\sigma_1)c(\sigma_2) = 0$, while if Θ is supported outside Σ we have $\Theta(c(\sigma_1)f_1 - c(\sigma_2)f_2) = 0 - 0 = 0$; in both cases, we used the definition of pseudocoefficients and the $\mathcal{O}'_\mathrm{M}$-invariance of Θ.
This proves that $c(\sigma_2)f_1 - c(\sigma_1)f_2$ is unstable. By [Kot88, Section 3, Proposition 1], we get $c(\sigma_2)f_1(1) - c(\sigma_1)f_2(1) = 0$, and since $\mathcal{O}'_\mathrm{M}$-averaging of functions preserves evaluation of functions at the identity, we get $c(\sigma_2)f_{\sigma_1}(1) - c(\sigma_1)f_{\sigma_2}(1) = 0$. But it is easy to see that $f_{\sigma_i}(1) = d(\sigma_i) \neq 0$ for $i = 1, 2$ (see [DKV84, Proposition A.3.g]), so we get $c(\sigma_2)d(\sigma_1) = c(\sigma_1)d(\sigma_2)$, as desired.
This proves (ii). Coming to (iii), (ii) and its proof now give us that $d(\sigma_2)f_1 - d(\sigma_1)f_2$ is unstable, applying which, along with the $\mathcal{O}'_\mathrm{M}$-invariance of Θ, we get:

$$c(\sigma_1)d(\sigma_2) = \Theta(d(\sigma_2)f_{\sigma_1}) = \Theta(d(\sigma_2)f_1) = \Theta(d(\sigma_1)f_2) = \Theta(d(\sigma_1)f_{\sigma_2}) = c(\sigma_2)d(\sigma_1),$$

which yields $c(\sigma_1)d(\sigma_1)^{-1} = c(\sigma_2)d(\sigma_2)^{-1}$, as desired. □

Proposition 3.3.7. *Let the quasi-split form M^* of F and various auxiliary data such as the inner twist ψ_M and the endoscopic datum $\underline{\mathrm{M}}^*$ be as in Notation 3.2.1(i). Let $\mathcal{O}'_{\mathrm{M}^*} \subset \mathrm{Aut}(\mathrm{M}^*)$ be a subgroup with finite image in $\mathrm{Out}(\mathrm{M}^*)$. Let Σ be an $\mathcal{O}'_\mathrm{M}$-atomically stable discrete series L-packet on $\mathrm{M}(F)$, and Σ^* an $\mathcal{O}'_{\mathrm{M}^*}$-atomically stable discrete series L-packet on $\mathrm{M}^*(F)$. Assume that $\mathcal{O}'_\mathrm{M}$ and $\mathcal{O}'_{\mathrm{M}^*}$ fix $\mathrm{A_M}$ and $\mathrm{A_{M^*}}$ pointwise, and that their finite images $\bar{\mathcal{O}}'_\mathrm{M}$ and $\bar{\mathcal{O}}'_{\mathrm{M}^*}$ in $\mathrm{Out}(\mathrm{M}) = \mathrm{Out}(\mathrm{M}^*)$ are equal. Let $\Theta_\Sigma \in SD_{\mathrm{ell}}(\mathrm{M})^{\mathcal{O}'_\mathrm{M}}$ and $\Theta_{\Sigma^*} \in SD_{\mathrm{ell}}(\mathrm{M}^*)^{\mathcal{O}'_{\mathrm{M}^*}}$ be as in Definition 3.3.2. Assume that the image of Θ_{Σ^*} under the isomorphism $SD_{\mathrm{ell}}(\mathrm{M}^*) \to SD_{\mathrm{ell}}(\mathrm{M})$ (see Corollary 3.2.7) is supported in Σ.*

(i) There exists a smooth character $\zeta : \mathrm{A_M}(F) = \mathrm{A_{M^*}}(F) \to \mathbb{C}^\times$ (the identification $\mathrm{A_M} = \mathrm{A_{M^*}}$ made using the inner twist ψ_M), such that each $\sigma \in \Sigma$ and $\sigma^* \in \Sigma^*$ has a central character restricting to ζ on $\mathrm{A_M}(F) = \mathrm{A_{M^*}}(F)$.

(ii) We normalize the transfer factors as in Remark 3.2.2(i), and give $\mathrm{M}^*(F)$ and $\mathrm{M}(F)$ compatible Haar measures (see Notation 3.2.1(iv)), and similarly with $\mathrm{A_{M^*}}(F)$ and $\mathrm{A_M}(F)$, so that we get compatible quotient measures on $(\mathrm{M}^*/\mathrm{A_{M^*}})(F)$ and $(\mathrm{M}/\mathrm{A_M})(F)$. If we use Proposition 3.3.6(ii) to choose $\Theta_\Sigma = \sum_{\sigma \in \Sigma} d(\sigma)\Theta_\sigma$ and $\Theta_{\Sigma^*} = \sum_{\sigma^* \in \Sigma^*} d(\sigma^*)\Theta_{\sigma^*}$, the image of Θ_{Σ^*} under the isomorphism $SD_{\mathrm{ell}}(\mathrm{M}^*) \to SD_{\mathrm{ell}}(\mathrm{M})$ equals $e(\mathrm{M})\Theta_\Sigma$.

(iii) Suppose we are in the situation of (ii). Suppose Θ is an $\mathcal{O}'_\mathrm{M}$-invariant stable distribution on $\mathrm{M}(F)$, and Θ^* an $\mathcal{O}'_{\mathrm{M}^*}$-invariant stable distribution on $\mathrm{M}^*(F)$, defined by infinite but well-defined (by [Wal03, Theorem VIII.1.2], *as explained in Remark 2.2.5) sums:*

$$\Theta = \sum_{\sigma \in \mathrm{Irr}_2(\mathrm{M})_\zeta} c(\sigma)\Theta_\sigma, \qquad \textit{and} \qquad \Theta^* = \sum_{\sigma \in \mathrm{Irr}_2(\mathrm{M}^*)_\zeta} c(\sigma^*)\Theta_{\sigma^*},$$

where each $c(\sigma), c(\sigma^*) \in \mathbb{C}$*. If further* Θ *is the image of* Θ^* *under the endoscopic transfer of distributions between* M *and* M^**, then we have* $c(\sigma) = e(\mathrm{M}) \cdot d(\sigma)c(\sigma^*) \cdot d(\sigma^*)^{-1}$ *for each* $\sigma \in \Sigma$ *and* $\sigma^* \in \Sigma^*$*.*

Proof of Proposition 3.3.7. (i) immediately follows from Lemma 3.2.3(i).
Now let us prove (ii), for which we write $\mathscr{Z} = \mathrm{A}_{\mathrm{M}}(F) = \mathrm{A}_{\mathrm{M}^*}(F)$, this identification made using ψ_{M}; it will not create confusion, though $\mathscr{Z}$ is being viewed as a subgroup of two different groups. Let '$\mathscr{Z}$-central character' stand for 'the restriction of the central character to $\mathscr{Z}$', so that ζ is the common $\mathscr{Z}$-central character of the elements of Σ as well as of σ^*. For $\sigma \in \Sigma$, let $f_\sigma \in C^\infty_{\mathscr{Z},\zeta}(\mathrm{M}(F))$ be a pseudocoefficient for σ from among those representations of $\mathrm{M}(F)$ with $\mathscr{Z}$-central character ζ, i.e., for every $\sigma' \in \mathrm{Irr}_{\mathrm{temp}}(F)$ with $\mathscr{Z}$-central character ζ, we have that $\operatorname{tr} \sigma'(f_\sigma)$ equals 0 if $\sigma' \not\cong \sigma$, and that it equals 1 otherwise (here $\sigma'(f_\sigma)$ is defined using an integral over $\mathrm{M}(F)/\mathscr{Z} = (\mathrm{M}/\mathrm{A}_{\mathrm{M}})(F)$). Similarly, we can talk of pseudocoefficients $f_{\sigma^*} \in C^\infty_{\mathscr{Z},\zeta}(\mathrm{M}^*(F))$ for each $\sigma^* \in \Sigma^*$. Let $f \in C^\infty_{\mathscr{Z},\zeta}(\mathrm{M}(F))$ be the average of the pseudocoefficients $f_\sigma \circ \beta^{-1}$ of the representations $\sigma \circ \beta^{-1} \in \Sigma$, as β runs over a set of representatives in $\mathcal{O}'_{\mathrm{M}}$ for the finite group $\mathcal{O}'_{\mathrm{M}} \cdot \operatorname{Int} \mathrm{M}(F)/\operatorname{Int} \mathrm{M}(F)$. Similarly, define f^* by averaging the $f_{\sigma^*} \circ \beta^{-1}$, as β runs over a set of representatives for $\mathcal{O}'_{\mathrm{M}^*} \cdot \operatorname{Int} \mathrm{M}^*(F)/\operatorname{Int} \mathrm{M}^*(F)$.
We can talk of elements in $C^\infty_{\mathscr{Z},\zeta}(\mathrm{M}(F))$ and $C^\infty_{\mathscr{Z},\zeta}(\mathrm{M}^*(F))$ having matching orbital integrals. Further, by Lemma 3.2.3(i), the map $SD(\mathrm{M}^*) \to SD(\mathrm{M})$ takes $SD_\zeta(\mathrm{M}^*)$ to $SD_\zeta(\mathrm{M})$. Note that the actions of $\mathcal{O}'_{\mathrm{M}}$ and $\mathcal{O}'_{\mathrm{M}^*}$ on $SD_\zeta(\mathrm{M})$ and $SD_\zeta(\mathrm{M}^*)$ (which are well-defined as $\mathcal{O}'_{\mathrm{M}}$ and $\mathcal{O}'_{\mathrm{M}^*}$ fix $\mathscr{Z}$ pointwise) each factor through $\bar{\mathcal{O}}'_{\mathrm{M}} = \bar{\mathcal{O}}'_{\mathrm{M}^*}$, and the map $SD_\zeta(\mathrm{M}) \to SD_\zeta(\mathrm{M}^*)$ is equivariant for $\bar{\mathcal{O}}'_{\mathrm{M}} = \bar{\mathcal{O}}'_{\mathrm{M}^*}$ by Lemma 3.2.3(ii). Thus, the image of Θ_{Σ^*} under $SD_{\mathrm{ell}}(\mathrm{M}^*) \to SD_{\mathrm{ell}}(\mathrm{M})$, which is supported in Σ by hypothesis, is also $\mathcal{O}'_{\mathrm{M}}$-invariant, and nonzero (by Corollary 3.2.7), and can hence be written as $a\Theta_\Sigma$ for some nonzero $a \in \mathbb{C}$.

Claim 1. $a^{-1} \cdot d(\sigma)^{-1} f \in C^\infty_{\mathscr{Z},\zeta}(\mathrm{M}(F))$ and $d(\sigma^*)^{-1} f^* \in C^\infty_{\mathscr{Z},\zeta}(\mathrm{M}^*(F))$ have matching orbital integrals.
Since $\mathrm{A}_{\mathrm{M}} = \mathrm{A}_{\mathrm{M}^*}$ is split and in particular induced, it is an easy consequence of [Art96, Lemma 6.3], as explained in [LM20, page 587] (see the equivalence of the conditions (A) and (B) there), that Claim 1 follows if we show that for every $\Theta^* \in SD_\zeta(\mathrm{M}^*)$ with image $\Theta \in SD_\zeta(\mathrm{M})$, we have

$$\Theta^*(d(\sigma^*)^{-1} f^*) = \Theta(a^{-1} d(\sigma)^{-1} f). \tag{36}$$

f and f^* are linear combinations of pseudocoefficients of discrete series representations. Therefore, using Proposition 3.2.8 and the compatibility between endoscopic transfer and parabolic induction (see Remark 3.2.2(iii)), we may assume without loss of generality that $\Theta^* \in SD_{\zeta,\mathrm{ell}}(\mathrm{M}^*)$ and $\Theta \in SD_{\zeta,\mathrm{ell}}(\mathrm{M})$. The image of f^* in the space $\mathcal{I}_{\mathscr{Z},\zeta}(\mathrm{M}^*)$ of $\operatorname{Int} \mathrm{M}^*(F)$-coinvariants for $C^\infty_{\mathscr{Z},\zeta}(\mathrm{M}^*(F))$ is $\mathcal{O}'_{\mathrm{M}^*}$-invariant. Combining this the analogous observation for f, the hypothesis $\bar{\mathcal{O}}'_{\mathrm{M}} = \bar{\mathcal{O}}_{\mathrm{M}^*}$, and Lemma 3.2.3(ii), we may and do replace Θ^* by its well-defined $\bar{\mathcal{O}}'_{\mathrm{M}^*}$-average and Θ by its $\bar{\mathcal{O}}'_{\mathrm{M}}$-average, to assume that $\Theta^* \in SD_{\zeta,\mathrm{ell}}(\mathrm{M}^*)^{\mathcal{O}'_{\mathrm{M}^*}}$ and $\Theta \in SD_{\zeta,\mathrm{ell}}(\mathrm{M})^{\mathcal{O}'_{\mathrm{M}}}$. Using Definition 3.3.2, we can write $\Theta^* = b\Theta_{\Sigma^*} + \Theta_1^*$, where $b \in \mathbb{C}$, and $\Theta_1^* \in SD_{\zeta,\mathrm{ell}}(\mathrm{M}^*)$ is supported outside Σ^*. Accordingly, we can write $\Theta = ab\Theta_\Sigma + \Theta_1$, where Θ_1 is the image of Θ_1^* under $SD_{\zeta,\mathrm{ell}}(\mathrm{M}^*) \to SD_{\zeta,\mathrm{ell}}(\mathrm{M})$. On the other hand, we can instead apply Definition 3.3.2 to $\Theta \in SD_{\zeta,\mathrm{ell}}(\mathrm{M})^{\mathcal{O}'_{\mathrm{M}}}$, to write $\Theta = b'\Theta_\Sigma + \Theta'_1$, where $b' \in \mathbb{C}$, and $\Theta'_1 \in SD_{\zeta,\mathrm{ell}}(\mathrm{M})$ is supported outside Σ. Using the $\mathcal{O}'_{\mathrm{M}^*}$-invariance of Θ^*, the $\mathcal{O}'_{\mathrm{M}}$-invariance of Θ and the definition of pseudocoefficients, we get:

$$b\Theta_{\Sigma^*}(d(\sigma^*)^{-1} f^*) = b\Theta_{\Sigma^*}(d(\sigma^*)^{-1} f_{\sigma^*}) = b = (ab\Theta_\Sigma)(a^{-1} d(\sigma)^{-1} f_\sigma) = (ab\Theta_\Sigma)(a^{-1} d(\sigma)^{-1} f).$$

From this, and recalling that $\Theta^* = b\Theta_{\Sigma^*} + \Theta_1^*$ and $\Theta = ab\Theta_\Sigma + \Theta_1$, (36), and hence also Claim 1, follows if we show that $\Theta_1^*(d(\sigma^*)^{-1} f^*) = 0 = \Theta_1(a^{-1} d(\sigma)^{-1} f)$. The definition of pseudocoefficients gives us $\Theta_1^*(d(\sigma^*)^{-1} f^*) = 0 = \Theta'_1(a^{-1} d(\sigma)^{-1} f)$ instead, so Claim 1 follows if we prove Claim 2 below.
Claim 2 . We have $\Theta_1 = \Theta'_1$ (and consequently we have $ab = b'$ as well).
Let us give a proof of Claim 2; it will involve some basic facts about the elliptic inner products on $SD_{\zeta,\mathrm{ell}}(\mathrm{M}^*)$ and $D_{\zeta,\mathrm{ell}}(\mathrm{M}) \supset SD_{\zeta,\mathrm{ell}}(\mathrm{M})$, about which more references and explanation are given in the proof of Proposition 3.4.11 below, which uses the same idea in a slightly more general setting. Claim 2 follows if we show that Θ_1 is orthogonal to Θ_Σ under the elliptic inner product

on $D_{\zeta,\mathrm{ell}}(\mathrm{M})$, a property that Θ_1' clearly satisfies (because Θ_Σ and Θ_1' have disjoint supports). But since Θ_1^* is orthogonal to Θ_{Σ^*} for the elliptic inner product on $SD_{\zeta,\mathrm{ell}}(\mathrm{M}^*)$ (as it is a multiple of the restriction of the elliptic inner product on $D_{\zeta,\mathrm{ell}}(\mathrm{M}^*)$, by [LMW18, Section 4.6, Lemma 3]), Claim 2 follows from the fact that the map $SD_{\zeta,\mathrm{ell}}(\mathrm{M}^*) \to D_{\zeta,\mathrm{ell}}(\mathrm{M})$ is known to take the elliptic inner product on the former space to a multiple of the elliptic inner product on the latter (again by [LMW18, Section 4.6, Lemma 3]).

Thus, we have proved Claim 2, and hence also Claim 1. By [Kot88, Section 3, Proposition 2], given that our choice of measures is compatible with that in [Kot88], we conclude that $e(\mathrm{M}) \cdot a^{-1} \cdot d(\sigma)^{-1} f(1) = d(\sigma^*)^{-1} f^*(1)$ (here, the Kottwitz sign $e(\mathrm{M})$ comes from the definition of singular stable orbital integrals in [Kot88, page 638]; the Kottwitz sign of M^* equals 1 since M^* is quasi-split). Since $\mathrm{Aut}(\mathrm{M})$ and $\mathrm{Aut}(\mathrm{M}^*)$ preserve evaluation at the identity element, we get $e(\mathrm{M}) \cdot a^{-1} \cdot d(\sigma)^{-1} f_\sigma(1) = d(\sigma^*)^{-1} f_{\sigma^*}(1)$. But it is easy to see that $f_\sigma(1) = d(\sigma) \neq 0$ and $f_{\sigma^*}(1) = d(\sigma^*) \neq 0$ (see [DKV84, Proposition A.3.g]), so we get $a = e(\mathrm{M})$, giving (ii).

To see (iii), apply Claim 1 to Θ and Θ^*; we then get:

$$c(\sigma)e(\mathrm{M})^{-1}d(\sigma)^{-1} = \Theta(e(\mathrm{M})^{-1}d(\sigma)^{-1}f_\sigma) = \Theta(e(\mathrm{M})^{-1}d(\sigma)^{-1}f) = \Theta^*(d(\sigma^*)^{-1}f^*) = \Theta^*(d(\sigma^*)^{-1}f_{\sigma^*}) = c(\sigma^*)d(\sigma^*)^{-1},$$

giving (iii). □

Lemma 3.3.8. *Suppose Σ, Σ' are $\mathcal{O}_\mathrm{M}'$-atomically stable discrete series L-packets on $\mathrm{M}(F)$. Assume that $\mathcal{O}_\mathrm{M}'$ fixes $\mathrm{A_M}$ pointwise. Then:*

- *(i) The space of nonzero stable $\mathcal{O}_\mathrm{M}'$-invariant virtual characters on $\mathrm{M}(F)$ supported on Σ is one-dimensional. Thus, Σ determines Θ_Σ up to a nonzero complex multiple.*
- *(ii) Σ and Σ' are either equal or disjoint.*
- *(iii) If M is a Levi subgroup of G, $\mathcal{O}_\mathrm{M}' = \mathcal{O}_\mathrm{M}$, and Hypothesis 2.7.1 is satisfied, then $\Sigma \in \Phi_2(\mathrm{M})$.*
- *(iv) Let θ be an F-rational automorphism of M normalizing $\mathcal{O}_\mathrm{M}'$, and suppose χ belongs to the group $\mathrm{Hom_{cts}}(\mathrm{M}(F), \mathbb{C}^\times)^{\mathcal{O}_\mathrm{M}'}$ of (quasi-)characters of $\mathrm{M}(F)$ fixed by $\mathcal{O}_\mathrm{M}'$. Assume that χ is unitary. Then*
$$(\Sigma \circ \theta) \otimes \chi := \{(\sigma \circ \theta) \otimes \chi \mid \sigma \in \Sigma\}$$
is an $\mathcal{O}_\mathrm{M}'$-atomically stable discrete series L-packet on $\mathrm{M}(F)$, supporting the stable $\mathcal{O}_\mathrm{M}'$-invariant virtual character $\Theta_{(\Sigma\circ\theta)\otimes\chi} := (\Theta_\Sigma \circ \theta)\chi$.

Proof. All assertions are easy. (i) is immediate from the definitions, and (iii) follows from Lemma 2.7.3(i). For (ii), if $\Sigma \cap \Sigma' \neq \emptyset$, then expanding $\Theta_{\Sigma'}$ as $c_1\Theta_\Sigma + c_2\Theta'$ as in Definition 3.3.2 gives the inclusion $\Sigma \subset \Sigma'$, where we use the consequence of Proposition 3.3.6(ii) that the coefficients of $\Theta_{\sigma'}$ in $\Theta_{\Sigma'}$ and Θ_σ in Θ_Σ are nonzero for all $\sigma \in \Sigma$ and $\sigma' \in \Sigma'$; similarly $\Sigma' \subset \Sigma$, so $\Sigma = \Sigma'$. For (iv), one uses that $\Sigma \circ \theta$ and χ are $\mathcal{O}_\mathrm{M}'$-invariant, the former since θ normalizes $\mathcal{O}_\mathrm{M}'$. □

Thanks to (ii) and (iv) of the above lemma, we get the following easy corollary:

Corollary 3.3.9. *Let $\mathcal{O}_\mathrm{M}''$ be some group of F-rational automorphisms of M normalizing $\mathcal{O}_\mathrm{M}'$. Let $\mathcal{F}_0$ be a set of $\mathcal{O}_\mathrm{M}'$-atomically stable discrete series L-packets on $\mathrm{M}(F)$ in the sense of Definition 3.3.2, and let*

$$\begin{aligned}\mathcal{F} &= \{(\Sigma \circ \theta) \otimes \chi \mid \Sigma \in \mathcal{F}_0, \theta \in \mathcal{O}_\mathrm{M}'', \chi \in \mathrm{Hom_{cts}}(\mathrm{M}(F), \mathbb{C}^\times)^{\mathcal{O}_\mathrm{M}'} \text{ is unitary}\} \\ &= \{(\Sigma \otimes \chi) \circ \theta \mid \Sigma \in \mathcal{F}_0, \theta \in \mathcal{O}_\mathrm{M}'', \chi \in \mathrm{Hom_{cts}}(\mathrm{M}(F), \mathbb{C}^\times)^{\mathcal{O}_\mathrm{M}'} \text{ is unitary}\}.\end{aligned}$$

Extend the definition of Θ_Σ to $\mathcal{F}$ as follows: for $\Sigma \in \mathcal{F}$, make a choice of $\Sigma_0 \in \mathcal{F}_0, \theta \in \mathcal{O}_\mathrm{M}''$ and $\chi \in \mathrm{Hom_{cts}}(\mathrm{M}(F), \mathbb{C}^\times)^{\mathcal{O}_\mathrm{M}'}$ such that $\Sigma = (\Sigma_0 \circ \theta) \otimes \chi$, and set $\Theta_\Sigma = (\Theta_{\Sigma_0} \circ \theta)\chi$. Then:

- *(i) The (distinct) elements of $\mathcal{F}$ are all disjoint.*
- *(ii) Given $\Theta \in SD_{\mathrm{ell}}(\mathrm{M})^{\mathcal{O}_\mathrm{M}'}$, write $\Theta = \Theta_1 + \Theta_2$, where Θ_1 (resp., Θ_2) is supported outside (resp., inside) the union of the members of $\mathcal{F}$. Then $\Theta_1, \Theta_2 \in SD_{\mathrm{ell}}(\mathrm{M})^{\mathcal{O}_\mathrm{M}'}$, and Θ_2 is uniquely a linear combination of the Θ_Σ, as Σ runs over $\mathcal{F}$.*

Proof. (i) follows from (ii) and (iv) of Lemma 3.3.8. Given (i), and using that each element of $\mathcal{F}$ is also an $\mathcal{O}_\mathrm{M}'$-atomically stable discrete series L-packet (by (iv) of Lemma 3.3.8), (ii) then follows by induction. □

3.4. **Arthur's formalism and atomically stable discrete series L-packets.** The main results of this subsection are Propositions 3.4.2 and 3.4.11, each of which gives a 'character theoretic' criterion intended to help verify whether a given 'candidate packet', in the form of a finite set of discrete series representations, forms an atomically stable discrete series L-packet. Proposition 3.4.2 is inspired by, and at least aspires to be a commentary on, a remark in [Mg14, Section 4.8]. We feel that it should be possible to check the criterion in this proposition whenever one can verify stability for the given candidate packet. The criterion of Proposition 3.4.11 is more involved, since it almost amounts to verifying the endoscopic character identities for the candidate packet, but has the advantage that it has already been verified by Kaletha for regular supercuspidal representations when $p \gg 0$. These two propositions should be well-known to experts, and the proof of Propsition 3.4.11 seems to have some similarities with that of [MY20, Proposition 4.2].

Notation 3.4.1. (i) In this subsection, given a twisted space $(\mathrm{M}, \tilde{\mathrm{M}})$ over F, with M reductive, $\tilde{\mathrm{M}}(F)_{\mathrm{ell}}$ will denote the set of strongly regular semisimple elliptic elements of $\tilde{\mathrm{M}}(F)$; it is an open subset of $\tilde{\mathrm{M}}(F)$. In this subsection, given a triple $(\mathrm{M}, \tilde{\mathrm{M}}, \mathbf{a})$ and the associated unitary character $\omega : \mathrm{M}(F) \to \mathbb{C}^\times$ as in Notation 3.1.1, an element $\Theta \in D_{\mathrm{ell}}(\tilde{\mathrm{M}}, \omega)$ will be viewed by restriction as a locally constant function $\tilde{\mathrm{M}}(F)_{\mathrm{ell}} \to \mathbb{C}$ (this restriction determines Θ, as follows from [MgW18, Theorem 7.3]). Thus, $D_{\mathrm{ell}}(\tilde{\mathrm{M}}, \omega)$ can be viewed as a collection of locally constant functions $\tilde{\mathrm{M}}(F)_{\mathrm{ell}} \to \mathbb{C}$.

(ii) If $(\mathrm{M}, \tilde{\mathrm{M}})$ is a twisted space over F, with M quasi-split reductive and $\tilde{\mathrm{M}}$ of inner torsion (i.e., $\mathrm{Int}\, \tilde{m}$ is an inner automorphism of M for each $\tilde{m} \in \tilde{\mathrm{M}}(F)$), then for each $\gamma \in \tilde{\mathrm{M}}(F)_{\mathrm{ell}}$, we let $\kappa(\gamma)$ be the number of conjugacy classes in the stable conjugacy class of γ. Moreover, given any virtual character $\Theta \in D_{\mathrm{ell}}(\tilde{\mathrm{M}}(F))$, we define Θ^{st} to be the function $\tilde{\mathrm{M}}(F)_{\mathrm{ell}} \to \mathbb{C}$, given by:

$$\Theta^{\mathrm{st}}(\gamma) = \kappa(\gamma)^{-1} \sum_{\gamma'} \Theta(\gamma'),$$

where γ' runs over a set of representatives for the $\mathrm{M}(F)$-conjugacy classes in the stable conjugacy class of γ.

(iii) For the rest of this subsection, let M be a connected reductive group over F; we will put ourselves in the situation of Notation 3.3.1, but with $\mathcal{O}'_{\mathrm{M}}$ trivial.

Now we can state the first main result of this subsection.

Proposition 3.4.2. *Let $\Sigma \subset \mathrm{Irr}_2(\mathrm{M})$ be a finite subset. Then Σ is an atomically stable discrete series L-packet (see Definition 3.3.2) if and only if the following conditions are satisfied:*

(a) The $\Theta^{\mathrm{st}}_\sigma$, as σ varies over Σ, are all proportional to each other; and

(b) There exist nonzero complex numbers c_σ for each $\sigma \in \Sigma$, such that $\Theta_\Sigma := \sum_{\sigma \in \Sigma} c_\sigma \Theta_\sigma$ is stable, i.e., belongs to $SD_{\mathrm{ell}}(\mathrm{M}) \subset D_{\mathrm{ell}}(\mathrm{M})$.

Moreover, when these conditions are satisfied, we have that $d(\sigma)^{-1}\Theta^{\mathrm{st}}_\sigma = d(\sigma')^{-1}\Theta^{\mathrm{st}}_{\sigma'}$ for any $\sigma, \sigma' \in \Sigma$, and that for any $\sigma_0 \in \Sigma$:

$$\left(\sum_{\sigma \in \Sigma} d(\sigma)^2\right)^{-1} \cdot \sum_{\sigma \in \Sigma} d(\sigma)\Theta_\sigma = d(\sigma_0)^{-1} \cdot \Theta^{\mathrm{st}}_{\sigma_0}.$$

Remark 3.4.3. It would be satisfying if Proposition 3.4.2 could be interpreted as giving a 'stable' version of the classical result that orbital integrals of pseudocoefficients at elliptic strongly regular elements yield character values (e.g., the much simpler elliptic untwisted case of [MgW18, Theorem 7.2]). However, we do not know if such an interpretation is appropriate.

We now proceed to do some preparations for the proof of Proposition 3.4.2.

Notation 3.4.4. Let a triple $(\mathrm{M}, \tilde{\mathrm{M}}, \mathbf{a})$ and the associated unitary character $\omega : \mathrm{M}(F) \to \mathbb{C}^\times$ be as in Notation 3.1.1, and assume notation from Notation 3.1.2. Note that for any closed subgroup $\mathscr{X} \subset \mathrm{Z}_{\mathrm{M}}(F)$ we have a decomposition

$$D_{\mathrm{ell}}(\tilde{\mathrm{M}}, \omega) = \bigoplus_{\zeta} D_{\mathscr{X},\zeta,\mathrm{ell}}(\tilde{\mathrm{M}}, \omega), \tag{37}$$

where ζ varies over unitary characters of $\mathscr{X}$. We define an inner product $\langle\cdot,\cdot\rangle$ on $D_{\mathrm{ell}}(\tilde{\mathrm{M}},\omega)$, using (37) with $\mathscr{X}$ taken to be by $\mathrm{A}_{\tilde{\mathrm{M}}}(F)$: we require $\langle\cdot,\cdot\rangle$ to restrict to the elliptic inner product (see [MgW18, Section 7.3]) on each $D_{\mathrm{A}_{\tilde{\mathrm{M}}}(F),\zeta,\mathrm{ell}}(\tilde{\mathrm{M}},\omega)$, and the $D_{\mathrm{A}_{\tilde{\mathrm{M}}}(F),\zeta,\mathrm{ell}}(\tilde{\mathrm{M}},\omega)$ for distinct ζ to be orthogonal to each other. If either $\tilde{\mathrm{M}}=\mathrm{M}$ or if M is quasi-split and $\tilde{\mathrm{M}}$ is of inner torsion, and if ω is trivial, we get by restriction an inner product on $SD_{\mathrm{ell}}(\tilde{\mathrm{M}})\subset D_{\mathrm{ell}}(\tilde{\mathrm{M}})$. We will use the following well-known property of this inner product (see [MgW18, Theorem 7.3(i)]): if $\tilde{\sigma}_1,\tilde{\sigma}_2$ are ω-representations of $\tilde{\mathrm{M}}(F)$, whose underlying $\mathrm{M}(F)$-representations σ_1,σ_2 belong to $\mathrm{Irr}_2(\mathrm{M})$, then $\langle\Theta_{\sigma_1},\Theta_{\sigma_2}\rangle=0$ unless $\sigma_1\cong\sigma_2$.

The inner product on $D_{\mathrm{ell}}(\mathrm{M})$ having been defined, we can now state the following lemma, modulo which Proposition 3.4.2 is almost formal.

Lemma 3.4.5. *If $\Theta\in D_{\mathrm{ell}}(\mathrm{M})$, then Θ^{st} is the image of Θ under the orthogonal projection $D_{\mathrm{ell}}(\mathrm{M})\to SD_{\mathrm{ell}}(\mathrm{M})$. In particular, $\Theta^{\mathrm{st}}\in SD_{\mathrm{ell}}(\mathrm{M})$ (when viewed as in Notation 3.4.1(i)).*

Proof of Proposition 3.4.2, assuming Lemma 3.4.5. Since the condition (b) concerning the stable virtual character Θ_Σ is clearly necessary, we may and do assume it.
It is easy to see from the definition of the elliptic inner product (see Notation 3.4.4) that Σ is an atomically stable discrete series L-packet if and only if each $\Theta\in SD_{\mathrm{ell}}(\mathrm{M})$ that is orthogonal to Θ_Σ in $SD_{\mathrm{ell}}(\mathrm{M})\subset D_{\mathrm{ell}}(\mathrm{M})$ is also orthogonal in $D_{\mathrm{ell}}(\mathrm{M})$ to Θ_σ for each $\sigma\in\Sigma$. For each $\Theta\in SD_{\mathrm{ell}}(\mathrm{M})$ and each $\Theta'\in D_{\mathrm{ell}}(\mathrm{M})$ with projection $\bar{\Theta}'$ in $SD_{\mathrm{ell}}(\mathrm{M})$, the elliptic inner product $\langle\Theta,\Theta'\rangle$ inside $D_{\mathrm{ell}}(\mathrm{M})$ equals $\langle\Theta,\bar{\Theta}'\rangle$ taken inside $SD_{\mathrm{ell}}(\mathrm{M})$. Therefore, using Lemma 3.4.5, we conclude that Σ is an atomically stable discrete series L-packet if and only if each $\Theta\in SD_{\mathrm{ell}}(\mathrm{M})$ that is orthogonal to Θ_Σ in $SD_{\mathrm{ell}}(\mathrm{M})$ is also orthogonal in $SD_{\mathrm{ell}}(\mathrm{M})$ to $\Theta_\sigma^{\mathrm{st}}$ for each $\sigma\in\Sigma$. This is clearly equivalent to Θ_Σ being proportional to $\Theta_\sigma^{\mathrm{st}}$ for each $\sigma\in\Sigma$, which is easily seen to be equivalent to the condition (a). Here, we note that each $\Theta_\sigma^{\mathrm{st}}$ is nonzero: this follows from Lemma 3.4.5 and [Mg14, Proposition 2.1].
For the last assertion, note from Proposition 3.3.6(ii) that $\sum d(\sigma)\Theta_\sigma$ is stable, where σ runs over Σ. Further, it is easy to see that $d(\sigma)^{-1}\Theta_\sigma^{\mathrm{st}}$ is independent of $\sigma\in\Sigma$: either use the argument of [Mg14, Proposition 2.1], or note that for $\sigma_1,\sigma_2\in\Sigma$, the constant of proportionality between the $\Theta_{\sigma_i}^{\mathrm{st}}$ equals that between the $\langle\Theta_{\sigma_i}^{\mathrm{st}},\sum d(\sigma)\Theta_\sigma\rangle=\langle\Theta_{\sigma_i},\sum d(\sigma)\Theta_\sigma\rangle=d(\sigma_i)$. Using these two observations:

$$\sum d(\sigma)\Theta_\sigma=(\sum d(\sigma)\Theta_\sigma)^{\mathrm{st}}=\sum d(\sigma)^2\cdot(d(\sigma)^{-1}\Theta_\sigma)^{\mathrm{st}}=\Big(\sum d(\sigma)^2\Big)\cdot d(\sigma_0)^{-1}\Theta_{\sigma_0}^{\mathrm{st}},$$

where each sum is over $\sigma\in\Sigma$. This gives the last assertion of the lemma. □

We still need to prove Lemma 3.4.5 to complete the proof of Proposition 3.4.2, for which we now make some further preparations.

Remark 3.4.6. In what follows, we will use a lot of observations from [LMW18]. In each case it will be implicitly left to the reader to verify that, though the setting of [LMW18] involves an unramified quasi-split group in place of our M, the observations that we will use do not depend on these assumptions.

The following remark will also introduce some notation.

Remark 3.4.7. Let a triple $(\mathrm{M},\tilde{\mathrm{M}},\mathbf{a})$ and the associated unitary character $\omega:\mathrm{M}(F)\to\mathbb{C}^\times$ be as in Notation 3.1.1, and assume notation from Notation 3.1.2. Suppose $\mathscr{X}$ is a closed subgroup of $\mathrm{Z}_{\tilde{\mathrm{M}}}(F)$, such that $\mathscr{X}\cap\mathrm{A}_{\tilde{\mathrm{M}}}(F)$ is of finite index in $\mathrm{A}_{\tilde{\mathrm{M}}}(F)$. Let $\zeta:\mathscr{X}\to\mathbb{C}^\times$ be a unitary character, and fix Haar measures on $\mathscr{X}$ and $\mathrm{M}(F)$, the latter also giving a measure on $\tilde{\mathrm{M}}(F)$.

(i) Let $C^\infty_{\zeta,\mathrm{cusp}}(\tilde{\mathrm{M}}(F))\subset C^\infty_\zeta(\tilde{\mathrm{M}}(F))$ be the subspace consisting of functions that are cuspidal in the sense of [MgW18, Sections 7.1 and 7.2] (i.e., whose nonelliptic strongly regular semisimple orbital integrals vanish), and let $\mathcal{I}_{\mathscr{X},\zeta,\mathrm{cusp}}(\tilde{\mathrm{M}},\omega)$ be the quotient of $C^\infty_{\zeta,\mathrm{cusp}}(\tilde{\mathrm{M}}(F))$ by the subspace consisting of functions whose strongly regular semisimple ω-twisted orbital integrals vanish. When M is quasi-split and $\tilde{\mathrm{M}}$ has inner torsion, and ω is trivial, we define $\mathcal{SI}_{\mathscr{X},\zeta,\mathrm{cusp}}(\tilde{\mathrm{M}})$ as the quotient of $C^\infty_{\zeta,\mathrm{cusp}}(\tilde{\mathrm{M}}(F))$ by the subspace consisting of functions whose strongly regular semisimple stable orbital integrals vanish (this is a priori slightly different from the 'variant with central character' of the definition in [MW16, page 57], but equivalent to it, thanks to the surjectivity of the obvious map $\mathcal{I}_{\mathscr{Z},\zeta,\mathrm{cusp}}(\tilde{\mathrm{M}}) \to \mathcal{SI}_{\mathscr{Z},\zeta,\mathrm{cusp}}(\tilde{\mathrm{M}})$; see [MW16, Proposition I.4.11], or rather its variant with central character, discussed below).

(ii) For $\gamma \in \tilde{\mathrm{M}}(F)_{\mathrm{ell}}$, we will normalize, in this subsection alone, the ω-twisted orbital integral of $f \in C^\infty_{\zeta,\mathrm{cusp}}(\tilde{\mathrm{M}}(F))$ at γ as:

$$O(\gamma,\omega,f) := \int_{\mathscr{Z}\backslash\mathrm{M}(F)} \omega(m) f(m^{-1}\gamma m)\, dm, \tag{38}$$

where we use the measures on $\mathrm{M}(F)$ and $\mathscr{Z}$ fixed in the present collection of notation (and recall that ω is trivial on $\mathrm{Z}_{\tilde{\mathrm{M}}}(F) \supset \mathscr{Z}$, since we are imposing the conditions of Notation 3.1.1). The integral defining $O(\gamma,\omega,f)$ is absolutely convergent by ellipticity. Recall that if ω is trivial, we write $O(\gamma,f) = O(\gamma,\omega,f)$. Recall that the stable orbital integrals $SO(\gamma,f)$, when M is quasi-split and $\tilde{\mathrm{M}}$ has inner torsion and ω is trivial, are defined by summing the $O(\gamma',f)$ as γ' runs over a set of representatives for the $\mathrm{M}(F)$-conjugacy classes in the stable conjugacy class of γ.

(iii) If $a \in \mathcal{I}_{\zeta,\mathrm{cusp}}(\tilde{\mathrm{M}},\omega)$ and $\gamma \in \tilde{\mathrm{M}}(F)_{\mathrm{ell}}$, $a(\gamma)$ will denote $O(\gamma,\omega,f)$, for any $f \in C^\infty_{\zeta,\mathrm{cusp}}(\tilde{\mathrm{M}}(F))$ with image a; it is independent of a by the density of strongly regular semisimple orbital integrals in the space of invariant distributions.

(iv) For each $\underline{\mathrm{H}} \in \mathcal{E}(\tilde{\mathrm{M}},\mathbf{a})$, we fix associated auxiliary data such as the z-extension H_1 of H (see Notation 3.1.2(iii)). Let us now recall from [LMW18, Section 4.5] a subset $\mathcal{E}(\tilde{\mathrm{M}},\mathbf{a})_\zeta \subset \mathcal{E}(\tilde{\mathrm{M}},\mathbf{a})$, and for each $\underline{\mathrm{H}} \in \mathcal{E}(\tilde{\mathrm{M}},\mathbf{a})$ a pair $(\mathscr{Z}_1,\zeta_1)$ consisting of a closed subgroup $\mathscr{Z}_1 \subset \mathrm{Z}_{\mathrm{H}_1}(F)$ and a unitary character $\zeta_1 : \mathscr{Z}_1 \to \mathbb{C}^\times$ ($\mathcal{E}(\tilde{\mathrm{M}},\mathbf{a})_\zeta$ is the $\mathfrak{E}_{\mathcal{Z},\mu}$ defined in [LMW18, Section 4.5, between (4) and (5)], while $(\mathscr{Z}_1,\zeta_1)$ is denoted $(\mathcal{Z}_1',\mu_1')$ in [LMW18]; we suppress the dependence of these on $\underline{\mathrm{H}}$ and the auxiliary choices for lightness of notation). $\mathscr{Z}_1$ is the inverse image in $\mathrm{Z}_{\mathrm{H}_1}(F)$ of the image $\mathscr{Z}_{\mathrm{H}}$ of $\mathscr{Z} \hookrightarrow \mathrm{Z}_{\mathrm{M}}(F) \to \mathrm{Z}_{\mathrm{H}}(F)$. $\mathcal{E}(\tilde{\mathrm{M}},\mathbf{a})_\zeta \subset \mathcal{E}(\tilde{\mathrm{M}},\mathbf{a})$ is defined to be the subset consisting of $\underline{\mathrm{H}}$ such that there exists a (necessarily unitary) character $\zeta_1 : \mathscr{Z}_1 \to \mathbb{C}^\times$ with the property that whenever strongly regular semisimple elements $\delta_1 \in \tilde{\mathrm{H}}_1(F)$ and $\tilde{\gamma} \in \tilde{\mathrm{M}}(F)$ match (such δ_1 and $\tilde{\gamma}$ exist as $\underline{\mathrm{H}}$ is relevant), we have an equality of transfer factors:

$$\Delta(z_1\delta_1, z\tilde{\gamma}) = \zeta_1(z_1)^{-1}\zeta(z)\Delta(\delta_1,\tilde{\gamma}),$$

for all $z_1 \in \mathscr{Z}_1$ and $z \in \mathscr{Z}$ with the same image in $\mathscr{Z}_{\mathrm{H}}$. In view of Remark 3.4.6, we remark that the '$\lambda_{\mathfrak{z}}$' of [LMW18, Section 4.5] is the 'λ_C' of [KS99, page 53]. Note that it follows from the ellipticity of $\underline{\mathrm{H}}$ that $\mathscr{Z}_1 \cap \mathrm{A}_{\mathrm{H}_1}(F)$ is of finite index in $\mathrm{A}_{\mathrm{H}_1}(F)$.

(v) As in [LMW18, Section 4.5, (5)], endoscopic transfer 'with $\mathscr{Z}$-central character ζ' defines an isomorphism of vector spaces:

$$\mathcal{I}_{\mathscr{Z},\zeta,\mathrm{cusp}}(\tilde{\mathrm{M}},\omega) \to \bigoplus_{\underline{\mathrm{H}} \in \mathcal{E}(\tilde{\mathrm{M}},\mathbf{a})_\zeta} \mathcal{SI}_{\mathscr{Z}_1,\zeta_1,\mathrm{cusp}}(\tilde{\mathrm{H}}_1)^{\mathrm{Aut}(\underline{\mathrm{H}})}. \tag{39}$$

Dually, as in [LMW18, Section 4.5, (6)], we get an isomorphism:

$$\bigoplus_{\underline{\mathrm{H}} \in \mathcal{E}(\mathrm{M},\mathbf{a})_\zeta} \mathbf{T}_{\underline{\mathrm{H}}} = \bigoplus_{\underline{\mathrm{H}} \in \mathcal{E}(\mathrm{M},\mathbf{a})_\zeta} \mathbf{T}_{\underline{\mathrm{H}},\mathrm{ell}} : SD_{\mathscr{Z}_1,\zeta_1,\mathrm{ell}}(\tilde{\mathrm{H}}_1)_{\mathrm{Aut}(\underline{\mathrm{H}})} \to D_{\mathscr{Z},\zeta,\mathrm{ell}}(\tilde{\mathrm{M}},\omega), \tag{40}$$

which is a restriction of (31) (any dependence on measures is at the level of the transfer of functions, and not at the level of the transfer of Harish-Chandra characters).

(vi) As in [LMW18, Section 4.4, just below (4)], we average with respect to $\mathrm{Aut}(\underline{\mathrm{H}})$ to identify each $SD_{\mathscr{Z}_1,\zeta_1,\mathrm{ell}}(\tilde{\mathrm{H}}_1)_{\mathrm{Aut}(\underline{\mathrm{H}})}$ with the space $SD_{\mathscr{Z}_1,\zeta_1,\mathrm{ell}}(\tilde{\mathrm{H}}_1)^{\mathrm{Aut}(\underline{\mathrm{H}})}$ of $\mathrm{Aut}(\underline{\mathrm{H}})$-invariants. This space gets an inner product from its inclusion in $D_{\mathscr{Z}_1,\zeta_1,\mathrm{ell}}(\tilde{\mathrm{H}}_1)$ (which has an inner product as in Notation 3.4.4), also denoted $\langle\cdot,\cdot\rangle$. For $\Theta_1^{\underline{\mathrm{H}}}, \Theta_2^{\underline{\mathrm{H}}} \in SD_{\mathscr{Z}_1,\zeta_1,\mathrm{ell}}(\tilde{\mathrm{H}}_1)^{\mathrm{Aut}(\underline{\mathrm{H}})}$, we have, by [LMW18, Section 4.6, (5) and Lemma 3], an equality

$$\langle \mathbf{T}_{\underline{\mathrm{H}}}(\Theta_1^{\underline{\mathrm{H}}}), \mathbf{T}_{\underline{\mathrm{H}}}(\Theta_2^{\underline{\mathrm{H}}})\rangle = c(\tilde{\mathrm{M}},\underline{\mathrm{H}})^{-1}\langle \Theta_1^{\underline{\mathrm{H}}}, \Theta_2^{\underline{\mathrm{H}}}\rangle, \tag{41}$$

describing the behavior of the inner products we have defined with respect to $\mathbf{T}_{\underline{\mathrm{H}}}$; here, $c(\tilde{\mathrm{M}}, \underline{\mathrm{H}})$ is the constant from [LMW18, Section 4.6, just before Lemma 2].

Here, in view of Remark 3.4.6, let us add that the key point is the inner product formula of [MW16, Proposition I.4.17], from which one deduces a 'variant with central character' involving (39) (see [LMW18, Section 4.6, Lemma 2]), which in turn by duality gives the inner product formula of [LMW18, Section 4.6, (5)] for virtual characters.

(vii) It follows from (vi) above that the components $\mathbf{T}_{\underline{\mathrm{H}}}(SD_{\mathscr{Z}_1,\zeta_1,\mathrm{ell}}(\tilde{\mathrm{H}}_1)_{\mathrm{Aut}(\underline{\mathrm{H}})})$ of the decomposition of $D_{\mathscr{Z},\zeta,\mathrm{ell}}(\tilde{\mathrm{M}}, \omega)$ given by (40) are orthogonal to each other.

(viii) Let us recall the antilinear isomorphism $\iota_{\mathscr{Z},\zeta} : D_{\mathscr{Z},\zeta,\mathrm{ell}}(\tilde{\mathrm{M}}(F), \omega) \to \mathcal{I}_{\mathscr{Z},\zeta,\mathrm{cusp}}(\tilde{\mathrm{M}}, \omega)$ described in [LMW18, Section 4.6, between (2) and (3)], and denoted by '$\iota_{\mathscr{Z},\mu}$' in that reference (here, an antilinear isomorphism refers to a bijective additive map that is semilinear for complex conjugation). $\iota_{\mathscr{Z},\zeta}$ is defined to satisfy:

$$\int_{\mathscr{Z}\backslash\tilde{\mathrm{M}}(F)} \Theta(\gamma) f_2(\gamma)(d\gamma/dz) =: \Theta(f_2) = (\iota_{\mathscr{Z},\zeta}(\Theta), f_2)_{\mathscr{Z},\zeta,\mathrm{ell}}, \tag{42}$$

where the $(\cdot,\cdot)_{\mathscr{Z},\zeta,\mathrm{ell}}$ on the right-hand side refers to the inner product on $\mathcal{I}_{\mathscr{Z},\zeta,\mathrm{cusp}}(\tilde{\mathrm{M}}, \omega)$ as in [LMW18, Section 4.6, (2)]; thus, if h_1, h_2 map to f_1, f_2 under the obvious map $C^\infty_{\mathrm{cusp}}(\tilde{\mathrm{M}}(F)) \to C^\infty_{\zeta,\mathrm{cusp}}(\tilde{\mathrm{M}}(F)) \to \mathcal{I}_{\mathscr{Z},\zeta,\mathrm{cusp}}(\tilde{\mathrm{M}})$ (see the map $p_{\mathscr{Z},\mu}$ of [LMW18, page 315]), then we have a formula of the form:

$$(f_1, f_2)_{\mathscr{Z},\zeta,\mathrm{ell}} := \int_{\mathscr{Z}} \int_{\tilde{\mathrm{M}}(F)_{\mathrm{ell}}/\,\mathrm{conj}} D^{\tilde{\mathrm{M}}}(\gamma)\, \mathrm{meas}(\mathscr{Z}\backslash \mathrm{M}^\gamma(F))^{-1} \overline{O(\gamma, \omega, h_1)} O(z\gamma, \omega, h_2) \zeta(z)\, d\gamma dz, \tag{43}$$

where M^γ denotes the centralizer of γ. To cross check this formula, use [LMW18, the discussion shortly before (1) in Section 4.6, and (4) in Section 4.3], and take into account various slight differences in notation (such as the definition of orbital integrals, in particular our using the unnormalized ones) between us and [LMW18]. [LMW18] itself refers to [MW16, Section I.4.17] for some of the notation, such as the measure on the set $\tilde{\mathrm{M}}(F)_{\mathrm{ell}}/\,\mathrm{conj}$ of $\mathrm{M}(F)$-conjugacy classes in $\tilde{\mathrm{M}}(F)_{\mathrm{ell}}$. Clearly, $\iota_{\mathscr{Z},\mu}$ depends on the chosen measures on $\mathrm{M}(F)$ and $\mathscr{Z}$.

(ix) The antilinear isomorphism $\iota_{\mathscr{Z},\zeta} : D_{\mathscr{Z},\zeta,\mathrm{ell}}(\tilde{\mathrm{M}}, \omega) \to \mathcal{I}_{\mathscr{Z},\zeta,\mathrm{cusp}}(\tilde{\mathrm{M}}, \omega)$ from (viii) above, we claim, is described as follows: $\iota_{\mathscr{Z},\zeta}(\Theta) = f_1$ if and only if for all $\gamma \in \tilde{\mathrm{M}}(F)_{\mathrm{ell}}$, we have:

$$\Theta(\gamma) = \overline{O(\gamma, \omega, f_1)}. \tag{44}$$

Write $f_1 = \iota_{\mathscr{Z},\zeta}(\Theta)$, and let h_1, h_2 map to f_1, f_2 under the obvious map $C^\infty_{\mathrm{cusp}}(\tilde{\mathrm{M}}(F)) \to C^\infty_{\zeta,\mathrm{cusp}}(\tilde{\mathrm{M}}(F)) \to \mathcal{I}_{\mathscr{Z},\zeta,\mathrm{cusp}}(\tilde{\mathrm{M}})$. Then (42) is equivalent to:

$$\Theta(h_2) = \int_{\mathscr{Z}} \int_{\tilde{\mathrm{M}}(F)_{\mathrm{ell}}/\,\mathrm{conj}} D^{\tilde{\mathrm{M}}}(\gamma)\, \mathrm{meas}(\mathscr{Z}\backslash \mathrm{M}^\gamma(F))^{-1} \overline{O(\gamma, \omega, h_1)} O(z\gamma, \omega, h_2) \zeta(z)\, d\gamma dz.$$

As far as the left-hand side is concerned, $\Theta(h_2)$, which is an integral over $\tilde{\mathrm{M}}(F)$, can be evaluated in terms of an integral over $\tilde{\mathrm{M}}(F)_{\mathrm{ell}}/\,\mathrm{conj}$ using an equality given in [MW16, Section I.4.17, shortly before (1)] (and keeping in mind that h_2 is a cuspidal function and that $\Theta(h_2)$ has an ω-equivariance on conjugation). On the right-hand side, we first replace γ by $z^{-1}\gamma$ and then change the order of integration and replace z by z^{-1}, and use the relation between h_1 and f_1 (and that $\zeta(z^{-1}) = \overline{\zeta(z)}$), to get that (42) is equivalent to:

$$\int_{\tilde{\mathrm{M}}(F)_{\mathrm{ell}}/\,\mathrm{conj}} D^{\tilde{\mathrm{M}}}(\gamma) m_{\mathscr{Z}}^{-1} \Theta(\gamma) O(\gamma, \omega, h_2)\, d\gamma = \int_{\tilde{\mathrm{M}}(F)_{\mathrm{ell}}/\,\mathrm{conj}} D^{\tilde{\mathrm{M}}}(\gamma) m_{\mathscr{Z}}^{-1} \overline{O(\gamma, \omega, f_1)} O(\gamma, \omega, h_2)\, d\gamma,$$

where $m_{\mathscr{Z}} = \mathrm{meas}(\mathscr{Z}\backslash \mathrm{M}^\gamma(F))$. From here, the claim involving (44) is easy to see.

(x) Suppose M is quasi-split and $\tilde{\mathrm{M}}$ has inner torsion, and assume that ω is trivial. In this case, one can view $\mathcal{SI}_{\mathscr{Z},\zeta,\mathrm{cusp}}(\tilde{\mathrm{M}}) = \mathcal{SI}_{\mathscr{Z},\zeta,\mathrm{cusp}}(\tilde{\mathrm{M}}, \omega)$ as a subspace of $\mathcal{I}_{\mathscr{Z},\zeta,\mathrm{cusp}}(\tilde{\mathrm{M}})$, using the isomorphism (39) and noting that one has a principal endoscopic datum $\underline{\mathrm{H}}$ with $\mathrm{H} = \mathrm{M}$ and $\tilde{\mathrm{H}} = \tilde{\mathrm{M}}$, for which $\mathrm{Aut}(\underline{\mathrm{H}}) = 1$ and $\mathcal{SI}_{\mathscr{Z}_1,\zeta_1,\mathrm{cusp}}(\tilde{\mathrm{H}}_1)$ identifies with $\mathcal{SI}_{\mathscr{Z},\zeta,\mathrm{cusp}}(\tilde{\mathrm{M}})$. As in the discussion in [LMW18, Section 4.6, between Lemma 1 and Lemma 2], this identifies $\mathcal{SI}_{\mathscr{Z},\zeta,\mathrm{cusp}}(\tilde{\mathrm{M}})$ as the subspace of $\mathcal{I}_{\mathscr{Z},\zeta,\mathrm{cusp}}(\tilde{\mathrm{M}})$ consisting of all a such that $a(\gamma) = a(\gamma')$ (using the notation of Notation 3.4.4(iii)) whenever γ and γ' are stably conjugate. From this perspective, if $a \in \mathcal{SI}_{\mathscr{Z},\zeta,\mathrm{cusp}}(\tilde{\mathrm{M}}) \subset \mathcal{I}_{\mathscr{Z},\zeta,\mathrm{cusp}}(\tilde{\mathrm{M}})$ is the image of $f \in C^\infty_{\zeta,\mathrm{cusp}}(\tilde{\mathrm{M}}(F))$, then for all $\gamma \in \tilde{\mathrm{M}}(F)_{\mathrm{ell}}$ we have:

$$SO(\gamma, f) = \kappa(\gamma)a(\gamma), \tag{45}$$

where, like in Notation 3.4.1(iii), $\kappa(\gamma)$ denotes the number of conjugacy classes in the stable conjugacy class of γ, and $SO(\gamma, f)$ and $a(\gamma)$ are as in (ii) and (iii) above.

(xi) Consider the setting of (v) above, but assume for simplicity that we are in the situation of standard endoscopy. Fix $\underline{\mathrm{H}} \in \mathcal{E}(\tilde{\mathrm{M}}, \mathbf{a})_\zeta = \mathcal{E}(\mathrm{M})_\zeta \subset \mathcal{E}(\mathrm{M})$, with associated data such as $\mathscr{Z}_1$ and ζ_1. The discussion of (x) above applies with $(\mathrm{M}, \tilde{\mathrm{M}}, \mathscr{Z}, \zeta)$ replaced by $(\mathrm{H}_1, \tilde{\mathrm{H}}_1 = \mathrm{H}_1, \mathscr{Z}_1, \zeta_1)$, so we have $\mathcal{SI}_{\mathscr{Z}_1,\zeta_1,\mathrm{cusp}}(\mathrm{H}_1) \subset \mathcal{I}_{\mathscr{Z}_1,\zeta_1,\mathrm{cusp}}(\mathrm{H}_1)$. Then, using (iii) above and (45), the endoscopic transfer map $\mathcal{I}_{\mathscr{Z},\zeta,\mathrm{cusp}}(\mathrm{M}) \to \mathcal{SI}_{\mathscr{Z}_1,\zeta_1,\mathrm{cusp}}(\mathrm{H}_1)$ can now be described as follows: $a \mapsto b$ if and only if for all strongly M-regular $\delta_1 \in \mathrm{H}_1(F)_{\mathrm{ell}}$, we have:

$$b(\delta_1) = \kappa(\delta_1)^{-1} \sum_\gamma D^{\mathrm{H}_1}(\delta_1)^{-1/2} D^{\mathrm{M}}(\gamma)^{1/2} \Delta(\delta_1, \gamma) a(\gamma), \tag{46}$$

where γ runs over a set of representatives for conjugacy classes in the (possibly empty) stable conjugacy class in $\mathrm{M}(F)$ matching δ. Here, we recall that our orbital integrals are unnormalized, and we have used the convention wherein "Δ_{IV}" is not part of Δ, but is accounted for separately using the discriminant factors. Moreover, we have used the discussion on the normalization of measures in [LMW18, the top of page 317], to justify our use of the definition of orbital integrals in (ii) above without adding any extra normalizing constants. Since we are in the case of standard endoscopy, the factors "$d_\theta^{1/2}$" and "d_γ^{-1}" from [LMW18, Section 4.5, (2)] are trivial, and the local isomorphism "$\mathscr{Z}\backslash G_\gamma(F) \to \mathscr{Z}_1'\backslash G'_{1,\delta_1}(F)$" from [LMW18, page 317] is an isomorphism. Another standard fact we have used is that for strongly regular semisimple elements $\delta_1 \in \mathrm{H}_1(F)$ and $\gamma \in \mathrm{M}(F)$, $\Delta(\delta_1, \gamma) \neq 0$ if and only if the stable conjugacy classes of δ_1 and γ match, in which case $\delta_1 \in \mathrm{H}_1(F)$ is elliptic if and only if $\gamma \in \mathrm{M}(F)$ is.

Lemma 3.4.8. *Let $\underline{\mathrm{H}} \in \mathcal{E}(\mathrm{M})$ (thus, we are considering standard endoscopy, not twisted endoscopy). Let $\mathrm{H}_1 = \tilde{\mathrm{H}}_1, \mathscr{Z}, \zeta, \mathscr{Z}_1, \zeta_1$ be as in Remark 3.4.7. Let $\Theta \in D_{\mathscr{Z},\zeta,\mathrm{ell}}(\mathrm{M})$, let $\Theta^{\underline{\mathrm{H}}}$ be its projection to $SD_{\mathscr{Z}_1,\zeta_1,\mathrm{ell}}(\mathrm{H}_1)_{\mathrm{Aut}(\underline{\mathrm{H}})} = SD_{\mathscr{Z}_1,\zeta_1,\mathrm{ell}}(\mathrm{H}_1)^{\mathrm{Aut}(\underline{\mathrm{H}})}$ as per (40), and write $\Theta^{\underline{\mathrm{H}},\mathrm{M}} := \mathbf{T}_{\underline{\mathrm{H}}}(\Theta^{\underline{\mathrm{H}}})$. We have, for all $\gamma \in \mathrm{M}(F)_{\mathrm{ell}}$:*

$$\Theta^{\underline{\mathrm{H}},\mathrm{M}}(\gamma) = c(\mathrm{M}, \underline{\mathrm{H}}) \sum_{\delta_1} \kappa(\delta_1)^{-1} \Delta(\delta_1, \gamma) \sum_{\gamma'} \overline{\Delta(\delta_1, \gamma')} \Theta(\gamma'),$$

where δ_1 ranges over a set of representatives in $\mathrm{H}_1(F)$ for the M*-regular stable conjugacy classes in $\mathrm{H}(F)_{\mathrm{ell}}$, γ' runs over a set of representatives for the $\mathrm{M}(F)$-conjugacy classes in $\mathrm{M}(F)_{\mathrm{ell}}$, and $c(\mathrm{M}, \underline{\mathrm{H}})$ is as in* (41)*, i.e., as in* [MW16, Section I.4.17] *or equivalently* [LMW18, Section 4.6].

Proof. In this proof, any sum over δ_1 will range over representatives in $\mathrm{H}_1(F)$ for M-regular stable conjugacy classes in $\mathrm{H}(F)_{\mathrm{ell}}$, and any sum over γ' will range over representatives for $\mathrm{M}(F)$-conjugacy classes in $\mathrm{M}(F)_{\mathrm{ell}}$.

The first step is to study $\Theta^{\underline{\mathrm{H}}}$. We claim that for all $\delta_1 \in \mathrm{H}_1(F)_{\mathrm{ell}}$ we have:

$$\Theta^{\underline{\mathrm{H}}}(\delta_1) = c(\mathrm{M}, \underline{\mathrm{H}}) \kappa(\delta_1)^{-1} \cdot \sum_{\gamma'} D^{\mathrm{H}_1}(\delta_1)^{-1/2} D^{\mathrm{M}}(\gamma')^{1/2} \cdot \overline{\Delta(\delta_1, \gamma')} \Theta(\gamma'). \tag{47}$$

Consider the isomorphism $\iota_{\mathscr{Z}_1,\zeta_1} : D_{\mathscr{Z}_1,\zeta_1,\mathrm{ell}}(\mathrm{H}_1) \to \mathcal{I}_{\mathscr{Z}_1,\zeta_1,\mathrm{cusp}}(\mathrm{H}_1)$ analogous to $\iota_{\mathscr{Z},\zeta} : D_{\mathscr{Z},\zeta,\mathrm{ell}}(\mathrm{M}) \to \mathcal{I}_{\mathscr{Z},\zeta,\mathrm{cusp}}(\mathrm{M})$. It follows from Remark 3.4.7(ix) (specifically, (44)), and the discussion of Remark 3.4.7(x), that $\iota_{\mathscr{Z}_1,\zeta_1}$ carries $SD_{\mathscr{Z}_1,\zeta_1,\mathrm{ell}}(\mathrm{H}_1)$ to $\mathcal{SI}_{\mathscr{Z}_1,\zeta_1,\mathrm{cusp}}(\mathrm{H}_1) \subset \mathcal{I}_{\mathscr{Z}_1,\zeta_1,\mathrm{cusp}}(\mathrm{H}_1)$. More is true: if $\iota_{\mathscr{Z},\zeta}(\Theta) = a \in \mathcal{I}_{\mathscr{Z},\zeta,\mathrm{cusp}}(\mathrm{M})$ and $\iota_{\mathscr{Z}_1,\zeta_1}(\Theta^{\underline{\mathrm{H}}}) = b \in \mathcal{SI}_{\mathscr{Z}_1,\zeta_1,\mathrm{cusp}}(\mathrm{H}_1) \subset \mathcal{I}_{\mathscr{Z}_1,\zeta_1,\mathrm{cusp}}(\mathrm{H}_1)$, then, as explained in [LMW18, Section 4.6, slightly before (5)], then we have $b = c(\mathrm{M}, \underline{\mathrm{H}})b'$, where b' is the image of a under the endoscopic transfer map $\mathcal{I}_{\mathscr{X},\zeta,\mathrm{cusp}}(\mathrm{M}) \to \mathcal{SI}_{\mathscr{X}_1,\zeta_1,\mathrm{cusp}}(\mathrm{H}_1)$. Therefore, we have, for all M-regular $\delta_1 \in \mathrm{H}_1(F)_{\mathrm{ell}}$:

$$\Theta^{\underline{\mathrm{H}}}(\delta_1) = \overline{b(\delta_1)} = c(\mathrm{M}, \underline{\mathrm{H}})\overline{b'(\delta_1)} = c(\mathrm{M}, \underline{\mathrm{H}})\kappa(\delta_1)^{-1} \sum_{\gamma'} \overline{\Delta(\delta_1, \gamma') D^{\mathrm{H}_1}(\delta_1)^{-1/2} D^{\mathrm{M}}(\gamma')^{1/2} a(\gamma')},$$

where we used Remark 3.4.7(ix) (specifically (44)) at the first step, and and (46) at the third. Noting that $\overline{a(\gamma')} = \Theta(\gamma')$ by (44), (47) follows.

The computation of $\Theta^{\underline{\mathrm{H}},\mathrm{M}} = \mathbf{T}_{\underline{\mathrm{H}}}(\Theta^{\underline{\mathrm{H}}})$ in terms of $\Theta^{\underline{\mathrm{H}}}$ can be done using [Art96, Lemma 2.3], analogously to how [Li13, Proposition 5.3.2] is proved from [Li13, Lemma 5.3.1], and is what is reflected in the 'character value' form of character identities found in, e.g., [Kal15, Theorem 6.6]; we merely state the result:

$$\Theta^{\underline{\mathrm{H}},\mathrm{M}}(\gamma) = \mathbf{T}_{\underline{\mathrm{H}}}(\Theta^{\underline{\mathrm{H}}})(\gamma) = \sum_{\delta_1} D^{\mathrm{H}_1}(\delta_1)^{1/2} D^{\mathrm{M}}(\gamma)^{-1/2} \Delta(\delta_1, \gamma) \Theta^{\underline{\mathrm{H}}}(\delta_1).$$

Thus, using (47), $\Theta^{\underline{\mathrm{H}}}(\gamma)$ equals:

$$c(\mathrm{M}, \underline{\mathrm{H}}) \cdot \sum_{\delta_1} \kappa(\delta_1)^{-1} D^{\mathrm{H}_1}(\delta_1)^{1/2} D^{\mathrm{M}}(\gamma)^{-1/2} \Delta(\delta_1, \gamma) \sum_{\gamma'} D^{\mathrm{H}_1}(\delta_1)^{-1/2} D^{\mathrm{M}}(\gamma')^{1/2} \overline{\Delta(\delta_1, \gamma')} \Theta(\gamma').$$

Since the set of elements in $\mathrm{M}(F)_{\mathrm{srss}}$ that match a given $\delta_1 \in \mathrm{H}(F)_{\mathrm{srss}}$ form a single stable conjugacy class, we have $D^{\mathrm{M}}(\gamma')^{1/2} = D^{\mathrm{M}}(\gamma)^{1/2}$ for each γ' occurring in the above sum, so that the above expression equals the one given in the lemma. □

Proof of Lemma 3.4.5. We apply Lemma 3.4.8 with $\underline{\mathrm{H}}$ replaced by the principal endoscopic datum $\underline{\mathrm{M}}^*$ attached to M as in Notation 3.2.1(i). It is easy to compute that (every factor in the definition, in [MW16], of) $c(\mathrm{M}, \underline{\mathrm{M}}^*)$ equals 1, and then, assuming transfer factors to be normalized as in Remark 3.2.2(i), that $\Theta^{\mathrm{st}}_\sigma = \Theta^{\underline{\mathrm{M}}^*,\mathrm{M}}_\sigma \in SD_{\mathrm{ell}}(\mathrm{M})$. It follows from Remark 3.4.7(vii), and the fact that $\mathbf{T}_{\underline{\mathrm{M}}^*}$ defines an isomorphism $SD_{\mathrm{ell}}(\mathrm{M}^*) \to SD_{\mathrm{ell}}(\mathrm{M})$ (see Corollary 3.2.7), that $\Theta^{\underline{\mathrm{M}}^*,\mathrm{M}}_\sigma$ is the projection of Θ_σ to $SD_{\mathrm{ell}}(\mathrm{M})$, and the lemma follows. □

The terminology in the following definition is ad hoc:

Definition 3.4.9. (i) By a discrete series L-packet for M equipped with an endoscopic decomposition, we refer to a finite set $\Sigma \subset \mathrm{Irr}_2(\mathrm{M})$ with the property that:

(a) There exists a nonzero complex number c_σ for each $\sigma \in \Sigma$, such that $\sum_{\sigma\in\Sigma} c_\sigma \Theta_\sigma$ is a stable distribution; and

(b) For each elliptic endoscopic datum $\underline{\mathrm{H}} \in \mathcal{E}(\mathrm{M})$, with underlying endoscopic group H, choosing auxiliary data and hence the 5-tuple $(\mathrm{H}_1 \to \mathrm{H}, \hat{\xi}_1, \tilde{\mathrm{H}}_1 \to \tilde{\mathrm{H}}, \mathrm{C}_1, \mu)$ as in Notation 3.1.2(iii), there exists a stable elliptic virtual character $\Theta^{\underline{\mathrm{H}}} \in SD_{\mu,\mathrm{ell}}(\mathrm{H}_1)$ on $\tilde{\mathrm{H}}_1(F) = \mathrm{H}_1(F)$, such that (letting $\underline{\mathrm{H}}$ vary in $\mathcal{E}(\mathrm{M})$ now) the following holds inside $D_{\mathrm{ell}}(\mathrm{M})$:

$$\sum_{\underline{\mathrm{H}}} \mathbb{C} \cdot \mathbf{T}_{\underline{\mathrm{H}}}(\Theta^{\underline{\mathrm{H}}}) = \sum_{\sigma\in\Sigma} \mathbb{C} \cdot \Theta_\sigma. \tag{48}$$

We will refer to (48) as an endoscopic decomposition for Σ.

(ii) Suppose that the triple $(\mathrm{M}, \tilde{\mathrm{M}}, \mathbf{a})$ and the associated character $\omega : \mathrm{M}(F) \to \mathbb{C}^\times$ are as in Notation 3.1.1, and that they satisfy the hypotheses there. By a *discrete series L-packet for* $(\tilde{\mathrm{M}}, \omega)$ *equipped with an endoscopic decomposition*, we refer to a pair $(\Sigma, \tilde{\Sigma})$ such that:

(a) Σ is a discrete series L-packet for M together with an endoscopic decomposition, in the sense of (i);

(b) $\tilde{\Sigma}$ is a finite set of (isomorphism classes of) representations of $(\tilde{\mathrm{M}}(F), \omega)$ such that the map that takes an $\tilde{\mathrm{M}}(F)$-representation to its underlying $\mathrm{M}(F)$-representation defines an injection $\tilde{\Sigma} \hookrightarrow \Sigma$; and

(c) For each $\underline{\mathrm{H}} \in \mathcal{E}(\tilde{\mathrm{M}}, \mathbf{a})$ with underlying endoscopic group H, choosing auxiliary data and hence the 5-tuple $(\mathrm{H}_1 \to \mathrm{H}, \hat{\xi}_1, \tilde{\mathrm{H}}_1 \to \tilde{\mathrm{H}}, \mathrm{C}_1, \mu)$ as in Notation 3.1.2(iii), there exists a stable elliptic virtual character $\Theta^{\underline{\mathrm{H}}} \in SD_{\mu,\mathrm{ell}}(\tilde{\mathrm{H}}_1)$ on $\tilde{\mathrm{H}}_1(F)$, such that the following holds inside $D_{\mathrm{ell}}(\tilde{\mathrm{M}},\omega)$:

$$\sum_{\underline{\mathrm{H}}} \mathbb{C}\cdot \mathbf{T}_{\underline{\mathrm{H}}}(\Theta^{\underline{\mathrm{H}}}) = \sum_{\tilde{\sigma}\in\tilde{\Sigma}} \mathbb{C}\cdot\Theta_{\tilde{\sigma}}. \tag{49}$$

Remark 3.4.10. By [Kal19, Theorem 6.3.4], it follows that when $p \gg 0$, the regular supercuspidal packets of Kaletha are equipped with an endoscopic decomposition in the sense of Definition 3.4.9(i). More generally, Kaletha-type results should give a large class of discrete series L-packets equipped with an endoscopic decomposition.

Now we prove that a discrete series L-packet equipped with an endoscopic decomposition is automatically atomically stable.

Proposition 3.4.11. *Suppose that the triple* $(\mathrm{M},\tilde{\mathrm{M}},\mathbf{a})$ *and the associated unitary character* $\omega : \mathrm{M}(F)\to\mathbb{C}^\times$ *are as in Notation 3.1.1, and that they satisfy the hypotheses there.*

- *(i) Suppose* $(\Sigma,\tilde{\Sigma})$ *is a discrete series L-packet for* $(\tilde{\mathrm{M}},\omega)$ *equipped with an endoscopic decomposition given by* (49) *(which in particular involves choosing auxiliary data, and involves some* $\Theta^{\underline{\mathrm{H}}} \in SD_{\mu,\mathrm{ell}}(\tilde{\mathrm{H}}_1)$ *for each* $\underline{\mathrm{H}} \in \mathcal{E}(\tilde{\mathrm{M}},\mathbf{a})$*). Fix* $\underline{\mathrm{H}}_0 \in \mathcal{E}(\tilde{\mathrm{M}},\mathbf{a})$*. Then any* $\Theta \in D_{\mathrm{ell}}(\tilde{\mathrm{M}},\omega)$ *that belongs to the image* $\mathbf{T}_{\underline{\mathrm{H}}_0}(SD_{\mu,\mathrm{ell}}(\tilde{\mathrm{H}}_{0,1})_{\mathrm{Aut}(\underline{\mathrm{H}}_0)})$ *of* $\mathbf{T}_{\underline{\mathrm{H}}_0}$ *can be uniquely written as* $c_0\mathbf{T}_{\underline{\mathrm{H}}_0}(\Theta^{\underline{\mathrm{H}}_0}) + c_\circ\Theta_\circ$*, where* $c_0, c_\circ \in \mathbb{C}$*, and* $\Theta_\circ$ *is supported on a set of representations each of whose underlying* $\mathrm{M}(F)$*-representations lies outside the image of* $\tilde{\Sigma}\hookrightarrow\Sigma$ *(note that* $\mathbf{T}_{\underline{\mathrm{H}}_0}(\Theta^{\underline{\mathrm{H}}_0})$ *is supported in* $\tilde{\Sigma}$ *by definition).*
- *(ii) Suppose* Σ *is a discrete series L-packet for* M *with an endoscopic decompostion* (48)*. Then* Σ *is an atomically stable discrete series L-packet.*

Proof. Let us see that (ii) follows from (i). By definition, (Σ,Σ) can be viewed as a discrete series packet for $(\mathrm{M},\mathbb{1})$ with an endoscopic decomposition. Fix $\underline{\mathrm{M}}^*$ and related objects as in Notation 3.2.1(i); thus, M^* is a quasi-split form of M. We also have $\Theta^{\underline{\mathrm{M}}^*} \in SD_{\mathrm{ell}}(\mathrm{M}^*)$ associated to Σ, as in (i), i.e., as in Definition 3.4.9(i). By the surjectivity assertion in Corollary 3.2.7, $SD_{\mathrm{ell}}(\mathrm{M}) = \mathbf{T}_{\underline{\mathrm{M}}^*}(SD_{\mathrm{ell}}(\mathrm{M}^*)) = \mathbf{T}_{\underline{\mathrm{H}}_0}(SD_{\mu,\mathrm{ell}}(\tilde{\mathrm{H}}_{0,1})_{\mathrm{Aut}(\underline{\mathrm{H}}_0)})$, where $\underline{\mathrm{H}}_0 = \underline{\mathrm{M}}^*$, for which we may and do take $\tilde{\mathrm{H}}_{0,1}$ to be $\mathrm{H}_{0,1}$ and μ to be trivial. Applying (i) to an arbitrary stable distribution in $\sum\mathbb{C}\cdot\Theta_\sigma$ (the sum being over $\sigma\in\Sigma$), and using the linear independence of characters, we see that the space of stable distributions in $\sum\mathbb{C}\cdot\Theta_\sigma$ is at most one dimensional, and spanned by $\mathbf{T}_{\underline{\mathrm{M}}^*}(\Theta^{\underline{\mathrm{M}}^*})$ if it is nonzero. But the requirement in Definition 3.4.9(i) that there exists a stable distribution $\sum c_\sigma\Theta_\sigma$ supported in Σ, with each c_σ nonzero, then forces that this space is indeed one dimensional with $\Theta_\Sigma := \sum c_\sigma\Theta_\sigma$ as a basis, which is therefore a scalar multiple of $\mathbf{T}_{\underline{\mathrm{M}}^*}(\Theta^{\underline{\mathrm{M}}^*})$. Applying (i) once again to an arbitrary stable virtual character $\Theta\in SD_{\mathrm{ell}}(\mathrm{M}) = \mathbf{T}_{\underline{\mathrm{M}}^*}(SD_{\mathrm{ell}}(\mathrm{M}^*))$ then shows that Σ satisfies the conditions of Definition 3.3.2, i.e., that it is atomically stable.

Thus, it remains to prove (i); we no longer have $\underline{\mathrm{H}}_0 = \underline{\mathrm{M}}^*$. We can write the given $\Theta \in \mathbf{T}_{\underline{\mathrm{H}}_0}(SD_{\mu,\mathrm{ell}}(\tilde{\mathrm{H}}_{0,1})_{\mathrm{Aut}(\underline{\mathrm{H}}_0)})$ of (49) as $c_0\mathbf{T}_{\underline{\mathrm{H}}_0}(\Theta^{\underline{\mathrm{H}}_0}) + \Theta_\circ$, where $c_0\in\mathbb{C}$, and $\Theta_\circ$ belongs to the image of $\mathbf{T}_{\underline{\mathrm{H}}_0}$ but is orthogonal in $D_{\mathrm{ell}}(\tilde{\mathrm{M}},\omega)$ to $\mathbf{T}_{\underline{\mathrm{H}}_0}(\Theta^{\underline{\mathrm{H}}_0})$ (even if $\mathbf{T}_{\underline{\mathrm{H}}_0}(\Theta^{\underline{\mathrm{H}}_0})$ is 0). By the definition of the inner product on $D_{\mathrm{ell}}(\tilde{\mathrm{M}},\omega)$ (see Notation 3.4.4), it is enough to see that $\Theta_\circ$ is orthogonal to $\Theta_{\tilde{\sigma}}$ for each $\tilde{\sigma}\in\tilde{\Sigma}$, or, equivalently by the decomposition (49), that it is orthogonal to each $\mathbf{T}_{\underline{\mathrm{H}}}(\Theta^{\underline{\mathrm{H}}})$ as $\underline{\mathrm{H}}$ varies through $\mathcal{E}(\tilde{\mathrm{M}},\mathbf{a})$. By definition, this is so for $\underline{\mathrm{H}} = \underline{\mathrm{H}}_0$, while, by Remark 3.4.7(vii), this is so for all $\underline{\mathrm{H}} \neq \underline{\mathrm{H}}_0$. This proves (i), as desired. □

Corollary 3.4.12. *If* $p \gg 0$*, the regular supercuspidal L-packets constructed in* [Kal19] *are atomically stable.*

Remark 3.4.13. If G is a quasi-split special orthogonal, symplectic or unitary group over F, so that Hypothesis 2.7.1 is satisfied by the work of Arthur and Mok ([Art13] and [Mok15]; see Proposition 7.2.2 for more details), then Corollary 3.4.12, in view of of Lemma 3.3.8(iii), implies that regular supercuspidal L-packets for G in the sense of Kaletha are also L-packets in the sense of Arthur and Mok (though we have no result comparing the relevant Langlands parametrizations).

An analogous comment applies with the work [Mg14] of Mœglin in place of [Art13] and [Mok15], provided one accounts for an outer automorphism in the even special orthogonal case.

Proof of Corollary 3.4.12. This follows from Remark 3.4.10 and Proposition 3.4.11. □

Remark 3.4.14. Lemma 3.4.8 is more involved than what was strictly needed to prove Lemma 3.4.5. The reason we went through Lemma 3.4.8 is to make the optimistic proposal that it might be possible in principle to start with the character of a single discrete series representation, and construct all the "unstable endoscopic characters" associated to the L-packet that contains it (for nice enough representations and packets). However, we do not know how far this can be used in practice to study, given the character of a single discrete series representation, the characters of the representations that belong to its L-packet.

4. The variety of infinitesimal characters and its harmonic analytic variant

4.1. **Cuspidal pairs and Bernstein varieties, and some variants.** Recall the collection $\{\mathcal{O}_{\mathrm{M}}\}_{\mathrm{M}}$ of groups, indexed by Levi subgroups $\mathrm{M} \subset \mathrm{G}$, from Notation 2.6.1.
In this section, we discuss three objects of relevance to us, $\Omega(\mathrm{G}), \Omega({}^L\mathrm{G})$, and (under Hypothesis 2.7.1) $\underline{\Omega}^{\mathrm{st}}(\mathrm{G})$: the first of these is the usual Bernstein variety, the second the variety of infinitesimal characters from [Hai14, Section 5], and the third a harmonic analytic variant of the second, the dependence on $\{\mathcal{O}_{\mathrm{M}}\}_{\mathrm{M}}$ of which (coming from Hypothesis 2.7.1) is suppressed but pointed to by the 'underline' in the notation.

Definition 4.1.1. (i) (see [Hai14, Section 3.3.1]) A cuspidal pair for G or for $\Omega(\mathrm{G})$ (which we will soon define) is a pair (M, σ), where $\mathrm{M} \subset \mathrm{G}$ is a Levi subgroup and σ is (the isomorphism class of) a supercuspidal representation of $\mathrm{M}(F)$. There is an obvious action of $\mathcal{O}_{\mathrm{G}}^+$ on the set of these cuspidal pairs, which we will refer to as 'conjugation'; it indeed restricts to the usual conjugation action on $\operatorname{Int} \mathrm{G}(F) \subset \mathcal{O}_{\mathrm{G}}^+$. The $\mathrm{G}(F)$-conjugacy class of a cuspidal pair (M, σ) will be denoted by $(\mathrm{M}, \sigma)_{\mathrm{G}}$, and its $\mathcal{O}_{\mathrm{G}}^+$-conjugacy class will be denoted by $\underline{(\mathrm{M}, \sigma)}$. The set of $\mathrm{G}(F)$-conjugacy classes of cuspidal pairs will be denoted by $\Omega(\mathrm{G})$, and the set of their $\mathcal{O}_{\mathrm{G}}^+$-conjugacy classes will be denoted by $\underline{\Omega}(\mathrm{G})$.

(ii) (Based on the discussion in [Hai14, Section 5.3]) A cuspidal pair for $\Omega({}^L\mathrm{G})$ (which we will soon define) is a pair $(\mathcal{M}, \lambda)$ consisting of a Levi subgroup $\mathcal{M} \subset {}^L\mathrm{G}$ and a homomorphism $\lambda : W_F \to \mathcal{M} \subset {}^L\mathrm{G}$ that is admissible (in the sense used in Definition 2.9.1), such that $\mathcal{M}$ is minimal among Levi subgroups of ${}^L\mathrm{G}$ that contain $\lambda(W_F)$. Denote its $\hat{\mathrm{G}}$-conjugacy class by $(\mathcal{M}, \lambda)_{\hat{\mathrm{G}}}$. It is easy to see that $\mathcal{O}_{\mathrm{G}}$ acts on the set of $\hat{\mathrm{G}}$-conjugacy classes of cuspidal pairs; we denote the $\mathcal{O}_{\mathrm{G}}$-orbit of $(\mathcal{M}, \lambda)_{\hat{\mathrm{G}}}$ by $\underline{(\mathcal{M}, \lambda)}$. Let $\Omega({}^L\mathrm{G})$ be the set of $\hat{\mathrm{G}}$-conjugacy classes of these cuspidal pairs, and $\underline{\Omega}({}^L\mathrm{G})$ the set of $\mathcal{O}_{\mathrm{G}}$-orbits in $\Omega({}^L\mathrm{G})$. This definition of $\Omega({}^L\mathrm{G})$ agrees with Definition 2.9.1; see Remark 4.1.3(a) below.

(iii) Assume Hypothesis 2.7.1. A cuspidal pair for $\underline{\Omega}^{\mathrm{st}}(\mathrm{G})$ (which we will soon define) is a pair (M, Σ), where $\mathrm{M} \subset \mathrm{G}$ is a Levi subgroup, and $\Sigma \in \Phi_2^+(\mathrm{M})$ (see Notation 2.7.6(i)) is a packet consisting entirely of supercuspidal representations. There is an obvious action of $\mathcal{O}_{\mathrm{G}}^+$ on this set, thanks to Lemma 2.7.3(ii), which we will refer to as 'conjugation'. Denote the $\mathcal{O}_{\mathrm{G}}^+$-conjugacy class of (M, Σ) in this sense by $\underline{(\mathrm{M}, \Sigma)}$. Let $\underline{\Omega}^{\mathrm{st}}(\mathrm{G})$ be the set of $\mathcal{O}_{\mathrm{G}}^+$-conjugacy classes of these analogues of cuspidal pairs. By $\underline{(\mathrm{M}, \Sigma)}_{\mathrm{M}}$ and $\underline{\Omega}^{\mathrm{st}}(\mathrm{M})$, we will refer to the analogues of $\underline{(\mathrm{M}, \Sigma)}$ and $\underline{\Omega}^{\mathrm{st}}(\mathrm{G})$, where G and $\mathcal{O}_{\mathrm{G}}$, are replaced by M and $\mathcal{O}_{\mathrm{M}}$, respectively. Note that for any Levi subgroup $\mathrm{M} \subset \mathrm{G}$, $\underline{(\mathrm{M}, \Sigma)}_{\mathrm{M}}$ is singleton: this is because, since Σ is $\mathcal{O}_{\mathrm{M}}$-invariant, (M, Σ) is fixed by any automorphism of M of the form $\beta \circ \operatorname{Int} m \in \mathcal{O}_{\mathrm{M}}^+$ with $\beta \in \mathcal{O}_{\mathrm{M}}$ and $m \in \mathrm{M}(F)$.

Notation 4.1.2. In this subsection, we will often refer to Case (i), Case (ii) or Case (iii) to describe the setting of (i), (ii) or (iii) of Definition 4.1.1 above. The term 'cuspidal pair' will refer to a notion in (i), (ii) or (iii) above, and these cases will be distinguished by the context. Note that Case (iii) only makes sense under Hypothesis 2.7.1, which will be implicitly assumed whenever we talk of Case (iii). We warn that $\underline{\Omega}^{\mathrm{st}}(\mathrm{G})$ may only be a meaningful construction (beyond technical validity) when G is quasi-split; we do not know if "$\underline{\Omega}^{\mathrm{st}}(\mathrm{G}^*)$" is the correct thing to consider in its place.

Remark 4.1.3. (a) In (ii) above, $\lambda : W_F \to \hat{\mathrm{G}}$ does not determine $\mathcal{M}$ uniquely, but it uniquely determines the conjugacy class $(\mathcal{M}, \lambda)_{\hat{\mathrm{G}}}$ — this is because, as discussed a bit before [Hai14, Lemma 5.3.1], any two possibilities for $\mathcal{M}$ associated to a given λ are conjugate under the centralizer of $\lambda(W_F)$ in $\hat{\mathrm{G}}$, by [Bor79, Proposition 3.6]. Thus $\Omega({}^L\mathrm{G})$ is also the set of infinitesimal characters for G, agreeing with Definition 2.9.1.

(b) If $\mathcal{O} = \mathcal{O}_{\mathrm{G}}$ is trivial, or if $\mathcal{O} \subset \mathrm{Int}\,\mathrm{G}(F)$, then we could set $\Omega^{\mathrm{st}}(\mathrm{G}) = \underline{\Omega}^{\mathrm{st}}(\mathrm{G})$ — this is the object we are really interested in. $\underline{\Omega}^{\mathrm{st}}(\mathrm{G})$ is what $\Omega^{\mathrm{st}}(\mathrm{G})/\mathcal{O} = \Omega^{\mathrm{st}}(\mathrm{G})/\mathcal{O}_{\mathrm{G}}^{+}$ would be, if we could also construct $\Omega^{\mathrm{st}}(\mathrm{G})$ by assuming Hypothesis 2.7.1 to hold with $\{\mathcal{O}_{\mathrm{M}}\}_{\mathrm{M}}$ replaced by a similar collection $\{\mathcal{O}'_{\mathrm{M}}\}_{\mathrm{M}}$, but satisfying that $\mathcal{O}'_{\mathrm{G}} \subset \mathrm{Int}\,\mathrm{G}(F)$.

The following proposition is the main result of this subsection:

Proposition 4.1.4. *(i)* $\Omega(\mathrm{G})$ *(resp.,* $\underline{\Omega}(\mathrm{G})$*) has a unique structure as a countable union of reduced affine varieties over* $\mathbb{C}$*, such that for any variety* X *over* $\mathbb{C}$*, a set-theoretic map* $f : \Omega(\mathrm{G}) \to \mathrm{X}$ *(resp.,* $f : \underline{\Omega}(\mathrm{G}) \to \mathrm{X}$*) is regular if and only if for each cuspidal pair* (M, σ)*,* $\chi \mapsto f((\mathrm{M}, \sigma \otimes \chi)_{\mathrm{G}})$ *(resp.,* $\chi \mapsto f(\underline{(\mathrm{M}, \sigma \otimes \chi)})$*) is a regular map* $X^{\mathrm{unr}}(\mathrm{M}) \to \mathrm{X}$ *(or equivalently, a regular map* $X^{\mathrm{unr}}(\mathrm{S_M}) \to \mathrm{X}$*).*

(ii) $\Omega({}^L\mathrm{G})$ *(resp.,* $\underline{\Omega}({}^L\mathrm{G})$*) has a unique structure as a countable union of reduced affine varieties over* $\mathbb{C}$*, such that for any variety* X *over* $\mathbb{C}$*, a set-theoretic map* $f : \Omega({}^L\mathrm{G}) \to \mathrm{X}$ *(resp.,* $f : \underline{\Omega}({}^L\mathrm{G}) \to \mathrm{X}$*) is regular if and only if for each cuspidal pair* $(\mathcal{M}, \lambda)$*,* $z \mapsto f((\mathcal{M}, z\cdot\lambda)_{\hat{\mathrm{G}}})$ *(resp.,* $z \mapsto f(\underline{(\mathcal{M}, z\cdot\lambda)})$*) is a regular map* $(\mathrm{Z}_{\mathcal{M}^0}^{I_F})^0_{\mathrm{Fr}} \to \mathrm{X}$ *(or equivalently, a regular map* $Z^0_{\mathcal{M}} \to \mathrm{X}$*), where the actions of* I_F *and* Fr *are as recalled in Notation 4.1.8 below, and* $z\cdot\lambda$ *is as in Notation 4.1.11 below.*

(iii) $\underline{\Omega}^{\mathrm{st}}(\mathrm{G})$ *has a unique structure as a countable union of reduced affine varieties over* $\mathbb{C}$*, such that for any variety* X *over* $\mathbb{C}$*, a set-theoretic map* $f : \underline{\Omega}^{\mathrm{st}}(\mathrm{G}) \to \mathrm{X}$ *is regular if and only if for each cuspidal pair* (M, Σ)*,* $\chi \mapsto f(\underline{(\mathrm{M}, \Sigma \otimes \chi)})$ *is a regular map* $X^{\mathrm{unr}}(\mathrm{M}) \to \mathrm{X}$ *(or equivalently, a regular map* $X^{\mathrm{unr}}(\mathrm{S_M}) \to \mathrm{X}$*).*

Remark 4.1.5. (i) The assertion (i) of Proposition 4.1.4 is due to Bernstein. (ii) is due to Vogan and Haines. The formulation of (ii) given above follows the latter more closely (see [Hai14, Section 5.3]), and is motivated by the isomorphisms $X^{\mathrm{unr}}(\mathrm{M}) \to (\mathrm{Z}_{\hat{\mathrm{M}}}^{I_F})^0_{\mathrm{Fr}}$ and $X^{\mathrm{unr}}(\mathrm{S_M}) \to (\mathrm{Z}_{\hat{\mathrm{M}}})^{\Gamma,0}$ recalled in (19).

(ii) In (i) and henceforth, for $\chi \in X^*(\mathrm{S_M})$, $\sigma \otimes \chi$ refers to the tensor product of σ and the image of χ under $X^{\mathrm{unr}}(\mathrm{S_M}) \to X^{\mathrm{unr}}(\mathrm{M})$. A similar comment applies to the objects $\Sigma \otimes \chi$ in (iii). In each of the assertions, we have written X for $\mathrm{X}(\mathbb{C})$, since X is a complex variety.

Before proving Proposition 4.1.4, we need to complete its statement by being precise about two of its ingredients (namely, Notation 4.1.8 and Notation 4.1.11 below). However, we can already derive two corollaries:

Corollary 4.1.6. *For any cuspidal pair* (M, σ) *(resp.,* $(\mathcal{M}, \lambda)$*; resp.,* (M, Σ)*) for Case (i) (resp., Case (ii); resp., Case (iii)), the maps from* $X^{\mathrm{unr}}(\mathrm{S_M})$ *to* $\Omega(\mathrm{G})$ *and* $\underline{\Omega}(\mathrm{G})$ *(resp., from* $\mathrm{Z}^0_{\mathcal{M}}$ *to* $\Omega({}^L\mathrm{G})$ *and* $\underline{\Omega}({}^L\mathrm{G})$*; resp., from* $X^{\mathrm{unr}}(\mathrm{S_M})$ *to* $\underline{\Omega}^{\mathrm{st}}(\mathrm{G})$*) given by* $\chi \mapsto (\mathrm{M}, \sigma\otimes\chi)_{\mathrm{G}}$ *and* $\chi \mapsto \underline{(\mathrm{M}, \sigma \otimes \chi)}$ *(resp.,* $z \mapsto (\mathcal{M}, z\cdot\lambda)_{\hat{\mathrm{G}}}$ *and* $z \mapsto \underline{(\mathcal{M}, z\cdot\lambda)}$*; resp.,* $\chi \mapsto \underline{(\mathrm{M}, \Sigma \otimes \chi)}$*) are regular. Moreover, the preceding statement also holds with* $X^{\mathrm{unr}}(\mathrm{S_M})$ *replaced by* $X^{\mathrm{unr}}(\mathrm{M})$ *(resp.,* $\mathrm{Z}^0_{\mathcal{M}}$ *replaced by* $(\mathrm{Z}_{\mathcal{M}^0}^{I_F})^0_{\mathrm{Fr}}$*; resp.,* $X^{\mathrm{unr}}(\mathrm{S_M})$ *replaced by* $X^{\mathrm{unr}}(\mathrm{M})$*).*

Proof. This follows from applying Proposition 4.1.4 to the identity map(s) $\Omega(\mathrm{G}) \to \Omega(\mathrm{G})$ and $\underline{\Omega}(\mathrm{G}) \to \underline{\Omega}(\mathrm{G})$ (resp., $\Omega({}^L\mathrm{G}) \to \Omega({}^L\mathrm{G})$ and $\underline{\Omega}({}^L\mathrm{G}) \to \underline{\Omega}({}^L\mathrm{G})$; resp., $\underline{\Omega}^{\mathrm{st}}(\mathrm{G}) \to \underline{\Omega}^{\mathrm{st}}(\mathrm{G})$). □

Corollary 4.1.7. *The actions of either of* $\mathcal{O}$ *or* $\mathcal{O}_{\mathrm{G}}^{+}$ *on* $\Omega(\mathrm{G})$ *and* $\Omega({}^L\mathrm{G})$ *factor through a finite quotient that acts algebraically. Moreover, the obvious maps* $\Omega(\mathrm{G}) \to \underline{\Omega}(\mathrm{G})$ *and* $\Omega({}^L\mathrm{G}) \to \underline{\Omega}({}^L\mathrm{G})$ *(given by* $(\mathrm{M}, \sigma)_{\mathrm{G}} \to \underline{(\mathrm{M}, \sigma)}$ *and* $(\mathcal{M}, \lambda)_{\hat{\mathrm{G}}} \to \underline{(\mathcal{M}, \lambda)}$*, respectively) identify with the quotients of their sources by this finite group.*

Proof. The existence of the finite quotient in the first assertion follows from the fact that $\mathcal{O}_{\mathrm{G}}^{+}$ has finite image in $\mathrm{Out}(\mathrm{G})$, and hence contains $\mathrm{Int}\,\mathrm{G}(F)$ with finite index. The algebraicity of this action is an easy consequence of Proposition 4.1.4 together with Corollary 4.1.6. Since quotients by finite abstract groups in characteristic zero are good and hence categorical, the second assertion follows if we prove that the maps $\Omega(\mathrm{G}) \to \underline{\Omega}(\mathrm{G})$ and $\Omega({}^{L}\mathrm{G}) \to \underline{\Omega}(\mathrm{G})$ identify the coordinate ring of their target with the ring of $\mathcal{O}_{\mathrm{G}}^{+}$-invariants of the coordinate ring of their source. This in turn follows from Proposition 4.1.4 and the set-theoretic version of the second assertion (note that this set-theoretic version is immediate from the definitions). □

Notation 4.1.8. For any Levi subgroup $\mathcal{M} \subset {}^{L}\mathrm{G}$, we will consider $\mathrm{Z}_{\mathcal{M}^0}$ as a W_F-module as in Remark 2.3.2, specifically, (8). This in fact makes $\mathrm{Z}_{\mathcal{M}^0}$ into a Γ-module, since the action of W_F on $\mathrm{Z}_{\mathcal{M}^0}$ can be realized by combining Proposition 2.4.5 with Definition 2.3.3(iv), and hence factors through W_F/W_E for a finite extension E/F of F in $\bar{F}$. Thus, W_F acts via a chain

$$W_F = \mathcal{M}/\mathcal{M}^0 \to \mathrm{Aut}(\mathcal{M}^0) \to \mathrm{Out}(\mathcal{M}^0) \to \mathrm{Aut}(\mathrm{Z}_{\mathcal{M}^0}).$$

This action does not arise 'naively' from some fixed identification ${}^{L}G = \hat{\mathrm{G}} \rtimes W_F$, but it does arise from a preferred section $W_F \to \mathcal{M}$ as provided by Proposition 2.4.5 (see Notation 2.4.6).

Remark 4.1.9. If we identify ${}^{L}\mathrm{G}$ with $\hat{\mathrm{G}} \rtimes W_F$ using a preferred section, and fix a pinning of $\hat{\mathrm{G}}$ preserved by W_F, then given a standard Levi subgroup $\mathcal{M}$ of ${}^{L}\mathrm{G}$, the action of W_F on $\mathrm{Z}_{\mathcal{M}^0}$, as defined using Notation 4.1.8, agrees with the 'usual' action of $W_F \subset \hat{\mathrm{G}} \rtimes W_F = {}^{L}\mathrm{G}$ on $\hat{\mathrm{G}} \supset \mathrm{Z}_{\mathcal{M}^0}$. Moreover, if $\mathcal{M}_1 \subset {}^{L}\mathrm{G}$ is a Levi subgroup, we can choose a (non-unique) standard Levi subgroup $\mathcal{M}_2 \subset {}^{L}\mathrm{G}$ that is conjugate to $\mathcal{M}_1$ under some $\hat{g} \in \hat{\mathrm{G}}$, so that $\mathrm{Int}\,\hat{g}$ induces an isomorphism from the W_F-module $\mathrm{Z}_{\mathcal{M}_1^0}$ to the W_F-module $\mathrm{Z}_{\mathcal{M}_2^0}$. Using these two observations, it is easy to see that in what follows, the considerations and definitions that we make concerning the action of W_F on $\mathrm{Z}_{\mathcal{M}^0}$ are consistent with analogous ones in [Hai14], which typically follows the approach of reducing to the case of standard Levi subgroups of ${}^{L}\mathrm{G}$, and then using the 'usual' action of W_F.

Notation 4.1.10. Let $\mathcal{M} \subset {}^{L}\mathrm{G}$ be a Levi subgroup. We write $\Omega(\mathcal{M})_0$ for the set of $\mathcal{M}^0$-conjugacy classes of cuspidal pairs of the form $(\mathcal{M}, \lambda)$, and $(\mathcal{M}, \lambda)_{\mathcal{M}^0}$ for the $\mathcal{M}^0$-conjugacy class of a given cuspidal pair of the form $(\mathcal{M}, \lambda)$. Moreover, we define an action of $H^1(W_F, \mathrm{Z}_{\mathcal{M}^0})$ on $\Omega(\mathcal{M})_0$ as in Notation 2.8.4(ii): if $\dot{\alpha} \in Z^1(W_F, \mathrm{Z}_{\mathcal{M}^0})$ is a 1-cocycle representing $\alpha \in H^1(W_F, \mathrm{Z}_{\mathcal{M}^0})$, then for any cuspidal pair of the form $(\mathcal{M}, \lambda)$, we have $\alpha \cdot (\mathcal{M}, \lambda)_{\mathcal{M}^0} = (\mathcal{M}, \dot{\alpha} \cdot \lambda)_{\mathcal{M}^0}$, where $\dot{\alpha} \cdot \lambda : W_F \to \mathcal{M} \hookrightarrow {}^{L}\mathrm{G}$ takes w to $\dot{\alpha}(w)\lambda(w) \in \mathrm{Z}_{\mathcal{M}^0} \cdot \mathcal{M} = \mathcal{M}$ — it is easy to check that $(\mathcal{M}, \dot{\alpha} \cdot \lambda)_{\mathcal{M}^0}$ is indeed a cuspidal pair, whose image in $\Omega(\mathcal{M})_0$ depends only on α and $(\mathcal{M}, \lambda)_{\mathcal{M}^0}$.

Notation 4.1.11. Given a Levi subgroup $\mathcal{M} \subset {}^{L}\mathrm{G}$, we get from Notation 4.1.10 actions of $\mathrm{Z}_{\mathcal{M}}^0 = \mathrm{Z}_{\mathcal{M}^0}^{\Gamma,0}$ and $(\mathrm{Z}_{\mathcal{M}^0}^{I_F})_{\mathrm{Fr}}^0 \subset H^1(W_F, \mathrm{Z}_{\mathcal{M}}^0)$ (recall that $(\mathrm{Z}_{\mathcal{M}^0}^{I_F})_{\mathrm{Fr}}^0$ is interpreted as in Remark 2.5.9) on $\Omega(\mathcal{M})_0$, via the chain of obvious maps:

$$H^1(W_F/I_F, \mathrm{Z}_{\mathcal{M}}^0) = \mathrm{Z}_{\mathcal{M}}^0 = \mathrm{Z}_{\mathcal{M}^0}^{\Gamma,0} \twoheadrightarrow (\mathrm{Z}_{\mathcal{M}^0}^{I_F})_{\mathrm{Fr}}^0 \hookrightarrow H^1(W_F/I_F, \mathrm{Z}_{\mathcal{M}^0}^{I_F}) \subset H^1(W_F, \mathrm{Z}_{\mathcal{M}^0}). \tag{50}$$

If $\mathcal{M} = \iota_{\mathrm{G},\mathrm{M}}({}^{L}\mathrm{M})$ for a Levi subgroup $\mathrm{M} \subset \mathrm{G}$, these maps can be identified with those in the top row of (19). The composite of this chain is the obvious map $H^1(W_F/I_F, \mathrm{Z}_{\mathcal{M}}^0) \to H^1(W_F, \mathrm{Z}_{\mathcal{M}^0})$. Note that the action of $\mathrm{Z}_{\mathcal{M}}^0$ is even well-defined on the set of admissible homomorphisms $\lambda : W_F \to \mathcal{M} \subset {}^{L}\mathrm{G}$, while that of $(\mathrm{Z}_{\mathcal{M}^0}^{I_F})_{\mathrm{Fr}}^0$ is only well-defined at the level of $\mathcal{M}^0$-conjugacy classes of such λ.

The following lemma is more or less [Hai14, Lemma 5.3.7], and will be used in the proof of Proposition 4.1.4.

Lemma 4.1.12. *Let $\mathcal{M} \subset {}^{L}\mathrm{G}$ be a Levi subgroup. Then the actions of $(\mathrm{Z}_{\mathcal{M}^0}^{I_F})_{\mathrm{Fr}}^0$ and $\mathrm{Z}_{\mathcal{M}}^0$ on $\Omega(\mathcal{M})_0$ (see Notation 4.1.11) have finite stabilizers.*

Proof. Since the map $\mathrm{Z}_{\mathcal{M}}^0 \to H^1(W_F, \mathrm{Z}_{\mathcal{M}^0})$ (see (50)) clearly has finite kernel, it suffices to show that the stabilizer of any element of $\Omega(\mathcal{M})_0$ in $H^1(W_F, \mathrm{Z}_{\mathcal{M}^0})$ is finite. Such a stabilizer is easily seen to be contained in the kernel of $H^1(W_F, \mathrm{Z}_{\mathcal{M}^0}) \to H^1(W_F, \mathcal{M}^0/(\mathcal{M}^0)_{\mathrm{der}})$ (e.g., one can use a preferred section to realize $\Omega(\mathcal{M})_0$ inside a suitably defined $H^1(W_F, \mathcal{M}^0)$, as in Remark 2.8.3), which is finite, since $\mathrm{Z}_{\mathcal{M}^0} \to \mathcal{M}^0/(\mathcal{M}^0)_{\mathrm{der}}$ is an isogeny. □

Proof of Proposition 4.1.4. The uniqueness assertions are clear, since the relevant rings of regular functions are determined by the given criterion. Thus, it is enough to prove the existence of the claimed structures.

Let us prove (i). It is enough to prove the assertion for $\underline{\Omega}(\mathrm{G})$ (the assertion for $\Omega(\mathrm{G})$ then follows by considering the case where $\mathcal{O}$ is trivial). Further, it is enough to prove the assertion with $X^{\mathrm{unr}}(\mathrm{S_M})$ in place of $X^{\mathrm{unr}}(\mathrm{M})$, the latter being a quotient of the former by a finite subgroup. We have a decomposition:

$$\underline{\Omega}(\mathrm{G}) = \bigsqcup_{\mathrm{M}} \Omega(\mathrm{M})_0/\mathcal{O}^+_{\mathrm{G,M}}, \tag{51}$$

where M runs over a set of representatives for the $\mathcal{O}^+_{\mathrm{G}}$-orbits of Levi subgroups of G, $\mathcal{O}^+_{\mathrm{G,M}}$ as in (the $\mathcal{O}^+_{\mathrm{M,L}}$ of (iii) of) Notation 2.6.1, and $\Omega(\mathrm{M})_0$ is the 'cuspidal' subset of $\Omega(\mathrm{M})$ consisting of the images of cuspidal pairs for $\Omega(\mathrm{M})$ that are of the form (M,σ). This reduces to defining, for a fixed Levi subgroup $\mathrm{M} \subset \mathrm{G}$, a structure on $\Omega(\mathrm{M})_0/\mathcal{O}^+_{\mathrm{G,M}}$ as a countable union of reduced affine varieties, satisfying a description analogous to that in the proposition: to see this reduction, use that for each cuspidal pair (M,σ), each $\beta \in \mathcal{O}^+_{\mathrm{G}}$ and each map $f : \underline{\Omega}(\mathrm{G}) \to \mathrm{Y}$ for some variety Y over $\mathbb{C}$, the map $X^{\mathrm{unr}}(\mathrm{S_M}) \to \mathrm{Y}$ given by $\chi \mapsto f(\underline{(\mathrm{M},\sigma\otimes\chi)})$ is regular if and only if the map $X^{\mathrm{unr}}(\mathrm{S}_{\beta(\mathrm{M})}) \to \mathrm{Y}$ given by $\chi \mapsto f(\underline{(\beta(\mathrm{M}),(\sigma\circ\beta^{-1})\otimes\chi)})$ is regular, since the former is the pull-back of the latter under the isomorphism of varieties $X^{\mathrm{unr}}(\mathrm{S_M}) \to X^{\mathrm{unr}}(\mathrm{S}_{\beta(\mathrm{M})})$ induced by β^{-1}.

$\Omega(\mathrm{M})_0$ identifies with the set of cuspidal pairs for $\Omega(\mathrm{M})$ of the form (M,σ), and has an action of $X^{\mathrm{unr}}(\mathrm{S_M})$ on it, given by $\chi\cdot(\mathrm{M},\sigma) = (\mathrm{M},\sigma\otimes\chi)$. $\Omega(\mathrm{M})_0$ is a countable union of orbits for this action of $X^{\mathrm{unr}}(\mathrm{S_M})$. The stabilizers for this action are all finite (since $\mathrm{Hom_{cts}}(\mathrm{S_M}(F),\mathbb{C}^\times) \to \mathrm{Hom_{cts}}(\mathrm{Z_M}(F),\mathbb{C}^\times)$ has finite kernel), giving each orbit the structure of an affine variety over $\mathbb{C}$ (in fact that of a torsor under a complex torus), and giving $\Omega(\mathrm{M_0})$ the structure of a countable union of reduced affine varieties (the countability follows, e.g., from [Wal03, Theorem VIII.1.2]). Since quotients by finite abstract groups over fields of characteristic zero are good and hence categorical, it follows that given any variety X over $\mathbb{C}$, a set-theoretic map $f : \Omega(\mathrm{M})_0 \to \mathrm{X}$ is regular if and only if for each cuspidal pair for $\Omega(\mathrm{M})$ of the form (M,σ), the map $X^{\mathrm{unr}}(\mathrm{S_M}) \to \mathrm{X}$ given by $\chi \mapsto f((\mathrm{M},\sigma\otimes\chi)_{\mathrm{M}})$ is regular. Recall that $\mathrm{Int}\,\mathrm{M}(F) \subset \mathcal{O}^+_{\mathrm{G,M}}$ is of finite index (see Lemma 2.6.3(ii)). Hence, again applying the fact that a quotient by a finite group in characteristic zero is a good and hence categorical quotient, (i) follows (the reducedness of the coordinate rings of these components follows from $X^{\mathrm{unr}}(\mathrm{S_M})$ being reduced).

The proofs of (ii) and (iii) are similar, where we make the following additional remarks concerning the proof of (ii). The analogue of (51) needed for that proof takes the form:

$$\underline{\Omega}({}^L\mathrm{G}) = \bigsqcup_{\mathcal{M}} \Omega(\mathcal{M})_0/\mathcal{O}^+_{\mathrm{G},\mathcal{M}}, \tag{52}$$

similar to (24) (but with the relevance condition absent), with $\mathcal{M}$ running over a set of representatives for the orbits of Levi subgroups of ${}^L\mathrm{G}$ under the group $\mathcal{O}^+_{\mathrm{G}}({}^L\mathrm{G},{}^L\mathrm{G})$ of all automorphisms of ${}^L\mathrm{G}$ that are dual to an element of $\mathcal{O}^+_{\mathrm{G}}$. Here, $\mathcal{O}^+_{\mathrm{G},\mathcal{M}}$ is as in Notation 2.6.1(iii), and is easily seen to have finite image in $\mathrm{Out}(\mathcal{M}^0)$, and hence to act on $\Omega(\mathcal{M})_0$ through a finite quotient. Moreover, for any Levi subgroup $\mathcal{M} \subset {}^L\mathrm{G}$, the finiteness of the stabilizers for the action of $\mathrm{Z}^0_{\mathcal{M}}$ on $\Omega(\mathcal{M})_0$ was proved in Lemma 4.1.12 above. □

Remark 4.1.13. Let $\mathcal{M} \subset {}^L\mathrm{G}$ be a Levi subgroup. For later use, we make a note of the variety structure on $\Omega(\mathcal{M})_0$ imposed in the above proof: the complex torus $\mathrm{Z}^0_{\mathcal{M}}$ acts on $\Omega(\mathcal{M})_0$ with finite stabilizers (by Lemma 4.1.12), realizing $\Omega(\mathcal{M})_0$ as a countable union of torsors for complex tori. Thus, given any complex variety Y, a map $f : \Omega(\mathcal{M})_0 \to \mathrm{Y}$ is regular if and only if for each cuspidal pair for $\Omega({}^L\mathrm{G})$ of the form $(\mathcal{M},\lambda)$, the map $\mathrm{Z}^0_{\mathcal{M}} \to \mathrm{Y}$ given by $z \mapsto f((\mathcal{M}, z\cdot\lambda)_{\mathcal{M}^0})$ is regular.

Definition 4.1.14. In Case (i) (resp., Case (ii); resp., Case (iii)) of Definition 4.1.1, the connected component of $\Omega(\mathrm{G})$ or $\underline{\Omega}(\mathrm{G})$ (resp., $\Omega({}^L\mathrm{G})$ or $\underline{\Omega}({}^L\mathrm{G})$; resp., $\underline{\Omega}^{\mathrm{st}}(\mathrm{G})$) containing the image of a cuspidal pair (M, σ) (resp., $(\mathcal{M}, \lambda)$; resp., (M, Σ)) will be denoted by $\Omega(\mathrm{M}, \sigma) = \Omega([\mathrm{M}, \sigma]_{\mathrm{G}})$ or $\underline{\Omega}(\mathrm{M}, \sigma) = \underline{\Omega}(\underline{[\mathrm{M}, \sigma]})$ (resp., $\Omega(\mathcal{M}, \lambda) = \Omega([\mathcal{M}, \lambda]_{\hat{\mathrm{G}}})$ or $\underline{\Omega}(\mathcal{M}, \lambda) = \underline{\Omega}(\underline{[\mathcal{M}, \lambda]})$; resp., $\underline{\Omega}(\mathrm{M}, \Sigma) = \underline{\Omega}(\underline{[\mathrm{M}, \Sigma]})$).

Now let us explain why the variety structures on $\Omega(\mathrm{G})$ and $\Omega({}^L\mathrm{G})$ described in Proposition 4.1.4 agree with the ones in [Hai14].

Remark 4.1.15. (i) In Case (i), given a cuspidal pair (M, σ), it is easy to see that:

$$\Omega(\mathrm{M}, \sigma) = \Omega([\mathrm{M}, \sigma]_{\mathrm{G}}) = \{(\mathrm{M}, \sigma \otimes \chi)_{\mathrm{G}} \mid \chi \in X^{\mathrm{unr}}(\mathrm{S}_{\mathrm{M}})\} = \{(\mathrm{M}, \sigma \otimes \chi)_{\mathrm{G}} \mid \chi \in X^{\mathrm{unr}}(\mathrm{M})\}.$$

A similar description applies to $\Omega([\mathrm{M}, \sigma]_{\mathrm{M}}) \subset \Omega(\mathrm{M})_0 \subset \Omega(\mathrm{M})$, which we easily see from the proof of Proposition 4.1.4 to be a torsor for a finite quotient of either of $X^{\mathrm{unr}}(\mathrm{S}_{\mathrm{M}})$ or $X^{\mathrm{unr}}(\mathrm{M})$. It is also easy to see from the same proof that the the obvious map $\Omega([\mathrm{M}, \sigma]_{\mathrm{M}}) \to \Omega([\mathrm{M}, \sigma]_{\mathrm{G}})$ realizes its target as a quotient of the former by the finite group $W^{\mathrm{G}}_{[\mathrm{M}, \sigma]_{\mathrm{M}}}$ obtained as the quotient by $\mathrm{M}(F)$ of

$$\{g \in \mathrm{G}(F) \mid {}^g(\mathrm{M}, \sigma) = (\mathrm{M}, \sigma \otimes \chi) \text{ for some } \chi \in X^{\mathrm{unr}}(\mathrm{M}) \text{ (or some } \chi \in X^{\mathrm{unr}}(\mathrm{S}_{\mathrm{M}}))\}.$$

This shows that the variety structure on $\Omega(\mathrm{G}) \supset \Omega([\mathrm{M}, \sigma]_{\mathrm{G}})$ as explained in Proposition 4.1.4 agrees with the one seen more commonly in literature, e.g., [Hai14, Section 3.3.1].

(ii) In Case (ii), given a cuspidal pair $(\mathcal{M}, \lambda)$, we define:

$$\Omega([\mathcal{M}, \lambda]_{\mathcal{M}^0}) = \{(\mathcal{M}, z\lambda)_{\mathcal{M}^0} \mid z \in \mathrm{Z}^0_{\mathcal{M}}\} = \{(\mathcal{M}, z\lambda)_{\mathcal{M}^0} \mid z \in (\mathrm{Z}^{I_F}_{\mathcal{M}^0})^0_{\mathrm{Fr}}\} \subset \Omega(\mathcal{M})_0.$$

This is a finite quotient of $\mathrm{Z}^0_{\mathcal{M}}$ or equivalently of $(\mathrm{Z}^{I_F}_{\mathcal{M}^0})^0_{\mathrm{Fr}}$, by Lemma 4.1.12. It is easy to see from the proof of Proposition 4.1.4 that the obvious map $\Omega([\mathcal{M}, \lambda]_{\mathcal{M}^0}) \to \Omega([\mathcal{M}, \lambda]_{\hat{\mathrm{G}}})$ taking $(\mathcal{M}, \lambda')_{\mathcal{M}^0}$ to $(\mathcal{M}, \lambda')_{\hat{\mathrm{G}}}$ realizes its target as the quotient of its source by the finite abstract group $W^{\hat{\mathrm{G}}}_{[\mathcal{M}, \lambda]_{\mathcal{M}^0}}$ obtained as the quotient by $\mathcal{M}^0$ of

$$\{n \in \hat{\mathrm{G}} \mid \operatorname{Int} n(\mathcal{M}) = \mathcal{M} \text{ and } ({}^n(\mathcal{M}, \lambda))_{\mathcal{M}^0} = (\mathcal{M}, z\lambda)_{\mathcal{M}^0} \text{ for some } z \in (\mathrm{Z}^{I_F}_{\mathcal{M}^0})^0_{\mathrm{Fr}} \text{ (or some } z \in \mathrm{Z}^0_{\mathcal{M}})\}.$$

This shows that the variety structure on $\Omega({}^L\mathrm{G})$ explained in Proposition 4.1.4 agrees with the one defined in [Hai14], just after Lemma 5.3.8 of that reference.

4.2. Aside: $\Omega({}^L\mathrm{G})$ and inertial infinitesimal characters.

Definition 4.2.1. Given an admissible homomorphism $\lambda : W_F \to {}^L\mathrm{G}$, we denote its restriction to I_F by $\lambda_i : I_F \to {}^L\mathrm{G}$, and the $\hat{\mathrm{G}}$-conjugacy class of λ_i by $(\lambda_i)_{\hat{\mathrm{G}}}$. Any such $(\lambda_i)_{\hat{\mathrm{G}}}$ will be called an inertial infinitesimal character, and the set of all $(\lambda_i)_{\hat{\mathrm{G}}}$ as λ varies over admissible homomorphisms $W_F \to {}^L\mathrm{G}$ will be referred to as the set $\Omega_i({}^L\mathrm{G})$ of inertial infinitesimal characters.

Remark 4.2.2. Note that restriction from W_F to I_F gives us a map $\Omega({}^L\mathrm{G}) \to \Omega_i({}^L\mathrm{G})$, which is clearly constant on each connected component of $\Omega({}^L\mathrm{G})$. It is well-known that if $\mathrm{G} = \mathrm{GL}_n$, the fibers of this map are precisely the connected components of $\Omega({}^L\mathrm{G})$ (we recover this in Corollary 4.2.4 below), but this is not true in general, e.g., it fails if G is an anisotropic torus over F whose Néron model is not connected (use [Hai14, (3.3.2)] and the discussion around there).

The following lemma is almost certainly present in the literature in some form, e.g., some sentence in [Dat17] may specialize to it, but we give a proof for the convenience of the reader.

Lemma 4.2.3. *If $\lambda_i : I_F \to {}^L\mathrm{G}$ is (a representative for) an inertial infinitesimal character, and if the centralizer of $\lambda_i(I_F)$ in $\hat{\mathrm{G}}$ is connected, then the fiber of $\Omega({}^L\mathrm{G}) \to \Omega_i({}^L\mathrm{G})$ over $(\lambda_i)_{\hat{\mathrm{G}}}$ is a single connected component of $\Omega({}^L\mathrm{G})$.*

Proof. Suppose $\lambda, \lambda' : W_F \to {}^L\mathrm{G}$ are admissible homomorphisms that restrict to λ_i. Recall that Fr also denotes a lift of itself to W_F. Let C denote the centralizer of $\lambda_i(I_F)$ in $\hat{\mathrm{G}}$, so that $\lambda(\mathrm{Fr}), \lambda'(\mathrm{Fr}) \in {}^L\mathrm{G}$ both normalize $\mathrm{C} = \mathrm{C}^0$. We fix a Borel pair in C, whose underlying maximal torus we denote by T. Since $\lambda(\mathrm{Fr}), \lambda'(\mathrm{Fr}) \in {}^L\mathrm{G}$ are semisimple and clearly normalize C, we may and do assume each of them to preserve the chosen Borel pair of $\mathrm{C} = \mathrm{C}^0$ (e.g., use Lemma 2.8.5 and [Ste68, Section 7]). Let θ denote the automorphism of T given by $\operatorname{Int} \lambda(\mathrm{Fr})$. The centralizer in ${}^L\mathrm{G}$ of the identity component $(\mathrm{T}^\theta)^0$ of the subgroup T^θ of T fixed by θ, contains $\lambda(W_F)$, and is hence a Levi subgroup $\mathcal{M}' \subset {}^L\mathrm{G}$ (use [Bor79, Proposition 3.5]). Thus, a Levi subgroup $\mathcal{M} \subset \mathcal{M}'$ of ${}^L\mathrm{G}$ that minimally contains $\lambda(W_F)$ satisfies that $(\mathrm{T}^\theta)^0 \subset \mathrm{Z}^0_{\mathcal{M}}$. Therefore, it suffices to show that there exists $z \in (\mathrm{T}^\theta)^0$ such that $\lambda'(\mathrm{Fr})$ and $z\lambda(\mathrm{Fr})$ are T-conjugate (and hence C-conjugate). Since $\mathrm{Int}\,\lambda(\mathrm{Fr})\circ\lambda_i$ and $\mathrm{Int}\,\lambda'(\mathrm{Fr})\circ\lambda_i$ both equal $\lambda_i\circ\mathrm{Int}\,\mathrm{Fr}$, we have that $x := \lambda(\mathrm{Fr})^{-1}\lambda'(\mathrm{Fr})$ belongs to $\mathrm{C} = \mathrm{C}^0$. Clearly x preserves the chosen Borel pair of C, so that $x \in \mathrm{T}$. Thus, the T-conjugacy of $\lambda'(\mathrm{Fr})$ and $\lambda(\mathrm{Fr})$, and hence also the lemma, follows from the fact that $(\mathrm{T}^\theta)^0 \times ((1-\theta)\mathrm{T}) \to \mathrm{T}$ is surjective. □

Corollary 4.2.4. *If* $\mathrm{G} = \mathrm{GL}_n$*, then each fiber of* $\Omega({}^L\mathrm{G}) \to \Omega_i({}^L\mathrm{G})$ *is a single connected component of* $\Omega({}^L\mathrm{G})$*.*

Proof. This follows from Lemma 4.2.3 and the fact that the centralizer of a finite subgroup of $\mathrm{GL}_n(\mathbb{C})$ is a product of smaller general linear groups over $\mathbb{C}$ (associated to the multiplicity spaces for the restriction of the standard representation of $\mathrm{GL}_n(\mathbb{C})$ to this finite subgroup) and hence connected. □

4.3. **Maps between the (variants of) Bernstein varieties,** $\Omega(\mathrm{G}), \Omega({}^L\mathrm{G})$ **and** $\underline{\Omega}^{\mathrm{st}}(\mathrm{G})$**.** Under appropriate hypotheses, one wishes to relate ($\mathbb{C}[\Omega({}^L\mathrm{G})]$, or rather) $\mathbb{C}[\underline{\Omega}({}^L\mathrm{G})]$ and $\mathbb{C}[\underline{\Omega}^{\mathrm{st}}(\mathrm{G})]$ to subrings of $\mathbb{C}[\Omega(\mathrm{G})]$ (which can be identified with the Bernstein center $\mathcal{Z}(\mathrm{G})$), by defining maps $p_1 : \Omega(\mathrm{G}) \to \underline{\Omega}({}^L\mathrm{G})$ and $p_2 : \Omega(\mathrm{G}) \to \underline{\Omega}^{\mathrm{st}}(\mathrm{G})$.

4.3.1. *A map* $p_1 : \Omega(\mathrm{G}) \to \underline{\Omega}({}^L\mathrm{G})$. For this subsubsection, we assume the LLC+ hypothesis, Hypothesis 2.10.3. Recall from Remark 4.1.3(a) that $\Omega({}^L\mathrm{G})$ can be thought of as the set of infinitesimal characters for G, so that $\underline{\Omega}({}^L\mathrm{G})$ can be thought of as the set of $\mathcal{O}_{\mathrm{G}}$-orbits of infinitesimal characters for G. Given a cuspidal pair (M, σ) for $\Omega(\mathrm{G})$, Theorem 2.10.10 associates to it a well-defined element $\lambda(\varphi_\sigma) \in \underline{\Omega}({}^L\mathrm{M}) := \Omega({}^L\mathrm{M})/\mathcal{O}_{\mathrm{M}}$, where $\lambda(\cdot)$ is as in Notation 2.9.2. Choosing $\iota_{\mathrm{G},\mathrm{M}}$, we get by Remark 2.10.2(i) a well-defined map $(\mathrm{M}, \sigma) \mapsto \underline{\iota_{\mathrm{G},\mathrm{M}} \circ \lambda(\varphi_\sigma)}$ from the set of cuspidal pairs (M, σ) to $\underline{\Omega}({}^L\mathrm{G})$ — here, for an infinitesimal character λ' valued in $\hat{\mathrm{G}}$, we denote its image in $\underline{\Omega}({}^L\mathrm{G})$ by $\underline{\lambda}'$. By Theorem 2.10.10(iii) and Remark 2.10.2(iii), $\underline{\iota_{\mathrm{G},\mathrm{M}} \circ \lambda(\varphi_\sigma)}$ depends only on the $\mathcal{O}^+_{\mathrm{G}}$-orbit $\underline{(\mathrm{M}, \sigma)}$ of (M, σ), so this map factors through $\underline{\Omega}(\mathrm{G})$, i.e., descends to a well-defined map of sets $p_1 = p_{1,\mathrm{G}} : \underline{\Omega}(\mathrm{G}) \to \underline{\Omega}({}^L\mathrm{G})$, which will be also viewed as a map of sets $\Omega(\mathrm{G}) \to \underline{\Omega}({}^L\mathrm{G})$.

It is now easy to see from Remark 4.1.3(a) that the above description can be summarized as follows:

Definition 4.3.1. Assume the LLC+ hypothesis, Hypothesis 2.10.3. We write $p_1 = p_{1,\mathrm{G}}$ for either of the maps $\Omega(\mathrm{G}) \to \underline{\Omega}({}^L\mathrm{G})$ or $\underline{\Omega}(\mathrm{G}) \to \underline{\Omega}({}^L\mathrm{G})$, such that for any cuspidal pair (M, σ):

$$p_1((\mathrm{M}, \sigma)_{\mathrm{G}}) = p_1(\underline{(\mathrm{M}, \sigma)}) = \underline{(\mathcal{M}, \iota_{\mathrm{G},\mathrm{M}} \circ \lambda(\varphi_\sigma))}, \tag{53}$$

where $\mathcal{M}$ is any Levi subgroup of ${}^L\mathrm{G}$ minimally containing the image of $\iota_{\mathrm{G},\mathrm{M}}(\lambda(\varphi_\sigma))$, viewed (up to choices) as a map $W_F \to {}^L\mathrm{G}$. By Proposition 4.3.2 below, $p_{1,\mathrm{G}}$ is a map of varieties and will be viewed as such.

Proposition 4.3.2. *(Under Hypothesis 2.10.3)* $p_1 = p_{1,\mathrm{G}}$ *is a map of varieties, which is surjective at the level of* $\mathbb{C}$*-points if* G *is quasi-split.*

Proof. By Proposition 4.1.4, the first assertion follows if we show that for any cuspidal pair (M, σ), $\chi \mapsto p_1((\mathrm{M}, \sigma \otimes \chi)_{\mathrm{G}}) = p_1(\underline{(\mathrm{M}, \sigma \otimes \chi)})$ is a regular map $X^{\mathrm{unr}}(\mathrm{S}_{\mathrm{M}}) \to \underline{\Omega}({}^L\mathrm{G})$.

Let $\mathcal{M}$ be as in (53), i.e., minimal among the Levi subgroups of ${}^L\mathrm{G}$ containing the image of (a representative for) $\iota_{\mathrm{G},\mathrm{M}}(\lambda(\varphi_\sigma)) : W_F \to {}^L\mathrm{G}$. Without loss of generality, we assume $\mathcal{M} \subset \iota_{\mathrm{G},\mathrm{M}}({}^L\mathrm{M})$, so that $\iota_{\mathrm{G},\mathrm{M}}$ gives an injective homomorphism of tori $\iota_{\mathrm{G},\mathrm{M}} : \mathrm{Z}^0_{{}^L\mathrm{M}} \cong \mathrm{Z}^0_{\iota_{\mathrm{G},\mathrm{M}}({}^L\mathrm{M})} \hookrightarrow \mathrm{Z}^0_{\mathcal{M}}$. Write $\chi \mapsto \alpha_\chi$ for the isomorphism $X^{\mathrm{unr}}(\mathrm{S}_{\mathrm{M}}) \to \mathrm{Z}^{\Gamma,0}_{\hat{\mathrm{M}}} = \mathrm{Z}^0_{{}^L\mathrm{M}}$ (see the left vertical arrow of (19)), and set $z_\chi = \iota_{\mathrm{G},\mathrm{M}}(\alpha_\chi) \in \mathrm{Z}^0_{\mathcal{M}}$. Thus, $\chi \mapsto z_\chi$ is an injective homomorphism of tori $X^{\mathrm{unr}}(\mathrm{S}_{\mathrm{M}}) \to \mathrm{Z}^0_{\mathcal{M}}$. Write λ' for $\iota_{\mathrm{G},\mathrm{M}}(\lambda(\varphi_\sigma))$ to lighten the notation. We have for all $\chi \in X^{\mathrm{unr}}(\mathrm{S}_{\mathrm{M}})$:

$$p_1((\mathrm{M}, \sigma \otimes \chi)_{\mathrm{G}}) = \underline{(\mathcal{M}, \iota_{\mathrm{G},\mathrm{M}}(\lambda(\varphi_{\sigma\otimes\chi})))} = \underline{(\mathcal{M}, \iota_{\mathrm{G},\mathrm{M}}(\alpha_\chi \cdot \lambda(\varphi_\sigma)))} = \underline{(\mathcal{M}, z_\chi \cdot \lambda')},$$

where the second equality uses the desideratum with respect to twisting by unramified characters, from Theorem 2.10.10(i). Now the fact that $\chi \mapsto z_\chi$ is an injective algebraic homomorphism of tori from $X^{\mathrm{unr}}(\mathrm{S_M})$ to $\mathrm{Z}^0_{\mathcal{M}}$, together with Corollary 4.1.6, shows that $\chi \mapsto p_1((\mathrm{M}, \sigma \otimes \chi)_\mathrm{G})$ is indeed a regular morphism $X^{\mathrm{unr}}(\mathrm{S_M}) \to \underline{\Omega}({}^L\mathrm{G})$, finishing the proof of the first assertion.
Now assume that G is quasi-split, and let $(\mathcal{M}, \lambda)$ be a cuspidal pair for $\underline{\Omega}({}^L\mathrm{G})$. Then $\mathcal{M}$ is automatically relevant, so $\mathcal{M} = \iota_{\mathrm{G,M}}({}^L\mathrm{M})$ for some Levi subgroup $\mathrm{M} \subset \mathrm{G}$ and some choice of $\iota_{\mathrm{G,M}}$ (use Corollary 2.4.12). Moreover, M is also quasi-split, and hence we can write $\lambda = \iota_{\mathrm{G,M}} \circ \varphi$ for some admissible relevant homomorphism $\varphi : W_F \to {}^L\mathrm{M}$. Since $\mathcal{M}$ contains λ minimally, φ represents an element of $\Phi_2^+(\mathrm{M})$, and hence by Theorem 2.10.10, we can write $\varphi = \varphi_\sigma \in \Phi_2^+(\mathrm{M})$ for some $\sigma \in \mathrm{Irr}_2^+(\mathrm{M})$ (viewing φ as a homomorphism $W'_F \to {}^L\mathrm{M}$ trivial on the $\mathrm{SL}_2(\mathbb{C})$-factor and confusing it with its $\hat{\mathrm{M}}$-conjugacy class). Using Theorem 2.10.10(ii), it is easy to see (as explained in Remark 2.11.2) that σ is a supercuspidal representation of $\mathrm{M}(F)$. Thus, $(\mathrm{M}, \sigma)_\mathrm{G} \in \Omega(\mathrm{G})$, and we have $p_1((\mathrm{M}, \sigma)_\mathrm{G}) = \underline{(\mathcal{M}, \lambda)}$, finishing the proof that p_1 is surjective at the level of $\mathbb{C}$-points when G is quasi-split. □

4.3.2. *A map* $p_2 = p_{2,\mathrm{G},\mathcal{O}} : \Omega(\mathrm{G}) \to \underline{\Omega}^{\mathrm{st}}(\mathrm{G})$. We now (for this subsubsection) restrict to the case where G is quasi-split, and Hypotheses 2.7.1, and 2.11.4 are satisfied, and use these define a map of varieties $p_2 = p_{2,\mathrm{G},\mathcal{O}} : \Omega(\mathrm{G}) \to \underline{\Omega}^{\mathrm{st}}(\mathrm{G})$.

Definition 4.3.3. Assume that G is quasi-split, and assume the hypotheses on the existence of tempered L-packets and stable cuspidal support (Hypotheses 2.7.1 and 2.11.4). We let $p_2 = p_{2,\mathrm{G},\mathcal{O}} : \Omega(\mathrm{G}) \to \underline{\Omega}^{\mathrm{st}}(\mathrm{G})$ be the unique map of sets such that for any cuspidal pair (M, σ), letting Σ be the unique element of $\mathit{\Phi}_2^+(\mathrm{M})$ containing σ, and letting (L, Υ) be a stable cuspidal support for (M, Σ) (which exists by Hypothesis 2.11.4), we have:

$$p_2((\mathrm{M}, \sigma)_\mathrm{G}) = \underline{(\mathrm{L}, \Upsilon)}.$$

Here we used that, by Remark 2.11.5(i), Υ consists entirely of supercuspidal representations, so that $\underline{(\mathrm{L}, \Upsilon)}$ makes sense as an element of $\underline{\Omega}^{\mathrm{st}}(\mathrm{G})$. It is easy to see from the definition of stable cuspidal support that p_2 is well-defined, and that it descends to a map of sets $\underline{\Omega}(\mathrm{G}) = \Omega(\mathrm{G})/\mathcal{O} \to \underline{\Omega}^{\mathrm{st}}(\mathrm{G})$, which we will also denote by p_2. By Proposition 4.3.4 below, p_2 is a map of varieties, and will be viewed as such.

Proposition 4.3.4. *(When* G *is quasi-split and under Hypotheses 2.7.1 and 2.11.4)* $p_2 : \underline{\Omega}(\mathrm{G}) \to \underline{\Omega}^{\mathrm{st}}(\mathrm{G})$ *(and hence also* $p_2 : \Omega(\mathrm{G}) \to \underline{\Omega}^{\mathrm{st}}(\mathrm{G})$*) is a map of varieties, which is surjective at the level of* $\mathbb{C}$*-points.*

Proof. The surjectivity assertion is clear: given $\underline{(\mathrm{L}, \Upsilon)} \in \underline{\Omega}^{\mathrm{st}}(\mathrm{G})$, since Υ consists entirely of supercuspidal representations as we observed in Remark 2.11.5(a), we have $p_2((\mathrm{L}, \upsilon)_\mathrm{G}) = p_2(\underline{(\mathrm{L}, \upsilon)}) = \underline{(\mathrm{L}, \Upsilon)}$ for any $\upsilon \in \Upsilon$.
Let (M, σ) be a cuspidal pair for $\Omega(\mathrm{G})$. Let $\Sigma \in \mathit{\Phi}_2^+(\mathrm{M})$ be the packet containing σ, and let (L, Υ) be a stable cuspidal support for (M, Σ). It is enough to show that p_2 restricts to a well-defined algebraic map $\Omega(\underline{[\mathrm{M}, \sigma]}) \to \Omega(\underline{[\mathrm{L}, \Upsilon]})$. Since maximal chains of the form (30) starting with a given Levi subgroup $\mathrm{M} \subset \mathrm{G}$ are stable under tensoring with some $\chi \in X^{\mathrm{unr}}(\mathrm{S_M})$, it is easy to see that $p_2(\underline{(\mathrm{M}, \sigma \otimes \chi)}) = \underline{(\mathrm{L}, \Upsilon \otimes (\chi|_{\mathrm{S_L}(F)}))}$, where $\chi|_{\mathrm{S_L}(F)}$ is the element of $X^{\mathrm{unr}}(\mathrm{S_L})$ obtained by pulling χ back under the homomorphism $\mathrm{S_L}(F) \to \mathrm{S_M}(F)$ obtained from the homomorphism $\mathrm{S_L} \to \mathrm{S_M}$ induced by $\mathrm{L} \hookrightarrow \mathrm{M} \to \mathrm{S_M}$.
Note that $\chi \mapsto \chi|_{\mathrm{S_L}(F)}$ is a homomorphism of complex tori $X^{\mathrm{unr}}(\mathrm{S_M}) \to X^{\mathrm{unr}}(\mathrm{S_L})$ (corresponding to the morphism $X_*(\mathrm{S_L}) \to X_*(\mathrm{S_M})$ at the level of their character groups). Therefore, by Corollary 4.1.6, $\chi \mapsto p_2(\underline{(\mathrm{M}, \sigma \otimes \chi)}) = \underline{(\mathrm{L}, \Upsilon \otimes (\chi|_{\mathrm{S_L}(F)}))}$ is a regular map $X^{\mathrm{unr}}(\mathrm{S_M}) \to \underline{\Omega}^{\mathrm{st}}(\mathrm{G})$. Since this is true for each cuspidal pair (M, σ), it follows from Proposition 4.1.4 that p_2 is a regular morphism of varieties. □

4.3.3. *A map* $p_{12} : \underline{\Omega}^{\mathrm{st}}(\mathrm{G}) \to \underline{\Omega}({}^L\mathrm{G})$.

Notation 4.3.5. Assume that G is quasi-split, and assume the hypotheses on the existence of tempered L-packets, LLC+, and LLC+ and stability (Hypotheses 2.7.1, 2.10.3 and 2.10.12). Define a set-theoretic map $p_{12} : \underline{\Omega}^{\mathrm{st}}(\mathrm{G}) \to \underline{\Omega}({}^L\mathrm{G})$ as follows. Let (M, Σ) be a cuspidal pair for $\underline{\Omega}^{\mathrm{st}}(\mathrm{G})$. Choose $\sigma \in \Sigma$ and define

$$p_{12}(\mathrm{M}, \Sigma) = p_1((\mathrm{M}, \sigma)_{\mathrm{G}}) = \underline{\iota_{\mathrm{G,M}} \circ \lambda(\varphi_\sigma)} \in \underline{\Omega}({}^L\mathrm{G}). \tag{54}$$

Then $p_{12}(\mathrm{M}, \Sigma)$ is independent of the choice of $\sigma \in \Sigma$, since φ_σ is (by Lemma 2.10.13, which applies as we are assuming Hypotheses 2.7.1, 2.10.3 and 2.10.12). Since p_1 is $\mathcal{O}$-invariant (see Proposition 4.3.2), it is easy from Lemma 2.7.3(ii) that $p_{12}(\mathrm{M}, \Sigma)$ depends only on $\underline{(\mathrm{M}, \Sigma)}$. Thus, p_{12} descends to a map $p_{12} : \underline{\Omega}^{\mathrm{st}}(\mathrm{G}) \to \underline{\Omega}({}^L\mathrm{G})$. By Proposition 4.3.6 below it is a regular map of varieties, and will be viewed as such.

Proposition 4.3.6. *Assume that* G *is quasi-split, and assume Hypotheses 2.7.1, 2.10.3 and 2.10.12, so that* p_{12} *is defined. Then* p_{12} *is a regular map of varieties* $\underline{\Omega}^{\mathrm{st}}(\mathrm{G}) \to \underline{\Omega}({}^L\mathrm{G})$.

Proof. One can imitate the relevant part of the proof of Proposition 4.3.7 below. □

Proposition 4.3.7. *Suppose that* G *is quasi-split, and assume Hypotheses 2.7.1, 2.10.3, and 2.10.12, together with the hypothesis on supercuspidal packets (Hypothesis 2.11.1). Note that* $p_1 : \Omega(\mathrm{G}) \to \underline{\Omega}({}^L\mathrm{G})$ *and* $p_2 : \Omega(\mathrm{G}) \to \underline{\Omega}^{\mathrm{st}}(\mathrm{G})$ *are defined, the latter since, by Proposition 2.11.6, Hypothesis 2.11.4 is also satisfied. Then* $p_{12} : \underline{\Omega}^{\mathrm{st}}(\mathrm{G}) \to \underline{\Omega}({}^L\mathrm{G})$ *is an isomorphism of varieties, and* $p_1 = p_{12} \circ p_2$.

Proof. The equality $p_1 = p_{12} \circ p_2$ holds set-theoretically, because for all cuspidal pairs (M, σ) for $\Omega(\mathrm{G})$, letting $\Sigma \in \Phi_2^+(\mathrm{M})$ be the packet containing σ, letting (L, Υ) be a stable cuspidal support for (M, Σ), and letting υ be any element of Υ, we have:

$$\begin{aligned} p_1((\mathrm{M}, \sigma)_{\mathrm{G}}) &= \underline{\iota_{\mathrm{G,M}} \circ \lambda(\varphi_\sigma)} = \underline{\iota_{\mathrm{G,M}} \circ \iota_{\mathrm{M,L}} \circ \lambda(\varphi_\upsilon)} \\ &= \underline{\iota_{\mathrm{G,L}} \circ \lambda(\varphi_\upsilon)} = p_{12}(\underline{(\mathrm{L}, \Upsilon)}) = p_{12}(p_2((\mathrm{M}, \sigma)_{\mathrm{G}})), \end{aligned}$$

where the second step uses that $\lambda(\varphi_\sigma) = \iota_{\mathrm{M,L}} \circ \lambda(\varphi_\upsilon)$ (by repeated applications of Lemma 2.10.13, Theorem 2.10.10(ii) and Proposition 2.4.15), and the third step uses Proposition 2.4.15.

If (M, Σ) is any cuspidal pair for $\underline{\Omega}^{\mathrm{st}}(\mathrm{G})$ and $\sigma \in \Sigma$, we have by (54) that $p_{12}(\underline{(\mathrm{M}, \sigma)}) = p_1((\mathrm{M}, \sigma)_{\mathrm{G}}) = \underline{\iota_{\mathrm{G,M}} \circ \lambda(\varphi_\sigma)} = \underline{\iota_{\mathrm{G,M}} \circ \varphi_\sigma} \in \underline{\Omega}({}^L\mathrm{G})$, where the latter equality uses that $\lambda(\varphi_\sigma)$ identifies with φ_σ, a consequence of Hypothesis 2.11.1.

Let us first show that p_{12} is bijective. Since G is quasi-split, p_{12} is surjective by Proposition 4.3.2 and the fact that $p_1 = p_{12} \circ p_2$. To see that p_{12} is injective, suppose that (M_1, Σ_1) and (M_2, Σ_2) are cuspidal pairs for $\underline{\Omega}^{\mathrm{st}}(\mathrm{G})$ such that $p_{12}(\underline{(\mathrm{M}_1, \Sigma_1)}) = p_{12}(\underline{(\mathrm{M}_2, \Sigma_2)})$. Let $\sigma_1 \in \Sigma_1$ and $\sigma_2 \in \Sigma_2$, so that $\iota_{\mathrm{G,M}_1} \circ \varphi_{\sigma_1}$ and $\iota_{\mathrm{G,M}_2} \circ \varphi_{\sigma_2}$ have the same image in $\Omega({}^L\mathrm{G})/\mathcal{O} = \underline{\Omega}({}^L\mathrm{G})$. By Corollary 2.10.7(i), there exists $\beta \in \mathcal{O}_{\mathrm{G}}^+$ such that $\beta(\mathrm{M}_1) = \mathrm{M}_2$, and such that for any map ${}^L(\beta|_{\mathrm{M}_1}) : {}^L\mathrm{M}_2 \to {}^L\mathrm{M}_1$ dual to $\beta|_{\mathrm{M}_1}$, we have ${}^L(\beta|_{\mathrm{M}_1}) \circ \varphi_{\sigma_2} = \varphi_{\sigma_1}$. Hence $\varphi_{\sigma_2} = {}^L(\beta|_{\mathrm{M}_1})^{-1} \circ \varphi_{\sigma_1}$, which by Theorem 2.10.10(iii) equals $\varphi_{\sigma_1 \circ (\beta|_{\mathrm{M}_1})^{-1}}$. By Lemma 2.10.13, $\Sigma_1 \circ (\beta|_{\mathrm{M}_1})^{-1} = \Sigma_2$, so that (M_1, Σ_1) and (M_2, Σ_2) are $\mathcal{O}_{\mathrm{G}}^+$-conjugate, and hence represent the same element of $\underline{\Omega}^{\mathrm{st}}(\mathrm{G})$. This proves the injectivity of p_{12}. Thus, p_{12} is bijective; let us show that p_{12} and p_{12}^{-1} are regular morphisms.

We fix a cuspidal pair (M, Σ) for $\underline{\Omega}^{\mathrm{st}}(\mathrm{G})$, with $\Sigma = \Sigma(\varphi_\Sigma)$ as in Notation 2.10.11(i) (as justified by Lemma 2.10.13). Recall that φ_Σ identifies with $\lambda(\varphi_\Sigma)$ by Hypothesis 2.11.1. Choose $\iota_{\mathrm{G,M}}$, write $\mathcal{M} = \iota_{\mathrm{G,M}}({}^L\mathrm{M})$, and let $\lambda = \iota_{\mathrm{G,M}} \circ \dot{\varphi}_\Sigma$, where $\dot{\varphi}_\Sigma : W_F \to {}^L\mathrm{M}$ is a representative for φ_Σ. Since $\varphi_\Sigma \in \Phi_2^+(\mathrm{M})$, $(\mathcal{M}, \lambda)$ is a cuspidal pair for $\underline{\Omega}({}^L\mathrm{G})$. Clearly, p_{12} takes $\underline{(\mathrm{M}, \Sigma)}$ to $\underline{(\mathcal{M}, \lambda)}$.

Recall that we have an isomorphism

$$\mathrm{Z}_{\hat{\mathrm{M}}}^{\Gamma,0} = H^1(W_F/I_F, \mathrm{Z}_{\hat{\mathrm{M}}}^{\Gamma,0}) \to X^{\mathrm{unr}}(\mathrm{S}_{\mathrm{M}}) \tag{55}$$

of tori denoted $\alpha \mapsto \chi_\alpha$. For $\alpha \in Z_{\hat{\mathrm{M}}}^{\Gamma,0}$, write $\Sigma_\alpha = \Sigma \otimes \chi_\alpha \in \Phi_2^+(\mathrm{M})$ and $\lambda_\alpha = \iota_{\mathrm{G,M}}(\alpha) \cdot \lambda : W_F \to \mathcal{M}$ (where $\iota_{\mathrm{G,M}}(\alpha)$ is viewed as an element of $\mathrm{Z}_{\mathcal{M}^0}^{\Gamma,0} = \mathrm{Z}_{\mathcal{M}}^0$, which acts on the set of $\mathcal{M}^0$-conjugacy classes of admissible homomorphisms $W_F \to \mathcal{M} \subset {}^L\mathrm{G}$ as in Notation 4.1.11). Thus, $\Omega(\underline{[\mathrm{M}, \Sigma]})$ consists of the $\underline{(\mathrm{M}, \Sigma_\alpha)}$, while $\Omega(\underline{[\mathcal{M}, \lambda]})$ consists of the $\underline{(\mathcal{M}, \lambda_\alpha)}$.

It follows from the 'twisting by unramified characters desideratum' (the part of Remark 2.10.4(ii) proved in Theorem 2.10.10(i)) that for each $\alpha \in \mathrm{Z}_{\hat{\mathrm{M}}}^{\Gamma,0}$, the bijection p_{12} takes $\underline{(\mathrm{M},\Sigma_\alpha)} \in \underline{\Omega}^{\mathrm{st}}(\mathrm{G})$ to $\underline{(\mathcal{M},\lambda_\alpha)} \in \underline{\Omega}({}^L\mathrm{G})$. In particular, p_{12} restricts to a bijection $\Omega(\underline{[\mathrm{M},\Sigma]}) \to \Omega(\underline{[\mathcal{M},\lambda]})$. It now suffices to show that this restriction, call it $p_{12,\Sigma}$, is an isomorphism of varieties.
We use $\alpha \mapsto \chi_\alpha$ to identify $\mathrm{Z}_{\hat{\mathrm{M}}}^{\Gamma,0}$ with $X^{\mathrm{unr}}(\mathrm{S}_\mathrm{M})$, and we use $\alpha \mapsto \iota_{\mathrm{G},\mathrm{M}}(\alpha)$ to identify $\mathrm{Z}_{\hat{\mathrm{M}}}^{\Gamma,0}$ with $\mathrm{Z}^0_{\mathcal{M}}$. By Corollary 4.1.6, the map $\alpha \mapsto \underline{(\mathcal{M},\lambda_\alpha)} = p_{12,\Sigma}(\underline{(\mathrm{M},\Sigma_\alpha)})$ is regular, so by Proposition 4.1.4, $p_{12,\Sigma}$ is regular. A similar argument gives that $p_{12,\Sigma}^{-1}$, which takes $\underline{(\mathcal{M},\lambda_\alpha)}$ to $\underline{(\mathrm{M},\Sigma_\alpha)}$ for each $\alpha \in \mathrm{Z}_{\hat{\mathrm{M}}}^{\Gamma,0}$, is regular. □

5. Different Bernstein centers and relations between them

5.1. Candidates for the stable Bernstein center, $\mathcal{Z}_1(\mathrm{G})$ and $\mathcal{Z}_2(\mathrm{G})$.

Notation 5.1.1. In this subsection, given $f \in C_c^\infty(\mathrm{G}(F))$, $f^\vee \in C_c^\infty(\mathrm{G}(F))$ will stand for the function $x \mapsto f(x^{-1})$. Given $g \in \mathrm{G}(F)$ and $f \in C_c^\infty(\mathrm{G}(F))$, we will let $l_g f(x) = f(g^{-1}x)$ and $r_g f(x) = f(xg)$. We fix a Haar measure on $\mathrm{G}(F)$, which will be used in the convolutions that follow.

We recall some facts on convolutions from [Hai14, Section 3.1]. For a distribution D on $\mathrm{G}(F)$ and $f \in C_c^\infty(\mathrm{G}(F))$, $D * f \in C^\infty(\mathrm{G}(F))$ is given by $g \mapsto D((r_g f)^\vee) = D(l_g f^\vee)$ — thus, this is defined so as to satisfy:

$$(D * r_g f) = r_g(D * f) \qquad \text{and} \qquad D(f) = (D * f^\vee)(1).$$

D is said to be essentially compact if $D * f \in C_c^\infty(\mathrm{G}(F))$ for all $f \in C_c^\infty(\mathrm{G}(F))$. If D' and D are distributions and D is essentially compact, we can convolve them by letting $(D' * D)(f) = D'((D * f^\vee)^\vee)$.
Now we recall the definition of the Bernstein center $\mathcal{Z}(\mathrm{G})$ of G.

Definition 5.1.2. The Bernstein center $\mathcal{Z}(\mathrm{G})$ of G is the $\mathbb{C}$-vector space of essentially compact invariant distributions on $C_c^\infty(\mathrm{G}(F))$, i.e., the space of ($\mathrm{G}(F)$-conjugation) invariant distributions $C_c^\infty(\mathrm{G}(F)) \to \mathbb{C}$ with the property that for all $f \in C_c^\infty(\mathrm{G}(F))$, $z * f \in C^\infty(\mathrm{G}(F))$ belongs to $C_c^\infty(\mathrm{G}(F))$. Convolution makes $\mathcal{Z}(\mathrm{G})$ into a commutative $\mathbb{C}$-algebra (see [Hai14, Corollary 3.1.2]), and the work of Bernstein gives the following alternate descriptions of the ring $\mathcal{Z}(\mathrm{G})$:

(i) Via $z \mapsto (f \mapsto z * f)$, $\mathcal{Z}(\mathrm{G})$ identifies with the ring of endomorphisms of $C_c^\infty(\mathrm{G}(F))$ that commute with left and right convolution.

(ii) One can uniquely make each $z \in \mathcal{Z}(\mathrm{G})$ act as an intertwining operator $\pi(z)$ on π, for each smooth representation π of $\mathrm{G}(F)$, such that:
- Denoting temporarily by l the left-regular representation of $\mathrm{G}(F)$ on $C_c^\infty(\mathrm{G}(F))$, we have $l(z)(f) = z * f$;
- $\pi \mapsto \pi(z)$ respects morphisms of representations.

$z \mapsto (\pi \mapsto \pi(z))$ defines a homomorphism from $\mathcal{Z}(\mathrm{G})$ to the ring of endomorphisms of the identity functor of the category of smooth representations of $\mathrm{G}(F)$, which Bernstein's work shows to be an isomorphism. The action of $\mathcal{Z}(\mathrm{G})$ on a smooth representation (π, V) can typically be computed using the following: given $v \in V$, we have a map $(l, C_c^\infty(\mathrm{G}(F))) \to (\pi, V)$ given by $f \mapsto \pi(f)v$, so that $\pi(z)(\pi(f)v) = \pi(l(z)f)(v) = \pi(z * f)(v)$. This also gives:

$$\pi(z * f) = \pi(z)\pi(f). \tag{56}$$

(iii) By (ii) and Schur's lemma, each $z \in \mathcal{Z}(\mathrm{G})$ acts on each irreducible admissible representation π of $\mathrm{G}(F)$ by multiplication by some scalar, which can be shown to depend only on the cuspidal support $(\mathrm{M},\sigma)_\mathrm{G} \in \Omega(\mathrm{G})$ of π. We denote this scalar by:

$$\hat{z}(\pi) = \hat{z}(\mathrm{M},\sigma) = \hat{z}((\mathrm{M},\sigma)_\mathrm{G}).$$

More generally, for any Levi subgroup $\mathrm{M}' \subset \mathrm{G}$ and $\sigma' \in \mathrm{Irr}(\mathrm{M}')$ such that the cuspidal supports of σ' and π are $\mathrm{G}(F)$-conjugate, we will write $\hat{z}(\pi) = \hat{z}(\sigma') = \hat{z}((\mathrm{M}',\sigma')) = \hat{z}((\mathrm{M}',\sigma')_\mathrm{G})$ when there is no scope for confusion, where in turn $(\mathrm{M}',\sigma')_\mathrm{G}$ denotes the

G(F)-conjugacy class of (M, σ). By Bernstein's work, sending $z \in \mathcal{Z}(\mathrm{G})$ to $\hat{z} : \Omega(\mathrm{G}) \to \mathbb{C}$ gives an isomorphism of rings $\mathcal{Z}(\mathrm{G}) \to \mathbb{C}[\Omega(\mathrm{G})]$.

It is clear that $\mathcal{O}$ acts on $\mathcal{Z}(\mathrm{G})$; we now explicate this action. $\mathcal{O}$ acts on $C_c^\infty(\mathrm{G}(F))$ and $C^\infty(\mathrm{G}(F))$, and on the space of distributions on $\mathrm{G}(F)$: $(\beta \cdot f)(x) = f(\beta^{-1}(x))$ and $(\beta \cdot D)(f) = D(\beta^{-1} \cdot f)$. For $\beta \in \mathcal{O}$, one verifies the following equalities for each distribution D on $\mathrm{G}(F)$, $f \in C_c^\infty(\mathrm{G}(F))$ and $\beta \in \mathcal{O}$:

$$(\beta D)(\beta f) = D(f), \qquad \text{and} \qquad \beta D * \beta f = \beta(D * f). \tag{57}$$

It is now clear that the action of $\mathcal{O}$ on the space of distributions on $\mathrm{G}(F)$ preserves the subspace $\mathcal{Z}(\mathrm{G})$. The action of $\mathcal{O}$ on $\mathrm{Irr}(\mathrm{G})$, given by $\beta \cdot \pi = \pi \circ \beta^{-1}$, is related to the action of $\mathcal{O}$ on $\mathcal{Z}(\mathrm{G})$ as follows:

$$\widehat{\beta \cdot z}(\pi) = \hat{z}(\pi \circ \beta). \tag{58}$$

Indeed, using the identity $\pi(z * f) = \hat{z}(\pi)\pi(f)$ (which follows from (56)), the identity $(\pi \circ \beta)(f) = \pi(f \circ \beta^{-1})$, and (57), this follows from:

$$\hat{z}(\pi\circ\beta)\cdot(\pi\circ\beta)(f) = \pi\circ\beta(z*f) = \pi((z*f)\circ\beta^{-1}) = \pi(\beta\cdot(z*f)) = \pi((\beta\cdot z)*(\beta\cdot f)) = \widehat{\beta \cdot z}(\pi)\pi(f\circ\beta^{-1}) = \widehat{\beta \cdot z}(\pi)(\pi\circ\beta)(f).$$

From (57) and (58), the following is easy to deduce:

Lemma 5.1.3. *The isomorphism $\mathcal{Z}(\mathrm{G}) \to \mathbb{C}[\Omega(\mathrm{G})]$ given by $z \mapsto \hat{z}$ is $\mathcal{O}$-equivariant, and restricts to an isomorphism $\mathcal{Z}(\mathrm{G})^{\mathcal{O}} \to \mathbb{C}[\underline{\Omega}(\mathrm{G})]$, where the inclusion $\mathbb{C}[\underline{\Omega}(\mathrm{G})] \subset \mathbb{C}[\Omega(\mathrm{G})]$ comes from the quotient map $\Omega(\mathrm{G}) \to \underline{\Omega}(\mathrm{G})$ of varieties (see Corollary 4.1.7). In particular, $\mathcal{Z}(\mathrm{G})^{\mathcal{O}} \subset \mathcal{Z}(\mathrm{G})$ is a subring.*

Notation 5.1.4. In this subsection, $\mathcal{I}(\mathrm{G})$ will denote the space of coinvariants for $\mathrm{G}(F)$-conjugation on $C_c^\infty(\mathrm{G}(F))$. We will consider the actions of $\mathcal{O}$ and $\mathcal{O}_{\mathrm{G}}^+$ on $\mathcal{I}(\mathrm{G})$ inherited from their actions on $C_c^\infty(\mathrm{G}(F))$. Note that the space of invariant distributions on $C_c^\infty(\mathrm{G}(F))$ identifies with $\mathrm{Hom}_{\mathbb{C}}(\mathcal{I}(\mathrm{G}), \mathbb{C})$.

Remark 5.1.5. Since $\mathrm{Int}\,\mathrm{G}(F)$ is of finite index in $\mathcal{O}_{\mathrm{G}}^+$ by Notation 2.6.1(iv) (see Lemma 2.6.3(ii)), $\mathcal{O}$ acts on $\mathcal{I}(\mathrm{G})$ through a finite quotient, and therefore, given $f \in C_c^\infty(\mathrm{G}(F))$, there exists $f' \in C_c^\infty(\mathrm{G}(F))$ such that:

- f' is a sum of finitely many $\mathcal{O}$-translates of f, and has $\mathcal{O}$-invariant image in $\mathcal{I}(\mathrm{G})$.

It follows from (57) that for any such f', and any $z \in \mathcal{Z}(\mathrm{G})^{\mathcal{O}}$, we have:

- $z * f'$ is a sum of finitely many $\mathcal{O}$-translates of $z * f$, and has $\mathcal{O}$-invariant image in $\mathcal{I}(\mathrm{G})$

(use the easy observation that, if $f'' \in C_c^\infty(\mathrm{G}(F))$ has $\mathcal{O}$-invariant image in $\mathcal{I}(\mathrm{G})$, then so does $z * f''$: this is because $f'' \mapsto z * f''$ is $\mathcal{O}$-equivariant by (57), and hence so is the map it induces from $\mathcal{I}(\mathrm{G})$ to itself). It will also help to note that for any such f', since $\mathcal{O}$ acts by algebraic automorphisms:

- If f (resp., $z * f$) is unstable, then so is f' (resp., $z * f'$).

Now we recall the spaces $\mathcal{Z}_1(\mathrm{G}), \mathcal{Z}_2(\mathrm{G}) \subset \mathcal{Z}(\mathrm{G})$ from the introduction.

Notation 5.1.6. (i) $\mathcal{Z}_1(\mathrm{G}) \subset \mathcal{Z}(\mathrm{G})$ is the (clearly $\mathcal{O}$-invariant) $\mathbb{C}$-vector subspace of $\mathcal{Z}(\mathrm{G})$ consisting of all $z \in \mathcal{Z}(\mathrm{G})$ that are stable as a distribution on $\mathrm{G}(F)$. We will also study the $\mathcal{O}$-fixed subspace $\mathcal{Z}_1(\mathrm{G})^{\mathcal{O}}$ of $\mathcal{Z}_1(\mathrm{G})$.

(ii)
- $\mathcal{Z}_2(\mathrm{G}) \subset \mathcal{Z}(\mathrm{G})$ is the $\mathbb{C}$-sublgebra of $\mathcal{Z}(\mathrm{G})$ consisting of all $z \in \mathcal{Z}(\mathrm{G})$ with the property that $z * f$ is unstable for every unstable function $f \in C_c^\infty(\mathrm{G}(F))$.
- More generally $\mathcal{Z}_{2,\mathcal{O}}(\mathrm{G}) \subset \mathcal{Z}_2(\mathrm{G})$ is the $\mathbb{C}$-subalgebra of $\mathcal{Z}(\mathrm{G})^{\mathcal{O}}$ consisting of all $z \in \mathcal{Z}(\mathrm{G})^{\mathcal{O}}$ such that for every unstable function $f \in C_c^\infty(\mathrm{G}(F))$ whose image in $\mathcal{I}(\mathrm{G})$ is fixed by $\mathcal{O}$, $z * f$ is unstable.

Note that if $\mathcal{O}$ is trivial, then $\mathcal{Z}_1(\mathrm{G}) = \mathcal{Z}_1(\mathrm{G})^{\mathcal{O}}$, and $\mathcal{Z}_2(\mathrm{G}) = \mathcal{Z}_{2,\mathcal{O}}(\mathrm{G})$.

Lemma 5.1.7. *We have $\mathcal{Z}_{2,\mathcal{O}}(\mathrm{G}) \subset \mathcal{Z}_1(\mathrm{G})^{\mathcal{O}}$. In particular, if $\mathcal{O}$ is trivial, then $\mathcal{Z}_2(\mathrm{G}) \subset \mathcal{Z}_1(\mathrm{G})$.*

Proof. Let $z \in \mathcal{Z}_{2,\mathcal{O}}(\mathrm{G}) \subset \mathcal{Z}(\mathrm{G})^{\mathcal{O}}$, and let $f \in C_c^\infty(\mathrm{G}(F))$ be unstable. It is enough to show that $z(f) := z * f^\vee(1)$ equals 0. Choose f' as in Remark 5.1.5. Then, by Remark 5.1.5, ${f'}^\vee$ is unstable. Since $z \in \mathcal{Z}_{2,\mathcal{O}}(\mathrm{G})$, we conclude that $z * {f'}^\vee$ is unstable, from which it follows that $z * {f'}^\vee(1) = 0$ (as $f'' \mapsto f''(1)$ is a stable distribution, by [Kot88, Proposition 1]).
Since $z * {f'}^\vee$ is a finite sum of $\mathcal{O}$-translates of $z * f^\vee$ (see Remark 5.1.5), and since the action of $\mathcal{O}_{\mathrm{G}}^+$ on $C_c^\infty(\mathrm{G}(F))$ preserves $f'' \mapsto f''(1)$, it follows that $z * f^\vee(1) = 0$, as desired. □

Proposition 5.1.8. *Let $z \in \mathcal{Z}(\mathrm{G})^{\mathcal{O}}$. Then the following are equivalent:*

(i) *$z \in \mathcal{Z}_{2,\mathcal{O}}(\mathrm{G})$.*
(ii) *If D is a stable $\mathcal{O}$-invariant distribution on $\mathrm{G}(F)$, then the distribution $f \mapsto D(z * f)$ is stable.*
(iii) *If $D \in SD(\mathrm{G})^{\mathcal{O}}$, then the distribution $f \mapsto D(z * f)$ is stable.*
(iv) *If D is the $\mathcal{O}$-average of $\mathrm{Ind}_{\mathrm{M}}^{\mathrm{G}} \Theta'$, where $\mathrm{M} \subset \mathrm{G}$ is a Levi subgroup and $\Theta' \in SD_{\mathrm{ell}}(\mathrm{M})^{\mathcal{O}_{\mathrm{M}}}$, then the distribution $f \mapsto D(z * f)$ is stable.*

Remark 5.1.9. Using the formula $D * z(f) = D\left((z * f^\vee)^\vee\right)$ (see just below Notation 5.1.1), one can show that each of the conditions (ii), (iii) and (iv) of the above proposition has an equivalent variant where the distribution $f \mapsto D(z * f)$ is replaced by the distribution $D * z$.

Proof of Proposition 5.1.8. Let us prove (i) ⇒ (ii). Let $z \in \mathcal{Z}_{2,\mathcal{O}}(\mathrm{G}) \subset \mathcal{Z}(\mathrm{G})^{\mathcal{O}}$, and let us show that if D is an $\mathcal{O}$-invariant distribution on $\mathrm{G}(F)$, and $f \in C_c^\infty(\mathrm{G}(F))$ is unstable, then $D(z*f) = 0$. Let f' be as in Remark 5.1.5, so that f' is unstable, its image in $\mathcal{I}(\mathrm{G})$ is $\mathcal{O}$-invariant, and $z * f'$ is a finite sum of $\mathcal{O}$-translates of $z * f$. Therefore, $z * f'$ is unstable (by the definition of $\mathcal{Z}_{2,\mathcal{O}}(\mathrm{G})$), so that $D(z * f') = 0$, while by the $\mathcal{O}$-invariance of D, $D(z * f')$ is a nonzero integer multiple of $D(z * f)$. Therefore, $D(z * f) = 0$, and the implication (i) ⇒ (ii) follows.
Now it is clear that (i) ⇒ (ii) ⇒ (iii) ⇒ (iv) (for (iii) ⇒ (iv), use that parabolic induction preserves stability of distributions, for which a nice reference is [KV16, Corollary 6.13]).
If $f \in C_c^\infty(\mathrm{G}(F))$ has $\mathcal{O}$-invariant image in $\mathcal{I}(\mathrm{G})$, then so does $z * f$ (we observed this in Remark 5.1.5). This fact together with Corollary 3.2.12 gives the implication (iv) ⇒ (i). □

5.2. **Using Shahidi's argument on the Plancherel μ-function.** Let $\mathrm{M} \subset \mathrm{G}$ be a Levi subgroup, and $\zeta : \mathrm{A}_{\mathrm{M}}(F) \to \mathbb{C}^\times$ a unitary character. One of the results that we will prove in this subsection is Corollary 5.2.11, part (i) of which says that the distribution $\sum d(\sigma)\mu(\sigma)\Theta_\sigma$ is stable, and more generally so is $\sum d(\sigma)\mu(\sigma)\hat{z}(\sigma)\Theta_\sigma$ for any $z \in \mathcal{Z}_1(\mathrm{G})$, where the sum ranges over the subset $\mathrm{Irr}_2(\mathrm{M})_\zeta \subset \mathrm{Irr}_2(\mathrm{M})$ of discrete series representations of $\mathrm{M}(F)$ whose central character restricts to ζ on $\mathrm{A}_{\mathrm{M}}(F)$. Part (ii) of the corollary says that these distributions transfer well across inner forms. These are weaker but unconditional results in the spirit of the constancy of the Plancherel measure on L-packets as proved by Shahidi (see [Sha90, Section 9]), and that of the transfer of Plancherel measures across inner forms as one sees in the works of Choiy (see, e.g., [Cho14]) and Heiermann (see [Hei16, Appendix A]). We then use these results in Corollary 5.2.12 to show, in part (i) of the corollary, that the Plancherel measure or rather the μ-function, is constant on atomically stable discrete series L-packets, and in part (ii) of the corollary that whenever an atomically stable discrete series L-packet Σ on M transfers to an atomically stable discrete series L-packet Σ^* on the quasi-split inner form M^* of M, we have $\mu(\sigma) = c\mu(\sigma^*)$ for all $\sigma \in \Sigma$ and $\sigma^* \in \Sigma^*$, where c is an explicit constant. We will use the Paley-Wiener theorem as stated in [Art96], so we begin by reviewing it.

5.2.1. *Review of the version of the Paley-Wiener theorem in* [Art96].

Notation 5.2.1. (i) For this subsection, we fix a maximal split torus $\mathrm{A}_0 \subset \mathrm{G}$, and let M_0 be the minimal Levi subgroup of G obtained as the centralizer of A_0 in G. Further, let $\mathcal{L}$ denote the set of Levi subgroups of G that are semistandard, i.e., contain A_0, or equivalently, M_0.
(ii) Set $W_0 = W(\mathrm{M}_0)$ (i.e., the $W_{\mathrm{G}}(\mathrm{M}_0)$ in the sense of Subsubsection 2.1.7).
(iii) W_0 acts on $\mathcal{L}$, and we write $\mathcal{L}/W_0$ for the set-theoretic quotient. It is easy to see that each $\mathrm{G}(F)$-conjugacy class of Levi subgroups of G intersects $\mathcal{L}$ in a single W_0-orbit, so

that $\mathcal{L}/W_0$ can be identified with the set of $\mathrm{G}(F)$-conjugacy classes of Levi subgroups of G.

(iv) We let the topological space $\tilde{T}(\mathrm{G})$, the topological space $\tilde{T}_{\mathrm{ell}}(\mathrm{M})$ for each Levi subgroup $\mathrm{M} \in \mathcal{L}$, and the decomposition

$$\tilde{T}(\mathrm{G}) = \bigsqcup_{\mathrm{M} \in \mathcal{L}/W_0} (\tilde{T}_{\mathrm{ell}}(\mathrm{M})/W(\mathrm{M})) \tag{59}$$

be as in [Art96, Section 4]. $\tilde{T}(\mathrm{G}) \supset \tilde{T}_{\mathrm{ell}}(\mathrm{G})$ is formed of certain tuples (L, σ, r), where L is a Levi subgroup of G and $\sigma \in \mathrm{Irr}_2(\mathrm{L})$; we will recall a few more details below. We will also occasionally use the quotients $T(\mathrm{G})$ and $T_{\mathrm{ell}}(\mathrm{G})$ of $\tilde{T}(\mathrm{G})$ and $\tilde{T}_{\mathrm{ell}}(\mathrm{G})$ as in [Art96, page 531]. For a smooth unitary character $\zeta : \mathrm{A_G}(F) \to \mathbb{C}^\times$, we also have subsets $\tilde{T}_\zeta(\mathrm{G}) \subset \tilde{T}(\mathrm{G}), \tilde{T}_{\zeta,\mathrm{ell}}(\mathrm{G}) \subset \tilde{T}_{\mathrm{ell}}(\mathrm{G}), T_\zeta(\mathrm{G}) \subset T(\mathrm{G})$ and $T_{\zeta,\mathrm{ell}}(\mathrm{G}) \subset T_{\mathrm{ell}}(\mathrm{G})$ represented by tuples (L, σ, r) such that the central character of σ restricts to ζ on $\mathrm{A_G}(F)$ (slightly differing in notation from [Art96, page 531]).

(v) If $\mathrm{M} \in \mathcal{L}$ and $\tau = (\mathrm{L}, \sigma, r) \in \tilde{T}(\mathrm{M})$, we let Θ^{M}_τ be the associated virtual character on $\mathrm{M}(F)$: for $\mathrm{M} = \mathrm{G}$, the 'normalized version' of Θ^{G}_τ, obtained by multiplying it by the discriminant factor $\gamma \mapsto |D(\gamma)|^{1/2}$ in the notation of [Art96], is what is denoted by $\gamma \mapsto I(\tau, \gamma)$ in [Art96, Section 4, near the top of page 532]. For $\tau \in \tilde{T}(\mathrm{G})$, let $\Theta_\tau = \Theta^{\mathrm{G}}_\tau$. If $\tau = (\mathrm{L}, \sigma, r) \in \tilde{T}_{\mathrm{ell}}(\mathrm{M})$, where $\mathrm{M} \in \mathcal{L}$, and Θ^{G}_τ is defined using the image of τ in $\tilde{T}(\mathrm{G})$, one knows, and we will use without further comment in what follows, that $\Theta^{\mathrm{G}}_\tau = \mathrm{Ind}^{\mathrm{G}}_{\mathrm{M}} \Theta^{\mathrm{M}}_\tau$: use [MgW18, Lemma 2.10] (which works in the twisted case). One also knows that for each $\mathrm{M} \in \mathcal{L}$, the Θ^{M}_τ with $\tau \in \tilde{T}_{\mathrm{ell}}(\mathrm{M})$ running over a set of representatives for $T_{\mathrm{ell}}(\mathrm{M})$ form a basis for $D_{\mathrm{ell}}(\mathrm{M})$.

(vi) In this section too, we will write $\mathcal{I}(\mathrm{G})$ for the space of $\mathrm{Int}\,\mathrm{G}(F)$-coinvariants of $C^\infty_c(\mathrm{G}(F))$; it is also the quotient of $C^\infty_c(\mathrm{G}(F))$ by the subspace consisting of those functions whose strongly regular semisimple orbital integrals all vanish. $\mathcal{SI}(\mathrm{G})$ will denote the quotient of $\mathcal{I}(\mathrm{G})$ such that the kernel of $C^\infty_c(\mathrm{G}(F)) \to \mathcal{I}(\mathrm{G}) \to \mathcal{SI}(\mathrm{G})$ is the subspace of functions whose strongly regular semisimple orbital integrals all vanish. Thus, $\mathcal{I}(\mathrm{G})^* = \mathrm{Hom}(\mathcal{I}(\mathrm{G}), \mathbb{C})$ identifies with the space of invariant distributions on $\mathrm{G}(F)$, and $\mathcal{SI}(\mathrm{G})^* \subset \mathcal{I}(\mathrm{G})^*$ with the subspace of stable distributions. Of course, similar notation will apply with G replaced by a Levi subgroup M or a quasi-split form G^*, etc. According to the Paley-Wiener theorem, as stated in [Art96] and recalled in Remark 5.2.3 below, sending $f \in C^\infty_c(\mathrm{G}(F))$ to the function $\tilde{T}(\mathrm{G}) \to \mathbb{C}$ given by $\tau \mapsto \Theta_\tau(f)$, induces an isomorphism from $\mathcal{I}(\mathrm{G})$ to a concrete space of functions on $\tilde{T}(\mathrm{G})$. Once we describe this isomorphism in Remark 5.2.3, it will be thought of as an identification.

We now partially recall (slightly more than) what we need concerning the objects of Notation 5.2.1(iv). The second page of [Art96, Section 4] defines the set $\tilde{T}(\mathrm{G})$ as the set of W_0-orbits of certain triples (L, σ, r). For each such triple (L, σ, r), L is an element of $\mathcal{L}$, σ is a discrete series representation of $\mathrm{L}(F)$, and r is an element belonging to a certain central extension of the R-group of (L, σ) in G (we will not need the exact definition of this group, and hence refer the reader to [Art96] for more details). We refer to [Art96] for the definition of the subset $\tilde{T}_{\mathrm{ell}}(\mathrm{G}) \subset \tilde{T}(\mathrm{G})$ of elliptic elements, and the fact that we have a map from $\tilde{T}_{\mathrm{ell}}(\mathrm{M})$ (the set obtained by substituting M for G in the definition of $\tilde{T}_{\mathrm{ell}}(\mathrm{G})$) to $\tilde{T}(\mathrm{G})$, giving a decomposition of $\tilde{T}(\mathrm{G})$ as in (59).
For each $\mathrm{M} \in \mathcal{L}$, $X^{\mathrm{unr-uni}}(\mathrm{M})$ acts on $\tilde{T}_{\mathrm{ell}}(\mathrm{M})$, where the action of a unitary character $\chi : \mathrm{M}(F) \to \mathbb{C}^\times$ in $X^{\mathrm{unr-uni}}(\mathrm{M})$ sends each (L, σ', r) to $(\mathrm{L}, \sigma' \otimes \chi, r)$. This action makes each orbit into a torsor for a finite quotient of $X^{\mathrm{unr-uni}}(\mathrm{M})$, which being a compact torus topologizes the orbit. The orbits obtained this way partition $\tilde{T}_{\mathrm{ell}}(\mathrm{M})$, which we topologize by requiring this partition to be topological. Moreover, $\tilde{T}_{\mathrm{ell}}(\mathrm{M})/W(\mathrm{M})$ is then given the quotient topology. Allowing M to vary, this topologizes $\tilde{T}(\mathrm{G})$ by requiring the partition (59) to be topological.

Notation 5.2.2. For any $\mathrm{M} \in \mathcal{L}$, we have an injection $\mathrm{Irr}_2(\mathrm{M}) \hookrightarrow \tilde{T}_{\mathrm{ell}}(\mathrm{M})$ given by $\sigma \mapsto (\mathrm{M}, \sigma, 1)$, which will be thought of as an inclusion. Note that $\mathrm{Irr}_2(\mathrm{M}) \subset \tilde{T}_{\mathrm{ell}}(\mathrm{M})$ is a disjoint union of connected components of $\tilde{T}_{\mathrm{ell}}(\mathrm{M})$.

We will need to know Θ_τ, where $\tau \in \tilde{T}(\mathrm{G})$, only when it is the image of some $(\mathrm{M}, \sigma, 1) \in \mathrm{Irr}_2(\mathrm{M}) \subset \tilde{T}_{\mathrm{ell}}(\mathrm{M})$, where $\mathrm{M} \in \mathcal{L}$ and $\sigma \in \mathrm{Irr}_2(\mathrm{M})$. In such a situation, Θ_τ is simply the Harish-Chandra character of $\mathrm{Ind}_{\mathrm{M}}^{\mathrm{G}} \sigma$, i.e., of $\mathrm{Ind}_{\mathrm{P}}^{\mathrm{G}} \sigma$ for any parabolic subgroup $\mathrm{P} \subset \mathrm{G}$ with M as a Levi subgroup.

Remark 5.2.3. According to the trace Paley-Wiener theorem, as interpreted by Arthur in [Art96, page 532], the map $f \mapsto (\tau \mapsto \Theta_\tau^{\mathrm{G}}(f))$, from $C_c^\infty(\mathrm{G}(F))$ to some space of functions on $\tilde{T}(\mathrm{G})$, quotients to an isomorphism from $\mathcal{I}(\mathrm{G})$ to the space of functions $g : \tilde{T}(\mathrm{G}) \to \mathbb{C}$ satisfying the following three conditions:

- (i) g is supported on finitely many connected components of $\tilde{T}(\mathrm{G})$ (Condition (i) on [Art96, page 532]);
- (ii) For any $\mathrm{M} \in \mathcal{L}$ and any $\tau \in \tilde{T}_{\mathrm{ell}}(\mathrm{M})$, the map $X^{\mathrm{unr-uni}}(\mathrm{M}) \to \mathbb{C}$ given by $\chi \mapsto g(\overline{\chi \cdot \tau})$, where $\overline{\chi \cdot \tau}$ denotes the image of $\chi \cdot \tau \in \tilde{T}_{\mathrm{ell}}(\mathrm{M})$ in $\tilde{T}(\mathrm{G})$, is a finite complex linear combination of continuous characters of $X^{\mathrm{unr-uni}}(\mathrm{M})$ (Condition (ii) on [Art96, page 532]); and
- (iii) Condition (iii) on [Art96, page 532], which only concerns the third component of a triple $\tau = (\mathrm{L}, \sigma, r)$, and is automatically satisfied for functions that are supported on the union over $\mathrm{M} \in \mathcal{L}/W_0$, of $\mathrm{Irr}_2(\mathrm{M})/W(\mathrm{M}) \subset \tilde{T}_{\mathrm{ell}}(\mathrm{M})/W(\mathrm{M}) \subset \tilde{T}(\mathrm{G})$.

As mentioned in Notation 5.2.1(vi), we will now start viewing $\mathcal{I}(\mathrm{G})$ also as the space of functions $\tilde{T}(\mathrm{G}) \to \mathbb{C}$ satisfying the three conditions above.

Remark 5.2.4. In fact, the original version of the trace Paley-Wiener theorem in [BDK86] was stated quite differently: it involved the set of cuspidal supports, rather than the triples $\tau = (\mathrm{L}, \sigma, r)$ above. It is the difference between these two formulations that necessitated the extra care taken in the proof of [Sha90, Proposition 9.3] (as Shahidi mentions in the remark after that Proposition), which the formulation of the Paley-Wiener theorem given by Arthur in [Art96] lets one avoid. According to [MgW18, Sections 6.1 and 6.2], the version we use follows from [LH17, Section 3.2] (which in fact handles the twisted case).

Notation 5.2.5.

- (i) We fix a Haar measure on $\mathrm{G}(F)$, and more generally on $\mathrm{M}(F)$ for each $\mathrm{M} \in \mathcal{L}$. For each $\mathrm{M} \in \mathcal{L}$, as in [MgW18, Section 1.2], we give $\mathrm{A}_{\mathrm{M}}(F)$ and $X^{\mathrm{unr-uni}}(\mathrm{A}_{\mathrm{M}})$ Haar measures such that $\mathrm{meas}(\mathrm{A}_{\mathrm{M}}(F)_c)\,\mathrm{meas}(X^{\mathrm{unr-uni}}(\mathrm{A}_{\mathrm{M}})) = 1$, where $\mathrm{A}_{\mathrm{M}}(F)_c \subset \mathrm{A}_{\mathrm{M}}(F)$ is the maximal compact subgroup. We give $\mathrm{M}(F)/\mathrm{A}_{\mathrm{M}}(F)$ the quotient measure, and use it to define the formal degree $d(\sigma)$ for each $\sigma \in \mathrm{Irr}_2(\mathrm{M})$.
- (ii) Unless otherwise stated, for any compact open subgroup $H \subset \mathrm{G}(F)$ and an algebraic subgroup $\mathrm{L} \subset \mathrm{G}$, H_{L} will denote $H \cap \mathrm{L}$.
- (iii) Fix a maximal compact subgroup $K = K_{\mathrm{G}} \subset \mathrm{G}(F)$, which is the stabilizer of a special point belonging to the apartment of A_0 in the Bruhat-Tits building of G. We let $I = I_{\mathrm{G}} \subset K$ be an Iwahori subgroup of $\mathrm{G}(F)$ associated to a chamber in the same apartment. Thus, I has an Iwahori decomposition $I = I_{\mathrm{N}} I_{\mathrm{M}} I_{\mathrm{N}^-}$, whenever $\mathrm{M} \subset \mathrm{G}$ is a semistandard Levi subgroup, and N and N^- are unipotent radicals of opposite parabolic subgroups of G that have M as a common Levi subgroup.
- (iv) To each semistandard Levi subgroup $\mathrm{M} \subset \mathrm{G}$, we attach constants $\gamma(\mathrm{G}|\mathrm{M}), \gamma'(\mathrm{G}|\mathrm{M})$ and $\gamma''(\mathrm{G}|\mathrm{M})$ (the latter two are, notationally, nonstandard and ad hoc) as follows. We choose opposite parabolic subgroups P and P^- having M as a common Levi subgroup, with N and N^- as their unipotent radicals, and let $\gamma(\mathrm{P}) = \gamma(\mathrm{G}|\mathrm{M})$ be as in [Wal03, page 241], using the choices of the measures as fixed in that reference. Moreover, we set (ad hoc and non-standard notation):

(60)
$$\gamma'(\mathrm{G}|\mathrm{M}) = \left(\prod_\alpha \gamma(\mathrm{M}_\alpha|\mathrm{M})^{-2}\right), \qquad \gamma''(\mathrm{G}|\mathrm{M}) = [K_{\mathrm{N}} : I_{\mathrm{N}}]^{-1}[K_{\mathrm{N}^-} : I_{\mathrm{N}^-}]^{-1}, \quad \text{and } \gamma'''(\mathrm{G}|\mathrm{M}) = \gamma'(\mathrm{G}|\mathrm{M})\gamma''(\mathrm{G}|\mathrm{M}),$$

where in the first product α runs over the set of reduced roots of A_{M} (outside M), taken up to a sign. It follows from [Wal03, Section I.1, (3)] that $\gamma(\mathrm{G}|\mathrm{M})$ and $\gamma'(\mathrm{G}|\mathrm{M})$ depend only on M, and not on P and P^-. That the same applies to $\gamma''(\mathrm{G}|\mathrm{M})$ and hence also to

$\gamma'''(\mathrm{G}|\mathrm{M})$ follows from the relation

$$\gamma(\mathrm{G}|\mathrm{M}) = \frac{[K : H]}{[K_{\mathrm{N}} : H_{\mathrm{N}}][K_{\mathrm{M}} : H_{\mathrm{M}}][K_{\mathrm{N}^-} : H_{\mathrm{N}^-}]}, \tag{61}$$

which we claim holds for any compact open subgroup $H \subset \mathrm{G}(F)$ with an Iwahori decomposition $H = H_{\mathrm{N}} H_{\mathrm{M}} H_{\mathrm{N}^-}$ (and in particular for $H = I$, independently of P and P^-). The formula (61) follows from the latter equality of [Wal03, Section I.1, (2)], upon taking the f there to be the characteristic function of H, and noting that our analogues of the measures $dg, d\bar{u}, dm$ and du as in that equality are obtained by dividing arbitrarily chosen Haar measures on $\mathrm{G}(F), \mathrm{N}^-(F), \mathrm{M}(F)$ and $\mathrm{N}(F)$ respectively by $\mathrm{meas}(K), \mathrm{meas}(K_{\mathrm{N}^-}), \mathrm{meas}(K_{\mathrm{M}})$ and $\mathrm{meas}(K_{\mathrm{N}})$.

(v) For each Levi subgroup $\mathrm{M} \in \mathcal{L}$, let $\bar{E}_2(\mathrm{M})$ denote the set of connected components of $\mathrm{Irr}_2(\mathrm{M}) \subset \tilde{T}_{\mathrm{ell}}(\mathrm{M})$, and for each $\sigma \in \mathrm{Irr}_2(\mathrm{M})$, let

$$\mathscr{O}_\sigma := X^{\mathrm{unr-uni}}(\mathrm{M}) \cdot \sigma = X^{\mathrm{unr-uni}}(\mathrm{M}) \cdot (\mathrm{M}, \sigma) = X^{\mathrm{unr-uni}}(\mathrm{M}) \cdot (\mathrm{M}, \sigma, 1) \subset \mathrm{Irr}_2(\mathrm{M}) \subset \tilde{T}_{\mathrm{ell}}(\mathrm{M})$$

be the element of $\bar{E}_2(\mathrm{M})$ containing the image of $\sigma \in \mathrm{Irr}_2(\mathrm{M}) \subset \tilde{T}_{\mathrm{ell}}(\mathrm{M})$. As in [Wal03, pages 239 and 302], we give each $\mathscr{O}_\sigma$ the unique measure such that the restriction map $X^{\mathrm{unr-uni}}(\mathrm{M}) \to X^{\mathrm{unr-uni}}(\mathrm{A}_{\mathrm{M}})$ and the obvious map $X^{\mathrm{unr-uni}}(\mathrm{M}) \to \mathscr{O}_\sigma$ locally preserve measures. In other words, the "$\mathrm{A}_{\mathrm{M}}(F)$-central character" map from $\mathrm{Irr}_2(\mathrm{M})$ to the set $X^{\mathrm{uni}}(\mathrm{A}_{\mathrm{M}})$ of unitary characters $\mathrm{A}_{\mathrm{M}}(F) \to \mathbb{C}^\times$ is locally measure preserving, where $X^{\mathrm{uni}}(\mathrm{A}_{\mathrm{M}})$ is given the topology and measure such that each orbit map $X^{\mathrm{unr-uni}}(\mathrm{A}_{\mathrm{M}}) \to X^{\mathrm{uni}}(\mathrm{A}_{\mathrm{M}})$ is a measure preserving homeomorphism.

Here, the $\mathscr{O}$ in the orbit $\mathscr{O}_\sigma$ is not to be confused with the groups $\mathcal{O}_{\mathrm{M}}$ of automorphisms.

(vi) If $\mathrm{M} \in \mathcal{L}$ and $\sigma \in \mathrm{Irr}_2(\mathrm{L})$, we let $\mu(\sigma) = \mu^{\mathrm{G}}(\sigma)$ be the Harish-Chandra μ-function evaluated on σ, as in [Wal03, Section 5.2].

Now let us recall the Plancherel formula as stated in [Wal03, Theorem VIII.1.1(3)], but in terms of our different choice of measures, and in a form suited to our purposes:

Lemma 5.2.6. *Let $g : \tilde{T}(\mathrm{G}) \to \mathbb{C}$ be an element of $\mathcal{I}(\mathrm{G})$ (identified as in Remark 5.2.3), and let $f_g \in C_c^\infty(\mathrm{G}(F))$ have image g; in other words, $g(\tau) = \Theta_\tau(f_g)$ for all $\tau \in \tilde{T}(\mathrm{G})$. Let $f_g^\vee$ be as in Notation 5.1.1 (like what $\check{f}_g$ would be in the notation of* [Wal03, page 236]*). Then we have:*

$$f_g(1) = f_g^\vee(1) = \sum_{\mathrm{M} \in \mathcal{L}/W_0} \frac{\gamma'''(\mathrm{G}|\mathrm{M})\,\mathrm{meas}(I_{\mathrm{M}})}{\mathrm{meas}(I) \cdot \#W(\mathrm{M})} \int_{\zeta \in X^{\mathrm{uni}}(\mathrm{A}_{\mathrm{M}})} \left(\sum_{\sigma \in \mathrm{Irr}_2(\mathrm{M})_\zeta} \mu(\sigma) d(\sigma) g(\sigma) \right) d\zeta, \tag{62}$$

where $g(\sigma) = g((\mathrm{M}, \sigma))$ refers to the value of g on the image of $\sigma \in \mathrm{Irr}_2(\mathrm{M}) \subset \tilde{T}_{\mathrm{ell}}(\mathrm{M})$ in $\tilde{T}(\mathrm{G})$.

Proof. Given the constraint $\mathrm{meas}(\mathrm{A}_{\mathrm{M}}(F)_c)\,\mathrm{meas}(X^{\mathrm{unr-uni}}(\mathrm{A}_{\mathrm{M}})) = 1$ (see Notation 5.2.5(i)), and because $d(\sigma)$ varies linearly with the measure on $\mathrm{A}_{\mathrm{M}}(F)$, we may and do assume that $\mathrm{A}_{\mathrm{M}}(F)_c$ and $X^{\mathrm{unr-uni}}(\mathrm{A}_{\mathrm{M}})$ are given the normalized Haar measure, as in [Wal03].
Suppose we can prove:

$$f_g(1) = f_g^\vee(1) = \sum_{\mathrm{M} \in \mathcal{L}/W_0} \frac{\gamma'(\mathrm{G}|\mathrm{M})\gamma(\mathrm{G}|\mathrm{M})\,\mathrm{meas}(K_{\mathrm{M}})}{\mathrm{meas}(K) \cdot \#W(\mathrm{M})} \sum_{\mathscr{O} \in \bar{E}_2(\mathrm{M})} \int_{(\mathrm{M},\sigma) \in \mathscr{O}} \mu(\sigma) d(\sigma) g(\sigma)\, d\sigma. \tag{63}$$

Since the "$\mathrm{A}_{\mathrm{M}}(F)$-central character map" from $\mathrm{Irr}_2(\mathrm{M}) \subset \tilde{T}_{\mathrm{ell}}(\mathrm{M})$ to $X^{\mathrm{uni}}(\mathrm{A}_{\mathrm{M}})$ preserves measures locally, the fiber measure on each fiber of this map is the counting measure. Moreover, the fibers have finite intersection with each $\mathscr{O} \in \bar{E}_2(\mathrm{M})$ (since $X^{\mathrm{unr}}(\mathrm{M}) \to X^{\mathrm{unr}}(\mathrm{A}_{\mathrm{M}})$ is an isogeny). Using this and the Fubini theorem (justified by g being continuous and supported on finitely many connected components of $\tilde{T}(\mathrm{G})$, together with the finiteness of the map $\tilde{T}_{\mathrm{ell}}(\mathrm{M}) \to \tilde{T}(\mathrm{G})$), and the equality

$$\frac{\gamma(\mathrm{G}|\mathrm{M})\,\mathrm{meas}(K_{\mathrm{M}})}{\mathrm{meas}(K)} = \frac{\gamma''(\mathrm{G}|\mathrm{M})\,\mathrm{meas}(I_{\mathrm{M}})}{\mathrm{meas}(I)}$$

that follows from (61), it is easy to see that (62) follows from (63). Therefore, it now suffices to prove (63).

Thus, it is now enough to deduce (63) from the formula in [Wal03, Theorem VIII.1.1(3)]. In [Wal03, Theorem VIII.1.1(3)], the sum is over a set of associate classes of pairs $(\mathscr{O}, \mathrm{P})$ as defined in [Wal03, Remark VII.2.4], where $\mathrm{P} \subset \mathrm{G}$ is a semistandard parabolic subgroup and $\mathscr{O} \in \bar{E}_2(\mathrm{M})$, with M the unique semistandard Levi subgroup of P. Instead, we can clearly sum over pairs $(\mathrm{M}, \mathscr{O})$ with M running over (a set of representatives for) $\mathcal{L}/W_0$, and $\mathscr{O}$ running over elements of $\bar{E}_2(\mathrm{M})$ up to the action of $W(\mathrm{M})$ (which is the $W(\mathrm{G}|\mathrm{M})$ of [Wal03]): this is because, given pairs $(\mathrm{P}_1, \mathscr{O}_1)$ and $(\mathrm{P}_2, \mathscr{O}_2)$, where both P_1 and P_2 have the same $\mathrm{M} \in \mathcal{L}$ as their unique semistandard Levi subgroup, these pairs are associate if and only if $\mathscr{O}_1$ and $\mathscr{O}_2$ are conjugate under $W(\mathrm{M})$. It is then easy to check that the expression of [Wal03, Theorem VIII.1.1(3)] agrees with that on the right-hand side of (63), which adds the factors $\mathrm{meas}(K_{\mathrm{M}})$ and $\mathrm{meas}(K)$ to account for not fixing the measures on $\mathrm{G}(F)$ and $\mathrm{M}(F)$ as in [Wal03] (and we have also used our having normalized $\mathrm{meas}(\mathrm{A}_{\mathrm{M}}(F)_c) = \mathrm{meas}(\mathrm{A}_{\mathrm{M}}(F) \cap K)$ and $\mathrm{meas}(X^{\mathrm{unr-uni}}(\mathrm{A}_{\mathrm{M}}))$ to 1). Note that the $c(G|M)^{-2}$ of [Wal03] equals our $\gamma(\mathrm{G}|\mathrm{M})^2\gamma'(\mathrm{G}|\mathrm{M})$. □

Remark 5.2.7. (i) We recall a decomposition of $\mathcal{I}(\mathrm{G})$ ((64) below) from the top of [Art96, page 533], to which we refer for more explanation. Recall the subspace $\mathcal{I}_{\mathrm{cusp}}(\mathrm{M}) \subset \mathcal{I}(\mathrm{M})$ defined to be the image of $C^\infty_{c,\mathrm{cusp}}(\mathrm{M}(F)) \subset C^\infty_c(\mathrm{M}(F))$ in $\mathcal{I}(\mathrm{M})$, as in the proof of Proposition 3.2.6 (and as in Remark 3.4.7(i)). One knows that $\mathcal{I}_{\mathrm{cusp}}(\mathrm{M})$ identifies via the Paley-Wiener theorem (i.e., as in Remark 5.2.3) with the space of those functions $\tilde{T}(\mathrm{M}) \to \mathbb{C}$ in $\mathcal{I}(\mathrm{M})$ that are supported on $\tilde{T}_{\mathrm{ell}}(\mathrm{M})$. Recall that for each Levi subgroup $\mathrm{M} \subset \mathrm{G}$, the map $\tilde{T}_{\mathrm{ell}}(\mathrm{M}) \to \tilde{T}(\mathrm{G})$ factors through an isomorphism from $\tilde{T}_{\mathrm{ell}}(\mathrm{M})/W(\mathrm{M})$ onto its image. Using this fact, we see that the trace Paley-Wiener theorem from [Art96] (recalled in Remark 5.2.3) gives us a decomposition for $\mathcal{I}(\mathrm{G})$ of the form:

$$\mathcal{I}(\mathrm{G}) = \bigoplus_{\mathrm{M} \in \mathcal{L}/W_0} (\mathcal{I}_{\mathrm{cusp}}(\mathrm{M}))^{W(\mathrm{M})}. \tag{64}$$

(ii) Concretely, given $g : \tilde{T}(\mathrm{G}) \to \mathbb{C}$ in $\mathcal{I}(\mathrm{G})$, its projection g_{M} to $\mathcal{I}_{\mathrm{cusp}}(\mathrm{M})^{W(\mathrm{M})}$ is the unique function $\tilde{T}(\mathrm{M}) \to \mathbb{C}$ that is supported on $\tilde{T}_{\mathrm{ell}}(\mathrm{M})$, and such that $\Theta_\tau(g_{\mathrm{M}}) = \Theta_{\mathrm{Ind}_{\mathrm{M}}^{\mathrm{G}} \tau}(g)$ for each $\tau \in D_{\mathrm{ell}}(\mathrm{M})$. This identifies $\mathcal{I}_{\mathrm{cusp}}(\mathrm{M})^{W(\mathrm{M})}$ with the subspace of $\mathcal{I}(\mathrm{G})$ consisting of the images of functions $f \in C^\infty_c(\mathrm{G}(F))$ such that $(\mathrm{Ind}_{\mathrm{L}}^{\mathrm{G}} \Theta)(f) = 0$ whenever L is not $\mathrm{G}(F)$-conjugate to M, and $\Theta \in D_{\mathrm{ell}}(\mathrm{L})$.

(iii) From (64), taking duals, we have a decomposition involving spaces of distributions:

$$\mathcal{I}(\mathrm{G})^* = \bigoplus_{\mathrm{M} \in \mathcal{L}/W_0} (\mathcal{I}_{\mathrm{cusp}}(\mathrm{M})^*)^{W(\mathrm{M})}. \tag{65}$$

Tautologically, the pairing of $\mathcal{I}(\mathrm{G})$ with $D(\mathrm{G})$, after using the identifications of (64) and (34), is obtained by taking a direct sum of the pairings between the $\mathcal{I}_{\mathrm{cusp}}(\mathrm{L})^{W(\mathrm{L})}$ and $D_{\mathrm{ell}}(\mathrm{L})^{W(\mathrm{L})}$, as L ranges over $\mathcal{L}/W_0$.

(iv) By Proposition 3.2.10, $\mathcal{SI}_{\mathrm{cusp}}(\mathrm{M})$ (resp., $\mathcal{SI}(\mathrm{G})$) is the quotient of $\mathcal{I}_{\mathrm{cusp}}(\mathrm{M})$ (resp., $\mathcal{I}(\mathrm{G})$) by its subspace consisting of elements on which elements of $SD_{\mathrm{ell}}(\mathrm{M})$ (resp., $SD(\mathrm{G})$) vanishes. Therefore (64) induces a decomposition

$$\mathcal{SI}(\mathrm{G}) = \bigoplus_{\mathrm{M} \in \mathcal{L}/W_0} (\mathcal{SI}_{\mathrm{cusp}}(\mathrm{M}))^{W(\mathrm{M})}. \tag{66}$$

This decomposition has a description analogous to that for (64) given in (ii) above. Taking duals, we get a decomposition

$$\mathcal{SI}(\mathrm{G})^* = \bigoplus_{\mathrm{M} \in \mathcal{L}/W_0} (\mathcal{SI}_{\mathrm{cusp}}(\mathrm{M})^{W(\mathrm{M})})^* \tag{67}$$

at the level of stable distributions, that extends (35), and clearly aligns with (65).

(v) Let us expand on (66) and (67), and their compatibility with (64) and (65). The terms of (66) identify with quotients of the corresponding terms of (64); the derivation of (66) tells us that (64) identifies the subspace of $\mathcal{I}(\mathrm{G})$ consisting of its unstable elements, with the direct sum, over $\mathrm{M} \in \mathcal{L}/W_0$, of the subspace of unstable elements of $\mathcal{I}_{\mathrm{cusp}}(\mathrm{M})^{W(\mathrm{M})}$.

Quotienting (64) by this restricted isomorphism yields (66). Dualizing, (67) is a restriction of (65) in an obvious way. While $\mathcal{SI}(\mathrm{G})^*$ identifies with the space of stable distributions on $\mathrm{G}(F)$, we can also interpret $(\mathcal{I}_{\mathrm{cusp}}(\mathrm{M})^{W(\mathrm{M})})^*$ and $(\mathcal{SI}_{\mathrm{cusp}}(\mathrm{M})^{W(\mathrm{M})})^*$ as spaces of distributions on $\mathrm{M}(F)$, using $W(\mathrm{M})$-averaging and the analogues of (65) and (67) with G replaced by M. Thus, $(\mathcal{I}_{\mathrm{cusp}}(\mathrm{M})^{W(\mathrm{M})})^*$ can be identified with the space of $W(\mathrm{M})$-invariant functionals on $\mathcal{I}(\mathrm{M})$ (i.e., invariant distributions on $\mathrm{M}(F)$) that vanish on "$\mathcal{I}_{\mathrm{cusp}}(\mathrm{L})^{W_{\mathrm{M}}(\mathrm{L})}$" for each proper Levi subgroup $\mathrm{L} \subset \mathrm{M}$. A similar interpretation applies to $(\mathcal{SI}_{\mathrm{cusp}}(\mathrm{M})^{W(\mathrm{M})})^*$. Clearly, $(\mathcal{SI}_{\mathrm{cusp}}(\mathrm{M})^{W(\mathrm{M})})^*$ is precisely the subspace of $(\mathcal{I}_{\mathrm{cusp}}(\mathrm{M})^{W(\mathrm{M})})^*$ consisting of elements that when, viewed as distributions on $\mathrm{M}(F)$, are stable. Now it is easy to see the following: if, according to (65), $\Theta \in \mathcal{I}(\mathrm{G})^*$ has component $\Theta_{\mathrm{M}} \in (\mathcal{I}_{\mathrm{cusp}}(\mathrm{M})^{W(\mathrm{M})})^*$ for each $\mathrm{M} \in \mathcal{L}/W_0$, then the distribution Θ on $\mathrm{G}(F)$ is stable if and only if each $\Theta_{\mathrm{M}} \in (\mathcal{I}_{\mathrm{cusp}}(\mathrm{M})^{W(\mathrm{M})})^* \subset (\mathcal{I}(\mathrm{M})^{W(\mathrm{M})})^*$ is stable as a distribution on $\mathrm{M}(F)$.

(vi) The compatibility between parabolic induction and endoscopic transfer (Remark 3.2.2(iii)) admits a slight generalization involving more general distributions than virtual characters, as we now review in the case of transfer to the quasi-split inner form; this can perhaps be viewed as an 'endoscopic version' of (v) above. Let G^* be a quasi-split inner form of G underlying an endoscopic datum $\underline{\mathrm{G}}^*$ for G, as in Notation 3.2.1(i). Let $\mathcal{L}^*$ and $\mathcal{L}^*/W_0^*$ be analogues, for G^*, of $\mathcal{L}$ and $\mathcal{L}/W_0$. Choosing representatives, we identify $\mathcal{L}/W_0$ and $\mathcal{L}^*/W_0^*$ with subsets of $\mathcal{L}$ and $\mathcal{L}^*$. Notation 3.2.1(v) gives us an injection $\mathcal{L}/W_0 \hookrightarrow \mathcal{L}^*/W_0^*$. For each $\mathrm{M} \in \mathcal{L}/W_0 \subset \mathcal{L}$ and its image $\mathrm{M}^* \in \mathcal{L}^*/W_0^* \subset \mathcal{L}^*$, M^* is a quasi-split form of M, and more precisely, a choice of 'Levi subgroup matching data' as in Notation 3.2.1(vi) gives an endoscopic datum $\underline{\mathrm{M}}^*$ realizing M^* as endoscopic to M. Now we make two easy but useful observations:
   (a) Let $\mathrm{M} \in \mathcal{L}/W_0 \subset \mathcal{L}$, and consider the corresponding $\mathrm{M}^* \in \mathcal{L}^*/W_0^* \subset \mathcal{L}^*$. Via (64) and the analogue of (66) for G^*, the endoscopic transfer map $\mathcal{I}(\mathrm{G}) \to \mathcal{SI}(\mathrm{G}^*)$ along $\underline{\mathrm{G}}^*$ takes $\mathcal{I}_{\mathrm{cusp}}(\mathrm{M})^{W(\mathrm{M})}$ to $\mathcal{SI}_{\mathrm{cusp}}(\mathrm{M}^*)^{W(\mathrm{M}^*)}$, and moreover, the resulting map $\mathcal{I}_{\mathrm{cusp}}(\mathrm{M})^{W(\mathrm{M})} \to \mathcal{SI}_{\mathrm{cusp}}(\mathrm{M}^*)^{W(\mathrm{M}^*)}$ is obtained by restricting the endoscopic transfer map $\mathcal{I}(\mathrm{M}) \to \mathcal{SI}(\mathrm{M}^*)$ along $\underline{\mathrm{M}}^*$. Indeed, using the concrete description in (ii) and its analogue for (66) (as applied to G^*), both these assertions follow from the compatibility between parabolic induction and endoscopic transfer (Remark 3.2.2(iii)), together with the fact that the endoscopic transfer maps $\mathcal{I}(\mathrm{G}) \to \mathcal{SI}(\mathrm{G}^*)$ and $\mathcal{I}(\mathrm{M}) \to \mathcal{SI}(\mathrm{M}^*)$ are uniquely determined as dual to the endsocopic transfer maps $SD(\mathrm{G}^*) \to SD(\mathrm{G})$ and $SD(\mathrm{M}^*) \to SD(\mathrm{M})$, by the density of characters in [Art96, Lemma 6.3] (or as recalled in Proposition 3.2.10).
   (b) The map $\mathcal{SI}(\mathrm{G}^*)^* \to \mathcal{I}(\mathrm{G})^*$, via (65) and the analogue of (67) for G^*, restricts as follows to each $(\mathcal{SI}_{\mathrm{cusp}}(\mathrm{M}^*)^{W(\mathrm{M}^*)})^*$: If $\mathrm{M}^* \in \mathcal{L}^*/W_0^*$ is not the image of any element of $\mathcal{L}/W_0$, then this restriction is zero; if not, say M^* is the image of $\mathrm{M} \in \mathcal{L}/W_0$, it is a map $(\mathcal{SI}_{\mathrm{cusp}}(\mathrm{M}^*)^{W(\mathrm{M}^*)})^* \to (\mathcal{I}_{\mathrm{cusp}}(\mathrm{M})^{W(\mathrm{M})})^* \subset \mathcal{I}(\mathrm{G})^*$ obtained as the restriction of the endoscopic transfer map $\mathcal{SI}(\mathrm{M}^*)^* \to \mathcal{I}(\mathrm{M})^*$ along $\underline{\mathrm{M}}^*$. This observation follows by dualizing the observation (a) above (applied with M replaced by each $\mathrm{L} \in \mathcal{L}/W_0 \subset \mathcal{L}$).

5.2.2. *Stability of certain distributions, and their transfer to inner forms.*

Proposition 5.2.8. *Suppose $\Theta \in \mathcal{I}(\mathrm{G})^*$, and that for each $\mathrm{L} \in \mathcal{L}$, $\mu_\Theta = \mu_{\Theta,\mathrm{L}} : \mathrm{Irr}_2(\mathrm{L}) \to \mathbb{C}$ is a continuous function that is invariant under $W(\mathrm{L})$. Suppose that for all $g : \tilde{T}(\mathrm{G}) \to \mathbb{C}$ in $\mathcal{I}(\mathrm{G})$ we have:*

$$\Theta(g) = \sum_{\mathrm{L} \in \mathcal{L}/W_0} \int_{\zeta \in X^{\mathrm{uni}}(\mathrm{A_L})} \Theta_{\mathrm{L},\zeta}(g)\, d\zeta, \tag{68}$$

where $\Theta_{\mathrm{L},\zeta} \in \mathcal{I}_{\mathrm{cusp}}(\mathrm{L})^* \subset \mathcal{I}(\mathrm{L})^*$ is a distribution of the form:

$$\Theta_{\mathrm{L},\zeta} = \sum_{\sigma \in \mathrm{Irr}_2(\mathrm{L})_\zeta} \mu_\Theta(\sigma) d(\sigma) \Theta^{\mathrm{L}}_\sigma, \tag{69}$$

and in (68) $\Theta_{\mathrm{L},\zeta}(g)$ *refers to* $\Theta_{\mathrm{L},\zeta}(g_{\mathrm{L}})$, $g_{\mathrm{L}} \in \mathcal{I}_{\mathrm{cusp}}(\mathrm{L})^{W(\mathrm{L})}$ *being the projection of* g *via* (64). *Suppose that* $\Theta \in \mathcal{I}(\mathrm{G})^*$ *is stable, and let* $\mathrm{M} \in \mathcal{L}$. *Then* $\Theta_{\mathrm{M},\zeta} \in \mathcal{I}(\mathrm{M})^*$, *for each* $\zeta \in X^{\mathrm{uni}}(\mathrm{A_M})$.

Proof. By the compatibility between (65) and (67), the projection Θ_{M} of $\Theta \in \mathcal{SI}(\mathrm{G})^*$ to $(\mathcal{I}_{\mathrm{cusp}}(\mathrm{M})^*)^{W(\mathrm{M})} \subset \mathcal{I}_{\mathrm{cusp}}(\mathrm{M})^* \subset \mathcal{I}(\mathrm{M})^*$ under (65) belongs to $(\mathcal{SI}_{\mathrm{cusp}}(\mathrm{M})^*)^{W(\mathrm{M})}$.

Note that for any $g : \tilde{T}(\mathrm{M}) \to \mathbb{C}$ in $\mathcal{I}(\mathrm{M})$, $\zeta \mapsto \Theta_{\mathrm{M},\zeta}(g)$ is the push-forward of $\sigma \mapsto \mu_\Theta(\sigma)d(\sigma)g(\sigma)$ along the $\mathrm{A_M}(F)$-central character map $\tilde{T}_{\mathrm{ell}}(\mathrm{M}) \supset \mathrm{Irr}_2(\mathrm{M}) \to X^{\mathrm{uni}}(\mathrm{A_M})$, which is a local homeomorphism, so that $\zeta \mapsto \Theta_{\mathrm{M},\zeta}(g)$ is continuous (use that g is supported on finitely many connected components of $\mathrm{Irr}_2(\mathrm{M}) \subset \tilde{T}_{\mathrm{ell}}(\mathrm{M})$). We claim that for all $f \in C_c^\infty(\mathrm{M}(F))$, we have:

$$\Theta_{\mathrm{M}}(f) = \int_{\zeta \in X^{\mathrm{uni}}(\mathrm{A_M})} \Theta_{\mathrm{M},\zeta}(f)\, d\zeta. \tag{70}$$

The right-hand side of (70) represents a distribution in f that belongs to $\mathcal{I}_{\mathrm{cusp}}(\mathrm{M})^* \subset \mathcal{I}(\mathrm{M})^*$, since the $\Theta_{\mathrm{M},\zeta}$ are "supported" in discrete series representations. Therefore, (70) is tautological once we see that the right-hand side of (70) is a $W(\mathrm{M})$-invariant distribution in $f \in C_c^\infty(\mathrm{M}(F))$, which in turn follows from the $W(\mathrm{M})$-invariance of $\sigma \mapsto \mu_\Theta(\sigma)$ (by hypothesis) and that of $\sigma \mapsto d(\sigma)$.
Using (70) and the stability of Θ_{M}, let us show that $\Theta_{\mathrm{M},\zeta}$ is stable for each $\zeta \in X^{\mathrm{uni}}(\mathrm{A_M})$. For all $h \in C_c^\infty(\mathrm{A_M}(F))$ and $f \in C_c^\infty(\mathrm{M}(F))$, let $h * f \in C_c^\infty(\mathrm{M}(F))$ denote the left-regular action of h on f. It is easy to see that $\sigma(h * f) = \hat{h}(\zeta_\sigma)\sigma(f)$ for all unitary representations $\sigma \in \mathrm{Irr}(\mathrm{M})$, where ζ_σ is the $\mathrm{A_M}(F)$-central character of σ and $\hat{h} \in C_0(X^{\mathrm{uni}}(\mathrm{A_M}))$ is the Fourier transform of h (C_0 stands for functions that 'vanish at ∞'). This implies that $\Theta_{\mathrm{M},\zeta}(h * f) = \hat{h}(\zeta)\Theta_{\mathrm{M},\zeta}(f)$, for all $h \in C_c^\infty(\mathrm{A_M}(F))$ and $f \in C_c^\infty(\mathrm{M}(F))$.
It is immediately verified that if $f \in C_c^\infty(\mathrm{M}(F))$ is unstable, then so is $h * f$ for all $C_c^\infty(\mathrm{A_M}(F))$. It follows from the stability of Θ_{M} that for unstable functions $f \in C_c^\infty(\mathrm{M}(F))$:

$$\int_{\zeta \in X^{\mathrm{uni}}(\mathrm{A_M})} \hat{h}(\zeta)\Theta_{\mathrm{M},\zeta}(f)\, d\zeta = 0.$$

Since the image of $C_c^\infty(\mathrm{A_M}(F))$ in $C_0(X^{\mathrm{uni}}(\mathrm{A_M})) \cap L^2(X^{\mathrm{uni}}(\mathrm{A_M}))$ under the Fourier transform is dense in $L^2(X^{\mathrm{uni}}(\mathrm{A_M}))$ by Pontrjagin duality, it follows that $(\zeta \mapsto \Theta_{\mathrm{M},\zeta}(f)) \in C_c(X^{\mathrm{uni}}(\mathrm{A_M}))$ vanishes as an element of $L^2(X^{\mathrm{uni}}(\mathrm{A_M}))$, and hence as an element of $C_c(X^{\mathrm{uni}}(\mathrm{A_M}))$. Since this is true for all unstable $f \in C_c^\infty(\mathrm{M}(F))$, the stability of $\Theta_{\mathrm{M},\zeta}$ follows. □

Remark 5.2.9.
(i) The above proof can probably be adapted to prove a more general version of the proposition, where $\mu_\Theta = \mu_{\Theta,\mathrm{M}}$ is allowed to be any continuous function on the larger space $\tilde{T}_{\mathrm{ell}}(\mathrm{M}) \supset \mathrm{Irr}_2(\mathrm{M})$ that has an equivariance property opposite to that in the [Art96, page 532, condition (iii)] (whose articulation we omitted from Remark 5.2.3). The sum defining $\Theta_{\mathrm{M},\zeta}$ will then have to be over a set $T_{\mathrm{ell}}(\mathrm{M})_\zeta \supset \mathrm{Irr}_2(\mathrm{M})_\zeta$, the '$\zeta$-part' of the quotient $T_{\mathrm{ell}}(\mathrm{M})$ of $\tilde{T}_{\mathrm{ell}}(\mathrm{M})$ as in [Art96, just before (4.2)].
(ii) The argument of the proof can be adapted to deduce a 'version with central character' of the Plancherel formula: if $\zeta : \mathrm{A_G}(F) \to \mathbb{C}^\times$ is a smooth unitary character, and $f \in C^\infty_{\mathrm{A_G}(F),\zeta}(\mathrm{G}(F))$, then for an appropriate choice of a measure on $X^{\mathrm{uni}}(\mathrm{A_M}/\mathrm{A_G})$ we have a formula analogous to that in Lemma 5.2.6:

$$f(1) = \sum_{\mathrm{M} \in \mathcal{L}/W_0} \frac{\gamma'''(\mathrm{G}|\mathrm{M})\,\mathrm{meas}(I_{\mathrm{M}})}{\mathrm{meas}(I) \cdot \#W(\mathrm{M})} \int_{\zeta' \in X^{\mathrm{uni}}(\mathrm{A_M}/\mathrm{A_G})} \left(\sum_{\sigma \in \mathrm{Irr}_2(\mathrm{M})_{\zeta\zeta'}} \mu(\sigma)d(\sigma)\Theta^{\mathrm{G}}_\sigma(f) \right) d\zeta'. \tag{71}$$

Proposition 5.2.10. *Let* G^* *be an inner form of* G. *Fix an endoscopic datum* $\underline{\mathrm{G}}^*$ *for* G *with underlying group* G^*, *as in Notation 3.2.1(i). Let* $\mathrm{M} \subset \mathrm{G}$ *be a Levi subgroup in* $\mathcal{L}$, *and* $\mathrm{M}^* \subset \mathrm{G}^*$ *a Levi subgroup in an analogous set* $\mathcal{L}^*$ *defined using a maximal split torus* $\mathrm{A}_0^* \subset \mathrm{G}^*$. *Assume that some choice of 'Levi subgroup matching data' as in Notation 3.2.1(vi) matches* M *and* M^*, *giving an endoscopic datum* $\underline{\mathrm{M}}^*$ *for* M *with underlying group* M^*. *In particular, we have identifications* $\mathrm{A}_{\mathrm{M}^*} = \mathrm{A_M}$ *and* $\mathrm{A}_{\mathrm{G}^*} = \mathrm{A_G}$. *Let* $\Theta = \Theta_{\mathrm{G}}$, *the* Θ_{L}, *the* $\mu_\Theta = \mu_{\Theta,\mathrm{L}}$ *and the* $\Theta_{\mathrm{L},\zeta}$ *be as in Proposition*

5.2.8. Suppose that $\Theta^* = \Theta_{G^*}$ the Θ_{L^*}, the $\mu_{\Theta^*} = \mu_{\Theta^*,L^*}$ and the

$$\Theta_{L^*,\zeta} = \sum_{\sigma^* \in \mathrm{Irr}_2(L^*)_\zeta} d(\sigma^*)\mu_{\Theta^*}(\sigma^*)\Theta^{L^*}_{\sigma^*}$$

are analogous objects associated to G^*; in particular, Θ and Θ^* are stable, and the $\mu_{\Theta,L}$ and the μ_{Θ^*,L^*} are invariant under the $W(L)$ and the $W(L^*)$, respectively. Assume that Θ^* has image Θ under the endoscopic transfer map $\mathcal{SI}(G^*)^* \to \mathcal{SI}(G)^*$. Then for each $\zeta \in X^{\mathrm{uni}}(A_M) = X^{\mathrm{uni}}(A_{M^*})$, the image of $\Theta_{M^*,\zeta}$ under the endoscopic transfer map $\mathcal{SI}(M^*)^* \to \mathcal{SI}(M)^*$ equals $\Theta_{M,\zeta}$.

Proof. We follow the proof of Proposition 5.2.8. As we saw in that proof, the projection Θ_M of Θ along $\mathcal{I}(G)^* \to (\mathcal{I}_{\mathrm{cusp}}(M)^{W(M)})^*$ is stable, and and similarly we get the projection $\Theta_{M^*} \in (\mathcal{SI}_{\mathrm{cusp}}(M^*)^{W(M^*)})^* \subset \mathcal{SI}(G^*)^*$ of $\Theta^* = \Theta_{G^*}$. By Proposition 5.2.8, $\Theta_{M,\zeta}$ and $\Theta_{M^*,\zeta}$ are stable for each $\zeta \in X^{\mathrm{uni}}(A_M) = X^{\mathrm{uni}}(A_{M^*})$. We have (70) expressing Θ_M in terms of the $\Theta_{M,\zeta}$, and a similar equation relates Θ_{M^*} to the $\Theta_{M^*,\zeta}$. Remark 5.2.7 (vi)(b) gives us the following claim:
Claim. The restriction of the endoscopic transfer map $\mathcal{SI}(G^*)^* \to \mathcal{SI}(G)^* \subset \mathcal{I}(G)^*$ to $(\mathcal{SI}_{\mathrm{cusp}}(M^*)^{W(M^*)})^*$ is a map $(\mathcal{SI}_{\mathrm{cusp}}(M^*)^{W(M^*)})^* \to (\mathcal{I}_{\mathrm{cusp}}(M)^{W(M)})^*$, obtained by restricting the endoscopic transfer map $\mathcal{SI}(M^*)^* \to \mathcal{I}(M)^*$.
By this claim, Θ_{M^*} has image Θ_M under $\mathcal{SI}(M^*)^* \to \mathcal{SI}(M)^*$. We then identify $C_c^\infty(A_{M^*}(F))$ and $C_c^\infty(A_M(F))$ with each other, and consider their left-regular actions on $C_c^\infty(M^*(F))$ and $C_c^\infty(M(F))$, as well as the induced actions on associated spaces such as $\mathcal{I}(M^*)$ and $\mathcal{I}(M)$ and $\mathcal{SI}(M^*)$ and $\mathcal{SI}(M)$. It is easy to see, using the arguments in the proof of Lemma 3.2.3(i), that this action respects the map $\mathcal{SI}(M^*) \to \mathcal{SI}(M)$.
Now assume that $f \in C_c^\infty(M(F))$ and $f^* \in C_c^\infty(M^*(F))$ have matching orbital integrals. We need to show that $\Theta_{M^*,\zeta}(f^*) = \Theta_{M,\zeta}(f)$ for all $\zeta \in X^{\mathrm{uni}}(A_M)$. For all $h \in C_c^\infty(A_M(F)) = C_c^\infty(A_{M^*}(F))$, we have that $h * f$ and $h * f^*$ have matching orbital integrals, and (as in the proof of Proposition 5.2.8) that $\Theta_{M^*,\zeta}(h * f^*) = \hat{h}(\zeta)\Theta_{M^*,\zeta}(f^*)$, and that $\Theta_{M,\zeta}(h * f) = \hat{h}(\zeta)\Theta_{M,\zeta}(f)$. Therefore,

$$\int_{X^{\mathrm{uni}}(A_{M^*}) = X^{\mathrm{uni}}(A_M)} \hat{h}(\zeta)\Theta_{M^*,\zeta}(f^*)\, d\zeta = \Theta_{M^*}(h*f^*) = \Theta_M(h*f) = \int_{X^{\mathrm{uni}}(A_{M^*}) = X^{\mathrm{uni}}(A_M)} \hat{h}(\zeta)\Theta_{M,\zeta}(f)\, d\zeta.$$

Using Pontrjagin duality on $X^{\mathrm{uni}}(A_{M^*}) = X^{\mathrm{uni}}(A_M)$ as in the proof of Proposition 5.2.8, it is now easy to see that $\Theta_{M^*,\zeta}(f^*) = \Theta_{M,\zeta}(f)$ for each $\zeta \in X^{\mathrm{uni}}(A_M)$, as desired. ☐

Corollary 5.2.11. *(i) Let $M \subset G$ be a Levi subgroup. Then for each $\zeta \in X^{\mathrm{uni}}(A_M)$ and $z \in \mathcal{Z}_1(G)$, the distribution*

$$\sum_{\sigma \in \mathrm{Irr}_2(M)_\zeta} d(\sigma)\mu(\sigma)\hat{z}(\sigma)\Theta^M_\sigma \in \mathcal{I}(M^*)$$

is stable. In particular, $\sum_{\sigma \in \mathrm{Irr}_2(M)_\zeta} d(\sigma)\mu(\sigma)\Theta^M_\sigma$ is stable.

(ii) Let G^* be a quasi-split inner form of G, and let $\underline{G}^*$ be as in Notation 3.2.1(i). As in Proposition 5.2.10, let $M^* \subset G^*$ and $M \subset G$ be 'compatible Levi subgroups', i.e., related by an endoscopic datum $\underline{M}^*$ obtained using 'Levi subgroup matching data' as in Notation 3.2.1(vi). Assume that the measures on $M^*(F)$ and $M(F)$ are compatible in the sense explained in [Kot88, page 631]*, and that the identification $A_{M^*}(F) = A_M(F)$ is measure preserving. Let $\mathbf{T}_{\underline{M}^*}$ denote the endoscopic transfer map $\mathcal{SI}(M^*)^* \to \mathcal{SI}(M)^*$. Assume that $z^* \in \mathcal{Z}_1(G^*)$ and $z \in \mathcal{Z}_1(G)$ are related by $\mathbf{T}_{\underline{G}^*}(z^*) = e(G)z$, where $\mathbf{T}_{\underline{G}^*}$ is the endoscopic transfer map $\mathcal{SI}(G^*)^* \to \mathcal{SI}(G)^*$, normalized using compatible measures on $G^*(F)$ and $G(F)$ as in* [Kot88, page 631]*. Then for each $\zeta \in X^{\mathrm{uni}}(A_{M^*}) = X^{\mathrm{uni}}(A_M)$ we have that, and $\mathcal{SI}(G^*)^* \to \mathcal{SI}(G)^*$ we have:*

$$\gamma'''(G^*|M^*)\cdot\mathbf{T}_{\underline{M}^*}\left(\sum_{\sigma^* \in \mathrm{Irr}_2(M^*)_\zeta} d(\sigma^*)\mu(\sigma^*)\hat{z^*}(\sigma^*)\Theta^{M^*}_{\sigma^*}\right) = e(G)\gamma'''(G|M)\left(\sum_{\sigma \in \mathrm{Irr}_2(M)_\zeta} d(\sigma)\mu(\sigma)\hat{z}(\sigma)\Theta^M_\sigma\right).$$

In particular, we have

$$\gamma'''(\mathrm{G}^*|\mathrm{M}^*)\cdot \mathbf{T}_{\underline{\mathrm{M}}^*}\left(\sum_{\sigma^*\in \mathrm{Irr}_2(\mathrm{M}^*)_\zeta} d(\sigma^*)\mu(\sigma^*)\Theta^{\mathrm{M}^*}_{\sigma^*}\right) = e(\mathrm{G})\gamma'''(\mathrm{G}|\mathrm{M})\left(\sum_{\sigma\in \mathrm{Irr}_2(\mathrm{M})_\zeta} d(\sigma)\mu(\sigma)\Theta^{\mathrm{M}}_{\sigma}\right).$$

Proof. In (i), the latter assertion (i.e., the one starting with 'In particular') can be deduced from the former, by letting $z \in \mathcal{Z}(\mathrm{G})$ be the Dirac delta distribution at the identity, which ensures that $\hat{z}(\sigma) = 1$ for all $\sigma \in \mathrm{Irr}_2(\mathrm{M})$. A similar comment applies to (ii): if we take $z^* \in \mathcal{Z}(\mathrm{G}^*)$ to be the Dirac delta measure at the identity, then by [Kot88, Proposition 2] (which assumes the compatibility of measures between $\mathrm{G}^*(F)$ and $\mathrm{G}(F)$), we can take z to be the Dirac delta at the identity too (this is the reason for adding the Kottwitz sign $e(\mathrm{G})$ in the condition $\mathbf{T}_{\underline{\mathrm{G}}^*}(z^*) = e(\mathrm{G})z$). Therefore, in both (i) and (ii), we will only prove the former assertion.
To prove (i), we first note that Lemma 5.2.6 and the equality $\Theta^{\mathrm{G}}_\tau(z * f) = \hat{z}(\sigma)\Theta^{\mathrm{G}}_\tau(f)$ for $\mathrm{L} \in \mathcal{L}$ and $\tau = (\mathrm{L}, \sigma, 1) \in \mathrm{Irr}_2(\mathrm{L}) \subset \tilde{T}(\mathrm{G})$ (use (56)) imply:
(72)

$$z(f^\vee) = z * f(1) = \sum_{\mathrm{L}\in\mathcal{L}/W_0} \frac{\gamma'''(\mathrm{G}|\mathrm{L})\,\mathrm{meas}(I_\mathrm{L})}{\mathrm{meas}(I)\cdot \#W(\mathrm{L})}\int_{\zeta\in X^{\mathrm{uni}}(\mathrm{A_L})}\left(\sum_{\sigma\in\mathrm{Irr}_2(\mathrm{L})_\zeta}\mu(\sigma)d(\sigma)\hat{z}(\sigma)\Theta^{\mathrm{G}}_\sigma(f)\right)d\zeta.$$

Let $\Theta = \Theta_\mathrm{G} \in \mathcal{I}(\mathrm{G})^*$ be given by $f \mapsto z(f^\vee) = z * f(1)$. We claim that the hypotheses of Proposition 5.2.8 are satisfied for Θ, if we take, for each $\mathrm{L} \in \mathcal{L}/W_0$ and $\sigma \in \mathrm{Irr}_2(\mathrm{L})$:

$$\mu_\Theta(\sigma) = \mu_{\Theta,\mathrm{L}}(\sigma) = \frac{\gamma'''(\mathrm{G}|\mathrm{L})\,\mathrm{meas}(I_\mathrm{L})}{\mathrm{meas}(I)\cdot\#W(\mathrm{L})}\hat{z}(\sigma)\mu(\sigma). \tag{73}$$

By Proposition 5.2.8 (and using the expression (69)), proving this claim will yield (i). Given (72), using that Θ is stable (since $z \in \mathcal{Z}_1(\mathrm{G})$), this follows from the following three observations applied to each $\mathrm{L} \in \mathcal{L}$:

- If $f \in C^\infty_c(\mathrm{G}(F))$ maps to g in $\mathcal{I}(\mathrm{G})$, then for each $\sigma \in \mathrm{Irr}_2(\mathrm{L})$, $\Theta^\mathrm{G}_\sigma(f) = \Theta^\mathrm{G}_\sigma(g) = \Theta^\mathrm{L}_\sigma(g_\mathrm{L})$, where g_L is, as in Proposition 5.2.8, the projection of g to $\mathcal{I}_{\mathrm{cusp}}(\mathrm{L})^{W(\mathrm{L})}$ as per (64).
- $\mu_{\Theta,\mathrm{L}}$ is continuous, since for each $\sigma \in \mathrm{Irr}_2(\mathrm{L})$, $\chi \mapsto \hat{z}(\sigma\otimes\chi)$ and $\chi \mapsto \mu(\sigma\otimes\chi)$ are rational functions on $X^{\mathrm{unr}}(\mathrm{L})$ that are regular on $X^{\mathrm{unr-uni}}(\mathrm{L})$ (for the latter, see [Wal03, Lemma V.2.1]).
- $\mu_{\Theta,\mathrm{L}}$ is $W(\mathrm{L})$-invariant, since $\sigma \mapsto \hat{z}(\sigma)$ and $\sigma \mapsto \mu(\sigma)$ are (for the latter, again use [Wal03, Lemma V.2.1]).

This gives (i). The proof of (ii) will implicitly use the observations made in the proof of (i). Without loss of generality, M^* belongs to the set $\mathcal{L}^*$ analogous to $\mathcal{L}$, defined using a chosen split maximal torus $\mathrm{A}^*_0 \subset \mathrm{G}^*$. We apply Proposition 5.2.10 with the $\Theta^* = \Theta_{\mathrm{G}^*} \in \mathcal{SI}(\mathrm{G}^*)$ of that proposition taken to be the distribution $f^* \mapsto z^*((f^*)^\vee) = (z^* * f^*)(1)$. It is easy to see from Remark 3.2.2(i) that $f^\vee$ and $(f^*)^\vee$ have matching orbital integrals whenever f and f^* do, so that the image $\Theta := \Theta_\mathrm{G} := \mathbf{T}_{\underline{\mathrm{G}}^*}(\Theta^*)$ of Θ^* under endoscopic transfer with respect to $\underline{\mathrm{G}}^*$ equals $f \mapsto e(\mathrm{G})z(f^\vee) = e(\mathrm{G})(z * f)(1)$.
Combining Proposition 5.2.10 with (73) and its analogue with G replaced by G^*, we get

$$\begin{aligned}\frac{\gamma'''(\mathrm{G}^*|\mathrm{M}^*)\,\mathrm{meas}(I_{\mathrm{M}^*})}{\mathrm{meas}(I_{\mathrm{G}^*})\cdot\#W(\mathrm{M}^*)}&\cdot\mathbf{T}_{\underline{\mathrm{M}}^*}\left(\sum_{\sigma^*\in\mathrm{Irr}_2(\mathrm{M}^*)_\zeta} d(\sigma^*)\mu(\sigma^*)\hat{z^*}(\sigma^*)\Theta^{\mathrm{M}^*}_{\sigma^*}\right)\\ &= \frac{\gamma'''(\mathrm{G}|\mathrm{M})\,\mathrm{meas}(I_\mathrm{M})}{\mathrm{meas}(I_\mathrm{G})\cdot\#W(\mathrm{M})}\cdot\left(\sum_{\sigma\in\mathrm{Irr}_2(\mathrm{M})_\zeta} d(\sigma)\mu(\sigma)\cdot e(\mathrm{G})\hat{z}(\sigma)\Theta^{\mathrm{M}}_\sigma\right),\end{aligned}$$

for each $\zeta \in X^{\mathrm{uni}}(\mathrm{A_M}) = X^{\mathrm{uni}}(\mathrm{A}_{\mathrm{M}^*})$, where $I_\mathrm{G} = I$ is as in Notation 5.2.5(iii), I_{G^*} is the analogous subgroup of $\mathrm{G}^*(F)$. Note that $I_\mathrm{M} = I \cap \mathrm{M}(F)$ and $I_{\mathrm{M}^*} = I_{\mathrm{G}^*} \cap \mathrm{M}^*(F)$ is are Iwahori subgroups of $\mathrm{M}(F)$ and $\mathrm{M}^*(F)$. This much is what we get without imposing any compatibility between the measures on $\mathrm{G}^*(F)$ and $\mathrm{G}(F)$, and between the ones on $\mathrm{M}^*(F)$ and $\mathrm{M}(F)$. Since

$\#W(\mathrm{M}^*) = \#W(\mathrm{M})$ by the discussion of Notation 3.2.1(viii), (ii) will follow if we show that, for our choices of measures, we have:

$$\frac{\mathrm{meas}(I_{\mathrm{M}})}{\mathrm{meas}(I_{\mathrm{G}})} = \frac{\mathrm{meas}(I_{\mathrm{M}^*})}{\mathrm{meas}(I_{\mathrm{G}^*})}. \tag{74}$$

As in [Kot88, page 632], we may and do choose the measures on $\mathrm{G}(F), \mathrm{M}(F), \mathrm{G}^*(F)$ and $\mathrm{M}^*(F)$ to be integral and with nonzero reduction for the integral models of the parahoric group scheme structures associated to $I_{\mathrm{G}}, I_{\mathrm{M}}, I_{\mathrm{G}^*}$ and I_{M^*}. It is then enough to show that $\mathrm{meas}(I_{\mathrm{G}}) = \mathrm{meas}(I_{\mathrm{M}})$; for then we will similarly have $\mathrm{meas}(I_{\mathrm{G}^*}) = \mathrm{meas}(I_{\mathrm{M}^*})$, and (74) will follow. But this equality is an easy consequence of the discussion in [Kot88, page 633]; one has a formula $|\mathrm{S}_1(\mathbb{F}_q)|q^{-\dim \mathrm{S}_1}$ describing both $\mathrm{meas}(I_{\mathrm{G}})$ and $\mathrm{meas}(I_{\mathrm{M}})$, where $\mathrm{S}_1 \subset \mathrm{M}$ is an F-torus that becomes a maximal split torus over the maximal unramified extension of F in $\bar{F}$ (this does not need that G is simply connected, and is implicitly used for general reductive groups in a discussion in [Gro97, page 295, near (4.11)]). □

5.2.3. *Consequences for atomically stable packets.*

Corollary 5.2.12. *(i) Let* $\mathrm{M} \subset \mathrm{G}$ *be a Levi subgroup. Let* $\mathcal{O}'_{\mathrm{M}} \subset \mathrm{Aut}(\mathrm{M})$ *be a subgroup that acts trivially on* A_{M}*, consists of elements that extend to automorphisms of* G*, and has finite image in* $\mathrm{Out}(\mathrm{M})$*. Let* Σ *be an* $\mathcal{O}'_{\mathrm{M}}$*-atomically stable discrete series L-packet on* $\mathrm{M}(F)$*. Then* μ *is constant on* Σ*, and for all* $z \in \mathcal{Z}_1(\mathrm{G})$ *such that* $\sigma \mapsto \hat{z}(\sigma)$ *is* $\mathcal{O}'_{\mathrm{M}}$*-invariant on* $\mathrm{Irr}_2(\mathrm{M})$*,* $\sigma \mapsto \hat{z}(\sigma)$ *is constant on* Σ*.*

(ii) Suppose we are in the situation of Corollary 5.2.11(ii), with various measures chosen as in that corollary. Assume that for some subgroups $\mathcal{O}'_{\mathrm{M}} \subset \mathrm{Aut}(\mathrm{M})$ *and* $\mathcal{O}'_{\mathrm{M}^*} \subset \mathrm{Aut}(\mathrm{M}^*)$ *that act trivially on* $\mathrm{A}_{\mathrm{M}} = \mathrm{A}_{\mathrm{M}^*}$ *and consist of elements that extend to automorphisms of* G *and* G^**, respectively, the images* $\bar{\mathcal{O}}'_{\mathrm{M}}$ *of* $\mathcal{O}'_{\mathrm{M}}$ *and* $\bar{\mathcal{O}}'_{\mathrm{M}^*}$ *of* $\mathcal{O}'_{\mathrm{M}^*}$ *in* $\mathrm{Out}(\mathrm{M}) = \mathrm{Out}(\mathrm{M}^*)$ *are finite and equal. Assume that* $\mathcal{O}'_{\mathrm{M}}$ *and* $\mathcal{O}'_{\mathrm{M}^*}$ *in* Σ *is an* $\mathcal{O}'_{\mathrm{M}}$*-atomically stable discrete series L-packet on* $\mathrm{M}(F)$ *and* Σ^* *is an* $\mathcal{O}'_{\mathrm{M}^*}$*-atomically stable discrete series L-packet on* $\mathrm{M}^*(F)$*. Assume that* Σ *is a transfer of* Σ^**, in the sense that some nonzero virtual character* $\Theta_{\Sigma^*} \in SD_{\mathrm{ell}}(\mathrm{M}^*)^{\mathcal{O}'_{\mathrm{M}^*}}$ *supported on* Σ^* *transfers to a virtual character* $\Theta_\Sigma \in SD_{\mathrm{ell}}(\mathrm{M})^{\mathcal{O}'_{\mathrm{M}}}$ *supported on* Σ*. Then for all* $\sigma^* \in \Sigma^*$ *and* $\sigma \in \Sigma$*, we have:*

$$\gamma'''(\mathrm{G}^*|\mathrm{M}^*)\mu(\sigma^*) = \gamma'''(\mathrm{G}|\mathrm{M})\mu(\sigma).$$

Moreover, for all $z^* \in \mathcal{Z}_1^*(\mathrm{G}^*)$ *and* $z \in \mathcal{Z}_1(\mathrm{G})$ *such that* z^* *maps to* $e(\mathrm{G})z$ *under* $\mathcal{SI}(\mathrm{G}^*)^* \to \mathcal{SI}(\mathrm{G})^*$*, and such that* $\sigma \mapsto \hat{z}(\sigma)$ *is* $\mathcal{O}'_{\mathrm{M}}$*-invariant on* $\mathrm{Irr}_2(\mathrm{M})$ *and* $\sigma^* \mapsto \hat{z^*}(\sigma^*)$ *is* $\mathcal{O}'_{\mathrm{M}^*}$*-invariant on* $\mathrm{Irr}_2(\mathrm{M})^*$*, we have* $\hat{z^*}(\sigma^*) = \hat{z}(\sigma)$*.*

Proof. Let us first prove (i). Since $\mathcal{O}'_{\mathrm{M}}$ acts trivially on A_{M}, Proposition 3.3.6 implies that the elements of Σ have a common $\mathrm{A}_{\mathrm{M}}(F)$-central character, which we denote by $\zeta \in X^{\mathrm{uni}}(\mathrm{A}_{\mathrm{M}})$.

It is easy to see from the definition (see [Wal03, Section V.2]) that the μ-function on $\mathrm{Irr}_2(\mathrm{M})$ is invariant under automorphisms of M that extend to automorphisms of G, and hence that $\sigma \mapsto d(\sigma)\mu(\sigma)\hat{z}(\sigma)$ is $\mathcal{O}'_{\mathrm{M}}$-invariant on $\mathrm{Irr}_2(\mathrm{M})$. By Corollary 5.2.11(i) and Proposition 3.3.6(iii), it follows that $\sigma \mapsto \mu(\sigma)\hat{z}(\sigma)$ is constant on Σ. Applying this with z replaced by the Dirac delta measure z_0 at 1, so that $\hat{z}_0(\sigma) = 1$ for all σ, we get the constancy of $\sigma \mapsto \mu(\sigma)$ on Σ. If $\mu(\sigma) \neq 0$ for $\sigma \in \Sigma$, the constancy of $\sigma \mapsto \hat{z}(\sigma)$ follows from that of $\sigma \mapsto \mu(\sigma)\hat{z}(\sigma)$. In general, since $\chi \mapsto \mu(\sigma \otimes \chi)\hat{z}(\sigma \otimes \chi)$ is meromorphic on $X^{\mathrm{unr}}(\mathrm{M})$ and holomorphic at points of $X^{\mathrm{unr-uni}}(\mathrm{M})$ by [Wal03, Lemma V.2.1], and since $\chi \mapsto \mu(\sigma \otimes \chi)$ is not identically zero on $X^{\mathrm{unr-uni}}(\mathrm{A}_{\mathrm{M}})$ (otherwise it would be so on $X^{\mathrm{unr}}(\mathrm{A}_{\mathrm{M}})$, contradicting that intertwining operators are holomorphic on a dense open subset of the vector space on which they are defined), it suffices to show that for all $\chi \in X^{\mathrm{unr-uni}}(\mathrm{M})$, $\sigma \mapsto \mu(\sigma \otimes \chi)\hat{z}(\sigma \otimes \chi)$ is constant on Σ. This in turn follows from applying the above considerations with Σ replaced by $\Sigma \otimes \chi$, which is an $\mathcal{O}'_{\mathrm{M}}$-atomically stable discrete series L-packet by Lemma 3.3.8, and the fact that $\mathcal{O}'_{\mathrm{M}}$ acts trivially on $X^{\mathrm{unr}}(\mathrm{M})$ (since it does so on $X^{\mathrm{unr}}(\mathrm{A}_{\mathrm{M}})$ and hence on $X^{\mathrm{unr}}(\mathrm{S}_{\mathrm{M}})$, which surjects to $X^{\mathrm{unr}}(\mathrm{M})$).

Now let us prove (ii). By Lemma 3.2.3(i), the common A_{M}-central character $\zeta \in X^{\mathrm{uni}}(\mathrm{A}_{\mathrm{M}}) = X^{\mathrm{uni}}(\mathrm{A}_{\mathrm{M}^*})$ of the elements of Σ is also the common $\mathrm{A}_{\mathrm{M}^*}$-central character of the elements of Σ^*. This time, one similarly has the $\mathcal{O}'_{\mathrm{M}}$-invariance of $\sigma \mapsto d(\sigma)\mu(\sigma)\hat{z}(\sigma)$ on $\mathrm{Irr}_2(\mathrm{M})$ and the

$\mathcal{O}'_{\mathrm{M}^*}$-invariance of $\sigma^* \mapsto d(\sigma^*)\mu(\sigma^*)\hat{z}(\sigma^*)$ on $\mathrm{Irr}_2(\mathrm{M}^*)$. Thus, we apply Corollary 5.2.11(ii) and Proposition 3.3.7(iii) to get:

$$e(\mathrm{M})\gamma'''(\mathrm{G}^*|\mathrm{M}^*)\mu(\sigma^*)\hat{z}^*(\sigma^*) = e(\mathrm{G})\gamma'''(\mathrm{G}|\mathrm{M})\mu(\sigma)\hat{z}(\sigma). \tag{75}$$

Applying this with z^* replaced by the Dirac measure at the identity, which transfers to $e(\mathrm{G})$ times the Dirac measure at the identity by [Kot88, Proposition 2], we get $\gamma'''(\mathrm{G}^*|\mathrm{M}^*)\mu(\sigma^*) = e(\mathrm{G})e(\mathrm{M})^{-1}\gamma'''(\mathrm{G}|\mathrm{M})\mu(\sigma)$, which gives the first assertion of (ii), since $e(\mathrm{G}) = e(\mathrm{M})$ (see [Kot83, Corollary, (6)]). If $\mu(\sigma^*) \neq 0$, so that $\mu(\sigma) \neq 0$ as well, the remaining assertion of (ii) follows from (75). The case where we allow $\mu(\sigma^*)$ to be 0 then follows by twisting by various $\chi \in X^{\mathrm{unr-uni}}(\mathrm{M}^*) = X^{\mathrm{unr-uni}}(\mathrm{M})$, as in the proof of (i). □

5.3. **Two applications.**

5.3.1. *Normalizing intertwining operators using Langlands-Shahidi L-functions.*

Lemma 5.3.1. *Let* $\mathrm{G}, \mathrm{M}, \mathrm{G}^*, \mathrm{M}^*, \Sigma, \Sigma^*$ *be as in Corollary 5.2.12(ii); in particular we used the discussion of (i) and (vi) of Notation 3.2.1 to fix inner twists* ψ_{G^*} *from* G^* *to* G *and* ψ_{M^*} *from* M^* *to* M*, using fixed parabolic subgroups, say* $\mathrm{P}^* \subset \mathrm{G}^*$ *and* $\mathrm{P} \subset \mathrm{G}$ *(the analogues of* Q^* *and* Q *in Notation 3.2.1(vi)), with* M^* *and* M *respectively as Levi subgroups. Without loss of generality, we assume that* ψ_{M^*} *is a restriction of* ψ_{G^*} *(and not just a restriction of an element of* $\mathrm{Int}\,\mathrm{G}(\bar{F})\circ\psi_{\mathrm{G}^*}$*). Let* $\mathrm{P}^{*,-} \subset \mathrm{G}^*$ *and* $\mathrm{P}^- \subset \mathrm{G}$ *be parabolic subgroups that are opposite to* P^* *and* P *and contain* M^* *and* M*. Let* $\mathrm{N}^*, \mathrm{N}^{*,-}, \mathrm{N}$ *and* N^- *be the unipotent radicals of* $\mathrm{P}^*, \mathrm{P}^{*,-}, \mathrm{P}$ *and* P^-*. Note that* ψ_{G^*} *takes* $\mathrm{N}^*_{\bar{F}}$ *to* $\mathrm{N}_{\bar{F}}$ *and* $\mathrm{N}^{*,-}_{\bar{F}}$ *to* $\mathrm{N}^-_{\bar{F}}$*, letting us transfer top-degree differential forms (defined over* $\bar{F}$*) between these groups, and therefore lets us transfer Haar measures from* $\mathrm{N}^*(F)$ *to* $\mathrm{N}(F)$ *and* $\mathrm{N}^{*,-}(F)$ *to* $\mathrm{N}^-(F)$ *(using either the absolute value on* $\bar{F}$ *or a nontrivial continuous additive character* $\psi_F : F \to \mathbb{C}^\times$*). We choose Haar measures on* $\mathrm{N}^*(F)$ *and* $\mathrm{N}^{*,-}(F)$*, and transfer them to* $\mathrm{N}(F)$ *and* $\mathrm{N}^-(F)$ *using* ψ_{G^*}*, as just explained. For* $\sigma^* \in \Sigma^*$ *and* $\sigma \in \Sigma$*, let the intertwining operators* $J_{\mathrm{P}^{*,-}|\mathrm{P}^*}(\sigma^*), J_{\mathrm{P}^*|\mathrm{P}^{*,-}}(\sigma^*), J_{\mathrm{P}^-|\mathrm{P}}(\sigma)$ *and* $J_{\mathrm{P}|\mathrm{P}^-}(\sigma)$ *be defined as in* [Art89, around (1.1)] *or equivalently as in* [Wal03, just before Theorem IV.1.1]*, but using the choices of measures just fixed. Then, as meromorphic functions in* $\chi \in X^{\mathrm{unr}}(\mathrm{M}^*) = X^{\mathrm{unr}}(\mathrm{M})$*, we have:*

$$J_{\mathrm{P}^*|\mathrm{P}^{*,-}}(\sigma^* \otimes \chi) \circ J_{\mathrm{P}^{*,-}|\mathrm{P}^*}(\sigma^* \otimes \chi) = J_{\mathrm{P}|\mathrm{P}^-}(\sigma \otimes \chi) \circ J_{\mathrm{P}^-|\mathrm{P}}(\sigma \otimes \chi) \tag{76}$$

— here, the operators on either side are scalar multiplications, and hence viewed as complex numbers, for a dense subset of $\chi \in X^{\mathrm{unr-uni}}(\mathrm{M}^*) = X^{\mathrm{unr-uni}}(\mathrm{M})$*, which is automatically Zariski dense in* $X^{\mathrm{unr}}(\mathrm{M}^*) = X^{\mathrm{unr}}(\mathrm{M})$*.*

Proof. Recall $K = K_{\mathrm{G}}, I = I_{\mathrm{G}}, K_{\mathrm{M}}, K_{\mathrm{N}}, K_{\mathrm{N}^-}, I_{\mathrm{M}}, I_{\mathrm{N}}$ and I_{N^-} from Notation 5.2.5(iii); here $H_{\mathrm{L}} = H \cap \mathrm{L}(F)$ for each compact open subgroup $H \subset \mathrm{G}(F)$ and algebraic subgroup $\mathrm{L} \subset \mathrm{G}$. We choose analogous objects for G^*: $K^* = K_{\mathrm{G}^*}, I^* = I_{\mathrm{G}^*}, K_{\mathrm{M}^*} = K^* \cap \mathrm{M}^*(F), K_{\mathrm{N}^*}, K_{\mathrm{N}^{*,-}}, I_{\mathrm{M}^*}, I_{\mathrm{N}^*}$ and $I_{\mathrm{N}^{*,-}}$. We give $\mathrm{M}^*(F)$ and $\mathrm{M}(F)$ Haar measures that are compatible under ψ_{M^*}. We give $\mathrm{G}^*(F)$ and $\mathrm{G}(F)$ the unique Haar measures such that the multiplication maps $\mathrm{N}^*(F) \times \mathrm{M}^*(F) \times \mathrm{N}^{*,-}(F) \to \mathrm{G}^*(F)$ and $\mathrm{N}(F) \times \mathrm{M}(F) \times \mathrm{N}^-(F) \to \mathrm{G}(F)$ are measure preserving near the identity. It is then easy to see that $\mathrm{G}^*(F)$ and $\mathrm{G}(F)$ have measures that are compatible under ψ_{G^*}. Therefore, the equality (74) proved in the proof of Corollary 5.2.11 holds, and gives:

$$\mathrm{meas}(I_{\mathrm{N}})\,\mathrm{meas}(I_{\mathrm{N}^-}) = \frac{\mathrm{meas}(I)}{\mathrm{meas}(I_{\mathrm{M}})} = \frac{\mathrm{meas}(I^*)}{\mathrm{meas}(I_{\mathrm{M}^*})} = \mathrm{meas}(I_{\mathrm{N}^*})\,\mathrm{meas}(I_{\mathrm{N}^{*,-}}). \tag{77}$$

By the definitions in [Wal03, Sections IV.3 and V.2] and (60), the reciprocal of the left-hand side (resp., the reciprocal of the right-hand side) of (76) equals

$$\gamma'(\mathrm{G}^*|\mathrm{M}^*)\mu(\sigma^*\otimes\chi)\,\mathrm{meas}(K_{\mathrm{N}^*})^{-1}\,\mathrm{meas}(K_{\mathrm{N}^{*,-}})^{-1} = \gamma'''(\mathrm{G}^*|\mathrm{M}^*)\mu(\sigma^*\otimes\chi)\,\mathrm{meas}(I_{\mathrm{N}^*})^{-1}\,\mathrm{meas}(I_{\mathrm{N}^{*,-}})^{-1}$$

(resp., $\gamma'(\mathrm{G}|\mathrm{M})\mu(\sigma \otimes \chi)\,\mathrm{meas}(K_{\mathrm{N}})^{-1}\,\mathrm{meas}(K_{\mathrm{N}^-})^{-1} = \gamma'''(\mathrm{G}|\mathrm{M})\mu(\sigma \otimes \chi)\,\mathrm{meas}(I_{\mathrm{N}})^{-1}\,\mathrm{meas}(I_{\mathrm{N}^-})^{-1}$).

Now the lemma follows from (77) and Corollary 5.2.12(ii), the latter applied with Σ^* and Σ replaced by $\Sigma^* \otimes \chi$ and $\Sigma \otimes \chi$, as justified by Lemmas 3.3.8 and 3.2.3(i) (and the fact that $\mathcal{O}'_{\mathrm{M}}, \mathcal{O}'_{\mathrm{M}^*}$ act trivially on $\mathrm{A}_{\mathrm{M}}, \mathrm{A}_{\mathrm{M}^*}$). □

Remark 5.3.2. In Lemma 5.3.1, it is an easy exercise to see that replacing ψ_{G^*} by a different inner twist, while yielding different measures on $\mathrm{N}(F)$ and $\mathrm{N}^-(F)$, yields the same product measure on $\mathrm{N}(F) \times \mathrm{N}^-(F)$, and hence does not change the right-hand side of (76).

Remark 5.3.3. Assume that we are in the setting of Lemma 5.3.1, and assume for simplicity that $\mathcal{O}'_{\mathrm{M}}$ and $\mathcal{O}'_{\mathrm{M}^*}$ are trivial. Let r_i^* denote the representations of ${}^L\mathrm{M}^* = {}^L\mathrm{M}$ associated to $\mathrm{M}^* \subset \mathrm{P}^* \subset \mathrm{G}^*$ as in [Sha90]. Assume also that Σ^* contains a generic representation σ^*. Thus, the definition of the Langlands-Shahidi L-functions and ϵ-factors extend to representations $\sigma \in \Sigma$, as explained in [Sha90, shortly before Theorem 9.5] (with the difference that we are stopping at discrete series packets and not invoking Langlands classification):

$$L(s, \sigma, r_i) = L(s, \sigma^*, r_i^*),$$

and for any continuous nontrivial additive character $\psi_F : F \to \mathbb{C}^\times$,

$$\epsilon(s, \sigma, r_i, \psi_F) = \epsilon(s, \sigma^*, r_i^*, \psi_F).$$

It should be possible to use Lemma 5.3.1 to deduce from [Sha90] that these L-functions and ϵ-factors give a normalization of intertwining operators as in [Art89, Theorem 2.1]. We will skip exploring the precise details.

5.3.2. *Consequences for depth preservation.*

Corollary 5.3.4. *Suppose the residue characteristic p of F is a very good prime for* G *in the sense of* [BKV16, Section 8.10]*. Let* $\mathrm{M} \subset \mathrm{G}$ *be a Levi subgroup. Let* $\mathcal{O}'_{\mathrm{M}} \subset \mathrm{Aut}(\mathrm{M})$ *be a subgroup that acts trivially on* A_{M}*, consists of elements that extend to automorphisms of* G*, and has finite image in* $\mathrm{Out}(\mathrm{M})$*.*

(i) The elements of Σ *all have the same depth in the sense of Moy and Prasad (see* [MP96]*).*
(ii) Assume that $\mathrm{G}^*, \mathcal{O}'_{\mathrm{M}}, \mathcal{O}'_{\mathrm{M}^*}, \Sigma$ *and* Σ^* *are as in the situation of Corollary 5.2.12(ii). Assume additionally that there exists a nice bilinear form B on* $\mathfrak{g}$*, in the sense of Definition 5.3.6(iii) below. Then for each* $\sigma \in \Sigma$ *and* $\sigma^* \in \Sigma^*$*, we have*

$$\mathrm{depth}(\sigma) = \mathrm{depth}(\sigma^*).$$

The proof of (ii) of the corollary will use:

Proposition 5.3.5. *Suppose p is a very good prime for* G *in the sense of* [BKV16, Section 8.10]*, and that there exists a nice bilinear form on* $\mathfrak{g}$*, in the sense of Definition 5.3.6(iii) below. Let* G^* *be a quasi-split inner form for* G*, underlying an endoscopic datum* $\underline{\mathrm{G}}^*$ *for* G *defined using an inner twist* ψ_{G^*} *as in Notation 3.2.1(i). Let $r \geq 0$, and let* $E_r \in \mathcal{Z}(\mathrm{G})$ *and* $E_r^* \in \mathcal{Z}(\mathrm{G}^*)$ *be the depth r projectors in the sense of* [BKV16]*. Then the distribution E_r on* $\mathrm{G}(F)$ *and the distribution* E_r^* *on* $\mathrm{G}^*(F)$ *are stable. Moreover,* E_r^* *transfers to the distribution* $e(\mathrm{G})E_r$ *on* $\mathrm{G}(F)$*, provided* $\mathrm{G}(F)$ *and* $\mathrm{G}^*(F)$ *are given measures that are compatible with respect to* ψ_{G^*}*.*

Now we make some preparations for the proof of Proposition 5.3.5.

Definition 5.3.6. (i) For a finite extension F_1/F, we will denote by $\mathcal{B}(\mathrm{G}/F_1)$ the reduced Bruhat-Tits building of G_{F_1}, and abbreviate $\mathcal{B}(\mathrm{G}/F)$ to $\mathcal{B}(\mathrm{G})$. F^{unr} will denote the maximal unramified extension of F in $\bar{F}$, and for each extension F_1 of F in F^{unr}, $\mathcal{B}(\mathrm{G}/F_1)$ will be realized as $\mathcal{B}(\mathrm{G}/F^{\mathrm{unr}})^{\mathrm{Gal}(F^{\mathrm{unr}}/F_1)}$. As usual, the notation that follows will be adapted to more general groups and fields in place of G and F.

(ii) For $x \in \mathcal{B}(\mathrm{G})$ and $r \geq 0$ (resp., $r \in \mathbb{R}$), we have the Moy-Prasad filtration subgroups $\mathrm{G}(F)_{x,r}, \mathrm{G}(F)_{x,r+} \subset \mathrm{G}(F)$, and the Moy-Prasad filtration lattices $\mathfrak{g}(F)_{x,r}, \mathfrak{g}(F)_{x,r+} \subset \mathfrak{g}(F)$ and $\check{\mathfrak{g}}(F)_{x,r}, \check{\mathfrak{g}}(F)_{x,r+} \subset \check{\mathfrak{g}}(F)$, where $\check{\mathfrak{g}}$ is the dual vector space of $\mathfrak{g}$, which is given the coadjoint action. We also have the Moy-Prasad G-domains $\mathrm{G}(F)_r = \bigcup_x \mathrm{G}(F)_{x,r}, \mathrm{G}(F)_{r+} = \bigcup_x \mathrm{G}(F)_{x,r+} \subset \mathrm{G}(F)$, $\mathfrak{g}(F)_r = \bigcup_x \mathfrak{g}(F)_{x,r}, \mathfrak{g}(F)_{r+} = \bigcup_x \mathfrak{g}(F)_{x,r+} \subset \mathfrak{g}(F)$ and $\check{\mathfrak{g}}_r = \bigcup_x \check{\mathfrak{g}}(F)_{x,r}, \check{\mathfrak{g}}_{r+} = \bigcup_x \check{\mathfrak{g}}(F)_{x,r+} \subset \check{\mathfrak{g}}(F)$, where each of these unions is over $x \in \mathcal{B}(\mathrm{G})$. Thus, $\mathrm{G}(F)_{x,r+} = \mathrm{G}(F)_{x,r+\epsilon}$ for all small enough $\epsilon > 0$, and similarly with $\mathfrak{g}(F)_{x,r+}, \check{\mathfrak{g}}(F)_{x,r+}$, $\mathrm{G}(F)_{r+}, \mathfrak{g}(F)_{r+}$ and $\check{\mathfrak{g}}(F)_{r+}$. Here and in the rest of this subsection, we will often write $\mathfrak{g}(F)$ despite it being also denoted by $\mathfrak{g}$, to distinguish it from $\mathfrak{g}(F_1)$ for another field F_1.

(iii) A bilinear form $B : \mathfrak{g}(F) \times \mathfrak{g}(F) \to F$ will be called nice if it is symmetric, nondegenerate, Ad G-invariant and identifies each Moy-Prasad filtration lattice $\mathfrak{g}(F)_{x,r}$ in $\mathfrak{g}(F)$ with $\check{\mathfrak{g}}(F)_{x,r}$; this translates to requiring that for all $x \in \mathcal{B}(\mathrm{G})$ and $r \in \mathbb{R}$ we have

$$\{X \in \mathfrak{g}(F) \mid B(X, \mathfrak{g}(F)_{x,(-r)+}) \subset \varpi\mathfrak{O}_F\} = \mathfrak{g}(F)_{x,r}. \tag{78}$$

(iv) For the rest of this subsection G^* will denote a quasi-split inner form of G, and ψ_{G^*} and the endoscopic datum $\underline{\mathrm{G}}^*$ will be as in Notation 3.2.1(i). Note that $\mathfrak{g}^* = \mathrm{Lie}\,\mathrm{G}^*$ should not be confused with $\check{\mathfrak{g}}$.

(v) If B is an Ad G-invariant bilinear form on $\mathfrak{g}(F)$, then its transport by ψ_{G^*} refers to the bilinear form B^* on $\mathfrak{g}^*$ such that for all $X^*, Y^* \in \mathfrak{g}^*(\bar{F})$, $B^*(X^*, Y^*) = B(\psi_{\mathrm{G}^*}(X^*), \psi_{\mathrm{G}^*}(Y^*))$: that this prescription descends to a bilinear form on $\mathfrak{g}^*$ follows from the Ad G-invariance of B and the fact that ψ_{G^*} is an inner twisting. In fact, B^* is the transfer of B to $\mathfrak{g}^*$ via the endoscopic datum $\underline{\mathrm{G}}^*$ as in [Wal95, Section VIII.6] (see also the discussion after the proof of Remark 2 of that reference).

(vi) For this subsection, given $r \geq 0$, E_r (resp., E^*_r) will denote the depth r projector for G (resp., G^*).

Lemma 5.3.7. *Assume that* G *is not 'bad' in the sense of* [BKV16, Section 3.13]*, i.e., either p is odd, or* $(\mathrm{G}_{\mathrm{sc}})_{F^{\mathrm{unr}}}$ *does not have a restriction of scalars of an odd special unitary group over* F^{unr} *as a factor. Given a symmetric nondegenerate* Ad G*-invariant bilinear form on* $\mathfrak{g}$*, the following are equivalent:*

(i) B is nice.

(ii) For some $x \in \mathcal{B}(\mathrm{G})$ *and all* $r \in \mathbb{R}$*,* (78) *holds.*

(iii) For some finite unramified extension F_1 *of* F*, the base-change of* B *to* F_1 *is nice with respect to* G_{F_1}*.*

Proof. (i) ⇒ (ii) is immediate. Let us prove (ii) ⇒ (i); we refer to the condition in (ii) as x-nice. For this, it is enough to show that if $x, y \in \mathcal{B}(\mathrm{G})$, and if B is x-nice, then B is y-nice as well. Choose an apartment in $\mathcal{B}(\mathrm{G})$ containing x and y, associated to some split maximal torus S in G, and let M_0 be the centralizer of S in G. Then we have (see [BKV16, Proposition 3.10(b)]) an expansion:

$$\mathfrak{g}(F)_{x,r} = \mathfrak{m}_0(F)_r \oplus \bigoplus_\alpha \mathfrak{u}_\alpha(F)_{x,r},$$

where α runs over the set of roots of S in G, and $\mathfrak{u}_\alpha(F)_{x,r}$ is the union of the affine root lattices $\mathfrak{u}_\psi \subset \mathfrak{u}_\alpha(F)$ as ψ runs over the affine roots associated to the apartment of S that have gradient α and satisfy $\psi(x) \geq r$. We have a similar expression for $\mathfrak{g}(F)_{x,r+}$, where the definition of $\mathfrak{u}_\alpha(F)_{x,r+}$ involves the condition $\psi(x) > r$.

Thus, it is clear, using the S-equivariance of B, that B is x-nice if and only if the following two conditions are satisfied:

- $\{X \in \mathfrak{m}_0 \mid B(X, \mathfrak{m}_0(F)_{(-r)+}) \subset \varpi\mathcal{O}\} = \mathfrak{m}_0(F)_r$ for all $r \in \mathbb{R}$; and
- $\{X \in \mathfrak{u}_\alpha(F) \mid B(X, \mathfrak{u}_{-\alpha}(F)_{x,(-r)+}) \subset \varpi\mathcal{O}\} = \mathfrak{u}_\alpha(F)_{x,r}$, for each root α of S in G and each $r \in \mathbb{R}$.

Since similar considerations apply to y, it suffices to show that the above conditions are satisfied if and only if they are satisfied with x replaced by y. This is clear since the first condition is x-agnostic, while if $y = x + \lambda$ with $\lambda \in X_*(\mathrm{S}) \otimes \mathbb{R}$, then it is easy to see that $\mathfrak{u}_{-\alpha}(F)_{y,(-r)+} = \mathfrak{u}_{-\alpha}(F)_{x,-(r-\langle\alpha,\lambda\rangle)+}$ and $\mathfrak{u}_\alpha(F)_{y,r} = \mathfrak{u}_\alpha(F)_{x,r-\langle\alpha,\lambda\rangle}$.

This gives the equivalence of (i) and (ii). Given this, for the equivalence of either of these notions with (iii), assuming without loss of generality that $F_1 \subset F^{\mathrm{unr}}$ so that $\mathcal{B}(\mathrm{G}) \subset \mathcal{B}(\mathrm{G}/F_1) \subset \mathcal{B}(\mathrm{G}/F_{\mathrm{unr}})$, it suffices to check that for some $x \in \mathcal{B}(\mathrm{G})$ and each $r \in \mathbb{R}$, (78) holds if and only if it does with F replaced by F_1. In turn, this is easy to see using 'dual bases' with respect to B if we can show that for each $r \in \mathbb{R}$, $\mathfrak{g}(F)_{x,r} \otimes_{\mathfrak{O}_F} \mathfrak{O}_{F_1} = \mathfrak{g}(F_1)_{x,r}$, where $\mathfrak{O}_F$ and $\mathfrak{O}_{F_1}$ are the rings of integers of F and F_1, respectively. Since $\mathfrak{g}(F)_{x,r}$ is the $\mathrm{Gal}(F^{\mathrm{unr}}/F)$-fixed points of $\mathfrak{g}(F_1)_{x,r}$ (see [Adl98, Proposition 1.4.1] and [BKV16, Lemma 3.14] and use that G is not 'bad'), this should in turn follow from some sort of unramified descent.

Being naive about this sort of descent, let us give an elementary argument instead; it is enough to show that for each finitely generated $\mathfrak{O}_{F_1}$-lattice L with a semilinear action of $\mathrm{Gal}(F_1/F)$, the map $L^{\mathrm{Gal}(F_1/F)} \otimes_{\mathfrak{O}_F} \mathfrak{O}_{F_1} \to L$ is surjective. Let $\mathrm{Gal}(F_1/F) = \{\sigma_1 = \mathrm{id}, \dots, \sigma_n\}$, and let $a_1, \dots, a_n$ be an $\mathfrak{O}_F$-basis for $\mathfrak{O}_{F_1}$. The matrix $A = [\sigma_i(a_j)]_{1 \le i,j \le n}$ has determinant in $\mathfrak{O}_{F_1}^\times$, since $\mathrm{tr}_{F_1/F}(a_i a_j)$ is the (i,j)-th entry of ${}^t AA$, and $\mathrm{tr}_{F_1/F}$ is a perfect pairing $\mathfrak{O}_{F_1} \times \mathfrak{O}_{F_1} \to \mathfrak{O}_F$. Therefore there exist $b_1, \dots, b_n \in \mathfrak{O}_{F_1}$ such that $\sum_{l=1}^n \sigma_i(a_l) b_l$ equals $\delta_{1,i}$ for each i, i.e., 1 if $i = 1$, and 0 otherwise. Thus, given $v \in L$,

$$v = \sigma_1(v) = \sum_{i=1}^n \delta_{1,i}\sigma_i(v) = \sum_{i=1}^n \left(\sum_{l=1}^n b_l \sigma_i(a_l) \right) \sigma_i(v) = \sum_{l=1}^n b_l \cdot \left(\sum_{i=1}^n \sigma_i(a_l v) \right),$$

which lies in the image of $L^{\mathrm{Gal}(F_1/F)} \otimes_{\mathfrak{O}_F} \mathfrak{O}_{F_1} \to L$. □

Corollary 5.3.8. *If B is a nice bilinear form on $\mathfrak{g}$, then the associated bilinear form B^* on $\mathfrak{g}^*$ (obtained by transporting B via ψ_{G^*}) is nice as well.*

Proof. We reduce to the situation where ψ_{G^*} is defined over the maximal unramified extension F^{unr} of F. Recall that the inner twists of G^* are parameterized by $H^1(F, \mathrm{G}^*_{\mathrm{ad}})$. By the inflation-restriction sequence and the theorem of Steinberg which says that $H^1(F_{\mathrm{unr}}, (\mathrm{G}^*_{\mathrm{ad}})_{F^{\mathrm{unr}}})$ is trivial, the class of the inner twist ψ_{G^*} arises from an element of $H^1(\mathrm{Gal}(F^{\mathrm{unr}}/F), \mathrm{G}^*_{\mathrm{ad}}(F^{\mathrm{unr}}))$. This has the consequence that we may modify ψ_{G^*} to ensure that it is defined over F^{unr}, and hence over a finite extension F_1 of F contained in F^{unr}. Since the condition 'niceness' behaves well under isomorphisms of algebraic groups, it follows that the base-change of B to F_1 is nice if and only if the base-change of B^* to F_1 is. Therefore, the corollary follows from Lemma 5.3.7. □

From [BKV16] we have:

Lemma 5.3.9. *Suppose $\mathrm{G}_1 \to \mathrm{G}_2$ is an isogeny of connected reductive groups over F, whose degree is prime to p. Then it induces analytic isomorphisms $\mathrm{G}_1(F)_{x,r+} \to \mathrm{G}_2(F)_{x,r+}$ and $\mathrm{G}_1(F)_{r+} \to \mathrm{G}_2(F)_{r+}$ for all $x \in \mathcal{B}(\mathrm{G}_1) = \mathcal{B}(\mathrm{G}_2)$ and $r \ge 0$. Moreover, the depth r projector for G_2, which (as a distribution) is supported on $\mathrm{G}_2(F)_{r+}$, when pulled back along $\mathrm{G}_1(F)_{r+} \to \mathrm{G}_2(F)_{r+}$, equals the depth r projector for G_1, which is supported on $\mathrm{G}_1(F)_{r+}$.*

Proof. The first assertion is [BKV16, Lemma 8.12]. The second assertion is implicit in the proof of [BKV16, Corollary 8.13], and follows from the first assertion together with the Euler-Poincare formula for the depth r projector given in [BKV16, Corollary 1.9]. □

Lemma 5.3.10. *Let $r > 0$ (resp., $r \in \mathbb{R}$). If strongly regular semisimple elements $\gamma^* \in \mathrm{G}^*(F)$ and $\gamma \in \mathrm{G}(F)$ (resp., regular semisimple elements $X^* \in \mathfrak{g}^*(F)$ and $X \in \mathfrak{g}(F)$) match with respect to $\underline{\mathrm{G}}^*$, we have $\gamma^* \in \mathrm{G}^*(F)_r$ if and only if $\gamma \in \mathrm{G}(F)_r$ (resp., $X^* \in \mathfrak{g}^*(F)_r$ if and only if $X \in \mathfrak{g}(F)_r$). Consequently, for $r \in \mathbb{R}$, the stable distribution $\mathbb{1}_{\mathfrak{g}^*(F)_r}$ transfers to $\mathbb{1}_{\mathfrak{g}(F)_r}$ under endoscopic transfer with respect to $\underline{\mathrm{G}}^*$ (a similar assertion involving the $\mathbb{1}_{\mathrm{G}^*(F)_r}$ and the $\mathbb{1}_{\mathrm{G}(F)_r}$ holds, but we will not need it).*

Proof. By [BKV16, Lemma B.3], $\mathfrak{g}(F)_r \subset \mathfrak{g}(F)$ and $\mathfrak{g}^*(F)_r \subset \mathfrak{g}^*(F)$ are stable, justifying the stability of $\mathbb{1}_{\mathfrak{g}^*(F)_r}$. Given the simple description of transfer factors for $\underline{\mathrm{G}}^*$ (Remark 3.2.2(i)), and since $\mathfrak{g}(F)_r \subset \mathfrak{g}(F)$ and $\mathfrak{g}^*(F)_r \subset \mathfrak{g}^*(F)$ are open and closed, the latter assertion of the lemma follows from the former, which is a "$\underline{\mathrm{G}}^*$-endoscopic" form of the stability of the $\mathrm{G}(F)_r$ and the $\mathfrak{g}(F)_r$. Thus, we will adapt the proof of the stability assertion in [BKV16, Lemma B.3].

We will prove the assertion involving γ and γ^*; the proof of the assertion involving X and X^* is similar. As in the proof of Corollary 5.3.8, we may and do assume that ψ_{G^*} is defined over a finite extension F_1 of F contained in F^{unr}. We have $\gamma = \mathrm{Ad}\, g(\psi_{\mathrm{G}^*}(\gamma^*))$ for some $g \in \mathrm{G}(\bar{F})$. Since $\gamma, \delta := \psi_{\mathrm{G}^*}(\gamma^*) \in \mathrm{G}(F_1)$, letting T be the centralizer of δ in G_{F_1} and using that $H^1(\mathrm{Gal}(F^{\mathrm{unr}}/F_1), \mathrm{T}(F^{\mathrm{unr}})) \to H^1(F_1, \mathrm{T})$ is an isomorphism (by Steinberg's theorem that $\mathrm{H}^1(F^{\mathrm{unr}}, \mathrm{T}_{F^{\mathrm{unr}}})$ is trivial) and that $H^1(\mathrm{Gal}(F^{\mathrm{unr}}/F_1), \mathrm{G}(F^{\mathrm{unr}})) \to H^1(F_1, \mathrm{G}_{F_1})$ is injective, we may assume without loss of generality that $g \in \mathrm{G}(F^{\mathrm{unr}})$. Thus, there exists a finite extension F_2 of F_1 (and hence of F) in F^{unr} such that $g \in \mathrm{G}(F_2)$. It follows that $\gamma^* \in \mathrm{G}^*(F_2)_r$ if and only if $\gamma \in \mathrm{G}(F_2)_r$.

Using [AD04, Lemma 2.2.3] and the fact that the finite extension F_2/F is unramified, we have $\mathrm{G}^*(F_2)_r \cap \mathrm{G}^*(F) = \mathrm{G}^*(F)_r$ and $\mathrm{G}(F_2)_r \cap \mathrm{G}(F) = \mathrm{G}(F)_r$, so that $\gamma^* \in \mathrm{G}^*(F)_r$ if and only if $\gamma \in \mathrm{G}(F)_r$, as desired. Let us remark that the assertion involving X and X^* uses the Lie algebra version of [AD04, Lemma 2.2.3] (for arbitrary r), which can be proved similarly (it is a simple application of the Bruhat-Tits fixed point theorem also found in the proof of [BKV16, Lemma B.3]). □

Proof of Proposition 5.3.5. Since p is a very good prime for G and hence also for G^*, the isogenies $\mathrm{Z}^0_{\mathrm{G}} \times \mathrm{G}_{\mathrm{sc}} \to \mathrm{G}$ and $\mathrm{Z}^0_{\mathrm{G}^*} \times \mathrm{G}^*_{\mathrm{sc}} \to \mathrm{G}^*$ have degrees prime to p, as observed in [BKV16, the proof of Theorem 1.23]. Therefore, by Lemma 5.3.9, $\mathrm{Z}^0_{\mathrm{G}} \times \mathrm{G}_{\mathrm{sc}} \to \mathrm{G}$ induces an isomorphism from $C^\infty_c(\mathrm{Z}_{\mathrm{G}^0}(F)_r \times \mathrm{G}_{\mathrm{sc}}(F)_r) \subset C^\infty_c(\mathrm{Z}_{\mathrm{G}^0}(F) \times \mathrm{G}_{\mathrm{sc}}(F))$ to $C^\infty_c(\mathrm{G}(F)_r) \subset C^\infty_c(\mathrm{G}(F))$, which clearly preserves stable orbital integrals (for compatible choices of measures). A similar comment applies to the isogeny $\mathrm{Z}^0_{\mathrm{G}^*} \times \mathrm{G}^*_{\mathrm{sc}} \to \mathrm{G}^*$. Since ψ_{G^*} induces an isomorphism $\mathrm{Z}^0_{\mathrm{G}^*} \to \mathrm{Z}^0_{\mathrm{G}}$ as well as determines a unique inner twist from $\mathrm{G}^*_{\mathrm{sc}}$ to G_{sc}, and since the transfer factors all have a simple description in our setting, it is now easy to reduce, using Lemma 5.3.9 and Lemma 5.3.10, to the case where G is simply connected.

Moreover, as in the proof of Corollary 5.3.8, we may and do assume that ψ_{G^*} is defined over a finite extension F_1 of F contained in F^{unr}.

Since G and G^* are simply connected, and since p is a very good prime for G, it follows from [BKV16, Corollary 8.11] that G admits an r-logarithm in the sense of [BKV16, Section 1.21], which is an $\mathrm{Ad}\,\mathrm{G}(F)$-equivariant homeomorphism $\mathrm{G}(F)_{r+} \to \mathfrak{g}(F)_{r+}$ restricting to a homeomorphism $\mathrm{G}(F)_{x,r+} \to \mathfrak{g}(F)_{x,r+}$ for each $x \in \mathcal{B}(\mathrm{G})$. The same applies to G^*. Moreover, by [BKV16, Corollary 1.9(b)], E_r and E^*_r are supported in $\mathrm{G}(F)_{r+}$ and $\mathrm{G}^*(F)_{r+}$, respectively, and by [BKV16, Corollary 1.22], the push-forwards of E_r and E^*_r to $\mathfrak{g}(F)_{r+} \subset \mathfrak{g}(F)$ and $\mathfrak{g}^*(F)_{r+} \subset \mathfrak{g}^*(F)$ are the restrictions of the Lie algebra versions $\mathcal{E}_r$ and $\mathcal{E}^*_r$ of the depth r projectors to $\mathfrak{g}(F)_{r+}$ and $\mathfrak{g}^*(F)_{r+}$, respectively. Again using Lemma 5.3.10 and the fact that the transfer factors are particularly simple in our situation, it suffices to show that the transfer of the distribution $\mathcal{E}^*_r$ on $\mathfrak{g}^*(F)$ equals the distribution $\mathcal{E}_r$ on $\mathfrak{g}(F)$, where we use measures on $\mathfrak{g}^*$ and $\mathfrak{g}$ that are compatible via ψ_{G^*} (which, being an innner twist, even induces an F-rational map at the level of top-degree differential forms).

By [BKV16, (b) of Section 1.19], $\mathcal{E}_r$ is the inverse Fourier transform of the characteristic function of $\check{\mathfrak{g}}(F)_{-r}$, where the Fourier transform is defined as in [BKV16, Section 1.18], using a fixed additive character $\Lambda : F \to \mathbb{C}^\times$ that is nontrivial on the ring $\mathfrak{O}_F$ of integers of F but trivial on the maximal ideal $\mathfrak{p}_F$ of $\mathfrak{O}_F$. We identify $\mathfrak{g}$ with $\check{\mathfrak{g}}$ using the nice bilinear form B, so that the Fourier transform is a map from the space of distributions on $\mathfrak{g}(F)$ to itself, and $\mathcal{E}_r$ is the inverse Fourier transform of the characteristic function $\mathbb{1}_{\mathfrak{g}(F)_{-r}}$ of $\mathfrak{g}(F)_{-r}$. The transfer B^* of B to $\mathfrak{g}^*$ is nice by Corollary 5.3.8, using which we similarly realize $\mathcal{E}^*_r$ as the inverse Fourier transform of the characteristic function $\mathbb{1}_{\mathfrak{g}^*(F)_{-r}}$. Moreover, we may and do use measures on $\mathfrak{g}(F)$ and $\mathfrak{g}^*(F)$ that are self-dual for $\Lambda \circ B$ and $\Lambda \circ B^*$: this is because B^* is the transfer of B, and hence this use of self-dual measures satisfies the constraint that the measure on $\mathfrak{g}^*(F)$ is the transfer of the measure on $\mathfrak{g}(F)$ via ψ_{G^*}.

Given this choice of measures, one knows the commutativity of endoscopic transfer and Fourier transform (see [Wal95, Conjecture 1], which has been proved since, as explained in [KV12, Theorem 4.1.3]): if a distribution Θ^* on $\mathfrak{g}^*(F)$ transfers to a distribution Θ on $\mathfrak{g}(F)$, then the Fourier transform (resp., the inverse Fourier transform) of Θ^* transfers to $\gamma_\Lambda(B)/\gamma_\Lambda(B^*)$ times the Fourier transform of Θ (resp., $\gamma_\Lambda(B^*)/\gamma_\Lambda(B)$ times the inverse Fourier transform of Θ), where $\gamma_\Lambda(B)$ and $\gamma_\Lambda(B^*)$ are Weil constants as in [Wal95, Section VIII.1].

Now we are reduced to showing that the distribution $\mathbb{1}_{\mathfrak{g}^*(F)_{-r}}$ on $\mathfrak{g}^*(F)$ transfers to the distribution $e(\mathrm{G})\gamma_\Lambda(B) \cdot \gamma_\Lambda(B^*)^{-1} \cdot \mathbb{1}_{\mathfrak{g}(F)_{-r}}$ on $\mathfrak{g}(F)$, or equivalently to $\mathbb{1}_{\mathfrak{g}(F)_{-r}}$, since one knows that $e(\mathrm{G})\gamma_\Lambda(B^*)/\gamma_\Lambda(B) = 1$: see [KV12, Proposition 4.2.2], whose restrictions are unnecessary in our setting as mentioned in the first sentence of [KV12, Section 4.2.9, the proof of Proposition 4.2.2], or use [Kal15, Lemma 4.8 and Proposition 4.3].

It remains to note that $\mathbb{1}_{\mathfrak{g}^*(F)_r}$ transfers to $\mathbb{1}_{\mathfrak{g}(F)_r}$, which we already know from Lemma 5.3.10. □

Proof of Corollary 5.3.4. Let us prove (i). Since the depth of a representation is the same as that of its cuspidal support by [MP96, Theorem 5.2(1)], it follows that for each $\sigma \in \mathrm{Irr}(\mathrm{M})$ and $r \geq 0$, $\hat{E}_r(\sigma) = \hat{E}_r((\mathrm{M}, \sigma))$ equals 1 if the depth of σ is at most r, and 0 otherwise. Thus, (i) follows if we prove that for each $r \geq 0$, $\hat{E}_r$ takes the same value on each element of Σ. But since $E_r \in \mathcal{Z}_1(\mathrm{G})$ by [BKV16, Theorem 1.23] and the hypothesis that p is a very good prime for G, and since $\sigma \mapsto \hat{E}_r(\sigma)$ is $\mathcal{O}'_{\mathrm{M}}$-invariant on $\mathrm{Irr}_2(\mathrm{M})$ (since the action of $\mathrm{Aut}(\mathrm{M})$ on $\mathrm{Irr}(\mathrm{M})$ preserves depth), this follows from Corollary 5.2.12(i).

For (ii), by the observation at the beginning of (i), it suffices to prove that for all $r \geq 0$, $\sigma \in \Sigma$ and $\sigma^* \in \Sigma^*$, we have $\hat{E}_r(\sigma) = \hat{E}^*_r(\sigma^*)$. But using the invariance of $\sigma \mapsto \hat{E}_r(\sigma)$ and $\sigma^* \mapsto \hat{E}^*_r(\sigma^*)$ under $\mathcal{O}'_{\mathrm{M}}$ and $\mathcal{O}'_{\mathrm{M}^*}$, this follows from combining Proposition 5.3.5 with Corollary 5.2.12(ii). □

5.4. **Consequences for $\mathcal{Z}_1(\mathrm{G})$ and $\mathcal{Z}_2(\mathrm{G})$.** Let us now deduce from Corollary 5.2.12(i) that $\mathcal{Z}_1(\mathrm{G})^{\mathcal{O}} = \mathcal{Z}_{2,\mathcal{O}}(\mathrm{G})$ (see Notation 5.1.6) when the hypothesis on the existence of tempered L-packets (Hypothesis 2.7.1) is satisfied. Let us begin by restating Corollary 5.2.12(i) in the special case that concerns us here.

Corollary 5.4.1. *Suppose Σ is an $\mathcal{O}_{\mathrm{M}}$-atomically stable discrete series L-packet (see Definition 3.3.2). Then:*

(i) The Plancherel measure $\sigma \mapsto \mu(\sigma)$ ([Wal03, Section V.2]*) is constant on Σ.*
(ii) For all $z \in \mathcal{Z}_1(\mathrm{G})^{\mathcal{O}}$ and $\sigma_1, \sigma_2 \in \Sigma$, and any parabolic subgroup P of G with M as a Levi subgroup, z acts by the same scalar on $\mathrm{Ind}_{\mathrm{P}}^{\mathrm{G}} \sigma_1$ and $\mathrm{Ind}_{\mathrm{P}}^{\mathrm{G}} \sigma_2$. In other words, if (M_1, σ_1) and (M_2, σ_2) are cuspidal supports of elements of Σ, then $\hat{z}((\mathrm{M}_1, \sigma_1)_{\mathrm{G}}) = \hat{z}((\mathrm{M}_2, \sigma_2)_{\mathrm{G}})$.

Proof. Each $\mathcal{O}_{\mathrm{M}}$ from Notation 2.6.1 satisfies the hypotheses of Corollary 5.2.12(i), by the conditions imposed in (iv) of Notation 2.6.1, and Lemma 2.6.3(ii). This also gives that for each $z \in \mathcal{Z}(\mathrm{G})^{\mathcal{O}}$, $\sigma \mapsto \hat{z}(\sigma)$ is $\mathcal{O}_{\mathrm{M}}$-invariant in $\mathrm{Irr}_2(\mathrm{M})$. Thus, the corollary follows from Corollary 5.2.12(i). □

We now prove Theorem 1.1.5, after restating it in a slightly more convenient way.

Theorem 5.4.2. *Assume the hypothesis on the existence of tempered L-packets (Hypothesis 2.7.1). Then for $z \in \mathcal{Z}(\mathrm{G})$, the following are equivalent:*

(i) $\hat{z}$ is constant on each $\Sigma \in \Phi_{\mathrm{temp}}(\mathrm{G})$ (see Notation 2.7.6 for the definition of $\Phi_{\mathrm{temp}}(\mathrm{G})$).
(ii) $z \in \mathcal{Z}_{2,\mathcal{O}}(\mathrm{G})$.
(iii) $z \in \mathcal{Z}_1(\mathrm{G})^{\mathcal{O}}$.

Proof. Let us assume (i) and prove (ii). Since $\Phi_{\mathrm{temp}}(\mathrm{G})$ partitions $\mathrm{Irr}_{\mathrm{temp}}(\mathrm{G})$ (see Lemma 2.7.7(i)), and since each $\Sigma \in \Phi_{\mathrm{temp}}(\mathrm{G})$ is stable under the action of $\mathcal{O}$, it follows that $\hat{z}(\pi) = \hat{z}(\pi \circ \beta^{-1})$, for all $\beta \in \mathcal{O}$ and $\pi \in \mathrm{Irr}_{\mathrm{temp}}(\mathrm{G})$. Thus, $\hat{z} \in \mathbb{C}[\Omega(\mathrm{G})]$ takes the same value on (M, σ) and $\beta \cdot (\mathrm{M}, \sigma)$ whenever (M, σ) is a cuspidal pair such that σ is unitary. But the images of such cuspidal pairs in $\Omega(\mathrm{G})$ is Zariski dense (since $X^{\mathrm{unr-uni}}(\mathrm{M}) \subset X^{\mathrm{unr}}(\mathrm{M})$ is Zariski dense), so that $\hat{z}$ factors through $\Omega(\mathrm{G}) \to \underline{\Omega}(\mathrm{G})$ (set-theoretically, and hence by Corollary 4.1.7 and the reducedness of $\Omega(\mathrm{G})$, as a morphism). From this and Lemma 5.1.3, we get that $z \in \mathcal{Z}(\mathrm{G})^{\mathcal{O}}$.

Now if $\mathrm{M} \subset \mathrm{G}$ is a Levi subgroup and $\Sigma \in \Phi_2(\mathrm{M})$, then the constancy of $\hat{z}$ on $\Sigma^{\mathrm{G}} \in \Phi_{\mathrm{temp}}(\mathrm{G})$ (see (ii) and (iii) of Notation 2.7.6) implies that $f \mapsto \Theta_{\Sigma^{\mathrm{G}}}(z * f)$ is a scalar multiple of $\Theta_{\Sigma^{\mathrm{G}}}$ (where $\Theta_{\Sigma^{\mathrm{G}}} = \mathrm{Avg}_{\mathcal{O}_{\mathrm{G}}}(\mathrm{Ind}_{\mathrm{M}}^{\mathrm{G}} \Theta_{\Sigma})$ as in Notation 2.7.6(iv), and we use the identity $\Theta_\pi(z * f) = \mathrm{tr}\, \pi(z * f) = \hat{z}(\pi)\Theta_\pi(f)$), which is stable as seen in Proposition 3.2.8. Hence by Hypothesis 2.7.1, if D is the $\mathcal{O}$-average of $\mathrm{Ind}_{\mathrm{M}}^{\mathrm{G}} \Theta'$, where $\mathrm{M} \subset \mathrm{G}$ is a Levi subgroup and $\Theta' \in SD_{\mathrm{ell}}(\mathrm{M})^{\mathcal{O}_{\mathrm{M}}}$, then $f \mapsto D(z * f)$ is stable. Therefore, by the implication (iv) ⇒ (i) of Proposition 5.1.8 (which applies as $z \in \mathcal{Z}(\mathrm{G})^{\mathcal{O}}$), we get $z \in \mathcal{Z}_{2,\mathcal{O}}(\mathrm{G})$, as desired.

The implication (ii) ⇒ (iii) is Lemma 5.1.7.

For any Levi subgroup $\mathrm{M} \subset \mathrm{G}$, Hypothesis 2.7.1 implies that the elements of $\Phi_2(\mathrm{M})$ are all $\mathcal{O}_{\mathrm{M}}$-atomically stable (use Lemma 2.7.3(i)). Therefore, the implication (iii) ⇒ (i) is immediate from Corollary 5.4.1(ii) and the $\mathcal{O}$-invariance of z (the latter is used to account for the fact that the description of $\Phi_{\mathrm{temp}}(\mathrm{G})$ as given by (ii) and (iii) of Notation 2.7.6 involves taking a union of $\mathcal{O}$-orbits of quotients of parabolically induced representations). □

Lemma 5.4.3. *Assume Hypothesis 2.7.1. Suppose* $z \in \mathcal{Z}(\mathrm{G})$ *is such that* $\hat{z}(\pi_1) = \hat{z}(\pi_2)$ *whenever* π_1, π_2 *are irreducible subquotients of* $\mathrm{Ind}_{\mathrm{M}}^{\mathrm{G}} \sigma_1, \mathrm{Ind}_{\mathrm{M}}^{\mathrm{G}} \sigma_2$*, respectively, for some Levi subgroup* $\mathrm{M} \subset \mathrm{G}$ *and representations* σ_1, σ_2 *that belong to the same element of* $\Phi_2(\mathrm{M})$*. Then* $\hat{z}(\pi_1) = \hat{z}(\pi_2)$ *whenever* π_1, π_2 *are irreducible subquotients of* $\mathrm{Ind}_{\mathrm{M}}^{\mathrm{G}} \sigma_1, \mathrm{Ind}_{\mathrm{M}}^{\mathrm{G}} \sigma_2$*, respectively, for some Levi subgroup* $\mathrm{M} \subset \mathrm{G}$ *and representations* σ_1, σ_2 *that belong to the same element of* $\Phi_2^+(\mathrm{M})$*.*

Proof. The proof is similar to the first step in that of the implication (i) $\Rightarrow$ (ii) of Theorem 5.4.2. Suppose $\mathrm{M} \subset \mathrm{G}$ is a Levi subgroup, and σ_1, σ_2 belong to the same element $\Sigma \in \Phi_2^+(\mathrm{M})$. Since z factors through the cuspidal support map, it is easy to see that $\chi \mapsto \hat{z}(\sigma_1 \otimes \chi)$ and $\chi \mapsto \hat{z}(\sigma_2 \otimes \chi)$ are regular on $X^{\mathrm{unr}}(\mathrm{M})$, where for $i = 1, 2$, $\hat{z}(\sigma_i \otimes \chi)$ is the scalar with which z acts on any irreducible subquotient of $\mathrm{Ind}_{\mathrm{M}}^{\mathrm{G}} \sigma_i \otimes \chi$. By hypothesis, we have $\hat{z}(\sigma_1 \otimes \chi) = \hat{z}(\sigma_2 \otimes \chi)$ whenever $\Sigma \otimes \chi$ is unitary (and hence belongs to $\Phi_2(\mathrm{M}) \subset \Phi_2^+(\mathrm{M})$). Since the set of such χ is a coset of $X^{\mathrm{unr-uni}}(\mathrm{M})$ in $X^{\mathrm{unr}}(\mathrm{M})$, and is hence Zariski dense in $X^{\mathrm{unr}}(\mathrm{M})$, it follows that $\hat{z}(\sigma_1 \otimes \chi) = \hat{z}(\sigma_2 \otimes \chi)$ for all $\chi \in X^{\mathrm{unr}}(\mathrm{M})$. In particular $\hat{z}(\sigma_1) = \hat{z}(\sigma_2)$. □

Corollary 5.4.4. *Assume Hypothesis 2.7.1. If* z *belongs to* $\mathcal{Z}_1(\mathrm{G})^{\mathcal{O}}$ *or* $\mathcal{Z}_{2,\mathcal{O}}(\mathrm{G})$*,* $\mathrm{M} \subset \mathrm{G}$ *is a Levi subgroup and* σ, σ' *belong to the same element of* $\Phi_2^+(\mathrm{M})$*, then* z *acts by the same scalar on any irreducible subquotient of* $\mathrm{Ind}_{\mathrm{M}(F)}^{\mathrm{G}(F)} \sigma$ *as it does on any irreducible subquotient of* $\mathrm{Ind}_{\mathrm{M}(F)}^{\mathrm{G}(F)} \sigma'$*.*

Proof. This follows from Lemma 5.4.3, whose hypothesis is satisfied by either the implication (ii) $\Rightarrow$ (i) (if $z \in \mathcal{Z}_{2,\mathcal{O}}(\mathrm{G})$) or (iii) $\Rightarrow$ (i) (if $z \in \mathcal{Z}_1(\mathrm{G})^{\mathcal{O}}$) of Theorem 5.4.2. □

Theorem 5.4.2 has the following corollary.

Corollary 5.4.5. *Assume* $z \in \mathcal{Z}(\mathrm{G})$*. Denote by* z_{M} *the image of* z *under what is called the Harish-Chandra homomorphism* $\mathcal{Z}(\mathrm{G}) \to \mathcal{Z}(\mathrm{M})$ *in* [BDK86, Section 2.4]*, i.e., the homomorphism of* $\mathbb{C}$*-algebras that is dual to the obvious finite morphism* $\Omega(\mathrm{M}) \to \Omega(\mathrm{G})$ *induced by inclusion at the level of cuspidal supports. If Hypothesis 2.7.1 holds and* $z \in \mathcal{Z}_1(\mathrm{G})^{\mathcal{O}}$*, then* $z_{\mathrm{M}} \in \mathcal{Z}_1(\mathrm{M})^{\mathcal{O}_{\mathrm{M}}}$*.*

Proof. If $\mathrm{L} \subset \mathrm{M}$ is a Levi subgroup and $\upsilon_1, \upsilon_2 \in \Upsilon$ for some $\Upsilon \in \Phi_2(\mathrm{L})$, then

$$\hat{z}_{\mathrm{M}}((\mathrm{L}, \upsilon_1)_{\mathrm{M}}) = \hat{z}((\mathrm{L}, \upsilon_1)_{\mathrm{G}}) = \hat{z}((\mathrm{L}, \upsilon_2)_{\mathrm{G}}) = \hat{z}_{\mathrm{M}}((\mathrm{L}, \upsilon_2)_{\mathrm{M}}),$$

where the middle equality holds by the implication (iii) $\Rightarrow$ (i) of Theorem 5.4.2. Here, as usual, $\hat{z}_{\mathrm{M}}((\mathrm{L}, \upsilon_1)_{\mathrm{M}})$ refers to $\hat{z}_{\mathrm{M}}((\mathrm{L}', \upsilon_1')_{\mathrm{M}})$, where $(\mathrm{L}', \upsilon_1')$ is a cuspidal support of (L, υ_1), and the other terms are similar. Therefore, the corollary follows from the implication (i) $\Rightarrow$ (iii) of Theorem 5.4.2 applied with $(\mathrm{M}, \{\mathcal{O}_{\mathrm{L}}\}_{\mathrm{L}})$ in place of $(\mathrm{G}, \{\mathcal{O}_{\mathrm{L}}\}_{\mathrm{L}})$ (L ranging over the set of Levi subgroups of M or G, as appropriate): this application is justified by Lemma 2.6.3(iii), which ensures the validity Hypothesis 2.7.1 for $(\mathrm{M}, \{\mathcal{O}_{\mathrm{L}}\}_{\mathrm{L}})$. □

As the phrasing of Corollary 5.4.5 above indicates, we are not able to prove that $z \mapsto z_{\mathrm{M}}$ sends $\mathcal{Z}_1(\mathrm{G})$ to $\mathcal{Z}_1(\mathrm{M})$, without using Hypothesis 2.7.1. In contrast we have:

Proposition 5.4.6. *If* $z \in \mathcal{Z}_2(\mathrm{G})$*, then its image* $z_{\mathrm{M}} \in \mathcal{Z}(\mathrm{M})$ *under the Harish-Chandra homomorphism belongs to* $\mathcal{Z}_2(\mathrm{M})$*.*

Proof. To make use of notation defined so far, we may and do assume that $\mathcal{O}_{\mathrm{L}}$ is trivial for each Levi subgroup $\mathrm{L} \subset \mathrm{G}$. Let $\mathrm{L} \subset \mathrm{M}$ be a Levi subgroup and $D = \mathrm{Ind}_{\mathrm{L}}^{\mathrm{M}} \Theta$, where $\Theta \in SD_{\mathrm{ell}}(\mathrm{L})$. By the implication (iv) $\Rightarrow$ (i) of Proposition 5.1.8, it suffices to show that the distribution $f \mapsto D(z_{\mathrm{M}} * f)$ on $\mathrm{M}(F)$ is stable.
Let $z_{\mathrm{L}} \in \mathcal{Z}(\mathrm{L})$ be the image of z_{M} under the Harish-Chandra homomorphism $\mathcal{Z}(\mathrm{M}) \to \mathcal{Z}(\mathrm{L})$; thus, it is also the image of z under the Harish-Chandra homomorphism $\mathcal{Z}(\mathrm{G}) \to \mathcal{Z}(\mathrm{L})$.
Claim 1. $f \mapsto D(z_{\mathrm{M}} * f)$ is obtained by applying $\mathrm{Ind}_{\mathrm{L}}^{\mathrm{M}}$ to the virtual character $(f \mapsto \Theta(z_{\mathrm{L}} * f)) \in D(\mathrm{L})$.
Claim 1 follows from its own variant with $(\Theta, D = \mathrm{Ind}_{\mathrm{L}}^{\mathrm{M}} \Theta)$ replaced by $(\Theta_\upsilon, \mathrm{Ind}_{\mathrm{L}}^{\mathrm{M}} \Theta_\upsilon)$ for an arbitrary $\upsilon \in \mathrm{Irr}(\mathrm{M})$, which in turn follows from the chain of equalities:

$$(\mathrm{Ind}_{\mathrm{L}}^{\mathrm{M}} \Theta_\upsilon)(z_{\mathrm{M}} * -) = \hat{z}_{\mathrm{M}}((\mathrm{L}, \upsilon)_{\mathrm{M}})(\mathrm{Ind}_{\mathrm{L}}^{\mathrm{M}} \Theta_\upsilon) = \mathrm{Ind}_{\mathrm{L}}^{\mathrm{M}}(\hat{z}_{\mathrm{L}}((\mathrm{L}, \upsilon)_{\mathrm{L}})\Theta_\upsilon) = \mathrm{Ind}_{\mathrm{L}}^{\mathrm{M}}(\Theta_\upsilon(z_{\mathrm{L}} * -)).$$

Thus, using that parabolic induction preserves stability, it now suffices to show that the distribution $f \mapsto \Theta(z_{\mathrm{L}} * f)$ on $\mathrm{L}(F)$ is stable. Therefore, we may now replace M by L and D by Θ if necessary, and assume that $D \in SD_{\mathrm{ell}}(\mathrm{M})$.

Without loss of generality, we may and do assume that $D \in SD_{\mathrm{ell},\zeta}(\mathrm{M})$ for some $\zeta \in X^{\mathrm{uni}}(\mathrm{A_M})$: this is because the projection $D_{\mathrm{ell}}(\mathrm{M}) \to D_{\mathrm{ell},\zeta}(\mathrm{M})$ (which vanishes on $D_{\mathrm{ell},\zeta'}(\mathrm{M})$ for all $\zeta' \neq \zeta$) restricts to the projection $SD_{\mathrm{ell}}(\mathrm{M}) \to SD_{\mathrm{ell},\zeta}(\mathrm{M})$; see Remark 2.2.4.

Special case, where the stabilizer of ζ in $W(\mathrm{M})$ is trivial. Let us prove the proposition in the special case where the stabilizer of ζ in $W(\mathrm{M})$ is trivial. In what follows, we will freely use that for any $D' \in D_{\mathrm{ell}}(\mathrm{M})$ and $z'_{\mathrm{M}} \in \mathcal{Z}(\mathrm{M})$, $f \mapsto D'(z'_{\mathrm{M}} * f)$ belongs to $D_{\mathrm{ell}}(\mathrm{M})$ — to see this, use that the irreducible components of a parabolically induced representation all have the same cuspidal support. The proof of Claim 1 above, applied with (M,L) replaced by (G,M), gives:

(79)

$$(f \mapsto (\mathrm{Ind}_{\mathrm{M}}^{\mathrm{G}} D)(z*f)) = \mathrm{Ind}_{\mathrm{M}}^{\mathrm{G}}(f \mapsto D(z_{\mathrm{M}}*f)) = \mathrm{Ind}_{\mathrm{M}}^{\mathrm{G}}\left(\#W(\mathrm{M})^{-1} \sum_{w \in W(\mathrm{M})} {}^{w}(f \mapsto D(z_{\mathrm{M}} * f))\right),$$

where ${}^{w}D'(f) := D'(f \circ \mathrm{Int}\,\tilde{w})$ for any $D' \in D_{\mathrm{ell}}(\mathrm{M})$ and any $\tilde{w} \in \mathrm{G}(F)$ representing w; note that any such ${}^{w}D'$ automatically belongs to $D_{\mathrm{ell}}(\mathrm{M})$. Since the distribution in the parentheses in the right-most term of (79) belongs to $D_{\mathrm{ell}}(\mathrm{M})^{W(\mathrm{M})}$, it follows from Proposition 3.2.8 and the fact that $(f \mapsto (\mathrm{Ind}_{\mathrm{M}}^{\mathrm{G}} D)(z * f))$ is stable (since $z \in \mathcal{Z}_2(\mathrm{G})$) that

$$\sum_{w \in W(\mathrm{M})} {}^{w}(f \mapsto D(z_{\mathrm{M}} * f)) \in SD_{\mathrm{ell}}(\mathrm{M}).$$

The condition imposed on ζ is easily seen to imply that the projection $D_{\mathrm{ell}}(\mathrm{M}) \to D_{\mathrm{ell},\zeta}(\mathrm{M})$ takes the above expression to $f \mapsto D(z_{\mathrm{M}} * f)$. Since this projection takes $SD_{\mathrm{ell}}(\mathrm{M})$ to $SD_{\mathrm{ell},\zeta}(\mathrm{M})$, we conclude that $f \mapsto D(z_{\mathrm{M}} * f)$ belongs to $SD_{\mathrm{ell},\zeta}(\mathrm{M}) \subset SD_{\mathrm{ell}}(\mathrm{M})$, as desired.

The general case, with ζ arbitrary. Note that for each $\chi \in X^{\mathrm{unr-uni}}(\mathrm{M})$, $D\chi$, by which we mean the distribution $f \mapsto D(f\chi^{-1})$, belongs to $D_{\mathrm{ell}}(\mathrm{M})$ and is stable (since χ factors through $\mathrm{M}(F) \to \mathrm{S_M}(F)$, which is constant on stable conjugacy classes). Thus, for each such χ, we have $D\chi \in SD_{\mathrm{ell}}(\mathrm{M})$. For a nonempty open set of $\chi \in X^{\mathrm{unr-uni}}(\mathrm{M})$, it is easy to see that $\zeta \cdot \chi|_{\mathrm{A_M}(F)}$ does not have a nontrivial stabilizer in $W(\mathrm{M})$. Therefore, the special case above applies with D replaced by $D\chi$, and we conclude that $f \mapsto (D\chi)(z_{\mathrm{M}} * f) = D((z_{\mathrm{M}} * f)\chi^{-1})$ is stable, for a nonempty open set of $\chi \in X^{\mathrm{unr-uni}}(\mathrm{M})$, which is automatically Zariski dense in $X^{\mathrm{unr}}(\mathrm{M})$. Therefore, it follows that for each unstable function $f \in C_c^\infty(\mathrm{M}(F))$, $f \mapsto D((z_{\mathrm{M}} * f)\chi^{-1})$ vanishes for a Zariski dense subset of $\chi \in X^{\mathrm{unr-uni}}(\mathrm{M})$. Since $\chi \mapsto D((z_{\mathrm{M}} * f)\chi^{-1})$ is readily seen to be a regular function of $\chi \in X^{\mathrm{unr}}(\mathrm{M})$ (it is a linear combination of evaluations), it follows that $D(z_{\mathrm{M}} * f) = 0$, for any unstable function $f \in C_c^\infty(\mathrm{M}(F))$. This finishes the proof of the proposition. □

5.5. **The images of $\mathbb{C}[\underline{\Omega}^{\mathrm{st}}(\mathrm{G})]$ and $\mathbb{C}[\underline{\Omega}({}^{L}\mathrm{G})]$ in $\mathcal{Z}(\mathrm{G})$.** By Propositions 4.3.2 and 4.3.4, under appropriate hypotheses we get pull-back maps:

$$p_2^* : \mathbb{C}[\underline{\Omega}^{\mathrm{st}}(\mathrm{G})] \to \mathbb{C}[\Omega(\mathrm{G})] = \mathcal{Z}(\mathrm{G}) \qquad \text{and} \qquad p_1^* : \mathbb{C}[\underline{\Omega}({}^{L}\mathrm{G})] \to \mathbb{C}[\Omega(\mathrm{G})] = \mathcal{Z}(\mathrm{G}).$$

One of the questions in [Hai14] and in the paper of Scholze and Shin (see [SS13, Section 6]) is to characterize the image of p_1^* as being $\mathcal{Z}_2(\mathrm{G})$ or $\mathcal{Z}_1(\mathrm{G})$, when G is quasi-split. We will show that both these characterizations are valid when G is quasi-split and various hypotheses hold — those on the existence of tempered L-packets, LLC+, LLC+ and stability, and supercuspidal packets (Hypotheses 2.7.1, 2.10.3, 2.10.12, and 2.11.1).

Proposition 5.5.1. *Assume that* G *is quasi-split, and assume the hypotheses on the existence of tempered L-packets and stable cuspidal support (Hypotheses 2.7.1 and 2.11.4). Then the image* $p_2^*(\mathbb{C}[\underline{\Omega}^{\mathrm{st}}(\mathrm{G})]) \subset \mathcal{Z}(\mathrm{G})$ *of* p_2^* *equals* $\mathcal{Z}_{2,\mathcal{O}}(\mathrm{G})$.

Proof. Since G is quasi-split and we are assuming Hypotheses 2.7.1 and 2.11.4, p_2 is well-defined. As we saw in Proposition 4.3.4, p_2 factors through $\Omega(\mathrm{G}) \to \underline{\Omega}(\mathrm{G})$.

Suppose that $z \in \mathcal{Z}(\mathrm{G}) = \mathbb{C}[\Omega(\mathrm{G})]$ factors through p_2, and let us show that $z \in \mathcal{Z}_{2,\mathcal{O}}(\mathrm{G})$; this will give us that $p_2^*(\mathbb{C}[\underline{\Omega}^{\mathrm{st}}(\mathrm{G})]) \subset \mathcal{Z}_{2,\mathcal{O}}(\mathrm{G})$. By the implication (i) ⇒ (ii) of Theorem 5.4.2 and the fact

that z factors through $\Omega(\mathrm{G}) \to \underline{\Omega}(\mathrm{G})$ and hence belongs to $\mathcal{Z}(\mathrm{G})^{\mathcal{O}}$, this will follow if we show:
Claim. If $\mathrm{M} \subset \mathrm{G}$ is a Levi subgroup, $\Sigma \in \Phi_2(\mathrm{M})$ and $\sigma, \sigma' \in \Sigma$, then the value taken by $\hat{z}$ on irreducible subquotients of $\mathrm{Ind}_{\mathrm{M}}^{\mathrm{G}} \sigma$ equals the value taken by $\hat{z}$ on irreducible subquotients of $\mathrm{Ind}_{\mathrm{M}}^{\mathrm{G}} \sigma'$.
Let (M_1, σ_1) and $(\mathrm{M}'_1, \sigma'_1)$ be cuspidal supports for σ and σ', respectively. The claim follows if we show that $\hat{z}((\mathrm{M}_1, \sigma_1)_{\mathrm{G}}) = \hat{z}((\mathrm{M}'_1, \sigma'_1)_{\mathrm{G}})$, which in turn follows if we show that $(\mathrm{M}_1, \sigma_1)_{\mathrm{G}}$ and $(\mathrm{M}'_1, \sigma'_1)_{\mathrm{G}}$ have the same image in $\underline{\Omega}^{\mathrm{st}}(\mathrm{G})$ under p_2. In turn, this follows if we show that any two given stable cuspidal supports (L, Υ) and (L', Υ') for (M_1, Σ_1) and $(\mathrm{M}'_1, \Sigma'_1)$ are conjugate under $\mathcal{O}_{\mathrm{M}} \circ \mathrm{Int}\, \mathrm{M}(F)$, where $\Sigma_1 \in \Phi_2^+(\mathrm{M}_1)$ and $\Sigma'_1 \in \Phi_2^+(\mathrm{M}'_1)$ are the packets containing σ_1 and σ'_1, respectively. But this is clear from Hypothesis 2.11.4, since (L, Υ) and (L', Υ') are also stable cuspidal supports for (M, Σ).
This proves the claim, and hence also the assertion that $p_2^*(\mathbb{C}[\underline{\Omega}^{\mathrm{st}}(\mathrm{G})]) \subset \mathcal{Z}_{2,\mathcal{O}}(\mathrm{G})$.
To finish the proof of the proposition, it suffices now to start with $z \in \mathcal{Z}_{2,\mathcal{O}}(\mathrm{G})$, and show that $\hat{z} \in \mathbb{C}[\Omega(\mathrm{G})]$ arises via pull-back from a regular function on $\mathbb{C}[\underline{\Omega}^{\mathrm{st}}(\mathrm{G})]$ with respect to p_2. Define $\hat{z}_{\mathrm{st}} : \underline{\Omega}^{\mathrm{st}}(\mathrm{G}) \to \mathbb{C}$ by $\hat{z}_{\mathrm{st}}(\underline{(\mathrm{L}, \Upsilon)}) = \hat{z}((\mathrm{L}, \upsilon)_{\mathrm{G}})$, where υ is any element of Υ. Then $\hat{z}_{\mathrm{st}}$ is well-defined by Corollary 5.4.4 and the fact that z is $\mathcal{O}$-invariant. If (L, Υ) is any cuspidal pair for $\underline{\Omega}^{\mathrm{st}}(\mathrm{G})$ and $\upsilon \in \Upsilon$, then $\chi \mapsto \hat{z}_{\mathrm{st}}(\underline{(\mathrm{L}, \Upsilon \otimes \chi)}) = \hat{z}((\mathrm{L}, \upsilon \otimes \chi)_{\mathrm{G}})$ is a regular function of $X^{\mathrm{unr}}(\mathrm{S}_{\mathrm{L}})$. Therefore, $\hat{z}_{\mathrm{st}} \in \mathbb{C}[\underline{\Omega}^{\mathrm{st}}(\mathrm{G})]$. Thus, it now suffices to show that $\hat{z} = p_2^*(\hat{z}_{\mathrm{st}})$, which is immediate from repeated applications of Corollary 5.4.4, using the definition of stable cuspidal support. □

Corollary 5.5.2. *(i) Assume the hypotheses on the existence of tempered L-packets, LLC+, and LLC+ and stability (Hypotheses 2.7.1, 2.10.3 and 2.10.12). Then the image* $p_1^*(\mathbb{C}[\underline{\Omega}({}^L\mathrm{G})]) \subset \mathcal{Z}(\mathrm{G})$ *of* p_1^* *is contained in* $\mathcal{Z}_{2,\mathcal{O}}(\mathrm{G})$.

(ii) Assume that G *is quasi-split, and assume the hypotheses on the existence of tempered L-packets, LLC+, LLC+ and stability, and supercuspidal packets (Hypotheses 2.7.1, 2.10.3, 2.10.12 and 2.11.1). Then the map* p_1^* *is injective, and its image* $p_1^*(\mathbb{C}[\underline{\Omega}({}^L\mathrm{G})]) \subset \mathcal{Z}(\mathrm{G})$ *equals* $\mathcal{Z}_{2,\mathcal{O}}(\mathrm{G})$.

Proof. In the setting of both (i) and (ii), we are assuming Hypothesis 2.10.3, so that p_1 is well-defined. Let us first prove (i), following the explanation given by Haines in [Hai14, Remark 5.5.4]. By Theorem 5.4.2, (i) follows if we show that p_1 is constant on each element of $\Phi_{\mathrm{temp}}(\mathrm{G}) \subset \Phi(\mathrm{G})$. But this follows from Lemma 2.10.13, which holds as Hypothesis 2.10.12 (which includes Hypotheses 2.7.1 and 2.10.3) is being assumed.
Now let us prove (ii). If G is quasi-split, the injectivity of p_1^* follows from the surjectivity of p_1 at the level of $\mathbb{C}$-points (see Proposition 4.3.2) and the fact that $\underline{\Omega}({}^L\mathrm{G})$ is reduced. If we assume Hypotheses 2.7.1, 2.10.3, 2.10.12, and 2.11.1, then Proposition 4.3.7 applies, so that $p_1^*(\mathbb{C}[\underline{\Omega}({}^L\mathrm{G})])$ equals $p_2^*(\mathbb{C}[\underline{\Omega}^{\mathrm{st}}(\mathrm{G})])$, which equals $\mathcal{Z}_{2,\mathcal{O}}(\mathrm{G})$ by Proposition 5.5.1 and the fact that Hypothesis 2.11.4 is satisfied by Proposition 2.11.6. □

Remark 5.5.3. When G is not quasi-split, the map p_1^* can fail to be injective, as is immediately seen by looking at the Bernstein component of the trivial representation of $\mathrm{G}(F)$, where G is the algebraic group associated to $D^\times$, D being a quaternion division algebra over F. Nevertheless, it does seem interesting to ask if p_1^* surjects onto $\mathcal{Z}_2(\mathrm{G})$ when G is not quasi-split and $\mathcal{O}$ is trivial. We do not know even a conjectural answer to that question.

6. Endoscopy and the stable Bernstein center

6.1. $\Omega(\mathrm{H}_1)_\mu$ **and its map into** $\Omega({}^L\mathrm{G})$**.** In [Hai14, Conjectures 6.2.2 and 6.2.3], Haines stated special cases of his $\mathcal{Z}$-transfer conjecture, that deals with how the stable Bernstein center should be related to endoscopic transfer of functions. It can be viewed as a framework in which sit results of the form "the fundamental lemma for unit elements of certain Hecke algebras implies the fundamental lemma for their centers" (see [Hai14, shortly after Conjecture 6.2.2], as well as our comments later in this section). At the beginning of [Hai14, Section 6.2.1], Haines remarks that, if one could assume certain properties that are not obvious, one could formulate a more general version of this conjecture. In this section, we use our study of the various Bernstein varieties to formulate and prove, of course only under several hypotheses, a general version of the $\mathcal{Z}$-transfer conjecture in the setting of twisted endoscopy. These considerations are too 'soft' to explicitly invoke any of the deep aspects of the theory of endoscopy, but they do require us to involve a lot of notation pertaining to it.

Notation 6.1.1. For this section, we fix the following objects:

(i) Let $\tilde{\mathrm{G}}, \mathbf{a}, \omega$ be as in Subsection 3.1. We impose the hypothesis of Notation 3.1.1 on $(\mathrm{G}, \tilde{\mathrm{G}}, \mathbf{a})$.

(ii) We realize ${}^L\mathrm{G}$ as $\hat{\mathrm{G}} \rtimes W_F$, and fix an automorphism $\hat{\theta}$ of $\hat{\mathrm{G}}$ that preserves a Γ-fixed pinning, which is dual to $\operatorname{Int}\delta$ for any $\delta \in \tilde{\mathrm{G}}(\bar{F})$ that is semisimple in the sense discussed at the beginning of [MW16, Section I.1.3] (the existence of such a δ is automatic; the 'θ^* is of finite order' hypothesis from [MW16, Section I.1.5], imposed in Notation 3.1.1 which was in turn imposed above, implies that the notion of 'semisimple' from [MW16, Section I.1.3], as remarked in that reference, coincides with the usual one).

(iii) Form the 'twisted space' $\hat{\mathrm{G}}\hat{\theta} \subset {}^L\tilde{\mathrm{G}} := {}^L\mathrm{G}\hat{\theta}$ as in [MW16, Section I.1.4].

(iv) Fix an endoscopic datum $\underline{\mathrm{H}} = (\mathrm{H}, \mathcal{H}, \tilde{s})$ for $(\mathrm{G}, \tilde{\mathrm{G}}, \mathbf{a})$ (the notation is chosen as in [MW16, Section I.1.5]), which is relevant in the sense described in [MW16, Section I.1.8] (i.e., there exist $\delta \in \mathrm{H}(F)$ and $\gamma \in \tilde{\mathrm{G}}(F)$, with γ strongly regular, such that their conjugacy classes match). We do not require this endoscopic datum to be elliptic (thus, it may not belong to $\mathcal{E}(\tilde{\mathrm{G}}, \mathbf{a})$ in the sense of Notation 3.1.2(i)). Here, the element $\tilde{s}$ belongs to the twisted space $\hat{\mathrm{G}}\hat{\theta} \subset {}^L\tilde{\mathrm{G}}$.

(v) Fix a continuous section $c : W_F \to \mathcal{H}$ to $\mathcal{H} \to W_F$; such a section exists since by the definition of an endoscopic datum, $\mathcal{H}$ is a topologically split extension of $\hat{\mathrm{H}}$ by W_F. Hence, mapping each $(\hat{h}, w) \in \hat{\mathrm{H}} \times W_F$ to the element $\hat{h}c(w) \in \mathcal{H}$ gives an isomorphism $\hat{\mathrm{H}} \rtimes_c W_F \to \mathcal{H}$ of topological groups, where '$\rtimes_c$' indicates that the action of W_F on $\hat{\mathrm{H}}$ is not 'the usual one' associated to a preferred section, but the one obtained by composing Int with c.

(vi) Choose auxiliary data $(\mathrm{H}_1 \to \mathrm{H}, \hat{\xi}_1, \tilde{\mathrm{H}}_1 \to \mathrm{H}_1, \mathrm{C}_1, \mu)$ as in Notation 3.1.2(iii), associated to this endoscopic datum; in particular, we impose the condition of [MW16, Section I.7.1, (3)] (which will be partially recalled in the proof of Lemma 6.1.10 below) using the fact that ω is unitary, with the consequence that μ is unitary. Further, write $j : \mathrm{C}_1 \hookrightarrow \mathrm{H}_1$ for the obvious inclusion, and ${}^Lj : {}^L\mathrm{H}_1 \to {}^L\mathrm{C}_1$ for a homomorphism dual to j, which is determined up to $\hat{\mathrm{C}}_1$-conjugation. We will have occasion to use that the following sequence is exact:

$$1 \to \hat{\mathrm{H}} \hookrightarrow {}^L\mathrm{H}_1 \xrightarrow{{}^Lj} {}^L\mathrm{C}_1 \to 1, \tag{80}$$

where $\hat{\mathrm{H}} \hookrightarrow {}^L\mathrm{H}_1$ is obtained by restricting *any* embedding ${}^L\mathrm{H} \hookrightarrow {}^L\mathrm{H}_1$ dual to $\mathrm{H}_1 \to \mathrm{H}$, and the map ${}^L\mathrm{H}_1 \to {}^L\mathrm{C}_1$, which could be taken to be *any* map dual to $\mathrm{C}_1 \hookrightarrow \mathrm{H}_1$, is being taken to be ${}^Lj$.

(vii) Later, we will assume that we are in one of the following two scenarios:
- *Scenario 1.* We assume given a collection of automorphisms as in Notation 2.6.1, but with G replaced by H_1: thus, in particular, for each Levi subgroup $\mathrm{M}_1 \subset \mathrm{H}_1$, we have a group $\mathcal{O}_{\mathrm{M}_1}$ of automorphisms of M_1 (in particular, this defines the group $\mathcal{O}_{\mathrm{H}_1}$ of automorphisms of H_1). We suppose, further, that each of these automorphisms acts as the identity on C_1.
- *Scenario 2.* In this scenario, μ is assumed to be trivial, $\mathcal{H}$ is assumed to be given an identification with ${}^L\mathrm{H}$ whose composite with some map ${}^L\mathrm{H} \hookrightarrow {}^L\mathrm{H}_1$ dual to $\mathrm{H}_1 \to \mathrm{H}$ equals the auxiliary datum $\hat{\xi} : \mathcal{H} \hookrightarrow {}^L\mathrm{H}_1$ (a little bit about which is recalled in Remark 6.1.2(i) below), $\tilde{\mathrm{H}}$ is assumed to be the trivial (H, H)-bitorsor H together with its usual left and right multiplication actions, and we assume given a collection of automorphisms as in Notation 2.6.1, but with G replaced by H: thus, in particular, for each Levi subgroup $\mathrm{M}_\mathrm{H} \subset \mathrm{H}$, we have a group $\mathcal{O}_{\mathrm{M}_\mathrm{H}}$ of automorphisms of M_H (in particular, this defines the group $\mathcal{O}_\mathrm{H}$ of automorphisms of H).

For most of what follows, our preparatory steps will assume that we are in Scenario 1 rather than Scenario 2 of Notation 6.1.1(vii) above, Scenario 2 being essentially a special case.

Remark 6.1.2. At this point, it helps us to recall from [MW16, Section I.2.1] what $\hat{\xi}_1$ is, and how one can obtain μ from it:

(i) $\hat{\xi}_1 : \mathcal{H} \hookrightarrow {}^L\mathrm{H}_1$ is an injective homomorphism compatible with the projections to W_F, and whose restriction to $\hat{\mathrm{H}} \subset \mathcal{H}$ is a homomorphism $\hat{\mathrm{H}} \hookrightarrow \hat{\mathrm{H}}_1$ that is dual to $\mathrm{H}_1 \to \mathrm{H}$.

(ii) The character $\mu : \mathrm{C}_1(F) \to \mathbb{C}^\times$ has Langlands parameter φ_μ represented by the homomorphism

$$W_F \cong \mathcal{H}/\hat{\mathrm{H}} \overset{\hat{\xi}_1}{\hookrightarrow} {}^L\mathrm{H}_1/\hat{\mathrm{H}} \overset{{}^Lj}{\to} {}^L\mathrm{C}_1 \tag{81}$$

(use (80)), which can also be described as:

$${}^Lj \circ \hat{\xi}_1 \circ c : W_F \to {}^L\mathrm{C}_1.$$

We would now like to define sets $\Omega(\mathcal{H})$ and $\Omega'(\mathcal{H}) = \Phi(\mathcal{H})$ of infinitesimal characters and Langlands parameters valued in $\mathcal{H}$, respectively. For this, we will need to study semisimple, elliptic and hyperbolic elements of $\mathcal{H}$.

Notation 6.1.3. (i) An element $x \in \mathcal{H}$ is said to be semisimple if the automorphism $\mathrm{Int}\, x$ of $\hat{\mathrm{H}}$ preserves a maximal torus of $\hat{\mathrm{H}}$ (thus, this is just like Definition 2.8.1(i)).

(ii) We will denote by $\Omega'(\mathcal{H}) = \Phi(\mathcal{H})$ the set of $\hat{\mathrm{H}}$-conjugacy classes of homomorphisms $\dot{\varphi} : W'_F = W_F \times \mathrm{SL}_2(\mathbb{C}) \to \mathcal{H}$ that are admissible in the sense of satisfying the following three properties: they are continuous, they restrict to an algebraic map valued in $\hat{\mathrm{H}}$ on the $\mathrm{SL}_2(\mathbb{C})$-factor, and they satisfy that for all $w \in W_F$, $\dot{\varphi}(w)$ is a semisimple element of $\mathcal{H}$ (in the sense of (i) above) that maps to w under $\mathcal{H} \to W_F$ (perhaps the 'relevance' condition can be considered automatic as H is quasi-split). Similarly, we define $\Omega(\mathcal{H})$, a collection of $\hat{\mathrm{H}}$-conjugacy classes of homomorphisms $W_F \to \mathcal{H}$ that are admissible in a similar sense.

Remark 6.1.4. The notation $\Omega'(\mathcal{H})$ is more technically appropriate, but the notation $\Phi(\mathcal{H})$ seems more relatable, and hence will be used in what follows.

Lemma 6.1.5. *Let $x \in \mathcal{H}$. Then x is semisimple if and only if its image $\hat{\xi}_1(x) \in {}^L\mathrm{H}_1$ is semisimple (in the sense of Definition 2.8.1(i), or equivalently by Lemma 2.8.5(i), in the usual sense). Moreover, if x is semisimple then its image under $\mathcal{H} \hookrightarrow {}^L\mathrm{G}$ is semisimple.*

Proof. The centralizer of a maximal torus of $\hat{\mathrm{H}}$ in $\hat{\mathrm{H}}_1$ (resp., $\hat{\mathrm{G}}$) is a maximal torus of $\hat{\mathrm{H}}_1$ (resp., $\hat{\mathrm{G}}$). Since the intersection of a maximal torus of $\hat{\mathrm{H}}_1$ with $\hat{\mathrm{H}}$ is a maximal torus of $\hat{\mathrm{H}}$ as well, the lemma follows. □

Lemma 6.1.6. *(i) The inclusion $\mathcal{H} \hookrightarrow {}^L\mathrm{G}$ induces a well-defined map $\Omega(\mathcal{H}) \to \Omega({}^L\mathrm{G})$, and a well-defined map $\Phi_{\mathrm{G-rel}}(\mathcal{H}) \to \Phi(\mathrm{G})$, where $\Phi_{\mathrm{G-rel}}(\mathcal{H}) \subset \Phi(\mathcal{H})$ is the subset represented by those admissible homomorphisms $W'_F \to \mathcal{H}$ whose composite with $\mathcal{H} \hookrightarrow {}^L\mathrm{G}$ is relevant.*

(ii) The inclusion $\hat{\xi}_1 : \mathcal{H} \hookrightarrow {}^L\mathrm{H}_1$ induces well-defined maps $\Omega(\hat{\xi}_1) : \Omega(\mathcal{H}) \to \Omega({}^L\mathrm{H}_1)_\mu$ and $\Phi(\hat{\xi}_1) : \Phi(\mathcal{H}) \to \Phi(\mathrm{H}_1)_\mu$, where $\Omega({}^L\mathrm{H}_1)_\mu \subset \Omega({}^L\mathrm{H}_1)$ and $\Phi(\mathrm{H}_1)_\mu \subset \Phi(\mathrm{H}_1)$ are respectively the preimages of the Langlands parameter $\varphi_\mu \in \Omega({}^L\mathrm{C}_1) = \Phi(\mathrm{C}_1)$ of μ under the maps $\Omega({}^Lj) : \Omega({}^L\mathrm{H}_1) \to \Omega({}^L\mathrm{C}_1)$ and $\Phi({}^Lj) : \Phi(\mathrm{H}_1) \to \Phi(\mathrm{C}_1)$, respectively (induced by composition with ${}^Lj : {}^L\mathrm{H}_1 \to {}^L\mathrm{C}_1$).

Proof. (i) follows easily from Lemma 6.1.5. Lemma 6.1.5 also easily gives the part of (ii) that asserts that composition with $\hat{\xi}_1$ induces well-defined maps $\Omega(\hat{\xi}_1) : \Omega(\mathcal{H}) \to \Omega({}^L\mathrm{H}_1)$ and $\Phi(\hat{\xi}_1) : \Phi(\mathcal{H}) \to \Phi(\mathrm{H}_1)$ (the condition of relevance is not relevant in (ii), since H_1 is quasi-split). Thus, it suffices to note that the images of $\Omega(\hat{\xi}_1)$ and $\Phi(\hat{\xi}_1)$ are contained in $\Omega(\mathrm{H}_1)_\mu$ and $\Phi(\mathrm{H}_1)_\mu$, respectively, which follows from the definition of μ (see Remark 6.1.2(ii)). □

The following records notation, including from Lemma 6.1.6, that will be used in what follows.

Notation 6.1.7. (i) Henceforth, $\Omega({}^L j) : \Omega({}^L \mathrm{H}_1) \to \Omega({}^L \mathrm{C}_1) = \Phi(\mathrm{C}_1)$ and $\Phi({}^L j) : \Phi(\mathrm{H}_1) \to \Phi(\mathrm{C}_1) = \Omega({}^L \mathrm{C}_1)$ will denote maps induced by composition with the homomorphism ${}^L j : {}^L \mathrm{H}_1 \to {}^L \mathrm{C}_1$.

(ii) Henceforth, $\Omega({}^L \mathrm{H}_1)_\mu$ and $\Phi(\mathrm{H}_1)_\mu$ will denote respectively the inverse images of the Langlands parameter $\varphi_\mu \in \Omega({}^L \mathrm{C}_1) = \Phi(\mathrm{C}_1)$ of μ under $\Omega({}^L j)$ and $\Phi({}^L j)$. It is easy to see from Proposition 4.1.4(ii) that the map $\Omega({}^L j) : \Omega({}^L \mathrm{H}_1) \to \Omega({}^L \mathrm{C}_1)$ of varieties is regular, so $\Omega({}^L \mathrm{H}_1)_\mu$ is a closed subvariety of $\Omega({}^L \mathrm{H}_1)$.

(iii) Henceforth, $\Omega(\hat{\xi}_1) : \Omega(\mathcal{H}) \to \Omega({}^L \mathrm{H}_1)_\mu \subset \Omega({}^L \mathrm{H}_1)$ and $\Phi(\hat{\xi}_1) : \Phi(\mathcal{H}) \to \Phi(\mathrm{H}_1)_\mu \subset \Phi(\mathrm{H}_1)$ will respectively denote the maps induced by composition with $\hat{\xi}_1$.

(iv) Henceforth, $\Omega(\mathcal{H}) \to \Omega({}^L \mathrm{G})$ and $\Phi_{\mathrm{G-rel}}(\mathcal{H}) \to \Phi(\mathrm{G})$ will denote the maps induced by the inclusion $\mathcal{H} \hookrightarrow {}^L \mathrm{G}$, where $\Phi_{\mathrm{G-rel}}(\mathcal{H})$ is the set of $\hat{\mathrm{H}}$-conjugacy classes of admissible homomorphisms $W_F' \to \mathcal{H}$ whose composite with $\mathcal{H} \to {}^L \mathrm{G}$ is relevant; see Lemma 6.1.6(i).

(v) Let $\Phi_{\mathrm{temp}}(\mathcal{H})$ be inverse image of $\Phi_{\mathrm{temp}}(\mathrm{H}_1)$ under $\Phi(\hat{\xi}_1) : \Phi(\mathcal{H}) \to \Phi(\mathrm{H}_1)$, and let $\Phi_{\mathrm{temp,G-rel}}(\mathcal{H}) = \Phi_{\mathrm{temp}}(\mathcal{H}) \cap \Phi_{\mathrm{G-rel}}(\mathcal{H}) \subset \Phi(\mathcal{H})$. We also set $\Phi_{\mathrm{temp}}(\mathrm{H}_1)_\mu := \Phi(\mathrm{H}_1)_\mu \cap \Phi_{\mathrm{temp}}(\mathrm{H}_1)$.

Remark 6.1.8. The object $\Phi_{\mathrm{temp}}(\mathcal{H})$ is not intrinsic to $\mathcal{H}$; it also depends on the embedding $\hat{\xi}_1 : \mathcal{H} \hookrightarrow {}^L \mathrm{H}_1$. Thus, we are resorting to an abuse of notation. Unlike semisimplicity (Notation 6.1.3(i)), we do not know how to define the notion of ellipticity intrinsically for elements of $\mathcal{H}$; we do not know how to make up for the absence of a preferred section. However, in Lemma 6.1.10 below, we will show that the image of $\Phi_{\mathrm{temp}}(\mathcal{H})$ under $\Phi(\mathcal{H}) \to \Phi(\mathrm{G})$ consists of bounded parameters; this will make crucial use of our having imposed [MW16, Section I.7.1, (3)], which had the consequence that μ is unitary.

Lemma 6.1.9. *The maps $\Omega(\hat{\xi}_1) : \Omega(\mathcal{H}) \to \Omega({}^L \mathrm{H}_1)_\mu$ and $\Phi(\hat{\xi}_1) : \Phi(\mathcal{H}) \to \Phi(\mathrm{H}_1)_\mu$ are bijections.*

Proof. We will prove the former assertion; the latter assertion can be proved similarly.
Note that given an admissible homomorphism $\lambda : W_F \to {}^L \mathrm{H}_1$,

(i) λ is obtained by composing a map $\lambda_{\mathcal{H}} : W_F \to \mathcal{H}$ with $\hat{\xi}_1$ if and only if ${}^L j \circ \lambda$ equals the homomorphism (81) representing μ (use Remark 6.1.2(ii)), in which case the factored map $\lambda_{\mathcal{H}} : W_F \to \mathcal{H}$ is uniquely determined and represents an element of $\Omega(\mathcal{H})$ (use Lemma 6.1.5 and the fact that $\mathcal{H} \hookrightarrow {}^L \mathrm{H}_1$ is topologically proper and hence a homeomorphism onto its image); and

(ii) λ represents an element of $\Omega({}^L \mathrm{H}_1)_\mu$ if and only if ${}^L j \circ \lambda$ parameterizes μ, i.e., is $\hat{\mathrm{C}}_1$-conjugate to (81).

Since $\mathrm{Z}_{\hat{\mathrm{H}}_1} \to \hat{\mathrm{C}}_1$ is surjective, any $\lambda : W_F \to {}^L \mathrm{H}_1$ as in (ii) can be $\mathrm{Z}_{\hat{\mathrm{H}}_1}$-conjugated so as to be as in (i). This gives the surjectivity of $\Omega(\hat{\xi}_1)$. Therefore, it suffices to show the injectivity of $\Omega(\hat{\xi}_1)$, i.e., to show that if two admissible homomorphisms $\lambda_1, \lambda_2 : W_F \to \mathcal{H}$ satisfy that $\hat{\xi}_1 \circ \lambda_2 = \mathrm{Int}\, \hat{h}_1 \circ (\hat{\xi}_1 \circ \lambda_1)$ for some $\hat{h}_1 \in \hat{\mathrm{H}}_1$, then $\lambda_2 = \mathrm{Int}\, \hat{h} \circ \lambda_1$ for some $\hat{h} \in \hat{\mathrm{H}}$.

If $\hat{z}_1$ is the image of $\hat{h}_1$ in $\hat{\mathrm{C}}_1$, then the equality ${}^L j \circ \hat{\xi}_1 \circ \lambda_2 = {}^L j \circ \hat{\xi}_1 \circ c = {}^L j \circ \hat{\xi}_1 \circ \lambda_1$ gives that $\hat{z}_1$ commutes with the image of ${}^L j \circ \hat{\xi}_1 \circ \lambda_1$. Since $\hat{\mathrm{C}}_1$ is abelian, we conclude that $\hat{z}_1 \in \hat{\mathrm{C}}_1$ belongs to the subgroup $(\hat{\mathrm{C}}_1)^{W_F}$ fixed by W_F. Therefore, it now suffices to show that $\mathrm{Z}_{\hat{\mathrm{H}}_1}^{W_F} \to (\hat{\mathrm{C}}_1)^{W_F}$ is surjective — since conjugation by $\mathrm{Z}_{\hat{\mathrm{H}}_1}^{W_F}$ centralizes $\hat{\xi}_1(\mathcal{H})$, this will allow us to modify $\hat{h}_1$ so as to have trivial image in $\hat{\mathrm{C}}_1$, i.e., to belong to $\hat{\mathrm{H}} \subset \hat{\mathrm{H}}_1$.

Since C_1 is an induced torus, it is easy to see that $\hat{\mathrm{C}}_1^{W_F} = \hat{\mathrm{C}}_1^{\mathrm{Gal}(\bar{F}/F)}$ is connected (this is a special case of the Kottwitz isomorphism for $H^1(W_F, \mathrm{C}_1)$). Thus, $\hat{\mathrm{C}}_1^{W_F}$ is a complex torus. Since the image of of $\mathrm{Z}_{\hat{\mathrm{H}}_1}^{W_F}$ in it is an algebraic subgroup, it is now enough to show that some finite power of every element of $\hat{\mathrm{C}}_1^{W_F}$ belongs to the image of $\mathrm{Z}_{\hat{\mathrm{H}}_1}^{W_F}$. Since the actions of $\mathrm{Gal}(\bar{F}/F)$ on $\hat{\mathrm{C}}_1$ and $\mathrm{Z}_{\hat{\mathrm{H}}_1}$ factor through a common finite quotient, say $\mathrm{Gal}(E/F)$, this follows from the surjectivity of $\mathrm{Z}_{\hat{\mathrm{H}}_1} \to \hat{\mathrm{C}}_1$ together with averaging by the action of $\mathrm{Gal}(E/F)$. □

Lemma 6.1.10. *If $\varphi \in \Phi(\mathcal{H})$, then φ belongs to $\Phi_{\mathrm{temp}}(\mathcal{H})$ if and only if its image in $\Phi(\mathrm{G})$ belongs to $\Phi_{\mathrm{temp}}(\mathrm{G})$.*

Proof. Let $\dot\varphi : W_F' \to \mathcal{H}$ represent φ. It is easy to check that the boundedness of an admissible homomorphism $W_F \to {}^L\mathrm{M}$ over W_F (see Definition 2.8.1(iii)), where M is any reductive group over F, can be tested after restricting to W_E for any choice of a finite extension E/F, in terms of a condition involving an expression of the form $\dot\varphi(W_E) \subset Cs(W_E)$. We choose $E \subset \bar F$ so that G and H_1 split over E, and hence so does H as well.
Suppose we can prove that there exists a subset $S \subset \mathcal{H}$ with the following properties:

- The image of S under the projection $\mathcal{H} \to W_F$ equals W_E; and
- For some (or equivalently any) preferred sections $s : W_F \to {}^L\mathrm{G}$ and $s_1 : W_F \to {}^L\mathrm{H}_1$ of ${}^L\mathrm{G}$ and ${}^L\mathrm{H}_1$, respectively, there exist compact subsets $C \subset \hat{\mathrm{G}}$ and $C_1 \subset \hat{\mathrm{H}}_1$ such that the image of S in ${}^L\mathrm{G}$ is contained in $C \cdot s(W_E)$ and the image $\hat\xi_1(S)$ of S in ${}^L\mathrm{H}_1$ is contained in $C_1 \cdot s_1(W_E)$.

Then, using restriction to W_E as mentioned above, it is easy to check that the boundedness of $\hat\xi_1 \circ \dot\varphi$ and that of $(\mathcal{H} \hookrightarrow {}^L\mathrm{G}) \circ \dot\varphi$ are each equivalent to the existence of a bounded subset $C_{\mathrm{H}} \subset \hat{\mathrm{H}}$ such that $\dot\varphi(W_E) \subset C_{\mathrm{H}} \cdot S$, and hence to each other as well.
Thus, it remains to prove the existence of S as above. For this, we begin by recalling some objects from the proof of [MW16, Lemma I.7.1]. Recall that in Notation 6.1.1(ii), we fixed a pinning of $\hat{\mathrm{G}}$ that is invariant under Γ and $\hat\theta$, realizing ${}^L\mathrm{G}$ as $\hat{\mathrm{G}} \rtimes W_F$. We assume without loss of generality that there exists an element s belonging to the maximal torus $\hat{\mathrm{T}} \subset \hat{\mathrm{G}}$ underlying that pinning, such that $\tilde s := s\hat\theta$ (recall that we wrote our endoscopic datum as $(\mathrm{H}, \mathcal{H}, \tilde s)$). Intersecting with $\hat{\mathrm{H}}$, we get a Borel pair of $\hat{\mathrm{H}}$, whose underlying maximal torus is $\hat{\mathrm{T}}_{\hat{\mathrm{H}}} := (\hat{\mathrm{T}}^{\hat\theta})^0$, and which we extend to a pinning of $\hat{\mathrm{H}}$. We may and do assume that ${}^L\mathrm{H}_1$ is realized as $\hat{\mathrm{H}}_1 \rtimes W_F$ using the transfer of this pinning via $\hat\xi_1$. The maximal torus of $\hat{\mathrm{H}}_1$ underlying this transferred pinning, which we will denote by $\hat{\mathrm{T}}_1$, is the centralizer of $\hat\xi_1(\hat{\mathrm{T}}_{\hat{\mathrm{H}}})$ in $\hat{\mathrm{H}}_1$.
For each $w \in W_E$, we fix some $g_w := (g(w), w) \in \hat{\mathrm{G}} \times W_E \subset \hat{\mathrm{G}} \rtimes W_F = {}^L\mathrm{G}$ that belongs to $\mathcal{H}$ and preserves the pinning of $\hat{\mathrm{H}}$ just fixed. Write $\hat\xi_1(g_w) = (\zeta_1(w), w)$ for some $\zeta_1(w) \in \hat{\mathrm{H}}_1$. Since $\hat\xi_1(g_w)$ fixes a pinning of $\hat{\mathrm{H}}_1$, and since W_E acts trivially on $\hat{\mathrm{H}}_1$, we have $\zeta_1(w) \in \mathrm{Z}_{\hat{\mathrm{H}}_1}$. This also implies that for all $w \in W_E$, $g(w)$ commutes with $\hat{\mathrm{T}}_{\hat{\mathrm{H}}}$ and hence belongs to $\hat{\mathrm{T}}$. We have the embedding $t \mapsto (\hat\xi_1(t)^{-1}, t)$ of $\hat{\mathrm{T}}_{\hat{\mathrm{H}}}$ in $\hat{\mathrm{T}}_1 \times \hat{\mathrm{T}}$.
Now [MW16, Lemma I.7.1] gives the following: for $w \in W_E$, there exist $t(w) \in \hat{\mathrm{T}}_{\hat{\mathrm{H}}}$, a hyperbolic element $z(w) \in \mathrm{Z}_{\hat{\mathrm{H}}_1}^{\Gamma,0}$ and a compact element $(u_1(w), u(w)) \in \hat{\mathrm{T}}_1 \times \hat{\mathrm{T}}$, such that $(\zeta_1(w), g(w)^{-1}) \in \hat{\mathrm{T}}_1 \times \hat{\mathrm{T}}$ can be written as $(z(w), 1)(\hat\xi_1(t(w))^{-1}, t(w))(u_1(w), u(w))$. We claim that $z(w) = 1$ for all $w \in W_E$. By (1) of [MW16, Section I.7.1], $w \mapsto z(w)$ extends to a unique homomorphism from W_F to the group of hyperbolic elements in $\mathrm{Z}_{\hat{\mathrm{H}}_1}^{\Gamma,0}$. Our having imposed [MW16, Section I.7.1, (3)] implies that this latter homomorphism is trivial, so that $z(w)$ is trivial for all $w \in W_E$. Therefore, we conclude that for all $w \in W_E$, the element $(u(w)^{-1}, w) = (g(w)t(w), w) = t(w)(g(w), w) \in \mathcal{H}$ (note that $g(w), t(w) \in \hat{\mathrm{T}}$ commute with each other) satisfies that:

$$\hat\xi_1(u(w)^{-1}, w) = \hat\xi_1(t(w))(\zeta_1(w), w) = (u_1(w), w).$$

Therefore, we can take $S = \{(u(w)^{-1}, w) \mid w \in W_E\} \subset \mathcal{H}$, and the given conditions are satisfied with C (resp., C_1) equal to the set of compact elements of $\hat{\mathrm{T}}$ (resp., $\hat{\mathrm{T}}_1$). □

Notation 6.1.11.
(i) Denote by $\Omega(\underline{\mathrm{H}}) : \Omega({}^L\mathrm{H}_1)_\mu \to \Omega({}^L\mathrm{G})$ the unique map such that $\Omega(\underline{\mathrm{H}}) \circ \Omega(\hat\xi_1)$ equals the map $\Omega(\mathcal{H}) \to \Omega(\mathrm{G})$ of Notation 6.1.7(iv); note that the existence and uniqueness of $\Omega(\underline{\mathrm{H}})$ follows from Lemma 6.1.9.
(ii) Let $\Phi_{\mathrm{G-rel}}(\mathrm{H}_1)_\mu \subset \Phi(\mathrm{H}_1)_\mu$ (resp., $\Phi_{\mathrm{temp,G-rel}}(\mathrm{H}_1)_\mu \subset \Phi_{\mathrm{temp}}(\mathrm{H}_1)_\mu$) denote the image of $\Phi_{\mathrm{G-rel}}(\mathcal{H})$ (resp., $\Phi_{\mathrm{temp,G-rel}}(\mathcal{H})$) under the bijection $\Phi(\hat\xi_1) : \Phi(\mathcal{H}) \to \Phi(\mathrm{H}_1)_\mu$ (see Lemma 6.1.9). Thus, using $\Phi(\hat\xi_1)$ and the map $\Phi_{\mathrm{G-rel}}(\mathcal{H}) \to \Phi(\mathrm{G})$ (see Notation 6.1.7(iv)), we get a map $\Phi(\underline{\mathrm{H}}) : \Phi_{\mathrm{G-rel}}(\mathrm{H}_1)_\mu \to \Phi(\mathrm{G})$.

(iii) It follows from Lemma 6.1.10 that the map $\Phi_{\mathrm{G-rel}}(\mathcal{H}) \to \Phi(\mathrm{G})$ takes $\Phi_{\mathrm{temp,G-rel}}(\mathcal{H})$ to $\Phi_{\mathrm{temp}}(\mathrm{G})$. Thus, $\Phi(\underline{\mathrm{H}})$ restricts to a map $\Phi_{\mathrm{temp,G-rel}}(\mathrm{H}_1)_\mu \to \Phi_{\mathrm{temp}}(\mathrm{G})$, which will also be denoted $\Phi(\underline{\mathrm{H}})$.

(iv) We transfer the variety structure on the closed subvariety $\Omega({}^L\mathrm{H}_1)_\mu \subset \Omega({}^L\mathrm{H}_1)$ (see Notation 6.1.7(ii)) to $\Omega(\mathcal{H})$ via the bijection $\Omega(\hat{\xi}_1) : \Omega(\mathcal{H}) \to \Omega({}^L\mathrm{H}_1)_\mu$.

We will see in Lemma 6.1.21 below that $\Omega({}^L\mathrm{H}_1)_\mu \to \Omega({}^L\mathrm{G})$ is a regular morphism of varities. For this, we now proceed to understand the variety structure on $\Omega(\mathcal{H})$ better, by giving a description for it along the lines of the variety structure on $\Omega({}^L\mathrm{G})$.

Notation 6.1.12. By a Levi subgroup of $\mathcal{H}$, we mean a subgroup of $\mathcal{H}$ of the form $\mathcal{M}_\mathcal{H} := \hat{\xi}_1^{-1}(\mathcal{M}_1)$ (i.e., "$\mathcal{M}_1 \cap \mathcal{H}$"), where $\mathcal{M}_1$ is a Levi subgroup of ${}^L\mathrm{H}_1$. Since ${}^L\mathrm{H}_1 = \mathrm{Z}_{\hat{\mathrm{H}}_1} \cdot \hat{\xi}_1(\mathcal{H})$, it follows that $\mathcal{M}_1 = \mathrm{Z}_{\hat{\mathrm{H}}_1} \cdot \hat{\xi}_1(\mathcal{M}_\mathcal{H})$, so that $\mathcal{M}_\mathcal{H} \to W_F$ is a surjection. Clearly, $\mathcal{M}_1 \mapsto \mathcal{M}_\mathcal{H} = \hat{\xi}_1^{-1}(\mathcal{M}_1)$ gives a bijection between Levi subgroups of ${}^L\mathrm{H}_1$ and Levi subgroups of $\mathcal{H}$, whose inverse takes $\mathcal{M}_\mathcal{H}$ to $\mathrm{Z}_{\hat{\mathrm{H}}_1} \cdot \hat{\xi}_1(\mathcal{M}_\mathcal{H})$. Note also that for any Levi subgroup $\mathcal{M}_\mathcal{H} \subset \mathcal{H}$, $\mathcal{M}_\mathcal{H}^0 := \mathcal{M}_\mathcal{H} \cap \hat{\mathrm{H}}$ is a Levi subgroup of $\hat{\mathrm{H}}$, and hence a connected reductive complex algebraic group.

Lemma 6.1.13. *Given an admissible homomorphism $\lambda_\mathcal{H} : W_F \to \mathcal{H}$ (see Notation 6.1.3(ii)), any two Levi subgroups of $\mathcal{H}$ that contain $\lambda_\mathcal{H}(W_F)$ minimally are conjugate under the centralizer of $\lambda_\mathcal{H}(W_F)$ in $\hat{\mathrm{H}}$.*

Proof. Note that if $\mathcal{M}_\mathcal{H}$ contains $\lambda_\mathcal{H}(W_F)$ minimally, then $\mathcal{M}_\mathcal{H} = \hat{\xi}_1^{-1}(\mathcal{M}_1)$ for some Levi subgroup $\mathcal{M}_1 \subset {}^L\mathrm{H}_1$ that contains $\lambda_1(W_F)$ minimally, where $\lambda_1 = \hat{\xi}_1 \circ \lambda_\mathcal{H}$. Thus, it is enough to show that any two Levi subgroups $\mathcal{M}_1, \mathcal{M}_1' \subset {}^L\mathrm{H}_1$ that contain $\lambda_1(W_F)$ minimally are conjugate to each other under $\hat{\xi}_1(\hat{h})$, for some $\hat{h} \in \hat{\mathrm{H}}$ that centralizes $\lambda_\mathcal{H}(W_F)$. By [Bor79, Proposition 3.6], $\mathcal{M}_1$ and $\mathcal{M}_1'$ are conjugate to each other under some $\hat{h}_1 \in \hat{\mathrm{H}}_1$ that centralizes $\lambda_1(W_F)$. The image of $\hat{h}_1$ in $\hat{\mathrm{C}}_1$ belongs to $(\hat{\mathrm{C}}_1)^{W_F}$, since this image centralizes ${}^Lj \circ \lambda_1$ and since $\hat{\mathrm{C}}_1$ is abelian. Since $\mathrm{Z}_{\hat{\mathrm{H}}_1}^{W_F} \to (\hat{\mathrm{C}}_1)^{W_F}$ is surjective, as we saw in the proof of Lemma 6.1.9, we can therefore modify $\hat{h}_1$ by an element of $\mathrm{Z}_{\hat{\mathrm{H}}_1}^{W_F}$ to have trivial image in $\hat{\mathrm{C}}_1$, i.e., $\hat{h}_1 \in \hat{\xi}_1(\hat{\mathrm{H}})$. □

Notation 6.1.14. (i) Let $\mathcal{M}_\mathcal{H} \subset \mathcal{H}$ be a Levi subgroup. By abuse of notation, we will refer to $(\mathcal{M}_\mathcal{H}, \lambda_\mathcal{H})$ as a cuspidal pair for $\Omega(\mathcal{H})$ if $\lambda_\mathcal{H} : W_F \to \mathcal{H}$ is an admissible homomorphism such that $\mathcal{M}_\mathcal{H}$ is minimal among the Levi subgroups of $\mathcal{H}$ that contain $\lambda_\mathcal{H}(W_F)$.

(ii) (Compare with Notation 4.1.10). Let $\mathcal{M}_\mathcal{H} \subset \mathcal{H}$ be a Levi subgroup. We write $\Omega(\mathcal{M}_\mathcal{H})_0$ for the set of $\mathcal{M}_\mathcal{H}^0$-conjugacy classes of cuspidal pairs of the form $(\mathcal{M}_\mathcal{H}, \lambda_\mathcal{H})$, and $(\mathcal{M}_\mathcal{H}, \lambda_\mathcal{H})_{\mathcal{M}_\mathcal{H}^0}$ (resp., $(\mathcal{M}_\mathcal{H}, \lambda_\mathcal{H})_{\hat{\mathrm{H}}}$) for the $\mathcal{M}_\mathcal{H}^0$-conjugacy class (resp., $\hat{\mathrm{H}}$-conjugacy class) of a given cuspidal pair $(\mathcal{M}_\mathcal{H}, \lambda_\mathcal{H})$.

(iii) (Compare with Remark 4.1.3(a)). By Lemma 6.1.13, we may and shall also view $\Omega(\mathcal{H})$ as the set of $\hat{\mathrm{H}}$-conjugacy classes of cuspidal pairs for $\Omega(\mathcal{H})$, by identifying the $\hat{\mathrm{H}}$-conjugacy class $(\lambda_\mathcal{H})_{\hat{\mathrm{H}}}$ of an admissible homomorphism $\lambda_\mathcal{H} : W_F \to \mathcal{H}$ with the $\hat{\mathrm{H}}$-conjugacy class $(\mathcal{M}_\mathcal{H}, \lambda_\mathcal{H})_{\hat{\mathrm{H}}}$ of any cuspidal pair of the form $(\mathcal{M}_\mathcal{H}, \lambda_\mathcal{H})$. With this identification, $\Omega(\hat{\xi}_1)$ takes $(\mathcal{M}_\mathcal{H}, \lambda_\mathcal{H})_{\hat{\mathrm{H}}}$ to $(\mathcal{M}_1, \lambda_1)_{\hat{\mathrm{H}}_1}$, where $\lambda_1 = \hat{\xi}_1 \circ \lambda_\mathcal{H}$, and $\mathcal{M}_1 \subset {}^L\mathrm{H}_1$ is the unique Levi subgroup with $\hat{\xi}_1^{-1}(\mathcal{M}_1) = \mathcal{M}_\mathcal{H}$.

(iv) If $\mathcal{M}_1 \subset {}^L\mathrm{H}_1$ is a Levi subgroup, we will also make use of the map $\Omega(\mathcal{M}_\mathcal{H})_0 \to \Omega(\mathcal{M}_1)_0$ (where $\Omega(\mathcal{M}_1)_0$ is as in Notation 4.1.10) induced by $\hat{\xi}_1$, where $\mathcal{M}_\mathcal{H} = \hat{\xi}_1^{-1}(\mathcal{M}_1)$.

(v) (Compare with Notation 4.1.8 and Notation 4.1.11). Let $\mathcal{M}_\mathcal{H} \subset \mathcal{H}$ be a Levi subgroup. One makes W_F act on $\mathrm{Z}_{\mathcal{M}_\mathcal{H}^0}$ exactly as in Notation 4.1.8 (i.e., as in Remark 2.3.2), via a chain $W_F = \mathcal{M}_\mathcal{H}/\mathcal{M}_\mathcal{H}^0 \to \mathrm{Out}(\mathcal{M}_\mathcal{H}^0) \to \mathrm{Aut}(\mathrm{Z}_{\mathcal{M}_\mathcal{H}^0})$. If $(\mathcal{M}_\mathcal{H}, \lambda_\mathcal{H})$ is a cuspidal pair for $\Omega(\mathcal{H})$ and $z \in \mathrm{Z}_{\mathcal{M}_\mathcal{H}^0}^{W_F,0} = (\mathrm{Z}_{\mathcal{M}_\mathcal{H}} \cap \mathcal{M}_\mathcal{H}^0)^0 = \mathrm{Z}_{\mathcal{M}_\mathcal{H}}^0 = H^1(W_F/I_F, \mathrm{Z}_{\mathcal{M}_\mathcal{H}}^0)$, we let $(\mathcal{M}_\mathcal{H}, z \cdot \lambda_\mathcal{H})_{\mathcal{M}_\mathcal{H}^0}$ be the $\mathcal{M}_\mathcal{H}^0$-conjugacy class of the pair $(\mathcal{M}_\mathcal{H}, w \mapsto \alpha(w)\lambda_\mathcal{H}(w))$, where $\alpha \in Z^1(W_F, \mathrm{Z}_{\mathcal{M}_\mathcal{H}}^0)$ is a cocycle representing $z \in Z_{\mathcal{M}_\mathcal{H}}^0 = H^1(W_F/I_F, \mathrm{Z}_{\mathcal{M}_\mathcal{H}}^0)$. This is readily verified to define an action of $\mathrm{Z}^0_{\mathcal{M}_\mathcal{H}}$ on $\Omega(\mathcal{M}_\mathcal{H})_0$, which is immediately verified to descend via $\mathrm{Z}^0_{\mathcal{M}_\mathcal{H}} = \mathrm{Z}^{W_F,0}_{\mathcal{M}^0_\mathcal{H}} \to (\mathrm{Z}^{I_F}_{\mathcal{M}^0_\mathcal{H}})^0_{\mathrm{Fr}}$.

Lemma 6.1.15. *Let $\mathcal{M}_\mathcal{H} \subset \mathcal{H}, \mathcal{M}_1 \subset {}^L\mathrm{H}_1$ be Levi subgroups, with $\mathcal{M}_\mathcal{H} = \hat{\xi}_1^{-1}(\mathcal{M}_1)$. Consider the variety structure on $\Omega(\mathcal{M}_1)_0$ as in Remark 4.1.13. The map $\Omega(\mathcal{M}_\mathcal{H})_0 \to \Omega(\mathcal{M}_1)_0$ induced by the map sending a cuspidal pair $(\mathcal{M}_\mathcal{H}, \lambda_\mathcal{H})$ to $(\mathcal{M}_1, \hat{\xi}_1 \circ \lambda_\mathcal{H})$, is a bijection from $\Omega(\mathcal{M}_\mathcal{H})_0$ to the closed subvariety $\Omega(\mathcal{M}_1)_{0,\mu} \subset \Omega(\mathcal{M}_1)_0$ consisting of $\mathcal{M}_1^0$-conjugacy classes of admissible homomorphisms $\lambda_1 : W_F \to \mathcal{M}_1 \subset {}^L\mathrm{H}_1$ such that ${}^Lj \circ \lambda_1$ represesents μ.*

Proof. It is easy to see that the map $\Omega(\mathcal{M}_1)_0 \to \Omega({}^L\mathrm{C}_1)$ induced by composition with ${}^Lj$ is a morphism of varieties, so that $\Omega(\mathcal{M}_1)_{0,\mu}$ is a closed subvariety of $\Omega(\mathcal{M}_1)_0$. That the given map is a bijection $\Omega(\mathcal{M}_\mathcal{H})_0 \to \Omega(\mathcal{M}_1)_{0,\mu}$ follows exactly as in the proof of the bijectivity of $\Omega(\mathcal{H}) \to \Omega({}^L\mathrm{H}_1)_\mu$ in Lemma 6.1.9: the surjectivity of the map uses the surjectivity of $\mathrm{Z}_{\hat{\mathrm{H}}_1} \to \hat{\mathrm{C}}_1$, while its injectivity uses the surjectivity of $\mathrm{Z}^{W_F}_{\hat{\mathrm{H}}_1} \to (\hat{\mathrm{C}}_1)^{W_F}$; both of these help here as well, since $\mathrm{Z}_{\hat{\mathrm{H}}_1} \subset \mathcal{M}_1^0$, and since $\mathcal{M}_1 = \mathrm{Z}_{\hat{\mathrm{H}}_1} \cdot \hat{\xi}_1(\mathcal{M}_\mathcal{H})$. □

Notation 6.1.16. For any Levi subgroup $\mathcal{M}_\mathcal{H} \subset \mathcal{H}$, give $\Omega(\mathcal{M}_\mathcal{H})_0$ the variety structure obtained by transport via its bijection with the closed subvariety $\Omega(\mathcal{M}_1)_{0,\mu} \subset \Omega(\mathcal{M}_1)_0$ as in Lemma 6.1.15.

In Corollary 6.1.19 below, we will describe the variety structure on $\Omega(\mathcal{M}_\mathcal{H})_0$ more explicitly and intrinsically, in the spirit of Remark 4.1.13, so as to use it later for a more explicit and intrinsic description of the variety structure on $\Omega(\mathcal{H})$.

Lemma 6.1.17. *Let $\mathcal{M}_1 \subset {}^L\mathrm{H}_1$ be a Levi subgroup, and let $\mathcal{M}_\mathcal{H} = \hat{\xi}_1^{-1}(\mathcal{M}_1)$. For simplicity, think of $\hat{\xi}_1$ and the map $\Omega(\mathcal{M}_\mathcal{H})_0 \to \Omega(\mathcal{M}_1)_0$ as inclusions (the latter as justified by Lemma 6.1.15). Then each $\mathrm{Z}^0_{\mathcal{M}_1}$-orbit in $\Omega(\mathcal{M}_1)_0$ intersects $\Omega(\mathcal{M}_\mathcal{H})_0$ in a finite (possibly empty) union of $\mathrm{Z}^0_{\mathcal{M}_\mathcal{H}}$-orbits.*

Remark 6.1.18. While we have not verified if the number of orbits in the statement of Lemma 6.1.17 is at most one, that seems unlikely to us, for the same reason that the composite $\mathrm{sgn}_{\mathrm{unr}} \circ \det$ of the unramified sign character and the determinant on $\mathrm{GL}_2(F)$ descends to a character of $\mathrm{PGL}_2(F)$ which is not unramified. The proof will be longer than necessary, so as to potentially help locate the possible failure.

Proof of Lemma 6.1.17. Without loss of generality assume that the $\mathrm{Z}^0_{\mathcal{M}_1}$-orbit contains the image of a cuspidal pair for $\Omega(\mathcal{H})$ of the form $(\mathcal{M}_\mathcal{H}, \lambda_\mathcal{H})$. Suppose an element $z_1 \in \mathrm{Z}^0_{\mathcal{M}_1}$ takes $(\mathcal{M}_\mathcal{H}, \lambda_\mathcal{H})_{\mathcal{M}^0_\mathcal{H}}$ to another element of $\Omega(\mathcal{M}_\mathcal{H})_0$. Applying ${}^Lj$, we find that ${}^Lj(z_1) \in (\hat{\mathrm{C}}_1)^{W_F}$ maps to the identity element of $H^1(W_F, \hat{\mathrm{C}}_1)$, under the chain $(\hat{\mathrm{C}}_1)^{W_F} = H^1(W_F/I_F, (\hat{\mathrm{C}}_1)^{W_F}) \to H^1(W_F/I_F, (\hat{\mathrm{C}}_1)^{I_F}) \subset H^1(W_F, \hat{\mathrm{C}}_1)$.
Now the sequence

$$1 \to \mathrm{Z}_{\mathcal{M}^0_\mathcal{H}} \to \mathrm{Z}_{\mathcal{M}^0_1} \to \hat{\mathrm{C}}_1 \to 1$$

is exact, and remains so after taking I_F-invariants, since $(\hat{\mathrm{C}}_1)^{I_F}$ is connected (as C_1 is an induced torus, so that $X_*(\hat{\mathrm{C}}_1) = X^*(\mathrm{C}_1)$ has a basis permuted by $W_F \supset I_F$). Applying $H^1(W_F/I_F, \cdot)$, it follows that the image of z_1 in $(\mathrm{Z}^{I_F}_{\mathcal{M}^0_1})^0_{\mathrm{Fr}}$ lies in the image of $(\mathrm{Z}^{I_F}_{\mathcal{M}^0_\mathcal{H}})_{\mathrm{Fr}} \to (\mathrm{Z}^{I_F}_{\mathcal{M}^0_1})_{\mathrm{Fr}}$, which is a finite union of translates of the image of $(\mathrm{Z}^{I_F}_{\mathcal{M}^0_\mathcal{H}})^0_{\mathrm{Fr}} \to (\mathrm{Z}^{I_F}_{\mathcal{M}^0_1})^0_{\mathrm{Fr}}$. Thus, we are done, since on the one hand the action of $\mathrm{Z}^0_{\mathcal{M}_\mathcal{H}}$ on $\Omega(\mathcal{M}_\mathcal{H})_0$ factors through the isogeny $\mathrm{Z}^0_{\mathcal{M}_\mathcal{H}} \to (\mathrm{Z}^{I_F}_{\mathcal{M}^0_\mathcal{H}})^0_{\mathrm{Fr}} \subset H^1(W_F/I_F, \mathrm{Z}^{I_F}_{\mathcal{M}^0_\mathcal{H}})$, while the action of $\mathrm{Z}^0_{\mathcal{M}_1}$ on $\Omega(\mathcal{M}_1)_0$ factors through the isogeny $\mathrm{Z}^0_{\mathcal{M}_1} \to (\mathrm{Z}^{I_F}_{\mathcal{M}^0_1})^0_{\mathrm{Fr}}$, and these actions are compatible in an obvious sense. □

Corollary 6.1.19. *Let $\mathcal{M}_\mathcal{H} \subset \mathcal{H}$ be a Levi subgroup. For any complex variety Y, a map $f : \Omega(\mathcal{M}_\mathcal{H})_0 \to \mathrm{Y}$ is regular if and only if for each cuspidal pair for $\Omega(\mathcal{H})$ of the form $(\mathcal{M}_\mathcal{H}, \lambda_\mathcal{H})$, the map $\mathrm{Z}^0_{\mathcal{M}_\mathcal{H}} \to \mathrm{Y}$ given by $z \mapsto f((\mathcal{M}_\mathcal{H}, z \cdot \lambda_\mathcal{H})_{\mathcal{M}^0_\mathcal{H}})$ is regular.*

Proof. We again view $\hat{\xi}_1$ as an inclusion, and identify $\Omega(\mathcal{M}_{\mathcal{H}})_0$, using Lemma 6.1.15, as the closed subvariety $\Omega(\mathcal{M}_1)_{0,\mu} \subset \Omega(\mathcal{M}_1)_0$, where $\mathcal{M}_1 \subset {}^L\mathrm{H}_1$ is the unique Levi subgroup with $\mathcal{M}_1 \cap \mathcal{H} = \mathcal{M}_{\mathcal{H}}$. Recall that, by Remark 4.1.13, $\Omega(\mathcal{M}_1)_0$ is a countable union of orbits of $\mathrm{Z}^0_{\mathcal{M}_1}$, with each orbit, say A, identifying via an orbit map as a torsor under a quotient $\mathrm{Z}^0_{\mathcal{M}_1}/(\mathrm{Z}^0_{\mathcal{M}_1})_A$ of $\mathrm{Z}^0_{\mathcal{M}_1}$ by a finite subgroup $(\mathrm{Z}^0_{\mathcal{M}_1})_A$.
Further, by Lemma 6.1.17, the intersection of each such orbit A with $\Omega(\mathcal{M}_{\mathcal{H}})_0$ is either empty or a finite union of torsors under a quotient $\mathrm{Z}^0_{\mathcal{M}_{\mathcal{H}}}/(\mathrm{Z}^0_{\mathcal{M}_{\mathcal{H}}})_B$, where $(\mathrm{Z}^0_{\mathcal{M}_{\mathcal{H}}})_B$ identifies with the finite subgroup $(\mathrm{Z}^0_{\mathcal{M}_1})_A \cap \mathrm{Z}^0_{\mathcal{M}_{\mathcal{H}}} \subset \mathrm{Z}^0_{\mathcal{M}_{\mathcal{H}}}$, making $\mathrm{Z}^0_{\mathcal{M}_{\mathcal{H}}}/(\mathrm{Z}^0_{\mathcal{M}_{\mathcal{H}}})_B$ a subtorus of $\mathrm{Z}^0_{\mathcal{M}_1}/(\mathrm{Z}^0_{\mathcal{M}_1})_A$. Since $\Omega(\mathcal{M}_{\mathcal{H}})_0$ gets its variety structure from its being a Zariski closed subset of $\Omega(\mathcal{M}_1)_0$, it follows that $\Omega(\mathcal{M}_{\mathcal{H}})_0$ with its $\mathrm{Z}^0_{\mathcal{M}_{\mathcal{H}}}$-action identifies with a countable disjoint union of torsors under finite quotients of $\mathrm{Z}^0_{\mathcal{M}_{\mathcal{H}}}$. Since quotients by finite abstract groups over fields of characteristic zero are good and hence categorical, the corollary follows. □

Now we can give a more explicit description of the variety structure on $\Omega(\mathcal{H})$.

Lemma 6.1.20. *The variety structure on $\Omega(\mathcal{H})$ in Notation 6.1.11(iv) has the following property: for any complex variety Y, a map $f : \Omega(\mathcal{H}) \to \mathrm{Y}$ is regular if and only if for each cuspidal pair $(\mathcal{M}_{\mathcal{H}}, \lambda_{\mathcal{H}})$ for $\Omega(\mathcal{H})$, the map $\mathrm{Z}^0_{\mathcal{M}_{\mathcal{H}}} \to \mathrm{Y}$ given by $z \mapsto f((\mathcal{M}_{\mathcal{H}}, z\cdot\lambda)_{\hat{\mathrm{H}}})$ is regular.*

Proof. For lightness of notation, we will again think of $\hat{\xi}_1$ as an inclusion $\mathcal{H} \subset {}^L\mathrm{H}_1$, and $\Omega(\mathcal{H})$ as the subvariety $\Omega({}^L\mathrm{H}_1)_\mu$ of $\Omega({}^L\mathrm{H}_1)$, etc. We write as in (52) (but with the trivial group taking the place of $\mathcal{O}^+_{\mathrm{G}}$):

$$\Omega({}^L\mathrm{H}_1) = \bigsqcup_{\mathcal{M}_1} \Omega(\mathcal{M}_1)_0/W(\mathcal{M}_1). \tag{82}$$

An analogous set-theoretic decomposition is easy to see from the discussion of Notation 6.1.14(iii):

$$\Omega(\mathcal{H}) = \bigsqcup_{\mathcal{M}_{\mathcal{H}}} \Omega(\mathcal{M}_{\mathcal{H}})_0/W(\mathcal{M}_{\mathcal{H}}), \tag{83}$$

where $\mathcal{M}_{\mathcal{H}}$ runs over a set of representatives for the $\hat{\mathrm{H}}$-conjugacy classes of Levi subgroups of $\mathcal{H}$, and $W(\mathcal{M}_{\mathcal{H}})$ is the quotient, of the normalizer of $\mathcal{M}_{\mathcal{H}}$ in $\hat{\mathrm{H}}$, by $\mathcal{M}^0_{\mathcal{H}}$. For each Levi subgroup $\mathcal{M}_{\mathcal{H}} = \mathcal{M}_1 \cap \mathcal{H} \subset \mathcal{H}$, where $\mathcal{M}_1 \subset {}^L\mathrm{H}_1$ is a Levi subgroup, it is easy to see, exactly as in the proof of Lemma 6.1.15 (or equivalently as in the proof of Lemma 6.1.9), that $\Omega(\mathcal{M}_{\mathcal{H}})_0/W(\mathcal{M}_{\mathcal{H}}) \subset \Omega(\mathcal{H})$ is the intersection of $\Omega(\mathcal{M}_1)/W(\mathcal{M}_1)$ with $\Omega(\mathcal{H})$ inside $\Omega({}^L\mathrm{H}_1)$, and thus that it is a closed subvariety of $\Omega(\mathcal{H}) \subset \Omega({}^L\mathrm{H}_1)$.
By Corollary 6.1.19 and the fact that any quotient by the finite group $W(\mathcal{M}_{\mathcal{H}})$ is good and hence categorical, it suffices to show that the set-theoretic map $\Omega(\mathcal{M}_{\mathcal{H}})_0 \to \Omega(\mathcal{M}_{\mathcal{H}})_0/W(\mathcal{M}_{\mathcal{H}})$ between algebraic varieties is a quotient map of its source by $W(\mathcal{M}_{\mathcal{H}})$: this reduction uses that for each $\hat{h} \in \hat{\mathrm{H}}$, $z \mapsto f((\operatorname{Int}\hat{h}(\mathcal{M}_{\mathcal{H}}), z\cdot(\operatorname{Int}\hat{h}\circ\lambda))_{\hat{\mathrm{H}}})$ is a regular map $\mathrm{Z}^0_{\operatorname{Int}\hat{h}(\mathcal{M}_{\mathcal{H}})} \to \mathrm{Y}$ if and only if $z \mapsto f((\mathcal{M}_{\mathcal{H}}, z\cdot\lambda)_{\hat{\mathrm{H}}})$ is a regular map $\mathrm{Z}^0_{\mathcal{M}_{\mathcal{H}}} \to \mathrm{Y}$: this is so because the latter is the pull-back of the former under the isomorphism $\operatorname{Int}\hat{h} : \mathrm{Z}^0_{\mathcal{M}_{\mathcal{H}}} \to \mathrm{Z}^0_{\operatorname{Int}\hat{h}(\mathcal{M}_{\mathcal{H}})}$ (this verification ensures that what we prove for a chosen set of Levi subgroups of $\mathcal{H}$ applies to all Levi subgroups of $\mathcal{H}$).
It is easy to see that the inclusion $W(\mathcal{M}_{\mathcal{H}}) \to W(\mathcal{M}_1)$ is an isomorphism (use that $\hat{\mathrm{H}}_1 = \mathrm{Z}_{\hat{\mathrm{H}}_1}\cdot\hat{\mathrm{H}}$), and that the closed subvariety $\Omega(\mathcal{M}_{\mathcal{H}})_0 \subset \Omega(\mathcal{M}_1)_0$ is invariant under the action of $W(\mathcal{M}_{\mathcal{H}}) = W(\mathcal{M}_1)$. Therefore, the required claim (that $\Omega(\mathcal{M}_{\mathcal{H}})_0 \to \Omega(\mathcal{M}_{\mathcal{H}})_0/W(\mathcal{M}_{\mathcal{H}})$ is a quotient map) follows from [Dré04, Proposition 2.18] and the fact that $\Omega(\mathcal{M}_1)_0 \to \Omega(\mathcal{M}_1)_0/W(\mathcal{M}_1)$ is a quotient map (as per the construction in the proof of Proposition 4.1.4). □

Lemma 6.1.21. *The map $\Omega(\underline{\mathrm{H}}) : \Omega({}^L\mathrm{H}_1)_\mu \to \Omega({}^L\mathrm{G})$ of sets is a regular morphism of algebraic varieties.*

Proof. Since $\Omega(\hat{\xi}_1) : \Omega(\mathcal{H}) \to \Omega({}^L\mathrm{H}_1)_\mu$ is an isomorphism of varieties, it is enough to show that the map $\Omega(\mathcal{H}) \to \Omega({}^L\mathrm{G})$ induced by the inclusion $\mathcal{H} \hookrightarrow {}^L\mathrm{G}$ is a regular morphism. Let $(\mathcal{M}_{\mathcal{H}}, \lambda_{\mathcal{H}})$ be a cuspidal pair for $\mathcal{H}$. Denote by $\Omega([\mathcal{M}_{\mathcal{H}}, \lambda_{\mathcal{H}}]_{\hat{\mathrm{H}}})$ the connected component of $\Omega(\mathcal{H})$ containing

$(\lambda_{\mathcal{H}})_{\hat{\mathrm{H}}}$. Then by Lemma 6.1.20, $\Omega([\mathcal{M}_{\mathcal{H}}, \lambda_{\mathcal{H}}]_{\hat{\mathrm{H}}})$ is the image of $\mathrm{Z}^0_{\mathcal{M}_{\mathcal{H}}}$ under the finite morphism $z \mapsto (\mathcal{M}_{\mathcal{H}}, z \cdot \lambda_{\mathcal{H}})_{\hat{\mathrm{H}}}$.
As in [MW16, Section I.3.4] (but without any renormalization of the W_F-action on $\hat{\mathrm{H}}$), let $\mathcal{M}$ be the centralizer of $\mathrm{Z}^0_{\mathcal{M}_{\mathcal{H}}} = \mathrm{Z}^{W_F,0}_{\mathcal{M}^0_{\mathcal{H}}}$ in ${}^L\mathrm{G} \supset \mathcal{H}$, where we think of the map $\mathcal{H} \to {}^L\mathrm{G}$ as an inclusion. By choosing a cocharacter that is in 'general position' among those valued in $\mathrm{Z}^0_{\mathcal{M}_{\mathcal{H}}}$, and using that the map $\mathcal{M} \hookrightarrow {}^L\mathrm{G} \to W_F$ is surjective (since $\mathcal{M} \supset \mathcal{M}_{\mathcal{H}} \supset \lambda_{\mathcal{H}}(W_F)$), it is easy to see that $\mathcal{M}$ is a Levi subgroup of ${}^L\mathrm{G}$ (which may not be relevant, but that does not concern us). It is obvious that $\mathrm{Z}^0_{\mathcal{M}_{\mathcal{H}}} \subset \mathrm{Z}^0_{\mathcal{M}}$. We will not need the more precise fact, which one can prove, that $\mathrm{Z}^0_{\mathcal{M}_{\mathcal{H}}} = \mathrm{Z}^{\tilde{s},0}_{\mathcal{M}}$. Write λ for $\lambda_{\mathcal{H}}$, when viewed as valued in ${}^L\mathrm{G} \supset \mathcal{H}$. Thus, $\lambda(W_F) = \lambda_{\mathcal{H}}(W_F) \subset \mathcal{M}_{\mathcal{H}} \subset \mathcal{M}$, so that $\mathcal{M}$ contains $\lambda(W_F)$, though not necessarily minimally. Let $\mathcal{M}' \subset \mathcal{M}$ be a Levi subgroup of ${}^L\mathrm{G}$ containing $\lambda(W_F)$ minimally.
Thus, the connected component $\Omega([\mathcal{M}_{\mathcal{H}}, \lambda_{\mathcal{H}}]_{\hat{\mathrm{H}}}) \subset \Omega(\mathcal{H})$ containing $(\mathcal{M}_{\mathcal{H}}, \lambda_{\mathcal{H}})_{\hat{\mathrm{H}}}$ and the connected component $\Omega([\mathcal{M}', \lambda]_{\hat{\mathrm{G}}}) \subset \Omega({}^L\mathrm{G})$ containing $(\mathcal{M}', \lambda)_{\hat{\mathrm{G}}}$ belong to the following diagram whose commutativity is clear:

$$\begin{array}{ccccc} \mathrm{Z}^0_{\mathcal{M}_{\mathcal{H}}} & \hookrightarrow & \mathrm{Z}^0_{\mathcal{M}} & \hookrightarrow & \mathrm{Z}^0_{\mathcal{M}'} \\ \downarrow & & & & \downarrow \\ \Omega([\mathcal{M}_{\mathcal{H}}, \lambda_{\mathcal{H}}]_{\hat{\mathrm{H}}}) & & \longrightarrow & & \Omega([\mathcal{M}', \lambda]_{\hat{\mathrm{G}}}) \end{array}, \tag{84}$$

where the vertical arrows are surjections, given by $z \mapsto (\mathcal{M}_{\mathcal{H}}, z \cdot \lambda_{\mathcal{H}})_{\hat{\mathrm{H}}}$ and $z \mapsto (\mathcal{M}', z \cdot \lambda)_{\hat{\mathrm{G}}}$, respectively. The right vertical arrow is algebraic by Corollary 4.1.6. Using this, it is easy to conclude from Lemma 6.1.20 that the bottom horizontal arrow is algebraic. This shows that the map $\Omega(\mathcal{H}) \to \Omega({}^L\mathrm{G})$ is an algebraic map, being algebraic on each component of $\Omega(\mathcal{H})$, as required. □

Remark 6.1.22. Note that the proof of Lemma 6.1.21 would be harder if we worked with $(\mathrm{Z}^{I_F}_{\mathcal{M}^0_{\mathcal{H}}})^0_{\mathrm{Fr}}$ and $(\mathrm{Z}^{I_F}_{\mathcal{M}'^0})^0_{\mathrm{Fr}}$ instead of with $\mathrm{Z}^0_{\mathcal{M}_{\mathcal{H}}}$ and $\mathrm{Z}^0_{\mathcal{M}'}$.

Since we can only map $\Omega({}^L\mathrm{H}_1)_\mu$, and not $\Omega({}^L\mathrm{H}_1)$, to $\Omega({}^L\mathrm{G})$, we can only expect to map $\mathcal{Z}_2(\mathrm{G})$ to a quotient of $\mathcal{Z}_2(\mathrm{H}_1)$; we now make some preparations to deal with this quotient.

Notation 6.1.23. Sending a cuspidal pair (M_1, σ_1) for $\Omega(\mathrm{H}_1)$ to the restriction of the central character of σ_1 to $\mathrm{C}_1(F)$, descends to well-defined set-theoretic map $\Omega(\mathrm{H}_1) \to \mathrm{Hom}_{\mathrm{cts}}(\mathrm{C}_1(F), \mathbb{C}^\times) = \Omega(\mathrm{C}_1)$. It is immediate from Proposition 4.1.4(i) that this map $\Omega(\mathrm{H}_1) \to \Omega(\mathrm{C}_1)$ is a regular morphism of algebraic varieties. We define $\Omega(\mathrm{H}_1)_\mu$ to be the scheme-theoretic inverse image of $\mu \in \Omega(\mathrm{C}_1)$ under this map. The closed subscheme $\Omega(\mathrm{H}_1)_\mu \subset \Omega(\mathrm{H}_1)$ is reduced, because using Remark 4.1.15(i), each cuspidal pair (M_1, σ_1) for $\Omega(\mathrm{H}_1)$ is seen to satisfy:

$$\mathbb{C}[\Omega([\mathrm{M}_1, \sigma_1]_{\mathrm{H}_1}) \cap \Omega(\mathrm{H}_1)_\mu] \subset \mathbb{C}[\Omega([\mathrm{M}_1, \sigma_1]_{\mathrm{M}_1}) \cap \Omega(\mathrm{M}_1)_\mu] \subset \mathbb{C}[\ker(X^{\mathrm{unr}}(\mathrm{S}_{\mathrm{M}_1}) \to X^{\mathrm{unr}}(\mathrm{C}_1))],$$

where the latter inclusion is obtained from the orbit map for the action of $X^{\mathrm{unr}}(\mathrm{S}_{\mathrm{M}_1})$ on $\Omega(\mathrm{M}_1)$, at $(\mathrm{M}_1, \sigma_1)_{\mathrm{M}_1}$. Since last ring above is reduced by Cartier's theorem, the reducedness of $\Omega(\mathrm{H}_1)_\mu$ follows. We let $\mathcal{Z}(\mathrm{H}_1)_\mu := \mathbb{C}[\Omega(\mathrm{H}_1)_\mu]$ be the corresponding reduced quotient of $\mathbb{C}[\Omega(\mathrm{H}_1)] = \mathcal{Z}(\mathrm{H}_1)$: in other words, it is the ring of functions $\Omega(\mathrm{H}_1)_\mu \to \mathbb{C}$ obtained by restricting elements of $\mathbb{C}[\Omega(\mathrm{H}_1)]$. Note that if μ is trivial, then we have $\mathcal{Z}(\mathrm{H}_1)_\mu = \mathbb{C}[\Omega(\mathrm{H}_1)_\mu] = \mathbb{C}[\Omega(\mathrm{H})] = \mathcal{Z}(\mathrm{H})$.

Lemma 6.1.24. *If V_1 is any smooth representation of $\mathrm{H}_1(F)$ on which $\mathrm{C}_1(F)$ acts via μ, then the action of $\mathcal{Z}(\mathrm{H}_1) = \mathbb{C}[\Omega(\mathrm{H}_1)]$ on V_1 factors through the quotient $\mathbb{C}[\Omega(\mathrm{H}_1)_\mu] = \mathcal{Z}(\mathrm{H}_1)_\mu$.*

Proof. Every element of V_1 is contained in a homomorphic image of $C^\infty_\mu(\mathrm{H}_1(F))$, which is given the left-regular action of $\mathrm{H}_1(F)$. Therefore, to prove the lemma, it suffices to do so in the case where V_1 equals $C^\infty_\mu(\mathrm{H}_1(F))$, and is given the left-regular action of $\mathrm{H}_1(F)$. Thus, suppose that the image of $z \in \mathcal{Z}(\mathrm{H}_1) = \mathbb{C}[\Omega(\mathrm{H}_1)]$ in $\mathbb{C}[\Omega(\mathrm{H}_1)_\mu]$ vanishes, and let us show that the left-regular action of z annihilates $V_1 = C^\infty_\mu(\mathrm{H}_1(F))$.
The assumption on z implies that $\hat{z}(\pi_1) = 0$ for every irreducible admissible representation π_1 of $\mathrm{H}_1(F)$ whose central character restricts to μ on $\mathrm{C}_1(F)$. Suppose there exists $f_1 \in C^\infty_\mu(\mathrm{H}_1(F)) = V_1$

such that $z * f_1 \neq 0$; let us derive a contradiction. We may right-translate f_1 to assume without loss of generality that $z * f_1(1) \neq 0$. By the Plancherel formula with central character (Remark 5.2.9(ii)), there exists an irreducible admissible representation π_1 of $\mathrm{H}_1(F)$, with central character restricting to μ on $\mathrm{C}_1(F)$, such that $0 \neq \pi_1(z * f_1) = \hat{z}(\pi_1)\pi_1(f_1)$, a contradiction since $\hat{z}(\pi_1) = 0$. □

Notation 6.1.25. (i) Suppose we are in Scenario 1 of Notation 6.1.1(vii). Then the action of $\mathcal{O}_{\mathrm{H}_1}$ on $\Omega({}^L\mathrm{H}_1)$ factors through a finite quotient and preserves $\Omega({}^L\mathrm{H}_1)_\mu$, letting us talk of the *closed* subvariety $\underline{\Omega}({}^L\mathrm{H}_1)_\mu := \Omega({}^L\mathrm{H}_1)_\mu/\mathcal{O}_{\mathrm{H}_1}$ of $\underline{\Omega}({}^L\mathrm{H}_1) := \Omega({}^L\mathrm{H}_1)/\mathcal{O}_{\mathrm{H}_1}$, and of the reduced quotient ring $\mathbb{C}[\underline{\Omega}({}^L\mathrm{H}_1)_\mu] = \mathbb{C}[\Omega({}^L\mathrm{H}_1)_\mu]^{\mathcal{O}_{\mathrm{H}_1}}$ of $\mathbb{C}[\underline{\Omega}({}^L\mathrm{H}_1)] = \mathbb{C}[\Omega({}^L\mathrm{H}_1)]^{\mathcal{O}_{\mathrm{H}_1}}$. Note also that $\Phi_{\mathrm{temp,G-rel}}(\mathrm{H}_1)_\mu \subset \Phi(\mathrm{H}_1)$ is $\mathcal{O}_{\mathrm{H}_1}$-invariant.

(ii) Continue with the setting of (i). Since $\mathcal{O}_{\mathrm{H}_1}$ fixes C_1 pointwise and hence preserves μ_1, and since $\hat{\mathrm{H}}_1 \to \hat{\mathrm{C}}_1$ is surjective, it is easy to see that any $\beta \in \mathcal{O}_{\mathrm{H}_1}$ has a dual ${}^L\beta : {}^L\mathrm{H}_1 \to {}^L\mathrm{H}_1$ which restricts to an automorphism of $\mathcal{H}$; we impose the hypothesis that some (or equivalently by the surjectivity of $\mathrm{Z}_{\hat{\mathrm{H}}_1}^{W_F} \to (\hat{\mathrm{C}}_1)^{W_F}$, any) such automorphism of $\mathcal{H}$ is induced by conjugation by an element of $\hat{\mathrm{G}}$. Then (using reducedness) the map $\Omega(\underline{\mathrm{H}}) : \Omega({}^L\mathrm{H}_1)_\mu \to \Omega({}^L\mathrm{G})$ of varieties and the map $\Phi(\underline{\mathrm{H}}) : \Phi_{\mathrm{temp,G-rel}}(\mathrm{H}_1)_\mu \to \Phi_{\mathrm{temp}}(\mathrm{G})$ of sets (see Notation 6.1.11(iii)) quotient respectively to a map

$$\underline{\Omega}(\underline{\mathrm{H}}) : \underline{\Omega}({}^L\mathrm{H}_1)_\mu = \Omega({}^L\mathrm{H}_1)_\mu/\mathcal{O}_{\mathrm{H}_1} \to \Omega({}^L\mathrm{G})$$

of varieties, and a map $\Phi_{\mathrm{temp,G-rel}}(\mathrm{H}_1)_\mu/\mathcal{O}_{\mathrm{H}_1} \to \Phi_{\mathrm{temp}}(\mathrm{G})$ of sets (note that $\mathcal{O}_{\mathrm{H}_1}$ clearly preserves $\Phi_{\mathrm{temp,G-rel}}(\mathrm{H}_1) \subset \Phi_{\mathrm{temp}}(\mathrm{H}_1)$).

(iii) Suppose we are in Scenario 2 of Notation 6.1.1(vii). Then we have the variety $\underline{\Omega}({}^L\mathrm{H}) := \Omega({}^L\mathrm{H})/\mathcal{O}_{\mathrm{H}}$ and the ring $\mathbb{C}[\underline{\Omega}({}^L\mathrm{H})] = \mathbb{C}[\Omega({}^L\mathrm{H})]^{\mathcal{O}_{\mathrm{H}}}$. In this situation (where μ is trivial), we can identify $\Omega({}^L\mathrm{H}_1)_\mu = \Omega({}^L\mathrm{H})$ and $\mathbb{C}[\Omega({}^L\mathrm{H}_1)_\mu] = \mathbb{C}[\Omega({}^L\mathrm{H})]$, by definition. If we additionally assume that $\mathcal{O}_{\mathrm{H}}$ is the group of automorphisms of H induced by a group $\mathcal{O}_{\mathrm{H}_1}$ as in Scenario 1 of Notation 6.1.1(vii), then we can identify $\underline{\Omega}({}^L\mathrm{H}_1)_\mu = \underline{\Omega}({}^L\mathrm{H})$ and $\mathbb{C}[\underline{\Omega}({}^L\mathrm{H}_1)_\mu] = \mathbb{C}[\underline{\Omega}({}^L\mathrm{H})]$. Note also that $\Phi_{\mathrm{temp,G-rel}}(\mathrm{H})_\mu \subset \Phi(\mathrm{H})$ is $\mathcal{O}_{\mathrm{H}}$-invariant.

(iv) Continue with the setting of (iii), and additionally assume that any $\beta \in \mathcal{O}_{\mathrm{H}}$ has a dual ${}^L\beta : {}^L\mathrm{H} \to {}^L\mathrm{H}$ which is induced by conjugation by an element of $\hat{\mathrm{G}}$. Then the map $\Omega(\underline{\mathrm{H}}) : \Omega({}^L\mathrm{H}) \to \Omega({}^L\mathrm{G})$ of varieties and the map $\Phi(\underline{\mathrm{H}}) : \Phi_{\mathrm{temp,G-rel}}(\mathrm{H}) \to \Phi_{\mathrm{temp}}(\mathrm{G})$ of sets defined in obvious analogy with Notation 6.1.11(iii) quotient respectively to a map

$$\underline{\Omega}(\underline{\mathrm{H}}) : \underline{\Omega}({}^L\mathrm{H}) = \Omega({}^L\mathrm{H})/\mathcal{O}_{\mathrm{H}} \to \Omega({}^L\mathrm{G})$$

of varieties, and a map $\Phi_{\mathrm{temp,G-rel}}(\mathrm{H})/\mathcal{O}_{\mathrm{H}} \to \Phi_{\mathrm{temp}}(\mathrm{G})$ of sets.

Proposition 6.1.26. *Suppose that we are in Scenario 1 of Notation 6.1.1(vii). Assume further that the LLC+ hypothesis (Hypothesis 2.10.3) is satisfied with* G *and the groups of Notation 2.6.1 replaced by* H_1 *and the groups of Scenario 1 of Notation 6.1.1(vii), such that the resulting map* $p_{1,\mathrm{H}_1} : \Omega(\mathrm{H}_1) \to \underline{\Omega}({}^L\mathrm{H}_1)$ *as in Definition 4.3.1 satisfies the following weak central character compatibility with respect to* μ*:*

$$p_{1,\mathrm{H}_1}(\Omega(\mathrm{H}_1)_\mu) \subset \underline{\Omega}({}^L\mathrm{H}_1)_\mu. \tag{85}$$

Let (π_1, V_1) *be a smooth representation of* $\mathrm{H}_1(F)$ *on which* $\mathrm{C}_1(F)$ *acts via* μ*. Consider the action of* $\mathbb{C}[\underline{\Omega}({}^L\mathrm{H}_1)]$ *on* V_1 *via* $p_{1,\mathrm{H}_1}^* : \mathbb{C}[\underline{\Omega}({}^L\mathrm{H}_1)] \to \mathbb{C}[\Omega(\mathrm{H}_1)] = \mathcal{Z}(\mathrm{H}_1)$*, i.e., sending* $(\tilde{z}_1, v_1)$ *to* $p_{1,\mathrm{H}_1}^*(\tilde{z}_1)\cdot v_1$*. Then this action quotients to an action of* $\mathbb{C}[\underline{\Omega}({}^L\mathrm{H}_1)_\mu]$ *on* V_1*.*

Proof. By Lemma 6.1.24, the action of $\mathbb{C}[\Omega(\mathrm{H}_1)] = \mathcal{Z}(\mathrm{H}_1)$ on V_1 factors through $\mathbb{C}[\Omega(\mathrm{H}_1)_\mu] = \mathcal{Z}(\mathrm{H}_1)_\mu$, so it suffices to show that the composite $\mathbb{C}[\underline{\Omega}({}^L\mathrm{H}_1)] \to \mathbb{C}[\Omega(\mathrm{H}_1)] \to \mathbb{C}[\Omega(\mathrm{H}_1)_\mu]$ factors through $\mathbb{C}[\underline{\Omega}({}^L\mathrm{H}_1)] \to \mathbb{C}[\underline{\Omega}({}^L\mathrm{H}_1)_\mu]$. This follows from the hypothesis that $\Omega(\mathrm{H}_1) \to \underline{\Omega}({}^L\mathrm{H}_1)$ takes $\Omega(\mathrm{H}_1)_\mu$ to $\underline{\Omega}({}^L\mathrm{H}_1)_\mu$, and the fact that $\mathbb{C}[\Omega(\mathrm{H}_1)_\mu]$ is reduced. □

6.2. The $\mathcal{Z}$-transfer conjecture, under several hypotheses.

Notation 6.2.1. (i) Suppose Hypothesis 2.10.3 is valid, so that $p_{1,\mathrm{G}} : \Omega(\mathrm{G}) \to \underline{\Omega}({}^L\mathrm{G})$ (see Definition 4.3.1) is defined. Although this map may not be injective (as G may not be quasi-split), given $z \in \mathbb{C}[\Omega({}^L\mathrm{G})/\mathcal{O}_{\mathrm{G}}] = \mathbb{C}[\underline{\Omega}({}^L\mathrm{G})]$ and $f \in C^\infty(\tilde{\mathrm{G}}(F))$, we denote by $z * f$

the element $p_{1,\mathrm{G}}^*(z) * f \in C^\infty(\tilde{\mathrm{G}}(F))$, i.e., the result of letting the element $p_{1,\mathrm{G}}^*(z) \in \mathcal{Z}(\mathrm{G})$ act on the element f of the smooth representation of $\mathrm{G}(F)$ given by the left-regular action of $\mathrm{G}(F)$ on $C^\infty(\tilde{\mathrm{G}}(F))$.

(ii) If we assume instead that Hypothesis 2.10.3 is satisfied with G and the groups of Notation 2.6.1 replaced by H_1 or H and groups of automorphisms as in Scenario 1 or Scenario 2 of Notation 6.1.1(vii), so that we have a map $p_{1,\mathrm{H}_1} : \Omega(\mathrm{H}_1) \to \underline{\Omega}({}^L\mathrm{H}_1)$ or $p_{1,\mathrm{H}} : \Omega(\mathrm{H}) \to \underline{\Omega}({}^L\mathrm{H})$, then we can similarly define $\tilde{z}_1 * f_1 = p_{1,\mathrm{H}_1}^*(\tilde{z}_1) * f_1$ or $z_\mathrm{H} * f_\mathrm{H} = p_{1,\mathrm{H}}^*(z_\mathrm{H}) * f_\mathrm{H}$, whenever $\tilde{z}_1 \in \mathbb{C}[\underline{\Omega}({}^L\mathrm{H}_1)]$ and $f_1 \in C^\infty(\tilde{\mathrm{H}}_1(F))$ or $z_\mathrm{H} \in \mathbb{C}[\underline{\Omega}({}^L\mathrm{H})]$ and $f_\mathrm{H} \in C^\infty(\mathrm{H}(F))$. In these situations, we will also write $\hat{\tilde{z}}_1(\pi_1) = \widehat{p_{1,\mathrm{H}_1}^*(\tilde{z}_1)}(\pi_1)$ or $\hat{z}_\mathrm{H}(\pi_\mathrm{H}) = \widehat{p_{1,\mathrm{H}}^*(z_\mathrm{H})}(\pi_\mathrm{H})$, depending on the case.

(iii) Now assume, like in (ii) above, that Hypothesis 2.10.3 is satisfied with G and the groups of Notation 2.6.1 replaced by H_1 and groups of automorphisms as in Scenario 1 of Notation 6.1.1(vii), and additionally that the weak central character compatibility with respect to μ as in (85) of Proposition 6.1.26 is satisfied. Then by Proposition 6.1.26, for $\tilde{z}_1 \in \mathbb{C}[\underline{\Omega}({}^L\mathrm{H}_1)]$, $f_1 \in C_\mu^\infty(\tilde{\mathrm{H}}_1(F))$ and any irreducible admissible representation π_1 of $\mathrm{H}_1(F)$ whose central character restricts to μ on $\mathrm{C}_1(F)$, we have that $\tilde{z}_1 * f_1 = p_{1,\mathrm{H}_1}^*(\tilde{z}_1) * f_1$ and $\hat{\tilde{z}}_1(\pi_1) := \widehat{p_{1,\mathrm{H}_1}^*(\tilde{z}_1)}(\pi_1)$ depend only on the image z_1 of $\tilde{z}_1$ in the quotient $\mathbb{C}[\underline{\Omega}({}^L\mathrm{H}_1)_\mu]$ of $\mathbb{C}[\underline{\Omega}({}^L\mathrm{H}_1)]$. This defines, for $z_1 \in \mathbb{C}[\underline{\Omega}({}^L\mathrm{H}_1)_\mu]$, for $f_1 \in C_\mu^\infty(\tilde{\mathrm{H}}_1(F))$, and for any irreducible admissible representation π_1 of $\mathrm{H}_1(F)$ whose central character restricts to μ on $\mathrm{C}_1(F)$, $z_1 * f_1$ and $\hat{z}_1(\pi_1)$: these equal $p_{1,\mathrm{H}_1}^*(\tilde{z}_1) * f_1$ and $\widehat{p_{1,\mathrm{H}_1}^*(\tilde{z}_1)}(\pi_1)$, respectively, where $\tilde{z}_1 \in \mathbb{C}[\underline{\Omega}({}^L\mathrm{H}_1)]$ is any lift of $z_1 \in \mathbb{C}[\underline{\Omega}({}^L\mathrm{H}_1)_\mu]$.

As we have recalled from [MW16, Corollary XI.5.1] before, pull-back under endoscopic transfer determines a map

$$SD_\mu(\tilde{\mathrm{H}}_1) \to D(\tilde{\mathrm{G}}, \omega) \tag{86}$$

(with $SD_\mu(\tilde{\mathrm{H}}_1)$ as in Notation 2.1.1(vi)). This map sends $\Theta_{\underline{\mathrm{H}}} \in SD_\mu(\tilde{\mathrm{H}}_1)$ to the unique $\Theta_{\tilde{\mathrm{G}}} \in D(\tilde{\mathrm{G}}, \omega)$ such that, for any $\tilde{f} \in C_c^\infty(\tilde{\mathrm{G}}(F))$ and $f_1 \in C_\mu^\infty(\tilde{\mathrm{H}}_1(F))$ with matching orbital integrals, $\Theta_{\tilde{\mathrm{G}}}(\tilde{f}) = \Theta_{\underline{\mathrm{H}}}(f_1)$.

The following proposition is well-known, but we learned it only relatively recently, from [LM20]:

Proposition 6.2.2. *Let $f \in C_c^\infty(\tilde{\mathrm{G}}(F))$ and $f_1 \in C_\mu^\infty(\tilde{\mathrm{H}}_1(F))$. The following are equivalent:*

(i) f and f_1 have matching orbital integrals.

(ii) For every $\Theta_{\underline{\mathrm{H}}} \in SD_\mu(\tilde{\mathrm{H}}_1)$ mapping to $\Theta_{\tilde{\mathrm{G}}} \in D(\tilde{\mathrm{G}}, \omega)$ under (86),

$$\Theta_{\underline{\mathrm{H}}}(f_1) = \Theta_{\tilde{\mathrm{G}}}(f).$$

Proof. This is exactly analogous to the equivalence of conditions (A) and (B) towards the end of [LM20, Section 2.6].

The implication (i) $\Rightarrow$ (ii) follows from definition, so it suffices to show the implication (ii) $\Rightarrow$ (i). Since we know the existence of transfer ([Wal08]), let f_1' be a transfer of f. It is enough to show that f_1' and f_1 have the same stable orbital integral at every strongly $\tilde{\mathrm{G}}$-regular element $h_1 \in \tilde{\mathrm{H}}_1(F)$.

In other words, we need to show that the function $f_1 - f_1' \in C_\mu^\infty(\tilde{\mathrm{H}}_1(F))$ is unstable. By Proposition 3.2.13 (or just [Art96, Lemma 6.3] in case $\tilde{\mathrm{H}}_1$ is isomorphic to H_1 as a twisted space for H_1), this is equivalent to requiring that $\Theta_{\underline{\mathrm{H}}}(f_1 - f_1') = 0$ for all $\Theta_{\underline{\mathrm{H}}} \in SD_\mu(\tilde{\mathrm{H}}_1)$. But this is clear from the fact that we have, with $\Theta_{\tilde{\mathrm{G}}}$ denoting the image of $\Theta_{\underline{\mathrm{H}}}$ under (86):

$$\Theta_{\underline{\mathrm{H}}}(f_1) = \Theta_{\tilde{\mathrm{G}}}(f) = \Theta_{\underline{\mathrm{H}}}(f_1').$$

□

Theorem 6.2.3. *Assume that we are in either Scenario 1 (resp., Scenario 2) of Notation 6.1.1(vii). We additionally make the following assumptions (I) and (II) below:*

(I) The groups $\mathcal{O} = \mathcal{O}_\mathrm{G}$, and $\mathcal{O}_{\mathrm{H}_1}$ or $\mathcal{O}_\mathrm{H}$ satisfy the following properties:

(a) *$\mathcal{O}$ is trivial;*

(b) *The action of $\mathcal{O}_{\mathrm{H}_1}$ on H_1 extends to an action of $\mathcal{O}_{\mathrm{H}_1}$ on $\tilde{\mathrm{H}}_1$ compatibly with its structure as an H_1-bitorsor, if we are in Scenario 1;*

(c) *In Scenario 1 (resp., Scenario 2), if $f_1 \in C_\mu^\infty(\tilde{\mathrm{H}}_1(F))$ (resp., $f_{\mathrm{H}} \in C_c^\infty(\mathrm{H}(F))$) is a transfer of $f \in C_c^\infty(\tilde{\mathrm{G}}(F))$, so is $f_1 \circ \beta$ (resp., $f_{\mathrm{H}} \circ \beta$) for any $\beta \in \mathcal{O}_{\mathrm{H}_1}$ (resp., $\beta \in \mathcal{O}_{\mathrm{H}}$); and*

(d) *The condition of (ii) or (iv) of Notation 6.1.25 holds, i.e., in Scenario 1 (resp., Scenario 2), any element $\beta \in \mathcal{O}_{\mathrm{H}_1}$ (resp., $\beta \in \mathcal{O}_{\mathrm{H}}$) has a dual ${}^L\beta : {}^L\mathrm{H}_1 \to {}^L\mathrm{H}_1$ (resp., ${}^L\beta : {}^L\mathrm{H} = \mathcal{H} \to \mathcal{H} = {}^L\mathrm{H}$) which restricts to an automorphism of $\mathcal{H}$ that is induced by conjugation by an element of $\hat{\mathrm{G}}$.*

(II) *$\tilde{\mathrm{G}}$ and $\tilde{\mathrm{H}}_1$ satisfy the following conditions, which, roughly speaking, amount to twisted versions of Hypotheses 2.7.1, 2.10.3 and 2.10.12 together with compatibility with the endoscopic datum $\underline{\mathrm{H}}$:*

(a) *The hypotheses on the existence of tempered L-packets and LLC+ (Hypotheses 2.7.1 and 2.10.3) are satisfied, and in addition, in Scenario 1 (resp., Scenario 2), their analogues with G and the groups of Notation 2.6.1 replaced by H_1 and the groups of Scenario 1 (resp., H and the groups of Scenario 2) are also satisfied, so that we have a map $p_{1,\mathrm{H}_1} : \Omega(\mathrm{H}_1) \to \underline{\Omega}({}^L\mathrm{H}_1)$ (resp., $p_{1,\mathrm{H}} : \Omega(\mathrm{H}) \to \underline{\Omega}({}^L\mathrm{H})$);*

(b) *If we are in Scenario 1, the weak central character compatibility with μ as in (85) is satisfied, i.e., p_{1,H_1} restricts to a map $p_{1,\mathrm{H}_1} : \Omega(\mathrm{H}_1)_\mu \to \underline{\Omega}({}^L\mathrm{H}_1)_\mu$;*

(c) *If we are in Scenario 1, there exists a basis $\{\Theta_{\phi_1} \mid \phi_1 \in \Phi_1\}$ for $SD_\mu(\tilde{\mathrm{H}}_1)^{\mathcal{O}_{\mathrm{H}_1}}$ indexed by some subset $\Phi_1 \subset \Phi_{\mathrm{temp}}(\mathrm{H}_1)_\mu/\mathcal{O}_{\mathrm{H}_1}$, with each Θ_{ϕ_1} ($\phi_1 \in \Phi_1$) a linear combination of characters of irreducible smooth representations of $\tilde{\mathrm{H}}_1(F)$ whose underlying $\mathrm{H}_1(F)$-representations belong to the tempered L-packet $\Sigma(\phi_1)$ of $\mathrm{H}_1(F)$ associated to ϕ_1 as in Notation 2.10.11(i) (which uses the assumption concerning Hypothesis 2.10.3 in (a)); if we are in Scenario 2, there exists a basis $\{\Theta_{\phi_{\mathrm{H}}} \mid \phi_{\mathrm{H}} \in \Phi_{\mathrm{temp}}(\mathrm{H})/\mathcal{O}_{\mathrm{H}}\}$ for $SD(\mathrm{H})^{\mathcal{O}_{\mathrm{H}}}$ indexed by $\Phi_{\mathrm{temp}}(\mathrm{H})/\mathcal{O}_{\mathrm{H}}$, with each $\Theta_{\phi_{\mathrm{H}}}$ a linear combination of characters of irreducible smooth representations of $\mathrm{H}(F)$ belonging to the tempered L-packet $\Sigma(\phi_{\mathrm{H}})$ of $\mathrm{H}(F)$ associated to ϕ_{H} as in Notation 2.10.11(i). We now fix such a collection of the Θ_{ϕ_1} or the $\Theta_{\phi_{\mathrm{H}}}$, according to the scenario; and*

(d) *For each $\phi_1 \in \Phi_1 \subset \Phi_{\mathrm{temp}}(\mathrm{H}_1)_\mu/\mathcal{O}_{\mathrm{H}_1}$ (resp., $\phi_{\mathrm{H}} \in \Phi_{\mathrm{temp}}(\mathrm{H})/\mathcal{O}_{\mathrm{H}}$), the image of Θ_{ϕ_1} (resp., $\Theta_{\phi_{\mathrm{H}}}$) under $SD_\mu(\tilde{\mathrm{H}}_1)^{\mathcal{O}_{\mathrm{H}_1}} \to D(\tilde{\mathrm{G}}, \omega)$ (which identifies with a map $SD(\mathrm{H})^{\mathcal{O}_{\mathrm{H}}} \to D(\tilde{\mathrm{G}}, \omega)$ in Scenario 2) is zero if $\phi_1 \notin \Phi_{\mathrm{temp,G-rel}}(\mathrm{H}_1)_\mu/\mathcal{O}_{\mathrm{H}_1}$ (resp., if $\phi_{\mathrm{H}} \notin \Phi_{\mathrm{temp,G-rel}}(\mathrm{H})/\mathcal{O}_{\mathrm{H}}$), and is a linear combination of characters of irreducible smooth representations of $(\tilde{\mathrm{G}}(F), \omega)$ whose underlying $\mathrm{G}(F)$-representations belong to the tempered L-packet $\Sigma(\phi)$ as in Notation 2.10.11(i), where ϕ is the image of ϕ_1 (resp., ϕ_{H}) under the map $\Phi_{\mathrm{temp,G-rel}}(\mathrm{H}_1)_\mu/\mathcal{O}_{\mathrm{H}_1} \to \Phi_{\mathrm{temp}}(\mathrm{G})$ of Notation 6.1.25(ii) (resp., the map $\Phi_{\mathrm{temp,G-rel}}(\mathrm{H})/\mathcal{O}_{\mathrm{H}} \to \Phi_{\mathrm{temp}}(\mathrm{G})$ of Notation 6.1.25(iv)), otherwise.*

If $z \in \mathbb{C}[\Omega({}^L\mathrm{G})]$, $z_1 := \underline{\Omega}(\underline{\mathrm{H}})^*(z) \in \mathbb{C}[\underline{\Omega}({}^L\mathrm{H}_1)_\mu]$ (resp., $z_{\mathrm{H}} := \underline{\Omega}(\underline{\mathrm{H}})^*(z) \in \mathbb{C}[\underline{\Omega}({}^L\mathrm{H})]$) is the pull-back of z under the map $\underline{\Omega}(\underline{\mathrm{H}})$ of (ii) or (iv) of Notation 6.1.25, and if $f \in C_c^\infty(\tilde{\mathrm{G}}(F))$ and $f_1 \in C_\mu^\infty(\tilde{\mathrm{H}}_1(F))^{\mathcal{O}_{\mathrm{H}_1}}$ (resp., $f_{\mathrm{H}} \in C_c^\infty(\mathrm{H}(F))^{\mathcal{O}_{\mathrm{H}}}$) have matching orbital integrals, then so do $z * f$ and $z_1 * f_1$ (resp., $z_{\mathrm{H}} * f_{\mathrm{H}}$). Here $z_1 * f_1$ and $z_{\mathrm{H}} * f_{\mathrm{H}}$ are defined as in Notation 6.2.1.

Proof. We will give the proof only for Scenario 1; the proof for Scenario 2 will be an obvious variant.

Let z, z_1, f, f_1 be as given. Since $f_1 \in C_\mu^\infty(\tilde{\mathrm{H}}_1(F))$ is fixed by $\mathcal{O}_{\mathrm{H}_1}$, it suffices by Proposition 6.2.2 to prove that for any $\Theta_{\underline{\mathrm{H}}}$ belonging to some fixed basis of $SD_\mu(\tilde{\mathrm{H}}_1)^{\mathcal{O}_{\mathrm{H}_1}}$, denoting by $\Theta_{\tilde{\mathrm{G}}}$ its transfer to $D(\tilde{\mathrm{G}}, \omega)$ under (86), we have $\Theta_{\underline{\mathrm{H}}}(z_1 * f_1) = \Theta_{\tilde{\mathrm{G}}}(z * f)$. By assumption (II)(c), it is enough to show this with $\Theta_{\underline{\mathrm{H}}} = \Theta_{\phi_1}$, for each $\phi_1 \in \Phi_1$.

We can write $\Theta_{\underline{\mathrm{H}}} = \Theta_{\phi_1}$ as $\sum_i c_i \Theta_{\tilde{\pi}_{1,i}}$, where $c_i \in \mathbb{C}$ for each i, and each $\tilde{\pi}_{1,i}$ is an irreducible smooth representation of $\tilde{\mathrm{H}}_1(F)$ whose underlying $\mathrm{H}_1(F)$-representation $\pi_{1,i}$ belongs to $\Sigma(\phi_1)$.

Moreover, for each i, it is easy to see that:

$$\Theta_{\tilde{\pi}_{1,i}}(z_1 * f_1) = \operatorname{tr} \tilde{\pi}_{1,i}(z_1 * f_1) = \operatorname{tr}(\pi_{1,i}(z_1)\tilde{\pi}_{1,i}(f_1)) = \hat{z}_1(\pi_{1,i}) \cdot \Theta_{\tilde{\pi}_{1,i}}(f_1) \tag{87}$$

— here $\pi_{1,i}(z_1)$ stands for the endomorphism of the space of $\pi_{1,i}$ defined by z_1, and $\hat{z}_1(\pi_{1,i})$ is as in Notation 6.2.1(iii); to see the middle equality, use that, if $f_2 = (\operatorname{meas} K)^{-1}\mathbb{1}_K$ for a small enough compact open subgroup $K \subset \mathrm{H}_1(F)$ and if $\tilde{z}_1$ is defined as in Notation 6.2.1(iii), then:

$$\tilde{\pi}_{1,i}(z_1 * f_2 * f_1) = \tilde{\pi}_{1,i}(\tilde{z}_1 * f_2 * f_1) = \pi_{1,i}(\tilde{z}_1 * f_2)\tilde{\pi}_{1,i}(f_1) = \widehat{p^*_{1,\mathrm{H}_1}(\tilde{z}_1)}(\pi_{1,i})\pi_{1,i}(f_2)\tilde{\pi}_{1,i}(f_1) = \hat{z}(\pi_{1,i})\tilde{\pi}_{1,i}(f_1).$$

We conclude from (87) and the definition of p_{1,H_1} ((53) in Definition 4.3.1):

$$\Theta_{\phi_1}(z_1 * f_1) = z_1(\lambda(\phi_1)) \cdot \Theta_{\phi_1}(f_1). \tag{88}$$

If ϕ_1 is not G-relevant, then $\Theta_{\tilde{\mathrm{G}}} = 0$ by the hypothesis in (d) of the condition (II), so that $\Theta_{\tilde{\mathrm{G}}}(z * f) = 0$, while we also have the equality $\Theta_{\phi_1}(f_1) = \Theta_{\tilde{\mathrm{G}}}(f) = 0$, which gives, using (88), that $\Theta_{\phi_1}(z_1 * f_1) = 0 = \Theta_{\tilde{\mathrm{G}}}(z * f)$, as desired.
Thus, assume now that ϕ_1 is G-relevant, and let ϕ be the image of ϕ_1 under $\Phi_{\mathrm{temp,G-rel}}(\mathrm{H}_1)_\mu \to \Phi_{\mathrm{temp}}(\mathrm{G})$, as in (d) of the condition (II). In this case, $\Theta_{\tilde{\mathrm{G}}}$ can be written as a linear combination Θ_ϕ of characters of $(\tilde{\mathrm{G}}(F), \omega)$ whose underlying $\mathrm{G}(F)$-representations belong to the tempered L-packet $\Sigma(\phi)$ associated to ϕ. As in (88), but with fewer complications, we have:

$$\Theta_\phi(z * f) = z(\lambda(\phi)) \cdot \Theta_\phi(f). \tag{89}$$

Given (88) and (89), the desired equality follows from the fact that $\Theta_\phi(f) = \Theta_{\phi_1}(f_1)$ (which holds since Θ_ϕ is the image of Θ_{ϕ_1} under (86)) and the equality $z_1(\lambda(\phi_1)) = z(\lambda(\phi))$ (which holds because $z_1 := \Omega(\underline{\mathrm{H}})^*(z)$). □

Remark 6.2.4. (i) For certain classes of endoscopic data, versions of the statement of Theorem 6.2.3 as applied to important classes of elements of $\mathcal{Z}_2(\mathrm{G})$ (whose relative sizes inside $\mathcal{Z}_2(\mathrm{G})$ we are ignorant of), have been stated among the conjectures of Haines in [Hai14] (see [Hai14, Conjectures 6.2.2 and 6.2.3]) and those of Scholze and Shin in [SS13] (see [SS13, Conjecture 7.2.2]). Needless to say, these sources do not make the strong assumptions that we do in (II) of the statement of Theorem 6.2.3.

(ii) Consider the case in which $(\tilde{\mathrm{G}}, \omega) = (\mathrm{G}, \mathbb{1})$, with G quasi-split and $\mathrm{H} = \mathrm{G}$. Suppose that all the hypotheses of the theorem for this case hold, with $\mathcal{O}$ and $\mathcal{O}_{\mathrm{H}_1}$ additionally assumed to be trivial. Subject to these (very strong) assumptions, using that μ is trivial so that $C^\infty_\mu(\tilde{\mathrm{H}}_1(F))$ identifies with $C^\infty_c(\mathrm{H}(F))$, the theorem specializes to the conjecture that, if $f \in C^\infty_c(\mathrm{G}(F))$ is unstable, then so is $z * f$ (defined as $p^*_{1,\mathrm{G}}(z) * f$) for all $z \in \mathbb{C}[\Omega({}^L\mathrm{G})]$. However, this can also be seen more easily under closely related assumptions: see Corollary 5.5.2(i). This is an expected property of the stable Bernstein center conjectured by Haines and (independently) by Scholze and Shin; see, e.g., [SS13, Conjecture 6.3].

(iii) When $\mathrm{H}_1 = \mathrm{H}$ equals GL_n/F and $(\tilde{\mathrm{G}}, \omega)$ equals $(\mathrm{G}, \mathbb{1})$ with G an inner form of GL_n/F, the hypotheses of Theorem 6.2.3 follow from the properties of the local Langlands correspondence (which is available for H and G in these cases), and we recover some of the results of Jonathan Cohen from [Coh18]. However, the results of [Coh18] are phrased slightly differently: while our formulation maps $\Omega({}^L\mathrm{H})$ to $\Omega({}^L\mathrm{G})$ and $\mathcal{Z}_2(\mathrm{G})$ to $\mathcal{Z}_2(\mathrm{H})$ in this case, the version in [Coh18] maps $\Omega(\mathrm{G})$ to $\Omega(\mathrm{H})$ and $\mathcal{Z}_2(\mathrm{H}) = \mathcal{Z}(\mathrm{H})$ to $\mathcal{Z}_2(\mathrm{G}) = \mathcal{Z}(\mathrm{G})$. See also Proposition 6.3.1 below.

Remark 6.2.5. Of course, Theorem 6.2.3 assumes a lot of deep results, but hopefully, at least in many simple situations, it could suggest results that can be proved using other methods that avoid these assumptions. For instance, while we have not worked out the details, we feel that Theorem 6.2.3 should give one way to motivate the specific form of a result of Lemaire and Mishra ([LM20]) (and in particular the 'Weyl averaging' featured in that result), generalizing the explanation given by Haines, following the statement of [Hai14, Conjecture 6.2.2], as to how that conjecture formally contains as a special case the 'fundamental lemma implies transfer for the spherical Hecke algebra' result.

6.3. **$\mathcal{Z}$-transfer for inner twists.** The main result of this subsection is the following proposition, which is a variant of Theorem 6.2.3 in the much simpler case of transfer between G and its quasi-split inner form, with less ponderous assumptions. It can also be considered as an adaptation of some of the results of [Coh18] to groups other than inner forms of general linear groups, with hypotheses thrown in to make up for our not knowing the existence of tempered L-packets or Jacquet-Langlands/Deligne-Kazhdan-Vigneras type correspondences for more general groups.

Proposition 6.3.1. *Let G^* be a quasi-split inner form of G, underlying an endoscopic datum $\underline{\mathrm{G}}^*$ for G defined using an inner twist ψ_{G^*} as in Notation 3.2.1(i). Give $\mathrm{G}(F)$ and $\mathrm{G}^*(F)$ Haar measures compatible with respect to ψ_{G^*}. For simplicity, we assume that each group $\mathcal{O}_{\mathrm{M}}$ from the collection $\{\mathcal{O}_{\mathrm{M}}\}_{\mathrm{M}}$ of Notation 2.6.1 is trivial, and consider an analogous collection $\{\mathcal{O}_{\mathrm{M}^*}\}_{\mathrm{M}^*}$ for G^*, with each $\mathcal{O}_{\mathrm{M}^*}$ trivial. Assume:*

- *(a) Both G and G^* satisfy the hypothesis on the existence of tempered L-packets (Hypothesis 2.7.1); and*
- *(b) If Levi subgroups $\mathrm{M} \subset \mathrm{G}$ and $\mathrm{M}^* \subset \mathrm{G}^*$ are related as in Notation 3.2.1(vi) (but with $\mathrm{G}, \mathrm{G}^*, \mathrm{M}, \mathrm{M}^*$ in place of $\mathrm{M}, \mathrm{M}^*, \mathrm{M}_1, \mathrm{M}_1^*$), with M^* underlying an endoscopic datum $\underline{\mathrm{M}}^*$ for M defined using 'Levi subgroup matching data' as in Notation 3.2.1(vi), there is a bijection $\Phi_2(\mathrm{M}^*) \to \Phi_2(\mathrm{M})$, under which $\Sigma^* \in \Phi_2(\mathrm{M}^*)$ and $\Sigma \in \Phi_2(\mathrm{M})$ correspond if and only if the endoscopic transfer map $SD_{\mathrm{ell}}(\mathrm{M}^*) \to SD_{\mathrm{ell}}(\mathrm{M})$ takes some nonzero stable distribution Θ_{Σ^*} supported on Σ^* to some stable distribution Θ_{Σ} supported on Σ; in this case, we say that Σ is the transfer of Σ^* to $\mathrm{M}(F)$ (depending on the choices defining $\underline{\mathrm{M}}^*$).*

As before, we write $e(\mathrm{G})$ for the Kottwitz sign of G. Then the map

$$\mathcal{Z}_1(\mathrm{G}^*) \subset \mathcal{SI}(\mathrm{G}^*)^* \to \mathcal{SI}(\mathrm{G})^*,$$

taking $z^* \in \mathcal{SI}(\mathrm{G}^*)^*$ to $z := e(\mathrm{G})z' \in \mathcal{SI}(\mathrm{G})^*$, where $z' \in \mathcal{SI}(\mathrm{G})^*$ is the endoscopic transfer of $z^* \in \mathcal{SI}(\mathrm{G}^*)^*$, is valued in $\mathcal{Z}_1(\mathrm{G})$ and defines a homomorphism of $\mathbb{C}$-algebras $\mathcal{Z}_1(\mathrm{G}^*) \to \mathcal{Z}_1(\mathrm{G})$, and satisfies the $\mathcal{Z}$-transfer property, i.e., whenever $f^* \in C_c^\infty(\mathrm{G}^*(F))$ and $f \in C_c^\infty(\mathrm{G}(F))$ have matching orbital integrals, so do $z^* * f^*$ and $z * f$.

Let us make some preparations for the proof of Proposition 6.3.1.

Notation 6.3.2. For any reductive group M over F, let $D_{\mathrm{ell}}^+(\mathrm{M})$ denote the span of the $\Theta\chi$, where Θ ranges over $D_{\mathrm{ell}}(\mathrm{M})$ and χ over (unramified, or equivalently all) smooth characters $\mathrm{M}(F) \to \mathbb{C}^\times$. Let $SD_{\mathrm{ell}}^+(\mathrm{M}) \subset D_{\mathrm{ell}}^+(\mathrm{M})$ denote the subspace of stable distributions. Let $D_{\mathrm{Irr}}(\mathrm{M})$ denote the space of virtual characters spanned by the Θ_π as π ranges over all of $\mathrm{Irr}(\mathrm{M})$, and $SD_{\mathrm{Irr}}(\mathrm{M}) \subset D_{\mathrm{Irr}}(\mathrm{M})$ the subspace of stable virtual characters.

Lemma 6.3.3. *Let M be a connected reductive group over F.*

- *(i) For each $\pi \in \mathrm{Irr}\,\mathrm{M}$, there exists a unique element $\Theta_{\pi,\mathrm{ell}} \in D_{\mathrm{ell}}^+(\mathrm{M})$ such that $\Theta_\pi - \Theta_{\pi,\mathrm{ell}}$ is a finite sum of virtual characters obtained by parabolic induction from proper Levi subgroups of M.*
- *(ii) Restriction to the subset $\mathrm{M}(F)_{\mathrm{ell}}$ of strongly regular elliptic semisimple elements in $\mathrm{M}(F)$ induces a well-defined $\mathbb{C}$-linear map $D_{\mathrm{Irr}}(\mathrm{M}) \to D_{\mathrm{ell}}^+(\mathrm{M})$, which is a map of $\mathcal{Z}(\mathrm{M})$-modules, where $\mathcal{Z}(\mathrm{M})$ acts by $z \cdot \Theta = (f \mapsto \Theta(z * f))$. This map takes $SD_{\mathrm{Irr}}^+(\mathrm{M})$ to $SD_{\mathrm{ell}}^+(\mathrm{M})$.*
- *(iii) Let $\pi \in \mathrm{Irr}\,\mathrm{M}$. Write the $\Theta_{\pi,\mathrm{ell}}$ of (i) as a finite linear combination $\sum_i c_i \Theta_{\pi_i}$, where each c_i is nonzero and the π_i are pairwise distinct elements of $\mathrm{Irr}(\mathrm{M})$. Then each π_i has the same cuspidal support as π.*

Proof. To prove the existence of the $\Theta_{\pi,\mathrm{ell}}$ as in (i), it suffices by (34) to show the existence of a virtual character Θ' that is a finite linear combination of virtual characters of the form $\Theta\chi$, with $\Theta \in D(\mathrm{M})$ and $\chi : \mathrm{M}(F) \to \mathbb{C}^\times$ a smooth character, such that $\Theta_\pi - \Theta'$ is a linear combination of virtual characters obtained by parabolic induction from proper Levi subgroups. But this in turn follows from the fact that characters of standard representations form a basis for $D_{\mathrm{Irr}}(\mathrm{M})$ (see [Art89, Proposition 5.1]). Alternatively, see the comment on Langlands theory in [BDK86, Section 5.2]. This proves the existence of the $\Theta_{\pi,\mathrm{ell}}$ as in (i).

Any element of $D_{\mathrm{ell}}(\mathrm{M})$ is determined by its restriction to $\mathrm{M}(F)_{\mathrm{ell}}$ (as reviewed in Notation 3.4.1(i)). The same then applies to $D^+_{\mathrm{ell}}(\mathrm{M})$, since $\mathrm{M}(F)_{\mathrm{ell}} \subset \mathrm{M}(F)$ is closed under multiplication by $\mathrm{Z}_{\mathrm{M}}(F)$, letting us separate out the contributions from different central characters. From this and the fact that characters of induced representations vanish on $\mathrm{M}(F)_{\mathrm{ell}}$, the uniqueness assertion in (i) follows.

The existence of the map $D_{\mathrm{Irr}}(\mathrm{M}) \to D^+_{\mathrm{ell}}(\mathrm{M})$ as in (ii) is immediate from (i). Its linearity for the given action of $\mathcal{Z}(\mathrm{M})$ follows from the equality $\Theta_\pi(z * f) = \hat{z}(\pi)\Theta_\pi(f)$, and the uniqueness assertion in (i). That the map carries $SD_{\mathrm{Irr}}(\mathrm{M})$ to $SD^+_{\mathrm{ell}}(\mathrm{M})$ is an immediate consequence of the fact that, by [Art96], an element of $D_{\mathrm{ell}}(\mathrm{M})$ whose restriction to $\mathrm{M}(F)_{\mathrm{ell}}$ is stable belongs to $SD_{\mathrm{ell}}(\mathrm{M})$, and hence a similar assertion holds with $D_{\mathrm{ell}}(\mathrm{M})$ replaced by $D^+_{\mathrm{ell}}(\mathrm{M})$ as well.

(iii) follows from (ii) and the fact that $\mathcal{Z}(\mathrm{M})$ separates the points on $\Omega(\mathrm{M})$. □

Lemma 6.3.4. *Let* M^* *be a quasi-split inner form of a connected reductive group* M *over* F*, underlying an endoscopic datum* $\underline{\mathrm{M}}^*$ *for* M *defined using an inner twist as in Notation 3.2.1(i). Assume that* $\mathrm{Q}^* \subset \mathrm{M}^*$ *and* $\mathrm{Q} \subset \mathrm{M}$ *are parabolic subgroups with* $\mathrm{L}^* \subset \mathrm{M}^*$ *and* $\mathrm{L} \subset \mathrm{M}$ *as Levi subgroups, and that our fixed inner twist takes* $(\mathrm{Q}^*, \mathrm{L}^*)$ *to* (Q, L) *(over* $\bar{F}$*), realizing* L^* *as endoscopic to* L*, as in Notation 3.2.1(vi). Write* $r^{\mathrm{M}^*}_{\mathrm{Q}^*} : D_{\mathrm{Irr}}(\mathrm{M}^*) \to D_{\mathrm{Irr}}(\mathrm{L}^*)$ *and* $r^{\mathrm{M}}_{\mathrm{Q}} : D_{\mathrm{Irr}}(\mathrm{M}) \to D_{\mathrm{Irr}}(\mathrm{L})$ *for the associated Jacquet module maps at the level of virtual characters. Then, whenever* $\Theta^* \in SD_{\mathrm{Irr}}(\mathrm{M}^*)$ *transfers endoscopically to* $\Theta \in D_{\mathrm{Irr}}(\mathrm{M})$*,* $r^{\mathrm{M}^*}_{\mathrm{Q}^*}(\Theta^*)$ *belongs to* $SD_{\mathrm{Irr}}(\mathrm{L}^*)$ *and transfers endoscopically to* $r^{\mathrm{M}}_{\mathrm{Q}}(\Theta) \in D_{\mathrm{Irr}}(\mathrm{L})$*.*

Proof. This is well-known (e.g., used in [MR18, Section 8.3.3]), and is a much easier variant of an analogous assertion proved in [Xu17, Appendix C]. □

Proof of Proposition 6.3.1. In this proof, we will write $\mathbf{T}_{\underline{\mathrm{G}}^*}$ for the endoscopic transfer map from stable distributions on $\mathrm{G}^*(F)$ to stable distributions on $\mathrm{G}(F)$. We will also write $f^* \leftrightarrow f$, where $f^* \in C^\infty_c(\mathrm{G}^*(F))$ and $f \in C^\infty_c(\mathrm{G}(F))$, to mean that f^* and f have matching orbital integrals. Given a Levi subgroup $\mathrm{M}^* \subset \mathrm{G}^*$, if M^* is relevant for G^*, then we will write $\mathrm{M}^* \sim \mathrm{M}$ to mean that M and M^* are related as in Notation 3.2.1(vi), and implicitly choose relevant "Levi subgroup matching data" data as in Notation 3.2.1(vi) (such as parabolic subgroups, an inner twist and an endoscopic datum $\underline{\mathrm{M}}^*$ for M with M^* as the underlying group).

Given any Levi subgroup $\mathrm{M}^* \subset \mathrm{G}^*$ and any $\Sigma^* \in \Phi^+_2(\mathrm{M}^*)$, we will write Θ_{Σ^*} for the nonzero stable character $\sum d(\sigma^*)\Theta^*_\sigma$, the sum running over $\sigma^* \in \Sigma^*$ (see Proposition 3.3.6). Given any Levi subgroup $\mathrm{M} \subset \mathrm{G}$ and any $\Sigma \in \Phi^+_2(\mathrm{M})$, Θ_Σ will denote the stable character $e(\mathrm{M})\sum d(\sigma)\Theta_\sigma = e(\mathrm{G})\sum d(\sigma)\Theta_\sigma$, the sum running over $\sigma \in \Sigma$.

If $\mathrm{M}^* \sim \mathrm{M}$, then by Lemma 3.2.3(i), the bijection $\Phi_2(\mathrm{M}^*) \to \Phi_2(\mathrm{M})$ from the condition (b) of the proposition extends to a bijection $\Phi^+_2(\mathrm{M}^*) \to \Phi^+_2(\mathrm{M})$ having an analogous description, which is equivariant for the action of $X^{\mathrm{unr}}(\mathrm{M}^*) = X^{\mathrm{unr}}(\mathrm{M})$. Given Levi subgroups $\mathrm{M}^* \subset \mathrm{G}^*$ and $\mathrm{M} \subset \mathrm{G}$, we will write $(\mathrm{M}^*, \Sigma^*) \sim (\mathrm{M}, \Sigma)$ to mean that $\mathrm{M}^* \sim \mathrm{M}$ and that Σ^* and Σ correspond under this extended bijection $\Phi^+_2(\mathrm{M}^*) \to \Phi^+_2(\mathrm{M})$. Note that if $(\mathrm{M}^*, \Sigma^*) \sim (\mathrm{M}, \Sigma)$, then Θ_Σ is an endoscopic transfer of Θ_{Σ^*}, by Proposition 3.3.7 (and Lemma 3.2.3(i)).

Claim 1. The $\mathbb{C}$-algebra homomorphisms $\mathcal{Z}_1(\mathrm{G}^*) \to \mathrm{End}_{\mathbb{C}}(SD(\mathrm{G}^*))$ and $\mathcal{Z}_1(\mathrm{G}) \to \mathrm{End}_{\mathbb{C}}(SD(\mathrm{G}))$, given by $z^* \mapsto a_{z^*} := (\Theta^* \mapsto (f^* \mapsto \Theta^*(z^* * f^*)))$ and $z \mapsto a_z := (\Theta \mapsto (f \mapsto \Theta(z * f)))$, are injective.

We will prove the assertion for the map $z \mapsto a_z$, the proof for $z^* \mapsto a_{z^*}$ being similar. This map is well-defined, since $\mathcal{Z}_1(\mathrm{G})$ equals $\mathcal{Z}_2(\mathrm{G})$ by Theorem 5.4.2 (and the fact that we are assuming Hypothesis 2.7.1), which preserves $SD(\mathrm{G})$ by the implication (iv) ⇒ (i) of Proposition 5.1.8. Let $z \in \mathcal{Z}_1(\mathrm{G})$ be such that $a_z = 0$. The claim will follow if we show that given any cuspidal pair (M, σ) for G, we have $\hat{z}((\mathrm{M}, \sigma)_{\mathrm{G}}) = 0$, where we recall that $\hat{z}((\mathrm{M}, \sigma)_{\mathrm{G}})$ is the value of $\hat{z}$ on any irreducible subquotient of $\mathrm{Ind}^{\mathrm{G}}_{\mathrm{M}}\, \sigma$. By analytic continuation, we may assume that σ is unitary. Let $\Sigma \in \Phi_2(\mathrm{M})$ be the packet containing σ, and Θ_Σ the associated stable character as chosen above. Then $\Theta := \mathrm{Ind}^{\mathrm{G}}_{\mathrm{M}}\, \Theta_\Sigma$ belongs to $SD(\mathrm{M})$ (see (35)), and is nonzero since $d(\sigma') > 0$ for all $\sigma' \in \Sigma$. We have $a_z(\Theta) = \hat{z}((\mathrm{M}, \sigma')_{\mathrm{G}})\Theta = \hat{z}((\mathrm{M}, \sigma)_{\mathrm{G}})\Theta$ for all $\sigma' \in \Sigma$ by Theorem 5.4.2. Therefore, the hypothesis that $a_z(\Theta) = 0$ forces $\hat{z}((\mathrm{M}, \sigma)_{\mathrm{G}}) = 0$, finishing the proof of Claim 1.

As a step preliminary to mapping $\mathcal{Z}_1(\mathrm{G}^*)$ to $\mathcal{Z}_1(\mathrm{G})$, we claim:

Claim 2. Any element in the image of the map $\mathcal{Z}_1(\mathrm{G}^*) \ni z^* \to a_{z^*} \in \mathrm{End}_{\mathbb{C}}(SD(\mathrm{G}^*))$, via the surjection $SD(\mathrm{G}^*) \to SD(\mathrm{G})$ (see Corollary 3.2.9 for this surjectivity), induces a well-defined element of $\mathrm{End}_{\mathbb{C}}(SD(\mathrm{G}))$. Thus, we get a $\mathbb{C}$-algebra homomorphism $\mathcal{Z}_1(\mathrm{G}^*) \to \mathrm{End}_{\mathbb{C}}(SD(\mathrm{G}))$.

Since each a_{z^*} preserves each component $\mathrm{Ind}_{\mathrm{M}^*}^{\mathrm{G}^*} SD_{\mathrm{ell}}(\mathrm{M}^*)^{W(\mathrm{M}^*)}$ of $SD(\mathrm{G}^*)$ as per the analogue of (35) for G^*, Claim 2 is an easy consequence of the compatibility of endoscopic transfer and parabolic induction (Remark 3.2.2(iii)), together with the fact that whenever a Levi subgroup $\mathrm{M}^* \subset \mathrm{G}^*$ is relevant for $\underline{\mathrm{M}}^*$, with $\mathrm{M}^* \sim \mathrm{M}$ for some Levi subgroup $\mathrm{M} \subset \mathrm{G}$, $SD_{\mathrm{ell}}(\mathrm{M}^*) \to SD_{\mathrm{ell}}(\mathrm{M})$ is an isomorphism, which is $W(\mathrm{M}^*) = W(\mathrm{M})$-equivariant by Lemma 3.2.3(ii).

Claim 3. The image of the homomorphism $\mathcal{Z}_1(\mathrm{G}^*) \ni z^* \mapsto a_{z^*} \in \mathrm{End}_{\mathbb{C}}(SD(\mathrm{G}))$ from Claim 2 is contained in the image of the injection $\mathcal{Z}_1(\mathrm{G}) \ni z \hookrightarrow a_z \in \mathrm{End}_{\mathbb{C}}(SD(\mathrm{G}))$, thus inducing a well-defined $\mathbb{C}$-algebra homomorphism $\mathcal{Z}_1(\mathrm{G}^*) \to \mathcal{Z}_1(\mathrm{G})$ under which $z^* \mapsto z$ if and only if $a_{z^*} \in \mathrm{End}(SD(\mathrm{G}^*))$ induces $a_z \in \mathrm{End}(SD(\mathrm{G}))$.

Before proving Claim 3, let us assume it and prove finish the proof of the proposition. Let $z^* \in \mathcal{Z}_1(\mathrm{G}^*)$ map to $z \in \mathcal{Z}_1(\mathrm{G})$ under the above well-defined $\mathbb{C}$-algebra homomorphism. Thus, whenever $\Theta \in SD(\mathrm{G})$ is an endoscopic transfer of $\Theta^* \in SD(\mathrm{G}^*)$, $a_z(\Theta) : f \mapsto \Theta(z * f)$ is an endoscopic transfer of $a_{z^*}(\Theta^*) : f^* \mapsto \Theta^*(z^* * f^*)$. It therefore follows from Proposition 6.2.2 that whenever $f \in C_c^\infty(\mathrm{G}(F))$ and $f^* \in C_c^\infty(\mathrm{G}^*(F))$ have matching orbital integrals, so do $z * f$ and $z^* * f^*$, as required by the proposition. Thus, the proposition follows (modulo Claim 3) if we show that $z = e(\mathrm{G})z'$, where z' is the endoscopic transfer of z^* viewed as a distribution on $\mathrm{G}^*(F)$, i.e., that $z^*(f^*) = e(\mathrm{G})z(f)$ whenever $f^* \in C_c^\infty(\mathrm{G}^*(F))$ and $f \in C_c^\infty(\mathrm{G}(F))$ have matching orbital integrals. Write $f^{*\vee}$ and $f^\vee$ for $g^* \mapsto f^*(g^{*-1})$ and $g \mapsto f(g^{-1})$, respectively. It is easy to see from Remark 3.2.2(i) that $f^\vee$ and $f^{*\vee}$ have matching orbital integrals. Therefore, so do $z * f^\vee$ and $z^* * f^{*\vee}$ (modulo Claim 3), giving $e(\mathrm{G})z(f) = e(\mathrm{G}) \cdot (z * f^\vee)(1) = z^* * f^{*\vee}(1) = z^*(f^*)$ by [Kot88, Proposition 2] (we are not sure if it is more appropriate to write $e(\mathrm{G})^{-1}$ instead); in this step, we used the compatibility of measures between $\mathrm{G}(F)$ and $\mathrm{G}^*(F)$.

It thus remains to prove Claim 3. For the rest of this proof, a pair (M, σ) consisting of a Levi subgroup $\mathrm{M} \subset \mathrm{G}$ and $\sigma \in \mathrm{Irr}_2^+(\mathrm{M})$ be called an essentially discrete pair for G, its $\mathrm{G}(F)$-conjugacy class will be denoted by $(\mathrm{M}, \sigma)_{\mathrm{G}}$, and every cuspidal pair for G will be viewed as an essentially discrete pair. Similar notation will apply with G^* in place of G.

Fixing $z^* \in \mathcal{Z}_1(\mathrm{G}^*)$, let us show that its image in $\mathrm{End}_{\mathbb{C}}(SD(\mathrm{G}))$ is of the form a_z for some $z \in \mathcal{Z}_1(\mathrm{G})$. Given an essentially discrete pair (M, σ), we now construct a complex number $\hat{z}_0((\mathrm{M}, \sigma)) = \hat{z}_0((\mathrm{M}, \sigma)_{\mathrm{G}})$ as follows (the symbol $\hat{z}_0$ will not have any meaning apart from this). Let $\Sigma \in \Phi_2^+(\mathrm{M})$ be the packet containing σ, and let $(\mathrm{M}^*, \Sigma^*) \sim (\mathrm{M}, \Sigma)$ with M^* a Levi subgroup of G^* and $\Sigma^* \in \Phi_2^+(\mathrm{M}^*)$. Choose any $\sigma^* \in \Sigma^*$, and set $\hat{z}_0((\mathrm{M}, \sigma)) = \hat{z^*}((\mathrm{M}^*, \sigma^*)_{\mathrm{G}^*})$ (where we recall that $\hat{z^*}((\mathrm{M}^*, \sigma^*)_{\mathrm{G}^*})$ refers to the value of $\hat{z^*}$ on any irreducible subquotient of $\mathrm{Ind}_{\mathrm{M}^*(F)}^{\mathrm{G}^*(F)} \sigma^*$).

It is easy to see using using Theorem 5.4.2 and analytic continuation that $\hat{z}_0((\mathrm{M}, \sigma))$ is independent of the choice of σ^* within Σ^*, and then using Lemma 2.4.8 and an argument as in Lemma 3.2.3(ii) that $\hat{z}_0((\mathrm{M}, \sigma))$ does not depend on the choice of the Levi subgroup $\mathrm{M}^* \subset \mathrm{G}^*$ or the associated data as in Notation 3.2.1(vi) either. It is immediate that $\hat{z}_0((\mathrm{M}, \sigma))$ only depends on the conjugacy class $(\mathrm{M}, \sigma)_{\mathrm{G}}$ of the pair (M, σ), and hence may and shall be written as $\hat{z}_0((\mathrm{M}, \sigma)_{\mathrm{G}})$.

Restricting to cuspidal pairs, we get a function $\hat{z} : \Omega(\mathrm{G}) \to \mathbb{C}$. Using Lemma 3.2.3(i), we see that $\chi \mapsto \hat{z}_0((\mathrm{M}, \sigma \otimes \chi)_{\mathrm{G}})$ is regular on $X^{\mathrm{unr}}(\mathrm{M})$ for each essentially discrete pair (M, σ). It follows from Proposition 4.1.4 that $\hat{z}$ is a regular function on $\Omega(\mathrm{G})$, and hence determines an element z of $\mathcal{Z}(\mathrm{G})$. Note that this also defines $\hat{z}((\mathrm{M}, \sigma)_{\mathrm{G}})$ for any essentially discrete pair (M, σ): it is the scalar with which $z \in \mathcal{Z}(\mathrm{G})$ acts on any irreducible subquotient of $\mathrm{Ind}_{\mathrm{M}}^{\mathrm{G}} \sigma$.

Let us note the following property of $\hat{z}_0$. If $\mathrm{M}^* \subset \mathrm{G}^*$ is a Levi subgroup, $\Sigma^* \in \Phi_2^+(\mathrm{M}^*)$ and $\Theta_{\Sigma^*}^{\mathrm{G}^*} = \mathrm{Ind}_{\mathrm{M}^*}^{\mathrm{G}^*} \Theta_{\Sigma^*}$, then $a_{z^*}(\Theta_{\Sigma^*}^{\mathrm{G}^*}) = \widehat{z^*}((\mathrm{M}^*, \sigma^*)_{\mathrm{G}^*})\Theta_{\Sigma^*}^{\mathrm{G}^*}$ for any $\sigma^* \in \Sigma^*$ (use Theorem 5.4.2). Therefore the transfer of $a_{z^*}(\Theta_{\Sigma^*}^{\mathrm{G}^*}) \in SD(\mathrm{G}^*)$ to $SD(\mathrm{G})$ equals:

$$\begin{cases} 0, & \text{if } \mathrm{M}^* \text{ is not } \underline{\mathrm{G}}^*\text{-relevant; and} \\ \hat{z}_0((\mathrm{M}, \sigma)_{\mathrm{G}})\Theta_{\Sigma}^{\mathrm{G}}, & \text{if } (\mathrm{M}^*, \Sigma^*) \sim (\mathrm{M}, \Sigma) \text{ and } \sigma \in \Sigma. \end{cases}.$$

Therefore, Claim 3 will follow if we show that for each essentially discrete pair (M, σ) we have:

$$\hat{z}_0((\mathrm{M}, \sigma)_{\mathrm{G}}) = \hat{z}((\mathrm{M}, \sigma)_{\mathrm{G}}). \tag{90}$$

Before proving this, we note that in the special case where (M, σ) is a cuspidal pair, this is immediate from the definitions. Let σ embed into $\mathrm{Ind}_{\mathrm{Q}}^{\mathrm{M}}\, \upsilon$, where (L, υ) is a cuspidal support for σ and the parabolic subgroup Q of M has L as a Levi subgroup. Let $\mathrm{L}^* \sim \mathrm{L}$ and $\mathrm{M}^* \sim \mathrm{M}$ with $\mathrm{L}^* \subset \mathrm{M}^* \subset \mathrm{G}^*$ being Levi subgroups, and assume without loss of generality that the extra "Levi subgroup matching data" involved in these choices have an obvious compatibility with each other; in particular, we assume that the inner twist from M^* to M takes L^* to L, and some parabolic subgroup $\mathrm{Q}^* \subset \mathrm{M}^*$ with L^* as a Levi subgroup, to Q (over $\bar{F}$). Write $r_{\mathrm{Q}}^{\mathrm{M}} : D_{\mathrm{Irr}}(\mathrm{M}) \to D_{\mathrm{Irr}}(\mathrm{L})$ and $r_{\mathrm{Q}^*}^{\mathrm{M}^*} : D_{\mathrm{Irr}}(\mathrm{M}^*) \to D_{\mathrm{Irr}}(\mathrm{L}^*)$ for the associated Jacquet module maps at the level of virtual characters.
Write $r_{\mathrm{Q}^*}^{\mathrm{M}^*}(\Theta_{\Sigma^*}) = \Theta_1^* + \Theta_2^*$ as in Lemma 6.3.3, where $\Theta_1^* = (r_{\mathrm{Q}^*}^{\mathrm{M}^*}(\Theta_{\Sigma^*}))_{\mathrm{ell}}$ belongs to $SD_{\mathrm{ell}}^+(\mathrm{L}^*)$ and Θ_2^* is a linear combination of virtual characters induced from proper Levi subgroups of M^*. Similarly, write $r_{\mathrm{Q}}^{\mathrm{M}}(\Theta_\Sigma) = \Theta_1 + \Theta_2$, where $\Theta_1 = (r_{\mathrm{Q}}^{\mathrm{M}}(\Theta_\Sigma))_{\mathrm{ell}} \in D_{\mathrm{ell}}^+(\mathrm{L})$.
By Frobenius reciprocity and the fact that $e(\mathrm{M})\Theta_\Sigma$ is a positive or negative linear combination of characters, Θ_υ contributes nontrivially to $r_{\mathrm{Q}}^{\mathrm{M}}\Theta_\Sigma$. Because υ is cuspidal, it follows from this that Θ_υ contributes nontrivially to $(r_{\mathrm{Q}}^{\mathrm{M}}\Theta_\Sigma)_{\mathrm{ell}} = \Theta_1$. In particular, $\Theta_1 \neq 0$. By Lemma 6.3.4, $r_{\mathrm{Q}^*}^{\mathrm{M}^*}(\Theta_{\Sigma^*}) \in SD_{\mathrm{Irr}}(\mathrm{L}^*)$ endoscopically transfers to $r_{\mathrm{Q}}^{\mathrm{M}}(\Theta_\Sigma) \in D_{\mathrm{Irr}}(\mathrm{L})$. By the compatibility between endoscopic transfer and parabolic induction (see Remark 3.2.2(i)), $\Theta_1^* \in SD_{\mathrm{ell}}^+(\mathrm{L})$ transfers endoscopically to $\Theta_1 \in D_{\mathrm{ell}}^+(\mathrm{L})$, which also gives us that $\Theta_1 \in SD_{\mathrm{ell}}^+(\mathrm{L})$ and that $\Theta_1^* \neq 0$. Let $\Upsilon \in \Phi_2^+(\mathrm{L})$ be the packet that contains υ, and let $\Upsilon^* \in \Phi_2^+(\mathrm{L}^*)$ be such that $(\mathrm{L}^*, \Upsilon^*) \sim (\mathrm{L}, \Upsilon)$. For any $\upsilon^* \in \Upsilon^*$ we have

$$\widehat{z^*}((\mathrm{L}^*, \upsilon^*)_{\mathrm{G}^*}) = \hat{z}_0((\mathrm{L}, \upsilon)_{\mathrm{G}}) = \hat{z}((\mathrm{L}, \upsilon)_{\mathrm{G}}) = \hat{z}((\mathrm{M}, \sigma)_{\mathrm{G}}), \tag{91}$$

where the last equality follows from the fact that (L, υ) is a cuspidal support for σ.
Since Θ_υ and hence also Θ_Υ contributes nontrivially to $\Theta_1 \in SD_{\mathrm{ell}}^+(\mathrm{L})$, the hypotheses of the proposition imply that Θ_{Υ^*} and hence also Θ_{υ^*} contributes nontrivially to $\Theta_1^* \in SD_{\mathrm{ell}}^+(\mathrm{L}^*)$. While υ^* may not be cuspidal, this implies by Lemma 6.3.3(iii) that υ^* has the same cuspidal support as some irreducible character contributing to $r_{\mathrm{Q}^*}^{\mathrm{M}^*}(\Theta_{\Sigma^*})$, and hence by a standard fact about Jacquet modules ([Cas95, Theorem 6.3.5]) the same cuspidal support as some $\sigma^* \in \Sigma^*$. Thus, we have:

$$\widehat{z^*}((\mathrm{L}^*, \upsilon^*)_{\mathrm{G}^*}) = \widehat{z^*}((\mathrm{M}^*, \sigma^*)_{\mathrm{G}^*}) = \hat{z}_0((\mathrm{M}, \sigma)_{\mathrm{G}}). \tag{92}$$

Now (90) follows from (91) and (92), finishing the proof of Claim 3 and hence also of the proposition. ☐

7. Classical groups and their inner forms

7.1. Some generalities about the hypothesis on tempered L-packets.

Proposition 7.1.1. *Assume for simplicity that $\mathcal{O}_{\mathrm{M}}$ is trivial for each Levi subgroup $\mathrm{M} \subset \mathrm{G}$. Each reductive group G_1 over F we consider in this proposition will be equipped with a similar collection $\{\mathcal{O}_{\mathrm{M}_1}\}_{\mathrm{M}_1}$, with each $\mathcal{O}_{\mathrm{M}_1}$ trivial.*

- *(i) Suppose $\mathrm{G}_1 \hookrightarrow \mathrm{G}$ is an inclusion of reductive groups that induces an isomorphism $\mathrm{G}_{1,\mathrm{ad}} \to \mathrm{G}_{\mathrm{ad}}$ of adjoint groups. Suppose that the hypothesis on the existence of tempered L-packets (Hypothesis 2.7.1) is satisfied by G. Then this hypothesis is satisfied by G_1 as well.*
- *(ii) Suppose $\mathrm{G}_1 \to \mathrm{G}$ is a homomorphism of reductive groups that induces an isomorphism $\mathrm{G}_{1,\mathrm{ad}} \to \mathrm{G}_{\mathrm{ad}}$ of adjoint groups. We additionally assume that $\mathrm{G}_1(F)$ and $\mathrm{G}(F)$ have the same image in $\mathrm{G}_{1,\mathrm{ad}}(F) = \mathrm{G}_{\mathrm{ad}}(F)$. Suppose that Hypothesis 2.7.1 is satisfied by G_1. Then this hypothesis is satisfied by G as well.*
- *(iii) Suppose that $\mathrm{G} = \mathrm{Res}_{E/F}\, \mathrm{G}_1$, where G_1 is a connected reductive group over a finite extension E of F. Then Hypothesis 2.7.1 is satisfied by G if and only if it is satisfied by G_1.*

Proof. Parts of the proof will only be sketched, but we will not use this proposition to prove any other lemma, proposition or theorem. First let us prove (i). Let $\mathrm{M}_1 \subset \mathrm{G}_1$ and $\mathrm{M} \subset \mathrm{G}$ be Levi subgroups with the same image in $\mathrm{G}_{1,\mathrm{ad}} = \mathrm{G}_{\mathrm{ad}}$. For each $\Sigma \in \Phi_2(\mathrm{M})$, let Σ_{M_1} be the set of irreducible constituents of the representations in $\Sigma|_{\mathrm{M}_1(F)} := \{\sigma|_{\mathrm{M}_1(F)} \mid \sigma \in \Sigma\}$. Let $\Phi_2(\mathrm{M}_1)$ be the set of all the $\Sigma_{\mathrm{M}_1} \subset \mathrm{Irr}(\mathrm{M}_1)$ as Σ varies over $\Phi_2(\mathrm{M})$. Note that $\Sigma_{\mathrm{M}_1} \subset \mathrm{Irr}_2(\mathrm{M}_1)$ for each $\Sigma \in \Phi_2(\mathrm{M})$.

Now let us show that distinct elements of $\Phi_2(\mathrm{M}_1)$ are disjoint. Suppose that $\sigma_1 \in \Sigma_1' \cap \Sigma_1''$, where $\Sigma_1' = (\Sigma')_{\mathrm{M}_1}, \Sigma_1'' = (\Sigma'')_{\mathrm{M}_1} \in \Phi_2(\mathrm{M}_1)$, with $\Sigma', \Sigma'' \in \Phi_2(\mathrm{M})$. Thus, there exist $\sigma' \in \Sigma'$ and $\sigma'' \in \Sigma''$, such that σ_1 is a constituent of both $\sigma'|_{\mathrm{M}_1(F)}$ and $\sigma''|_{\mathrm{M}_1(F)}$. Our disjointness claim will follow if we show that $\sigma'' = \sigma' \otimes \chi$ for some smooth character $\chi : \mathrm{M}(F) \to \mathbb{C}^\times$ that is trivial on the image of $\mathrm{M}_1(F)$: for, any such χ is automatically unitary, and Lemma 2.7.3(ii) will give $\Sigma'' = \Sigma' \otimes \chi$. Thus, the disjointness claim for $\Phi_2(\mathrm{M}_1)$ follows if we show that any two irreducible admissible representations σ', σ'' of $\mathrm{M}(F)$ having a common constituent σ_1 in their (necessarily semisimple) restrictions to $\mathrm{M}_1(F)$ are twists of each other by some character $\chi : \mathrm{M}(F) \to \mathbb{C}^\times$ that is trivial on $\mathrm{M}_1(F)$.

This fact follows from [GK82, Lemma 2.4] under a ‘multiplicity one’ assumption, but let us prove it without that assumption. Since the image of $\mathrm{M}_{\mathrm{sc}}(F) \to \mathrm{M}(F)$ contains the derived group of $\mathrm{M}(F)$ (use a z-extension), so does the image of $\mathrm{M}_1(F) \to \mathrm{M}(F)$. Therefore, we can twist σ' by a character of $\mathrm{M}(F)$ that is trivial on $\mathrm{M}_1(F)$ to assume that σ' and σ'' have the same central character, and then notice that the well-defined action of $\mathrm{M}(F)$ on the space $\mathrm{Hom}_{\mathrm{M}_1(F)}(\sigma', \sigma'')$, which is nonzero because σ is contained in both $\sigma'|_{\mathrm{M}_1(F)}$ and $\sigma''|_{\mathrm{M}_1(F)}$, is trivial on $\mathrm{Z}_{\mathrm{M}}(F)$ and on $\mathrm{M}_1(F)$, and hence factors through a finite abelian quotient, which therefore has an eigenvector.

This finishes the proof of the disjointness of $\Phi_2(\mathrm{M}_1)$. This, together with the fact that every element of $\mathrm{Irr}_2(\mathrm{M}_1)$ is a constituent of the restriction of some element of $\mathrm{Irr}_2(\mathrm{M})$ to $\mathrm{M}_1(F)$, implies that $\Phi_2(\mathrm{M}_1)$ is a partition of $\mathrm{Irr}_2(\mathrm{M}_1)$. Note that the proof of the claim also shows that the fibers of the map $\Phi_2(\mathrm{M}) \to \Phi_2(\mathrm{M}_1)$ given by $\Sigma \mapsto \Sigma_{\mathrm{M}_1}$ are acted on transitively by the group of continuous unitary characters $\mathrm{M}(F) \to \mathbb{C}^\times$ that are trivial on $\mathrm{M}_1(F)$.

For $\Sigma_1 \in \Phi_2(\mathrm{M}_1)$, we choose $\Sigma \in \Phi_2(\mathrm{M})$ with $\Sigma_1 = \Sigma_{\mathrm{M}_1}$, and set $\Theta_{\Sigma_1} := \Theta_\Sigma|_{\mathrm{M}_1(F)} = \sum c_\sigma \Theta_\sigma|_{\mathrm{M}_1(F)}$ (at the level of Harish-Chandra characters), where we write $\Theta_\Sigma = \sum c_\sigma \Theta_\sigma$, with σ running over Σ. Θ_{Σ_1} is independent of the choice of Σ, because by the previous paragraph, any other choice is of the form $\Sigma \otimes \chi$ with $\chi : \mathrm{M}(F) \to \mathbb{C}^\times$ a smooth character trivial on $\mathrm{M}_1(F)$. It is standard and easy that $\Theta_\sigma|_{\mathrm{M}_1(F)} = \Theta_{\sigma|_{\mathrm{M}_1(F)}}$ for each $\sigma \in \mathrm{Irr}(\mathrm{M})$, so that for each $\Sigma_1 \in \Phi_2(\mathrm{M}_1)$, Θ_{Σ_1} is a virtual character on $\mathrm{M}_1(F)$ supported on Σ_1 (two different representations in Σ can collapse to the same representation in Σ_1, at least a priori). Since either of Θ_Σ or Θ_{Σ_1} is stable if and only if it is constant as a function on each strongly regular semisimple stable conjugacy class in $\mathrm{M}(F)$ or $\mathrm{M}_1(F)$ (and since strongly regular semisimple elements of $\mathrm{M}_1(F)$ map to strongly regular semisimple elements of $\mathrm{M}(F)$), it follows that Θ_{Σ_1} is stable, for each $\Sigma_1 \in \Phi_2(\mathrm{M}_1)$.

To finish the proof of (i), it remains to show that any $\Theta_1 \in SD_{\mathrm{ell}}(\mathrm{M}_1)$ is a linear combination of the Θ_{Σ_1} as Σ_1 runs over $\Phi_2(\mathrm{M}_1)$ (which are automatically linearly independent). By Remark 2.2.4(i), we may assume without loss of generality that $\Theta_1 = \sum c_{\sigma_1} \Theta_{\sigma_1}$ where the σ_1 are tempered representations all having the same central character, say ζ_1. Choose a character ζ of $\mathrm{Z}_{\mathrm{M}}(F)$ extending ζ_1. We have a well-defined smooth function Θ on the set of strongly regular semisimple elements of $\mathrm{M}(F)$, supported on $\mathrm{Z}_{\mathrm{M}}(F) \cdot \mathrm{M}_1(F)$, whose value at zm equals $\zeta(z)\Theta_1(m_1)$ for any $z \in \mathrm{Z}_{\mathrm{M}}(F)$ and $m_1 \in \mathrm{M}_1(F)$.

It is easy to see that the $\mathrm{Int}\,\mathrm{M}(F)$-average $\bar{\Theta}$ of Θ equals $\sum c_{\tilde{\sigma}_1} \Theta_{\tilde{\sigma}_1}$, where for each σ_1, $\tilde{\sigma}_1$ is the direct sum of all irreducible representations of $\mathrm{M}(F)$ with central character ζ and having σ_1 in its restriction to $\mathrm{M}_1(F)$, and where $c_{\tilde{\sigma}_1} = c_{\sigma_1} \cdot a^{-1}$, a being the length of $\tilde{\sigma}_1|_{\mathrm{M}_1(F)}$. Thus, $\bar{\Theta} \in D(\mathrm{M})$. Since $\Theta_1 \in SD(\mathrm{M}_1)$, Θ is $\mathrm{Int}\,\mathrm{M}(F)$-invariant, and hence $\Theta = \bar{\Theta} \in D(\mathrm{M})$. Since Θ as a function is constant on each strongly regular semisimple conjugacy class in $\mathrm{M}(F)$, we have $\Theta \in SD(\mathrm{M})$.

However, it is not immediate that $\Theta \in SD_{\mathrm{ell}}(\mathrm{M})$, because the theory of R-groups can work more nontrivially for M_1 than for M (e.g., as happens when $\mathrm{M} = \mathrm{GL}_2$ and $\mathrm{M}_1 = \mathrm{SL}_2$). We take a lazy short-cut using a deep result from [Art96]. By (35), we can write $\Theta = \Theta' + \Theta''$, where $\Theta' \in SD_{\mathrm{ell}}(\mathrm{M})$, and Θ'' is a linear combination of virtual characters fully induced from proper

Levi subgroups of M. By an obvious compatibility of parabolic induction with restriction from $\mathrm{M}(F)$ to $\mathrm{M}_1(F)$, it follows that $\Theta''|_{\mathrm{M}_1(F)} = 0$, so that $\Theta_1 = \Theta'|_{\mathrm{M}_1(F)}$. Since we are assuming Hypothesis 2.7.1 for G, Θ' is a finite linear combination of the Θ_Σ as Σ runs over $\Phi_2(\mathrm{M})$. Since $\Theta_1 = \Theta'|_{\mathrm{M}(F)}$, it follows that Θ_1 is a finite linear combination of the Θ_{Σ_1} as Σ_1 runs over $\Phi_2(\mathrm{M}_1)$, as desired. This argument is unsatisfactory as it proves something very elementary using an input that depends on global methods, and also on the fundamental lemma, but it allows us to avoid the mess of having to deal with distinct R-groups. This finishes the proof of (i).

Now to prove (ii), a part of whose proof will only be sketched, assume that the kernel of $\mathrm{G}_1 \to \mathrm{G}$ is central and that $\mathrm{G}_1(F)$ and $\mathrm{G}(F)$ have the same image in $\mathrm{G}_{\mathrm{ad}}(F)$, i.e., $\mathrm{G}(F)$ is the product of $\mathrm{Z}_\mathrm{G}(F)$ and the image of $\mathrm{G}_1(F)$. This property passes to Levi subgroups. Let $\mathrm{M}_1 \subset \mathrm{G}_1$ be a Levi subgroup with image $\mathrm{M} \subset \mathrm{G}$. Thus, elements of $\mathrm{Irr}(\mathrm{M}), \mathrm{Irr}_2(\mathrm{M})$ and $\mathrm{Irr}_{\mathrm{temp}}(\mathrm{M})$ pull back to elements of $\mathrm{Irr}(\mathrm{M}_1), \mathrm{Irr}_2(\mathrm{M}_1)$ and $\mathrm{Irr}_{\mathrm{temp}}(\mathrm{M}_1)$, and in each case the image of the pull-back map consists of representations of $\mathrm{M}_1(F)$ whose central character is trivial on the kernel of $\mathrm{Z}_{\mathrm{M}_1}(F) \to \mathrm{Z}_\mathrm{M}(F)$.

Fix a smooth unitary character $\zeta : \mathrm{Z}_\mathrm{M}(F) \to \mathbb{C}^\times$. Since $SD_{\mathrm{ell}}(\mathrm{M})$ is the direct sum of the $SD_{\mathrm{ell},\zeta'}(\mathrm{M})$ as ζ' varies over (smooth) unitary characters of $\mathrm{Z}_\mathrm{M}(F)$, it suffices to define a partition $\Phi_2(\mathrm{M})_\zeta$ of the subset $\mathrm{Irr}_2(\mathrm{M})_\zeta$ of $\mathrm{Irr}_2(\mathrm{M})$ consisting of representations with central character ζ, and a nonzero stable virtual character Θ_Σ for each $\Sigma \in \Phi_2(\mathrm{M})_\zeta$ that is supported on Σ, such that the Θ_Σ form a basis for $SD_{\mathrm{ell},\zeta}(\mathrm{M})$.

Let ζ_1 be the pull-back of ζ under $\mathrm{Z}_{\mathrm{M}_1}(F) \to \mathrm{Z}_\mathrm{M}(F)$. Since $\mathrm{M}(F)$ is generated by $\mathrm{Z}_\mathrm{M}(F)$ and the image of $\mathrm{M}_1(F)$, it is immediate that pull-back gives us bijections $\mathrm{Irr}(\mathrm{M})_\zeta \to \mathrm{Irr}(\mathrm{M}_1)_{\zeta_1}, \mathrm{Irr}_2(\mathrm{M})_\zeta \to \mathrm{Irr}_2(\mathrm{M}_1)_{\zeta_1}$ and $\mathrm{Irr}_{\mathrm{temp}}(\mathrm{M})_\zeta \to \mathrm{Irr}_{\mathrm{temp}}(\mathrm{M}_1)_{\zeta_1}$. We let $\Phi_2(\mathrm{M})_\zeta$ be the partition of $\mathrm{Irr}_2(\mathrm{M})_\zeta$ corresponding under the second of these bijections to the partition of $\mathrm{Irr}_2(\mathrm{M}_1)_{\zeta_1}$ given by the subset $\Phi_2(\mathrm{M}_1)_{\zeta_1} \subset \Phi_2(\mathrm{M}_1)$ (as made sense of using Lemma 2.7.3(ii)).

Writing a for the bijection $\mathrm{Irr}_2(\mathrm{M})_\zeta \to \mathrm{Irr}_2(\mathrm{M}_1)_{\zeta_1}$, if $\Sigma \in \Phi_2(\mathrm{M})$ pulls back to $\Sigma_1 \in \Phi_2(\mathrm{M}_1)$, write $\Theta_{\Sigma_1} = \sum c_{\sigma_1}\Theta_{\sigma_1}$, and define $\Theta_\Sigma = \sum c_{a(\sigma)}\Theta_\sigma$. It is immediate that Θ_Σ is supported on Σ and is nonzero, and that Θ_Σ and Θ_{Σ_1} are related as follows (and, informally, 'determining' each other): at the level of Harish-Chandra characters, Θ_Σ pulls back to Θ_{Σ_1}, and has central character ζ. From this it is easy to see that Θ_Σ is stable, so that $\Theta_\Sigma \in SD_{\mathrm{ell},\zeta}(\mathrm{M})$ (more generally, this argument shows that pull-back of virtual characters with a given '$\mathrm{Z}_\mathrm{M}(F)$-central character' from $\mathrm{M}(F)$ to $\mathrm{M}_1(F)$ respects stability). The only remaining assertion is that the inclusion

$$SD_2(\mathrm{M}) := \mathrm{span}\{\Theta_\Sigma \mid \Sigma \in \Phi_2(\mathrm{M})_\zeta\} \subset SD_{\mathrm{ell},\zeta}(\mathrm{M}) \tag{93}$$

is an equality. By (35) and looking at character values on the elliptic set, this follows if we show that $SD_2(\mathrm{M})$ is a complement for the span of the $\mathrm{Ind}_\mathrm{L}^\mathrm{M} SD_{\mathrm{temp}}(\mathrm{L})$, as L ranges over the proper Levi subgroups of M. The analogous assertion with M replaced by M_1 is immediate (since we know Hypothesis 2.7.1 for M_1), so we are done using the easy observation that for each Levi subgroup $\mathrm{L} \subset \mathrm{M}$ with preimage $\mathrm{L}_1 \subset \mathrm{M}_1$ (including for $\mathrm{L} = \mathrm{M}$), pull-back defines a linear isomorphism $\mathrm{Ind}_\mathrm{L}^\mathrm{M} D_{\mathrm{temp},\zeta}(\mathrm{L}) \to \mathrm{Ind}_{\mathrm{L}_1}^{\mathrm{M}_1} D_{\mathrm{temp},\zeta_1}(\mathrm{L}_1)$, that restricts to a linear isomorphism $\mathrm{Ind}_\mathrm{L}^\mathrm{M} SD_{\mathrm{temp},\zeta}(\mathrm{L}) \to \mathrm{Ind}_{\mathrm{L}_1}^{\mathrm{M}_1} SD_{\mathrm{temp},\zeta_1}(\mathrm{L}_1)$. This finishes the proof of (ii).

The proof of (iii) is easy but cumbersome to write down, so we will skip the details. Briefly, $\mathrm{Res}_{E/F}$ gives a bijection between the Levi subgroups of G_1 and those of G, and if $\mathrm{M}_1 \subset \mathrm{G}_1$ and $\mathrm{M} \subset \mathrm{G}$ are Levi subgroups with $\mathrm{M} = \mathrm{Res}_{E/F}\,\mathrm{M}_1$, one shows that $\mathrm{Irr}_2, D_{\mathrm{ell}}$ and SD_{ell} all pull back well under the identification $\mathrm{M}(F) = \mathrm{M}_1(E)$, the case of D_{ell} entailing appropriate compatibilities involving intertwining operators and their normalizations. Therefore, a definition of either of $\Phi_2(\mathrm{M}_1)$ or $\Phi_2(\mathrm{M})$ can be transported to a definition for the other, and made to work. □

7.2. **Verification of various hypotheses for many 'classical' groups.** The following proposition describes how to go from Arthur's endoscopic classification, which is stated in terms of discrete series and tempered L-packets, to the statement of Hypothesis 2.7.1, which is stated in terms of elliptic representations.

Proposition 7.2.1. *Suppose, for each Levi subgroup* $\mathrm{M} \subset \mathrm{G}$*, we are given a partition* $\Phi_{\mathrm{temp}}(\mathrm{M})$ *of* $\mathrm{Irr}_{\mathrm{temp}}(\mathrm{M})$ *by* $\mathcal{O}_\mathrm{M}$*-invariant subsets, such that the following properties are satisfied:*

(a) *Some subset* $\Phi_2(\mathrm{M}) \subset \Phi_{\mathrm{temp}}(\mathrm{M})$ *partitions* $\mathrm{Irr}_2(\mathrm{M})$*. Moreover, for each Levi subgroup* $\mathrm{M} \subset \mathrm{G}$ *and each* $\Sigma \in \Phi_{\mathrm{temp}}(\mathrm{M})$*, there exists a Levi subgroup* $\mathrm{L} \subset \mathrm{M}$ *and some* $\Upsilon \in \Phi_2(\mathrm{L})$*, such that* Σ *equals* Υ^{M} *as defined in Notation 2.7.6(ii), i.e.,* Σ *is the union of the sets of* $\mathcal{O}_{\mathrm{M}}$*-conjugates of the irreducible constituents of the unitary representation* $\mathrm{Ind}_{\mathrm{L}}^{\mathrm{M}} \upsilon$*, as* υ *ranges over* Υ*.*

(b) *Let* $\mathrm{M} \subset \mathrm{G}$ *be a Levi subgroup and* $\Sigma \in \Phi_{\mathrm{temp}}(\mathrm{M})$*, and choose* (L, Υ) *as in (a). Then:*
- *(Compare with Definition 3.4.9(i)) For each (not necessarily elliptic) relevant endoscopic datum* $\underline{\mathrm{H}}$*, with underlying endoscopic group* H*, choosing auxiliary data and hence the* 5*-tuple* $(\mathrm{H}_1 \to \mathrm{H}, \hat{\xi}_1, \tilde{\mathrm{H}}_1 = \mathrm{H}_1 \to \tilde{\mathrm{H}} = \mathrm{H}, \mathrm{C}_1, \mu)$ *as in Notation 3.1.2(iii), there exists a stable tempered virtual character* $\Theta^{\underline{\mathrm{H}}} \in SD_\mu(\mathrm{H}_1)$ *on* $\tilde{\mathrm{H}}_1(F) = \mathrm{H}_1(F)$*, such that the following holds inside* $D(\mathrm{M})$*:*

$$\sum_{\underline{\mathrm{H}}} \mathbb{C} \cdot \mathbf{T}_{\underline{\mathrm{H}}}(\Theta^{\underline{\mathrm{H}}}) = (\sum_{\sigma \in \Sigma} \mathbb{C} \cdot \Theta_\sigma)^{\mathcal{O}_{\mathrm{M}}}. \tag{94}$$

- *In* (94)*, the contribution* $\mathbf{T}_{\underline{\mathrm{M}}^*}(\Theta^{\underline{\mathrm{M}}^*})$ *from the 'principal' endoscopic datum* $\underline{\mathrm{M}}^*$ *as in Notation 3.2.1(i) equals the* $\mathcal{O}_{\mathrm{M}}$*-average* $\mathrm{Avg}_{\mathcal{O}_{\mathrm{M}}}(\mathrm{Ind}_{\mathrm{L}}^{\mathrm{M}} \Theta_\Upsilon)$ *of a character induced from some* $\Theta_\Upsilon \in SD(\mathrm{L})^{\mathcal{O}_{\mathrm{L}}}$*.*

Then the hypothesis on the existence of tempered L-packets, namely Hypothesis 2.7.1, is satisfied.

Proof. Let $\mathrm{M} \subset \mathrm{G}$ be a Levi subgroup. For any $\Sigma \in \Phi_2(\mathrm{M})$, note that any (L, Υ) as in (a) equals (M, Σ), and that the $\Theta_\Sigma := \Theta_\Upsilon \in SD(\mathrm{M})$ as in (b) is supported on Σ. It suffices to show that the Θ_Σ, Σ varying over $\Phi_2(\mathrm{M})$, form a basis for $SD_{\mathrm{ell}}(\mathrm{M})^{\mathcal{O}_{\mathrm{M}}}$. Each such Θ_Σ is clearly contained in $SD(\mathrm{M}) \cap D_{\mathrm{ell}}(\mathrm{M}) = SD_{\mathrm{ell}}(\mathrm{M})$, and it is also clear that the Θ_Σ form a linearly independent set as Σ varies over $\Phi_2(\mathrm{M})$. It remains to show that their span $SD_{\mathrm{ell}}(\mathrm{M})'$, which is contained in $SD_{\mathrm{ell}}(\mathrm{M})^{\mathcal{O}_{\mathrm{M}}}$, equals all of $SD_{\mathrm{ell}}(\mathrm{M})^{\mathcal{O}_{\mathrm{M}}}$.

Let $SD(\mathrm{M})' \subset SD(\mathrm{M})^{\mathcal{O}_{\mathrm{M}}}$ denote the span of the contributions $\mathbf{T}_{\underline{\mathrm{M}}^*}(\Theta^{\underline{\mathrm{M}}^*})$ from $\underline{\mathrm{M}}^*$ in (94) as Σ varies over $\Phi_{\mathrm{temp}}(\mathrm{M})$. Write $D_{\text{non-ell}}(\mathrm{M}) \subset D(\mathrm{M})$ for the span of tempered virtual characters fully induced from proper Levi subgroups of M, and $D_{\text{ell,non-st}}(\mathrm{M})$ for the span of the $\mathbf{T}_{\underline{\mathrm{H}}}(SD_{\mu,\mathrm{ell}}(\mathrm{H}_1))$ as $\underline{\mathrm{H}}$ varies over the elliptic endoscopic data for M distinct from $\underline{\mathrm{M}}^*$. It follows from the second condition of (b) that:

$$SD(\mathrm{M})' \subset SD_{\mathrm{ell}}(\mathrm{M})' + D_{\text{non-ell}}(\mathrm{M}). \tag{95}$$

The first condition of (b) gives us an expression of the form:

$$D(\mathrm{M})^{\mathcal{O}_{\mathrm{M}}} \subset SD(\mathrm{M})' + \sum_{\underline{\mathrm{H}} \neq \underline{\mathrm{M}}^*} \mathbf{T}_{\underline{\mathrm{H}}}(SD_\mu(\mathrm{H}_1)), \tag{96}$$

where $\underline{\mathrm{H}}$ runs over a set of relevant endoscopic data for M (taken up to isomorphism), and where for each $\underline{\mathrm{H}}$ we have implicitly chosen and fixed auxiliary data including μ and $\mathrm{H}_1 = \tilde{\mathrm{H}}_1$.

For any given endoscopic datum $\underline{\mathrm{H}}$ for M, using (35), together with the compatibility between endoscopic transfer and parabolic induction in the form of Remark 3.1.4(i) (if $\underline{\mathrm{H}}$ is elliptic) or Remark 3.1.4(ii) (otherwise), we have that :

$$\mathbf{T}_{\underline{\mathrm{H}}}(SD_\mu(\mathrm{H}_1)) \subset D_{\text{non-ell}}(\mathrm{M}) + D_{\text{ell,non-st}}(\mathrm{M}).$$

Combining this with (95) and 96, we get:

$$SD_{\mathrm{ell}}(\mathrm{M})^{\mathcal{O}_{\mathrm{M}}} \subset D(\mathrm{M})^{\mathcal{O}_{\mathrm{M}}} \cap SD_{\mathrm{ell}}(\mathrm{M}) \subset (SD_{\mathrm{ell}}(\mathrm{M})' + D_{\text{non-ell}}(\mathrm{M}) + D_{\text{ell,non-st}}(\mathrm{M})) \cap SD_{\mathrm{ell}}(\mathrm{M}) = SD_{\mathrm{ell}}(\mathrm{M})',$$

where the last equality uses (34) and (31), as desired. □

Proposition 7.2.2. *Suppose* G *is a quasi-split symplectic, special orthogonal, unitary, general symplectic, even general special orthogonal or odd general spin group:* $\mathrm{Sp}_{2n}, \mathrm{SO}_n, \mathrm{U}_n, \mathrm{GSp}_{2n}, \mathrm{GSO}_{2n}$ *or* GSpin_{2n+1}*. Except in the* GSO_{2n} *case, assume that each* $\mathcal{O}_{\mathrm{M}}$ *is trivial. When* $\mathrm{G} = \mathrm{GSO}_{2n}(F)$*, for any Levi subgroup* $\mathrm{M} \subset \mathrm{G}$ *with the 'GSO-part' equal to* G_{M}*, if* G_{M} *is nonabelian (resp., abelian), assume that* $\mathcal{O}_{\mathrm{M}}$ *is a two element group contained in the restriction of* $\mathrm{Int}\, \mathrm{O}_{2n}(F)$ *to* M*, with nontrivial image in* $\mathrm{Out}(\mathrm{G}_{\mathrm{M}})$ *(resp., that* G_{M} *is trivial). Then the hypothesis on the existence of tempered L-packets (Hypothesis 2.7.1) is satisfied.*

Proof. We will consider multiple cases, but these cases will overlap.
First assume that G is symplectic, special orthogonal, or unitary; in these cases we will use [Art13] (in the special orthogonal and symplectic cases) or [Mok15] (in the unitary case). In these cases, partitions $\Phi(\mathrm{M})$ as in Proposition 7.2.1 have been constructed by Arthur and Mok in [Art13] and [Mok15]. Here, for the case of even special orthogonal groups, see [Art13, Theorem 8.4.1]. To see that the latter assertions in (a) and (b) of Proposition 7.2.1 are satisfied, note that the tempered L-packets on these groups and their endoscopic decompositions are defined in [Art13] or [Mok15] starting from the discrete series case, using parabolic induction and the local intertwining relation: see [Art13, the proof of Proposition 2.4.3 and Sections 6.5, 6.6 and 8.4] and [Mok15, the proof of Proposition 3.4.4 and Section 7.6]. Thus, we are done in these cases by Proposition 7.2.1.
Now suppose that G equals GSp_{2n} or GSO_{2n} (and is quasi-split). Then partitions $\Phi(\mathrm{M})$ as in Proposition 7.2.1 have been constructed by Xu in [Xu18], associated to the given collection $\{\mathcal{O}_\mathrm{M}\}_\mathrm{M}$ (the analogue of $\mathcal{O}_\mathrm{G}$ for [Xu18] is the group Σ_0 of [Xu18, Introduction, page 73]). Here, to see that the latter assertions in (a) and (b) of the proposition are satisfied, note that the tempered L-packets on these groups and their endoscopic decompositions are defined in [Xu18] starting from the discrete series case, using parabolic induction and the local intertwining relation: see [Xu18, Lemma 4.10 and Section 6.4]. Thus, we are done in these cases by Proposition 7.2.1.
Thirdly, assume that G is odd special orthogonal, symplectic, unitary or odd general spin, i.e., $\mathrm{SO}_{2n+1}, \mathrm{Sp}_{2n}, \mathrm{U}_n$, or GSpin_{2n+1}; in these cases, we will use [Mg14]. For use in Proposition 7.2.4 below, we will also allow G to be an even special orthogonal group $\mathrm{SO}(V,q)$, but take $\mathcal{O}_\mathrm{G}$ to be $\mathrm{Int}\,\mathrm{O}(V,q)(F)$ and each $\mathcal{O}_\mathrm{M}$ to be the set of elements of $\mathcal{O}_\mathrm{G}^+$ that preserve M and act trivially on its center. Each Levi subgroup $\mathrm{M} \subset \mathrm{G}$ can be written as $\mathrm{GL}_\mathrm{M} \times \mathrm{G}_\mathrm{M}$, where GL_M is isomorphic to a product of groups of the form $\mathrm{Res}_{E/F}\,\mathrm{GL}_m$ for some trivial or quadratic extension E/F, and G_M is a group of the same type as G but of smaller rank (these groups can be possibly trivial). Hypothesis 2.7.1 is trivial for GL_M, because it is standard that $SD_{\mathrm{ell}}(\mathrm{GL}_\mathrm{M})$ equals $D_{\mathrm{ell}}(\mathrm{GL}_\mathrm{M})$ and is spanned by the characters of discrete series representations of $\mathrm{GL}_\mathrm{M}(F)$. Therefore, the construction of $\Phi_2(\mathrm{M})$ as in Hypothesis 2.7.1 reduces to such a construction for $\Phi_2(\mathrm{G}_\mathrm{M})$. The latter construction is trivial if G_M is abelian, while if G_M is nonabelian it follows from [Mg14, Corollary 4.11], noting that what is denoted $I^{\underline{G}}_{cusp,st}$ in that corollary is also what is noted $I^{G}_{cusp,st}$ in that reference, since the group Aut of automorphisms of the endoscopic datum $\underline{G}$ of that corollary is trivial in the cases that we are currently considering. □

Remark 7.2.3. In Proposition 7.2.2, we have avoided discussing the case of quasi-split even general spin groups GSpin_{2n}. This is because, for our arguments to work in this case, we need the relevant transfer factors $\Delta(\cdot,\cdot)$ to be invariant under the conjugation action of $\mathrm{O}_{2n}(F)$ on the first factor, so that the action of the group 'Aut' on the space noted $I^{G}_{cusp,st}$ in [Mg14, Section 2.3], where $G = \mathrm{GSpin}_{2n}(F)$, can be defined simply through its action on G, without the more complicated involvement of transfer factors as in [MW16, Section I.2.6]. It seems to us that the relevant invariance property is likely to hold, and hence should give Hypothesis 2.7.1 up to the action of the obvious outer automorphism group, but we have not verified it.

Proposition 7.2.4. *Let* $\mathrm{G} = \mathrm{G}(V,q)$ *be a quasi-split symplectic, special orthogonal, unitary, or odd general spin group, associated to a space* (V,q) *that is symplectic (if* G *is symplectic), quadratic (if* G *is special orthogonal or odd general spin) or* E/F*-Hermitian (if* G *is unitary, associated to a quadratic extension* E/F*). Let* $\mathcal{O} = \mathcal{O}_\mathrm{G}$ *be trivial if* G *is not even special orthogonal, and let it be generated by a reflection associated to an element of order two in* $\mathrm{O}(V,q)(F) \setminus \mathrm{SO}(V,q)(F)$*, when* $\mathrm{G} = \mathrm{G}(V,q) = \mathrm{SO}(V,q)$ *is even special orthogonal. Assume that, for each Levi subgroup* $\mathrm{M} \subset \mathrm{G}$*,* $\mathcal{O}_\mathrm{M}^+$ *consists of those elements of* $\mathcal{O}_\mathrm{G}^+ = \mathcal{O}_\mathrm{G} \circ \mathrm{Int}\,\mathrm{G}(F)$ *that fix the center of* M *pointwise (e.g.,* $\mathcal{O}_\mathrm{M}$ *itself could be the group of all such elements, as in Remark 2.6.2(i)). Then* G*, with respect to this collection* $\{\mathcal{O}_\mathrm{M}\}_\mathrm{M}$*, satisfies the hypotheses on the existence of tempered L-packets, LLC+, LLC+ and stability, supercuspidal packets and stable cuspidal support (Hypotheses 2.7.1, 2.10.3, 2.10.12, 2.11.1, and 2.11.4).*

Proof. The assertion concerning Hypothesis 2.7.1 follows from Proposition 7.2.2, so let us focus on the remaining hypotheses. For uniformity, we will use [Mg14], though for the symplectic and

orthogonal cases (resp., the unitary case), one could also use [Art13] (resp., [Mok15]). Since much of the proposition is present in [Mg14], we will occasionally be brief. We let $E = F$ unless we are in the unitary case, where E is the quadratic extension of F splitting G.
As in [Mg14, Section 2.1], we choose an elliptic endoscopic datum $\underline{\mathrm{G}}$ realizing G as endoscopic to a twisted space $\widetilde{\mathrm{GL}}_W$ whose underlying group GL_W is isomorphic to $\mathrm{GL}(W)$ (in the symplectic and orthogonal cases), $\mathrm{GL}(W)\times\mathrm{GL}_1$ (in the odd general spin case) and $\mathrm{Res}_{E/F}\,\mathrm{GL}(W)$ (in the unitary case). Note that there are multiple possibilities for $\underline{\mathrm{G}}$ in the symplectic and unitary cases; any of those will suffice for our purposes. For the convenience of the reader, here is a brief description of the case where $\mathrm{G} = \mathrm{GSpin}_{2n+1}$. In this case, one considers the automorphism $\theta : (g,\lambda) \mapsto (\theta_0(g), \lambda \det g)$ of $\mathrm{GL}_W = \mathrm{GL}(W) \times \mathrm{GL}_1 = \mathrm{GL}_{2n} \times \mathrm{GL}_1$, where θ_0 is a conjugate of the transpose inverse on GL_{2n} that preserves a pinning. Letting $\hat{\theta}_0 \in \mathrm{Aut}(\mathrm{GL}_{2n}(\mathbb{C})) = \mathrm{Aut}(\widehat{\mathrm{GL}(W)})$ to be dual to it and to preserve a pinning, a dual $\hat{\theta}$ to θ can be taken to be given by $(\hat{g},\hat{\lambda}) \mapsto (\hat{\theta}_0(\hat{g})\hat{\lambda},\hat{\lambda})$. Thus, $\mathrm{GL}_{2n}(\mathbb{C})^{\hat{\theta}}$ identifies with $\mathrm{GSp}_{2n}(\mathbb{C})$, which is dual to $\mathrm{GSpin}_{2n+1}(\mathbb{C})$. Coming back to the general case, the realization of these endoscopic data involve choices of Borel pairs for GL_W and G, which we fix, to talk of standard parabolic and Levi subgroups of these groups.
Given a standard Levi subgroup $\mathrm{M} \subset \mathrm{G}$, we can uniquely identify it with a group of the form $\mathrm{GL}_{\mathrm{M}}\times \mathrm{G}_{\mathrm{M}}$, where $\mathrm{GL}_{\mathrm{M}} = \prod_i \mathrm{Res}_{E/F}\,\mathrm{GL}(V_{\mathrm{M},i})$ with each $V_{\mathrm{M},i} \subset V$ an isotropic subspace, and G_{M} is the group of the same type as G associated to a nondegenerate subspace $(V_{\mathrm{M}}, q|_{V_{\mathrm{M}}}) \subset (V,q)$. Whenever $\mathcal{O}_{\mathrm{M}}$ is nontrivial, so that $\mathrm{G} = \mathrm{SO}(V,q)$, we may and do assume without loss of generality that $\mathcal{O}_{\mathrm{M}}$ equals $\mathcal{O}_{\mathrm{M}}^+$, so that it identifies with $\mathrm{Int}((\mathrm{GL}_{\mathrm{M}}\times \mathrm{O}(V_{\mathrm{M}}, q|_{V_{\mathrm{M}}}))(F))$, and equals $\mathrm{Int}\,\mathrm{M}(F)\,\mathrm{Int}\{1, g_{\mathrm{M}}\}$, where $g_{\mathrm{M}} \in \mathrm{GL}_E(V)$ has determinant -1, stabilizes $(V_{\mathrm{M}}, q_{\mathrm{M}})$ and acts as the identity on its orthogonal complement. In all cases, any Levi subgroup $\mathrm{M} \subset \mathrm{G}$ is automatically relevant in the sense of [MW16, Section I.3.4], and accordingly there is an elliptic endoscopic datum realizing M as an endoscopic group of a standard Levi subspace of $(\mathrm{GL}_W, \widetilde{\mathrm{GL}}_W)$, which can be computed to be of the form $(\mathrm{G}_1\times\mathrm{G}_2, \tilde{\mathrm{G}}_1\times\tilde{\mathrm{G}}_2)$, consisting of a realization of the factor $\prod_i \mathrm{Res}_{E/F}\,\mathrm{GL}(V_{\mathrm{M},i})$ of M as endoscopic to $(\mathrm{G}_1, \tilde{\mathrm{G}}_1)$, and of the factor G_{M} of M as endoscopic to $(\mathrm{G}_2, \tilde{\mathrm{G}}_2)$, the latter in a manner exactly analogous to the realization of G as endoscopic to $(\mathrm{GL}_W, \widetilde{\mathrm{GL}}_W)$. Here, $\mathrm{G}_2 \subset \mathrm{GL}_W$ is the product of the GL_1 factor if it exists (i.e., if $\mathrm{G} \cong \mathrm{GSpin}_{2n+1}$), and a subgroup $\mathrm{GL}_{W_{\mathrm{M}}} \subset \mathrm{GL}_W$ acting as the identity on a suitable complement of a subspace W_{M} in W. When $\mathrm{G} = \mathrm{GSpin}_{2n+1}$, this reflects the fact that the similitude character on the dual $\hat{\mathrm{G}} = \mathrm{GSp}_{2n}(\mathbb{C})$, when restricted to a Levi subgroup, is supported on the 'classical part' of the Levi subgroup. In all cases, these considerations let one make sense of 'extended cuspidal support' as in [Mg14, Section 4].
Let us consider Hypothesis 2.10.3. Let $\mathrm{M} \subset \mathrm{G}$ be a standard Levi subgroup, and let us specify the map $\sigma \mapsto \varphi_\sigma$, which is a map

$$\mathrm{Irr}_2(\mathrm{GL}_{\mathrm{M}}) \times \mathrm{Irr}_2(\mathrm{G}_{\mathrm{M}}) = \mathrm{Irr}_2(\mathrm{M}) \to \Phi_2(\mathrm{M})/\mathcal{O}_{\mathrm{M}} = \Phi_2(\mathrm{GL}_{\mathrm{M}}) \times (\Phi_2(\mathrm{G}_{\mathrm{M}})/\mathcal{O}_{\mathrm{M}})$$

($\mathcal{O}_{\mathrm{M}}$ acts on G_{M} by restriction). We take it to be the product of the map $\mathrm{Irr}_2(\mathrm{GL}_{\mathrm{M}}) \to \Phi_2(\mathrm{GL}_{\mathrm{M}})$ given by the usual local Langlands correspondence for GL-type groups, and an analogous map $\mathrm{Irr}_2(\mathrm{G}_{\mathrm{M}}) \to \Phi_2(\mathrm{G}_{\mathrm{M}})/\mathcal{O}_{\mathrm{M}}$, which we will also denote as $\sigma \mapsto \varphi_\sigma$. The latter can be described as follows:

- If G_{M} is not abelian or $\mathcal{O}_{\mathrm{G}}$ is trivial, we define this map as in [Mg14]. Specifically, any $\sigma \in \mathrm{Irr}_2(\mathrm{G}_{\mathrm{M}})$ has what [Mg14, Section 4] defines as its extended cuspidal support. By [Mg14, Remark 4.1 and Corollary 4.3], this extended cuspidal support is the cuspidal support of exactly one tempered representation of G_2, say σ^{G_2}. By [Mg14, Corollary 4.11 and Theorem 6.4], the local Langlands parameter of σ^{G_2} is the image in $\Phi_{\mathrm{temp}}(\mathrm{G}_2)$ of a unique element of $\Phi_2(\mathrm{G}_{\mathrm{M}})$ under the inclusion ${}^L\mathrm{G}_{\mathrm{M}} \hookrightarrow {}^L\mathrm{G}_2$ which is part of the realization of G_{M} as endoscopic to G_2. This element of $\Phi_2(\mathrm{G}_{\mathrm{M}})$ is what we take to be φ_σ.
- Suppose G_{M} is abelian and $\mathcal{O}_{\mathrm{G}}$ is nontrivial. Then G_{M} is a torus and $\mathcal{O}_{\mathrm{M}}$ is trivial, and we let the map $\mathrm{Irr}_2(\mathrm{G}_{\mathrm{M}}) \to \Phi_2(\mathrm{M})$ to be given by the local Langlands correspondence for tori. One can check that this refines the prescription in the nonabelian case, which gives the local Langlands correspondence only up to the action of a group of order two.

This defines $\mathrm{Irr}_2(\mathrm{M}) \to \Phi_2(\mathrm{M})/\mathcal{O}_{\mathrm{M}}$ when M is a standard Levi subgroup. It is $\mathcal{O}_{\mathrm{M}}$-invariant, because $\mathcal{O}_{\mathrm{M}}$ restricts to inner automorphisms on GL_{M}, and the same applies to G_{M} except possibly in the even special orthogonal case, where the fibers of $\sigma \mapsto \varphi_\sigma$ are preserved under $\mathcal{O}_{\mathrm{M}}|_{\mathrm{G}_{\mathrm{M}}} = \mathrm{Int}\,\mathrm{O}(V_{\mathrm{M}}, q|_{V_{\mathrm{M}}})(F)$ as mentioned in [Mg14, Theorem 6.5]. In order to extend this to more general M, let us first verify:

Claim. Whenever $\beta \in \mathcal{O}^+_{\mathrm{G}}$ takes a *standard* Levi subgroup M' to a *standard* Levi subgroup M, the condition $\varphi_\sigma = {}^L\beta \circ \varphi_{\sigma\circ\beta}$ from (iii) of Hypothesis 2.10.3 is satisfied for all $\sigma \in \mathrm{Irr}_2(\mathrm{M})$.

Since each automorphism in $\mathcal{O}^+_{\mathrm{G}}$ is induced by an isometry of (V, q), it is easy to see that $\mathrm{G}_{\mathrm{M}} = \mathrm{G}_{\mathrm{M}'}$ and that β normalizes this group. Similarly, β also takes each of the factors $\mathrm{Res}_{E/F}\,\mathrm{GL}(V_{\mathrm{M}',j})$ of $\mathrm{GL}_{\mathrm{M}'}$ to a factor $\mathrm{Res}_{E/F}\,\mathrm{GL}(V_{\mathrm{M},i})$ of GL_{M}, and the verification of the equality $\varphi_\sigma = {}^L\beta \circ \varphi_{\sigma\circ\beta}$ reduces to analogous verifications separately for the 'general linear' factors $\mathrm{Res}_{E/F}\,\mathrm{GL}(V_{\mathrm{M}',j})$ and $\mathrm{Res}_{E/F}\,\mathrm{GL}(V_{\mathrm{M},i})$ on the one hand, and for the 'classical' factors $\mathrm{G}_{\mathrm{M}'}$ and G_{M} on the other.

β takes each factor $\mathrm{Res}_{E/F}\,\mathrm{GL}(V_{\mathrm{M}',j})$ to its image $\mathrm{Res}_{E/F}\,\mathrm{GL}(V_{\mathrm{M},i})$ by an isomorphism of the form $\mathrm{Res}_{E/F}\,\beta_0$, where β_0 is an isomorphism $\mathrm{GL}(V_{\mathrm{M}',j}) \to \mathrm{GL}(V_{\mathrm{M},i})$ of groups over E. Therefore, the '$\mathrm{GL}(V_{\mathrm{M},i})$'-part of required equality follows from the functoriality of the local Langlands correspondence for isomorphisms of general linear groups, namely, [Hai14, Proposition 5.2.5]. On the other hand, the 'G_{M}-part' of the verification follows from the functoriality of local Langlands correspondence for tori if G_{M} is abelian, and from the fact that $\beta|_{\mathrm{G}_{\mathrm{M}}} \in \mathcal{O}_{\mathrm{M}}|_{\mathrm{G}_{\mathrm{M}}}$ otherwise, given that we have already seen the $\mathcal{O}_{\mathrm{M}}$-invariance of $\mathrm{Irr}_2(\mathrm{G}_{\mathrm{M}}) \to \Phi_2(\mathrm{M})/\mathcal{O}_{\mathrm{M}}$ above. This finishes the proof of the claim.

Thanks to this claim, we can now define the map $\sigma \mapsto \varphi_\sigma$ for all Levi subgroups $\mathrm{M} \subset \mathrm{G}$ by forcing the condition $\varphi_\sigma = {}^L\beta \circ \varphi_{\sigma\circ\beta}$ from (iii) of Hypothesis 2.10.3.

This completes the construction of the maps $\sigma \mapsto \varphi_\sigma$, for which we still need to verify the conditions in Hypothesis 2.10.3. The surjectivity of $\sigma \mapsto \varphi_\sigma$ follows from separately considering the $\mathrm{Res}_{E/F}\,\mathrm{GL}(V_{\mathrm{M},i})$ and the G_{M}, the latter of which follows from the 'if' part of [Mg14, Theorem 6.4].

Now we come to the condition involving twists by unramified unitary characters; this is a special case of a general fact describing the behaviour of twisted endoscopy with respect to character twists, which is very possibly somewhere in the literature. But since we cannot find a reference, let us sketch an argument for our specific situation which has some simplifying features. This question can be addressed separately for the 'general linear parts' $\mathrm{Res}_{E/F}\,\mathrm{GL}(V_{\mathrm{M},i})$ of M, and for the 'classical part' G_{M}. The former is well-known, while G_{M} does not have nontrivial unramified characters unless $\mathrm{G} \cong \mathrm{GSpin}_{2n+1}$. To handle this case, we assume without loss of generality for now that $\mathrm{G} = \mathrm{M} = \mathrm{GSpin}_{2n+1}, \mathrm{GL}_W = \mathrm{GL}_{2n} \times \mathrm{GL}_1$, and $\hat{\mathrm{G}} = \mathrm{GSp}_{2n}(\mathbb{C}) = (\mathrm{GL}_{2n}(\mathbb{C}) \times \mathrm{GL}_1(\mathbb{C}))^{\hat\theta} = \widehat{\mathrm{GL}}_W^{\hat\theta}$. Let $\alpha \in \mathrm{Z}_{\hat{\mathrm{G}}} = \mathrm{Z}_{\hat{\mathrm{G}}}^{\Gamma,0}$ be such that the corresponding $\chi_\alpha \in X^{\mathrm{unr}}(\mathrm{S}_{\mathrm{G}})$ (use Lemma 2.5.10) is unitary. The map $\mathrm{Z}^{\hat\theta}_{\widehat{\mathrm{GL}}_W} = (\mathrm{Z}^{\hat\theta,\Gamma}_{\widehat{\mathrm{GL}}_W})^0 \to \mathrm{Z}_{\hat{\mathrm{G}}}$ is an isomorphism (reflecting the ellipticity of $\underline{\mathrm{G}}$), giving us an isomorphism $\mathrm{S}_{\mathrm{G}} \to \mathrm{S}' := \mathrm{S}_{\mathrm{GL}_W}/(1-\theta)\mathrm{S}_{\mathrm{GL}_W}$. Let $\alpha' \in \mathrm{Z}^{\hat\theta}_{\widehat{\mathrm{GL}}_W}$ map to α. Note that the character $\chi_{\alpha'} \in X^{\mathrm{unr}}(\mathrm{S}') \subset X^{\mathrm{unr}}(\mathrm{S}_{\mathrm{GL}_W})$ associated to α' is unitary as well.

The composite homomorphism $\mathrm{GL}_W \to \mathrm{S}_{\mathrm{GL}_W} \to \mathrm{S}'$, together with the GL_W-bitorsor $\tilde{\mathrm{GL}}_W$, determines an S'-bitorsor $\tilde{\mathrm{S}}'$ together with a map $\tilde{\mathrm{GL}}_W \to \tilde{\mathrm{S}}'$ that is constant on stable conjugacy classes. To proceed, we find it simpler to use [Wal08] than [MW16]. We choose a base-point $\tilde\theta \in \tilde{\mathrm{GL}}_W(F)$ such that $\mathrm{Int}\,\tilde\theta$ equals θ, which preserves a pinning. In the language of [Wal08, Section 1.2], we can identify "$\tilde{G}$" (our $\tilde{\mathrm{GL}}_W$) with $\tilde{G}^* = G^*\theta^*$, with our $\tilde\theta$ taking the place of the θ^* of [Wal08]. The isomorphism $\mathrm{S}_{\mathrm{G}} \to \mathrm{S}'$ gives a compatible *homeomorphism* $\mathrm{S}_{\mathrm{G}} \to \tilde{\mathrm{S}}'$, where $1 \in \mathrm{S}_{\mathrm{G}}(F)$ maps to the image $x_{\tilde\theta} \in \tilde{\mathrm{S}}'(F)$ of $\tilde\theta$. $\chi_{\alpha'}$ extends to a representation $(\chi_{\alpha'}, \tilde\chi_{\alpha'})$ of $(\mathrm{S}'(F), \tilde{\mathrm{S}}'(F))$, with $\tilde\chi_{\alpha'}(x_{\tilde\theta}) = 1$. Looking at the transfer of semisimple conjugacy classes as defined in [Wal08, Section 1.3] one finds that if semisimple $\delta \in \mathrm{G}(F)$ and $\gamma \in \tilde{\mathrm{GL}}_W(F)$ have their $\bar{F}$-conjugacy classes match, the image of δ in $\mathrm{S}_{\mathrm{G}}(F)$ maps to the image of γ in $\tilde{\mathrm{S}}'(F)$, and thus, $\chi_\alpha(\delta) = \tilde\chi_{\alpha'}(\gamma)$ (this uses our identifying the θ^* of [Wal08] with $\tilde\theta$). Thus, if $f_W \in C_c^\infty(\tilde{\mathrm{GL}}_W(F))$ and $f \in C_c^\infty(\mathrm{G}(F))$ have matching orbital integrals, it is easy to see that so do $f_W\tilde\chi_{\alpha'}^{-1}$ and $f\chi_\alpha^{-1}$. In other words, if the corresponding transfer of distributions maps Θ to $\tilde\Theta_W$, it also maps $\Theta\chi_\alpha$

to $\Theta_W \tilde{\chi}_{\alpha'}$. From this, the compatibility of the local Langlands correspondence of [Mg14] for $\mathrm{G_M}$, with twisting by unramified characters, is easy to see.

Now we return to the general G of the proposition. The condition (ii) of Hypothesis 2.10.3 again breaks down into an assertion involving the 'general linear part' and the 'classical part', of which the former is standard while the latter is immediate from the definition of the extended cuspidal support in [Mg14] (see [Mg14, just before Remark 4.1]), together with the fact that the map $\sigma \mapsto \varphi_\sigma$ was defined using the extended cuspidal support. The condition (iii) was imposed in the construction of the maps $\sigma \mapsto \varphi_\sigma$, and is hence automatic, completing the verification of Hypothesis 2.10.3.

Noting from the proof of the third case of Proposition 7.2.2 that Hypothesis 2.7.1 is satisfied in our setting, and noting also from that proof that $\Phi_2(\mathrm{M})$ is simply the partition determined as the fibers of the map that takes each element of $\mathrm{Irr}_2(\mathrm{M})$ to its extended cuspidal support (see [Mg14, Corollary 4.11]), Hypothesis 2.10.12 is automatic from the definition of $\sigma \mapsto \varphi_\sigma$.

Now we come to Hypothesis 2.11.1. The implication (b) $\Rightarrow$ (a) of that hypothesis follows from Remark 2.11.2, so let us see the implication (a) $\Rightarrow$ (b). Thus, suppose $\Sigma(\varphi)$ (defined as in Notation 2.10.11(i)) consists entirely of supercuspidal representations, where $\varphi \in \Phi_2^+(\mathrm{M})/\mathcal{O}_\mathrm{M}$ and $\mathrm{M} \subset \mathrm{G}$ is a Levi subgroup, and let us show that φ is trivial on $\mathrm{SL}_2(\mathbb{C})$. This clearly reduces to an analogous question for $\mathrm{G_M}$, so that we may in fact assume $\mathrm{M} = \mathrm{G}$. Thus, $\Sigma(\varphi) \in \Phi_2^+(\mathrm{M}) = \Phi_2^+(\mathrm{G}) = \Phi_2(\mathrm{G})$ by Hypothesis 2.10.12. Let $\Theta_{\Sigma(\varphi)}$ be a nonzero $\mathcal{O}_\mathrm{G}$-invariant stable character supported on $\Sigma(\varphi)$ (as in Hypothesis 2.7.1), which is unique up to scalar multiplication.

Since $\Sigma(\varphi)$ consists of supercuspidal representations, Aubert-Zelevinsky involution preserves each element of $\Sigma(\varphi)$ up to a sign that depends only on G, and hence (by the uniqueness of $\Theta_{\Sigma(\varphi)}$) also preserves $\Theta_{\Sigma(\varphi)}$ up to a sign. Further, Aubert-Zelevinsky duality is compatible with twisted endoscopy by the discussion around and below [Art13, (7.1.5)]. Therefore, the tempered character of $\tilde{\mathrm{GL}}_W(F)$ to which $\Sigma(\varphi)$ transfers too is stable under the Aubert-Zelevinsky duality, up to a sign, forcing the representation supporting it to be supercuspidal (it is well-known that for general linear groups, the Aubert-Zelevinsky duality exchanges the two copies of $\mathrm{SL}_2(\mathbb{C})$ at the level of Arthur parameters). Thus, φ is a supercuspidal parameter, as desired.

Since we have seen Hypotheses 2.7.1, 2.10.3, 2.10.12 and 2.11.1, and since G is quasi-split, Hypothesis 2.11.4 follows by Proposition 2.11.6. □

7.3. **Inner forms of classical groups.** In this subsection, we give a proof of the existence of stable discrete series packets (essentially Hypothesis 2.7.1) for inner forms of symplectic, odd special orthogonal, unitary and odd general spin groups, as well as of the transfer of these packets to their quasi-split inner forms, using which we get a stability-compatible LLC+ for these groups (more specifically, Hypotheses 2.10.3 and 2.10.12). Similar results, but up to outer automorphisms, are proved for those inner forms of even special orthogonal groups that Arthur calls 'symmetric' in [Art13, Chapter 9], namely, those that are either associated to a quadratic space (without involving a division algebra) or whose quasi-split inner form is not split; a weaker variant is proved for the ones that are not symmetric. We hasten to add that, except possibly for odd general spin groups, all these results and much more are already known from the work of Arthur ([Art13, Chapter 9]), Mœglin and Renard ([MR18]), Kaletha, Minguez, Shin and White ([KMSW14]), and Ishimoto ([Ish23]), and that there is also an approach by Chen and Zou ([CZ20]) to the local Langlands correspondence for non-quasi-split unitary groups using the theta correspondence. However, our proof is different from the proofs in all these sources: we build on Shahidi's work on the constancy of the Plancherel μ-function on discrete series L-packets on Levi subgroups, as revisited in Corollary 5.2.12.

Notation 7.3.1. (i) In this subsection, we will follow the convention of denoting by M^* the quasi-split inner form of a connected reductive group M over F. We will often work with a (implicitly) fixed choice of an inner twist ψ_{G^*} from G^* to G, and an endoscopic datum $\underline{\mathrm{G}}^*$ as in Notation 3.2.1(i). If further M is a Levi subgroup of G, we will often consider a choice of $\mathrm{M}^*, \psi_{\mathrm{M}^*}$ and $\underline{\mathrm{M}}^*$ as in Notation 3.2.1(vi), defined using 'Levi subgroup matching data' as discussed there: this means we can fix, and have fixed, parabolic subgroups $\mathrm{P} \subset \mathrm{G}$

and $P^* \subset G^*$ having M and M^* respectively as Levi subgroups, and such that ψ_{G^*} takes $(P^*_{\bar{F}}, M^*_{\bar{F}})$ to $(P_{\bar{F}}, M_{\bar{F}})$.

(ii) Following [Art13], we will call G an inner form of a classical or odd general spin group if G is an inner form of a quasi-split symplectic, special orthogonal, unitary or odd general spin group. We call G symmetric if it additionally satisfies the following condition: if the quasi-split inner form G^* of G is even special orthogonal, then either G^* is not split, or the local index of G is not of type [Art13, (9.1.5) or (9.1.6)]. In other words, the condition is that either G is not an inner form of a split even special orthogonal group, or G is associated to a quadratic space over F rather than to a Hermitian form over a division algebra (thus, G is automatically symmetric if it is quasi-split).

(iii) Let G be an inner form of a classical or odd general spin group. If G (or equivalently G^*) is not unitary, we will let $E = F$; otherwise, we will let E be such that G (or equivalently G^*) is a unitary group associated to a quadratic extension E/F. If G is quasi-split, or if the root datum of $G_{\bar{F}}$ is of type A_n or B_n, or if G is even special orthogonal and of type ${}^1D_n$ associated to a quadratic space as opposed to a division algebra (see the '$d = 1$' subcase of the two cases discussed in [Art13, shortly above 9.1.5(i)]), then let $\mathbb{N}_G = \mathbb{N}$, and let $H_m = \mathrm{Res}_{E/F}\, \mathrm{GL}_m$ for every nonnegative integer m. In the remaining situations, i.e., if G is non-quasi-split and falls into one of the following three cases: it is of type C_n, it is of type ${}^2D_n$, or it is of type ${}^1D_n$ and is associated to a division algebra (see the '$d = 2$' subcase of the two cases discussed in [Art13, shortly above 9.1.5(i)]), then let $\mathbb{N}_G$ be the set of all even nonnegative integers, and for $m \in \mathbb{N}_G$, let H_m be the linear algebraic group associated to $\mathrm{GL}_{m/2}(D)$, where D is a quaternion division algebra over F. Note that in either case, associating H^*_m to G^* the same way H_m is associated to G, so that $H^*_m \cong \mathrm{Res}_{E/F}\, \mathrm{GL}_m$, we may and do fix an inner twist $H^*_m \to H_m$, which we will denote by $\psi_{H^*_m}$ in what follows, and take to be the identity map in the former case.

(iv) Suppose G is an inner form of a classical or odd general spin group. We will sometimes write $G = G_n$, where n is the dimension of the standard representation of $(G^*)_E$ (E as in (iii) above), and think of it as a member of an obvious series $\{G_m\}_m$ of groups 'of the same type'. We can realize G as the 'classical part' of a Levi subgroup of a bigger group of the same type as G: namely, for every $d_1, \ldots, d_r \in \mathbb{N}_G$, we can fix a realization of

$$M(d_1, \ldots, d_r; n) := H_{d_1} \times \cdots \times H_{d_r} \times G_n = H_{d_1} \times \cdots \times H_{d_r} \times G \tag{97}$$

as a Levi subgroup of the analogous but bigger group $G^+ := G_{2(d_1+\cdots+d_r)+n}$. In such a situation, we will also fix a realization of $M(d_1, \ldots, d_n)^* := H^*_{d_1} \times \cdots \times H^*_{d_r} \times G^*$ as a Levi subgroup of $(G^+)^* := G^*_{2(d_1+\cdots+d_r)+n}$. We will also fix (as we may in our cases; use, e.g., the discussion on inner forms vs inner twists in [Art13, Section 9.1]) an inner twist $\psi_{(G^+)^*}$ from $(G^+)^*$ to G^+, such that some inner twist from $M(d_1, \ldots, d_r; n)^*$ to $M(d_1, \ldots, d_r; n)$ obtained from $\psi_{(G^+)^*}$ using 'Levi subgroup matching data' as in Notation 3.2.1(vi), equals $\psi_{H^*_{d_1}} \times \cdots \times \psi_{H^*_{d_r}} \times \psi_{G^*}$. We may then change $\psi_{(G^+)^*}$ within its equivalence class to assume that it restricts to $\psi_{H^*_{d_1}} \times \cdots \times \psi_{H^*_{d_r}} \times \psi_{G^*}$. $\psi_{(G^+)^*}$ also defines an endoscopic datum, $\underline{(G^+)^*}$, for G^+ with underlying group $(G^+)^*$.

(v) Suppose $\underline{G} = G_n, G^+ = G_{2(d_1+\cdots+d_r)+n}$ and $H_{d_1} \times \cdots \times H_{d_r} \times G = M(d_1, \ldots, d_r; n) \subset G^+$ are as in (97). If we are given $\sigma \in \mathrm{Irr}_2(G)$ and $\upsilon_i \in \mathrm{Irr}_2(H_{d_i})$ for $1 \leq i \leq r$, then we will let $\mu(\upsilon_1 \otimes \cdots \otimes \upsilon_r \otimes \sigma)$ denote the Plancherel measure associated to the parabolic induction of $\upsilon_1 \otimes \cdots \otimes \upsilon_r \otimes \sigma$ from $M(d_1, \ldots, d_r; n)(F)$ to $G^+(F) = G_{2(d_1+\cdots+d_r)+n}(F)$, and set

$$\mu'''(\upsilon_1 \otimes \cdots \otimes \upsilon_r \otimes \sigma) = \gamma'''(G_{2(d_1+\cdots+d_r)+n} | M(d_1, \ldots, d_r; n)) \mu(\upsilon_1 \otimes \cdots \otimes \upsilon_r \otimes \sigma)$$

(see (60) for the definition of $\gamma'''(\cdot|\cdot)$). If we refer to these functions when $r = 0$, it will be understood that they will be ignored, i.e., replaced by 1.

(vi) If G is a symmetric inner form of a classical or odd general spin group, we will let $\mathcal{O}_G$ be trivial unless G^* is even special orthogonal and nonabelian (i.e., of rank greater than one), in which case we will often let $\mathcal{O}_G \subset \mathrm{Aut}(G)$ be the group generated by some F-rational automorphism of G whose image in $\mathrm{Out}(G)$ is the unique nontrivial element of

Out(G): such an automorphism exists, by the first sentence of the paragraph containing [Art13, (9.1.9)]. For each Levi subgroup $\mathrm{M} \subset \mathrm{G}$, we then let $\mathcal{O}_{\mathrm{M}}$ be as in Remark 2.6.2(i), i.e., the set of elements of $\mathcal{O}_{\mathrm{G}}^{+}$ that act trivially on Z_{M} (and hence automatically preserve M). It is easy to verify that $\mathcal{O}_{\mathrm{M}}$ consists of inner automorphisms of M if G^* is not even special orthogonal or if the 'classical part' of M is abelian, and it is easy to verify using the facts from [Art13] we just alluded to, that whenever G^* is even special orthogonal and the 'classical part' G_{M} of M is not abelian, $\mathcal{O}_{\mathrm{M}}$ is generated by $\operatorname{Int}\mathrm{M}(F)$ and an automorphism that restricts to an outer automorphism on G_{M}.

(vii) If G is an inner form of a classical or odd general spin group, which may not be symmetric, we will let $\bar{\mathcal{O}}_{\mathrm{M}}$ be the subgroup of Out(M) consisting of elements that induce the trivial automorphism of Z_{M}. If G is symmetric and $\mathcal{O}_{\mathrm{M}}$ is as in (vi) above, it is easy to see that the image of $\mathcal{O}_{\mathrm{M}}$ in Out(M) equals $\bar{\mathcal{O}}_{\mathrm{M}}$.

Remark 7.3.2. (i) Recall the objects defined in Notation 5.2.1(v). We abbreviate $\mathrm{M}^+ = \mathrm{M}(d_1, \ldots, d_r; n)$ and $\mathrm{G}^+ = \mathrm{G}_{2(d_1+\cdots+d_r)+n}$. It is easy to see from the definitions, the multiplicativity of the Plancherel measure ([Wal03, Lemma V.2.1]), and the multiplicativity of the expression $[K_{\mathrm{N}} : I_{\mathrm{N}}]^{-1}[K_{\mathrm{N}^-} : I_{\mathrm{N}^-}]^{-1}$ occurring in (60) (a general property of how parahoric groups intersect with unipotent radicals of appropriately positioned parabolic subgroups), that we have an expression:

$$\mu'''(\upsilon_1 \otimes \cdots \otimes \upsilon_r \otimes \sigma) = \prod_{\alpha} \mu_{\alpha}'''(\upsilon_1 \otimes \cdots \otimes \upsilon_r \otimes \sigma) = \left(\prod_{i=1}^{r} \mu'''(\upsilon_i \otimes \sigma)\right) \cdot {\prod_{\alpha}}' \mu_{\alpha}'''(\upsilon_1, \ldots, \upsilon_r). \tag{98}$$

Here, $\mu_{\alpha}'''(\cdot)$ denotes the μ'''-function associated to parabolic induction from M^+ to the Levi subgroup $\mathrm{M}_{\alpha}^{+} \subset \mathrm{G}^+$ obtained as the centralizer in G^+ of the connected kernel of α, and in the product in $\prod_{\alpha}$, α runs over all reduced roots of $\mathrm{A}_{\mathrm{M}^+}$ in G^+ up to a sign. Moreover, the last product in (98) is the sub-product over those α such that G is a direct factor of M_{α}^{+}: for such α, $\mu_{\alpha}'''(\upsilon_1 \otimes \cdots \otimes \upsilon_r \otimes \sigma)$ is independent of σ (and depends on only one or two of the υ_i's), and hence has been written as $\mu_{\alpha}'''(\upsilon_1, \ldots, \upsilon_r)$.

(ii) Now we additionally consider $(\mathrm{G}^+)^*$ and an inner twist $\psi_{(\mathrm{G}^+)^*}$ as in Notation 7.3.1(iv). Let $(\mathrm{M}^+)^* = \mathrm{M}(d_1, \ldots, d_r; n)^*$. Since $\psi_{(\mathrm{G}^+)^*}$ restricts to $\psi_{\mathrm{H}_{d_1}^*} \times \cdots \times \psi_{\mathrm{H}_{d_r}^*} \times \psi_{\mathrm{G}^*}$, its induces a bijection $\alpha \mapsto \alpha^*$ from the set of reduced roots of $\mathrm{A}_{\mathrm{M}^+}$ in G^+ to the set of reduced roots of $\mathrm{A}_{(\mathrm{M}^+)^*}$ in $(\mathrm{G}^+)^*$, and for each such α, restricts to an inner twist $((\mathrm{M}^+)_{\alpha^*}^*)_{\bar{F}} \to (\mathrm{M}_{\alpha}^+)_{\bar{F}}$ of the associated Levi subgroups. Note that we have an expression analogous to (98):

$$\mu'''(\upsilon_1^* \otimes \cdots \otimes \upsilon_r^* \otimes \sigma^*) = \prod_{\alpha^*} \mu_{\alpha^*}'''(\upsilon_1^* \otimes \cdots \otimes \upsilon_r^* \otimes \sigma^*) = \left(\prod_{i=1}^{r} \mu'''(\upsilon_i^* \otimes \sigma^*)\right) \cdot {\prod_{\alpha^*}}' \mu_{\alpha^*}'''(\upsilon_1^*, \ldots, \upsilon_r^*). \tag{99}$$

For each α, G is a direct factor of the associated Levi subgroup $\mathrm{M}_{\alpha}^{+} \supset \mathrm{M}$ if and only if G^* is a direct factor of the Levi subgroup $(\mathrm{M}^+)_{\alpha^*}^*$; for such an α, if we further assume that υ_i^* and υ_i are related by the Deligne-Kazhdan-Vigneras correspondence for each i, which is known to respect Plancherel measures (or use Corollary 5.2.12(ii)), we have:

$$\mu_{\alpha}'''(\upsilon_1, \ldots, \upsilon_r) = \mu_{\alpha^*}'''(\upsilon_1^*, \ldots, \upsilon_r^*). \tag{100}$$

When G is a symmetric inner form of a quasi-split classical or odd general spin group, the following is the main result of this subsection.

Theorem 7.3.3. *Let* G *be a symmetric inner form of a quasi-split classical or odd general spin group (see Notation 7.3.1(ii)), and let* $\{\mathcal{O}_{\mathrm{M}}\}_{\mathrm{M}}$ *be as in Notation 7.3.1(vi). Then the hypothesis on the existence of tempered L-packets, the LLC+ hypothesis, and the compatibility of LLC+ and stability (Hypotheses 2.7.1, 2.10.3 and 2.10.12) are satisfied by* G.

Before preparing to formulate a version of this theorem that applies to inner forms of classical groups that may not be symmetric, let us make some general remarks about our method and its limitations.

Remark 7.3.4. Our proof of Theorem 7.3.3 builds on the work of Shahidi from [Sha90], concerning the constancy of the Plancherel μ-function on discrete series L-packets on Levi subgroups. Specifically, we will apply Corollary 5.2.11 to bigger groups G^+ of the same type as G: as we have seen above, these have Levi subgroups M^+ that in turn have G as a factor. This will yield us a ring R of Plancherel measure based multipliers on spaces slightly bigger than $SD_{\zeta,\mathrm{ell}}(\mathrm{G})$ and $SD_{\zeta,\mathrm{ell}}(\mathrm{G}^*)$ for a fixed unitary character ζ of $\mathrm{A}_{\mathrm{G}}(F) = \mathrm{A}_{\mathrm{G}^*}(F)$ (this identification made using ψ_{G^*}). The simultaneous eigendecomposition for these actions will allow us to transfer stable packets from G^* to G. We now make a couple of observations on the scope of this method.

(i) Once the assertion concerning transfer factors associated to quasi-split even general spin groups mentioned in Remark 7.2.3 is verified, the proof of Theorem 7.3.3 given below should extend to this case, yielding similar assertions for those inner forms of even general spin groups that are symmetric in the sense that the inner forms of even special orthogonal groups obtained from them are symmetric.

(ii) Perhaps most of the arguments involved in the proof of Theorem 7.3.3 can also be adapted to inner forms of general symplectic and even general special orthogonal groups. However, for these groups, as for their quasi-split inner forms, the Plancherel measure cannot capture twists by characters that are trivial on both the derived group and the center. Thus, for these groups, these methods do not seem to yield anything more than what can be obtained using coarse methods from what is known about their derived groups. In the quasi-split case, Xu used global methods in [Xu18] to disentangle these twists and get finer decompositions from the coarser packets obtained naively from the derived groups.

Now we come to the task of formulating a version of Theorem 7.3.3 that makes sense in the case where G may not be symmetric: this will need us to have a version of Hypotheses 2.7.1, 2.10.3 and 2.10.12 that involves only outer automorphisms of Levi subgroups. We start with the following proposition, which helps us make outer automorphisms act on stable discrete series packets, essentially using [Art96] and the fact that elliptic tori transfer to all inner forms:

Proposition 7.3.5. *Let* M *be a connected reductive group over* F.

(i) View elements of $SD_{\mathrm{ell}}(\mathrm{M})$ *as functions on the space* $\overline{\mathrm{M}(F)_{\mathrm{ell}}/\sim}$ *of strongly regular elliptic stable conjugacy classes of* M*, viewed inside the adjoint quotient of* M *(this is the stable version of Notation 3.4.1(i), and a consequence of* [Art96] *explicated in the twisted setting in* [MW16, Theorem XI.3]*). Then any element of* $\mathrm{Out}(\mathrm{M})^\Gamma$ *(acting on the adjoint quotient) preserves* $\overline{\mathrm{M}(F)_{\mathrm{ell}}/\sim}$*, so that for any* $\Theta \in SD_{\mathrm{ell}}(\mathrm{M})$ *and* $\bar\beta \in \mathrm{Out}(\mathrm{M})^\Gamma$*,* $\Theta \circ \bar\beta$ *makes sense as such a function. Moreover, for any* $\Theta \in SD_{\mathrm{ell}}(\mathrm{M})$ *and* $\bar\beta \in \mathrm{Out}(\mathrm{M})^\Gamma$*, we have* $\Theta \circ \bar\beta \in SD_{\mathrm{ell}}(\mathrm{M})$.

(ii) Let $\bar{\mathcal{O}}_{\mathrm{M}} \subset \mathrm{Out}(\mathrm{M})^\Gamma$ *be a subgroup that is finite, and such that its image in* $\mathrm{Aut}(\mathrm{A}_{\mathrm{M}})$ *is trivial. Consider collections* $\Phi_2(\mathrm{M})$ *of pairwise disjoint subsets of* $\mathrm{Irr}_2(\mathrm{M})$ *such that for each* $\Sigma \in \Phi_2(\mathrm{M})$*, there exists an element* $\Theta_\Sigma \in SD_{\mathrm{ell}}(\mathrm{M})^{\bar{\mathcal{O}}_{\mathrm{M}}}$ *that is a (nonzero multiple of a) positive linear combination of the* Θ_σ *as* σ *varies over* Σ*, and such that* $\{\Theta_\Sigma \mid \Sigma \in \Phi_2(\mathrm{M})\}$ *forms a basis for the space* $SD_{\mathrm{ell}}(\mathrm{M})^{\bar{\mathcal{O}}_{\mathrm{M}}}$ *(which makes sense by (i) above). If such a collection* $\Phi_2(\mathrm{M})$ *exists, then it is unique and is a partition of* $\mathrm{Irr}_2(\mathrm{M})$.

Proof. If M is quasi-split, (i) follows from the fact that $\mathrm{Aut}(\mathrm{M}) \to \mathrm{Out}(\mathrm{M})^\Gamma$ is surjective (use a Γ-fixed pinning). To deal with the general case, we choose $\mathrm{M}^*, \underline{\mathrm{M}}^*$ and ψ_{M^*} as in Notation 3.2.1(i). Then the first assertion of (i) follows from the quasi-split case and the fact that elliptic tori in p-adic reductive groups transfer across inner forms (combine [Kot86, Lemmas 10.1 and 10.2]). For the second assertion, consider an automorphism $\beta^* \in \mathrm{Aut}(\mathrm{M}^*)$ mapping to $\bar\beta$ under $\mathrm{Out}(\mathrm{M})^\Gamma = \mathrm{Out}(\mathrm{M}^*)^\Gamma$, and let $\Theta^* \in SD_{\mathrm{ell}}(\mathrm{M}^*)$ map to Θ under the isomorphism $SD_{\mathrm{ell}}(\mathrm{M}^*) \to SD_{\mathrm{ell}}(\mathrm{M})$. Using Remark 3.2.2(i), it is easy to see that $\Theta \circ \bar\beta$ is the image of $\Theta^* \circ \beta^*$ under $SD_{\mathrm{ell}}(\mathrm{M}^*) \to SD_{\mathrm{ell}}(\mathrm{M})$, and hence belongs to $SD_{\mathrm{ell}}(\mathrm{M})$.

The uniqueness of $\Phi_2(\mathrm{M})$ follows exactly as with Hypothesis 2.7.1. Namely, if $\Phi_2(\mathrm{M})$ and $\Phi_2(\mathrm{M})'$ are two such collections, then the given conditions (including the positivity) imply that each element of $\Phi_2(\mathrm{M})$ (resp., $\Phi_2(\mathrm{M})'$) is a union of elements of $\Phi_2(\mathrm{M})'$ (resp., $\Phi_2(\mathrm{M})$). Now let us assume that such a collection $\Phi_2(\mathrm{M})$ exists, and sketch, following [Mg14, Proposition 2.4], a

proof that it partitions $\mathrm{Irr}_2(\mathrm{M})$. Let $\sigma_0 \in \mathrm{Irr}_2(\mathrm{M})$ have central character restricting to, say, ζ on $\mathrm{A}_\mathrm{M}(F) = \mathrm{A}_{\mathrm{M}^*}(F)$; it suffices to show that Θ_{σ_0} is not orthogonal to $SD_{\zeta,\mathrm{ell}}(\mathrm{M})^{\bar{\mathcal{O}}_\mathrm{M}}$ under the elliptic inner product on $D_{\zeta,\mathrm{ell}}(\mathrm{M})$ (see Notation 3.4.4). Choose a (possibly infinite) group $\mathcal{O}_{\mathrm{M}^*} \subset \mathrm{Aut}(\mathrm{M}^*)$ mapping onto $\bar{\mathcal{O}}_\mathrm{M} \subset \mathrm{Out}(\mathrm{M})^\Gamma = \mathrm{Out}(\mathrm{M}^*)^\Gamma$. It is easy to see from Remark 3.2.2(i) that the isomorphism $SD_{\zeta,\mathrm{ell}}(\mathrm{M}^*) \to SD_{\zeta,\mathrm{ell}}(\mathrm{M})$ respects the action of $\bar{\mathcal{O}}_\mathrm{M} \subset \mathrm{Out}(\mathrm{M})^\Gamma = \mathrm{Out}(\mathrm{M}^*)^\Gamma$. Therefore, since (31) respects inner products up to scalars (by [LMW18, Section 4.6, Lemma 3]), it suffices to show that the image of Θ_{σ_0} in $SD_{\zeta,\mathrm{ell}}(\mathrm{M}^*)$ according to (31) is not orthogonal to $SD_{\zeta,\mathrm{ell}}(\mathrm{M}^*)^{\mathcal{O}_{\mathrm{M}^*}}$. By the given hypotheses on $\bar{\mathcal{O}}_\mathrm{M}$ and the fact $\mathcal{O}_{\mathrm{M}^*}$ acts through a finite quotient on $D_{\zeta,\mathrm{ell}}(\mathrm{M}^*) \supset SD_{\zeta,\mathrm{ell}}(\mathrm{M}^*)$ preserving its inner product, we see that the orthogonal projection $SD_{\zeta,\mathrm{ell}}(\mathrm{M}^*) \to SD_{\zeta,\mathrm{ell}}(\mathrm{M}^*)^{\mathcal{O}_{\mathrm{M}^*}}$ is given by $\mathcal{O}_{\mathrm{M}^*}$-averaging.
Thus, it suffices to show that there exists a linear map $D_{\zeta,\mathrm{ell}}(\mathrm{M}) \to \mathbb{C}$ that factors as the composite of $D_{\zeta,\mathrm{ell}}(\mathrm{M}) \to SD_{\zeta,\mathrm{ell}}(\mathrm{M}^*)$ and an $\mathcal{O}_{\mathrm{M}^*}$-invariant map $SD_{\zeta,\mathrm{ell}}(\mathrm{M}^*) \to \mathbb{C}$, and is nonzero on Θ_{σ_0}. We claim that the map sending $\Theta \in D_{\zeta,\mathrm{ell}}(\mathrm{M})$ to the coefficient $c_0(\Theta)$ of the trivial nilpotent orbit in the character expansion of Θ at the identity satisfies these properties. This map is nonzero on Θ_{σ_0}, and factors as the composite of the projection $D_{\zeta,\mathrm{ell}}(\mathrm{M}) \to SD_{\zeta\,\mathrm{ell}}(\mathrm{M}^*)$ and the map $SD_{\mathrm{ell}}(\mathrm{M}^*) \to \mathbb{C}$ given by an analogous formula $\Theta^* \mapsto c_0(\Theta^*)$. This latter map is clearly $\mathcal{O}_{\mathrm{M}^*}$-invariant as well, proving our claim and hence the proposition. □

Remark 7.3.6. In [Art13, Section 9.4, around (9.4.11)], in the troublesome case of inner forms of even special orthogonal groups that are not symmetric, Arthur gives a considerably more sophisticated description of the action of the outer automorphism without a rational lift, on the quotient $\mathcal{I}(\mathrm{M}) = C_c^\infty(\mathrm{M}(F))/\operatorname{Int}\mathrm{M}(F)$. In fact, it operates via an involution on $\mathrm{Irr}_{\mathrm{temp}}(\mathrm{M})$. In contrast, we have only made $\bar{\mathcal{O}}_\mathrm{M}$ act on $SD_{\mathrm{ell}}(\mathrm{M})$, and our prescription is much more simple-minded, as it involves only naive considerations involving stable elliptic conjugacy classes.

Next, we consider systems $\{\bar{\mathcal{O}}_\mathrm{M}\}_\mathrm{M}$ as in Notation 2.6.1, but involving outer automorphisms in place of automorphisms.

Notation 7.3.7. Let $\{\bar{\mathcal{O}}_\mathrm{M}\}_\mathrm{M}$ be a collection indexed by Levi subgroups $\mathrm{M} \subset \mathrm{G}$, where $\bar{\mathcal{O}}_\mathrm{M} \subset \mathrm{Out}(\mathrm{M})^\Gamma$ for each M. We will call this collection nice if the following conditions somewhat analogous to the conditions in (iv) of Notation 2.6.1 are satisfied:

(a) For each M, under the well-defined map $\mathrm{Out}(\mathrm{M})^\Gamma \to \mathrm{Aut}(\mathrm{A}_\mathrm{M})$ induced by restriction, $\bar{\mathcal{O}}_\mathrm{M}$ has trivial image;
(b) Let $\mathrm{L}, \mathrm{M} \subset \mathrm{G}$ be Levi subgroups.
  - If $\beta \in \mathrm{Aut}(\mathrm{G})$ has image inside $\bar{\mathcal{O}}_\mathrm{G}$, then $\beta|_\mathrm{M}$ transports $\bar{\mathcal{O}}_\mathrm{M} \subset \mathrm{Out}(\mathrm{M})^\Gamma$ to $\bar{\mathcal{O}}_{\beta(\mathrm{M})} \subset \mathrm{Out}(\beta(\mathrm{M}))^\Gamma$;
  - If $\mathrm{L} \subset \mathrm{M}$ and $\beta \in \bar{\mathcal{O}}_\mathrm{L}$, then the well-defined $\operatorname{Int}\hat{\mathrm{M}}$-orbit of inclusions $\iota_{\mathrm{M},\mathrm{L}} \circ {}^L\beta : {}^L\mathrm{L} \to {}^L\mathrm{M}$ (see Notation 2.4.6(ii) for $\iota_{\mathrm{M},\mathrm{L}}$, and note that ${}^L\beta$ is an L-automorphism of ${}^L\mathrm{L}$ that is well-defined up to $\operatorname{Int}\hat{\mathrm{L}}$), is contained in the well-defined $\operatorname{Int}\hat{\mathrm{M}}$-orbit of inclusions ${}^L\tilde{\beta} \circ \iota_{\mathrm{M},\mathrm{L}}$, for some $\tilde{\beta} \in \bar{\mathcal{O}}_\mathrm{M}$ (so that ${}^L\tilde{\beta}$ is an L-automorphism of ${}^L\mathrm{M}$ that is well-defined up to $\operatorname{Int}\hat{\mathrm{M}}$);
(c) Each $\bar{\mathcal{O}}_\mathrm{M}$ is finite.

Lemma 7.3.8. *Let $\{\bar{\mathcal{O}}_\mathrm{M}\}_\mathrm{M}$ be a nice collection of automorphisms of Levi subgroups of* G*, as in Notation 7.3.7.*

(i) Let $\mathrm{L} \subset \mathrm{M} \subset \mathrm{G}$ be Levi subgroups. Then $\varphi \mapsto \lambda(\varphi)$ induces a well-defined map $\Phi(\mathrm{M})/\bar{\mathcal{O}}_\mathrm{M} \to \Omega({}^L\mathrm{M})/\bar{\mathcal{O}}_\mathrm{M}$.
(ii) The actions of $\mathrm{Z}_{\hat{\mathrm{M}}}^{\Gamma,0} = H^1(W_F/I_F, \mathrm{Z}_{\hat{\mathrm{M}}}^{\Gamma,0})$ and $(\mathrm{Z}_{\hat{\mathrm{M}}}^{I_F})_{\mathrm{Fr}}^0$ on $\Phi(\mathrm{M}), \Phi_2(\mathrm{M}), \Omega(\mathrm{M})$ etc. descend to actions on $\Phi(\mathrm{M})/\bar{\mathcal{O}}_\mathrm{M}, \Phi_2(\mathrm{M})/\bar{\mathcal{O}}_\mathrm{M}, \Omega(\mathrm{M})/\bar{\mathcal{O}}_\mathrm{M}$ etc.
(iii) (Any choice of) $\iota_{\mathrm{M},\mathrm{L}} : {}^L\mathrm{L} \to {}^L\mathrm{M}$ induces well-defined maps $\Phi(\mathrm{L})/\bar{\mathcal{O}}_\mathrm{L} \to \Phi(\mathrm{M})/\bar{\mathcal{O}}_\mathrm{M}$ and $\Omega({}^L\mathrm{L})/\bar{\mathcal{O}}_\mathrm{L} \to \bar{\Omega}({}^L\mathrm{M})/\bar{\mathcal{O}}_\mathrm{M}$.

Proof. (i) is immediate. (ii) follows by fixing an inner twist to M from a quasi-split form M^*, noting that the elements of $\bar{\mathcal{O}}_{\mathrm{M}^*} = \bar{\mathcal{O}}_\mathrm{M}$ inside $\mathrm{Out}(\mathrm{M}) = \mathrm{Out}(\mathrm{M}^*)$ lift to $\mathrm{Aut}(\mathrm{M}^*)$, applying the

proof of Lemma 2.6.5 to these lifts (which applies because $\bar{\mathcal{O}}_{\mathrm{M}} \hookrightarrow \mathrm{Out}(\mathrm{M}) \to \mathrm{Aut}(\mathrm{A}_{\mathrm{M}})$ is trivial), and transporting to the assertions on M claimed in (ii) via the inner twist.
(iii) is immediate from the second condition in (b) of Notation 7.3.7. □

Hypothesis 7.3.9. (Compare with Hypotheses 2.10.3 and 2.10.12) This hypothesis is stated with respect to a nice collection $\{\bar{\mathcal{O}}_{\mathrm{M}}\}_{\mathrm{M}}$ of outer automorphisms of Levi subgroups of G, as in Notation 7.3.7. Given this, this hypothesis assumes that the following conditions are satisfied:

(i) For each Levi subgroup $\mathrm{M} \subset \mathrm{G}$, there exists a partition $\mathit{\Phi}_2(\mathrm{M})$ of $\mathrm{Irr}_2(\mathrm{M})$ into finite subsets, and for each $\Sigma \in \mathit{\Phi}_2(\mathrm{M})$ some $\Theta_\Sigma \in SD_{\mathrm{ell}}(\mathrm{M})^{\bar{\mathcal{O}}_{\mathrm{M}}}$ that is a (nonzero multiple of a) positive linear combination of the Θ_σ with $\sigma \in \Sigma$, such that the Θ_Σ form a basis for $SD_{\mathrm{ell}}(\mathrm{M})^{\bar{\mathcal{O}}_{\mathrm{M}}}$ (here, $SD_{\mathrm{ell}}(\mathrm{M})^{\bar{\mathcal{O}}_{\mathrm{M}}}$ makes sense by Proposition 7.3.5).

(ii) For each Levi subgroup $\mathrm{M} \subset \mathrm{G}$, there exists a finite-to-one surjective map $\mathrm{Irr}_2(\mathrm{M}) \to \Phi_2(\mathrm{M})/\bar{\mathcal{O}}_{\mathrm{M}}$, denoted $\sigma \mapsto \varphi_\sigma$, such that the following conditions are satisfied:
  (a) For each Levi subgroup $\mathrm{M} \subset \mathrm{G}$, each $\sigma \in \mathrm{Irr}_2(\mathrm{M})$ and $\alpha \in H^1(W_F/I_F, \mathrm{Z}_{\hat{\mathrm{M}}}^{\Gamma,0})$ such that χ_α is unitary, $\varphi_{\sigma\otimes\chi_\alpha} = \alpha \cdot \varphi_\sigma$ (where $\alpha \cdot \varphi_\sigma$ makes sense by Lemma 7.3.8(ii)).
  (b) Let $\mathrm{L} \subset \mathrm{M} \subset \mathrm{G}$ be Levi subgroups, and suppose that $\sigma \in \mathrm{Irr}_2(\mathrm{M})$ is an irreducible subquotient of $\mathrm{Ind}_{\mathrm{L}}^{\mathrm{M}}(\upsilon \otimes \chi_\alpha)$ for some supercuspidal representation $\upsilon \in \mathrm{Irr}_2(\mathrm{L})$ and some $\alpha \in \mathrm{Z}_{\hat{\mathrm{L}}}^{\Gamma,0}$. Then $\iota_{\mathrm{M},\mathrm{L}} \circ \lambda(\alpha \cdot \varphi_\upsilon) = \lambda(\varphi_\sigma) \in \Omega({}^L\mathrm{M})/\bar{\mathcal{O}}_{\mathrm{M}}$ (this equality makes sense by (i), (ii) and (iii) of Lemma 7.3.8).
  (c) Suppose $\mathrm{M} \subset \mathrm{G}$ is a Levi subgroup, and suppose $\beta \in \mathrm{Aut}(\mathrm{G})$ has image in $\bar{\mathcal{O}}_{\mathrm{G}}$. Then for all $\sigma \in \mathrm{Irr}_2(\mathrm{M})$ we have ${}^L(\beta|_{\mathrm{M}}) \circ \varphi_{\sigma\circ\beta^{-1}} = \varphi_\sigma$.

(iii) (Compare with Hypothesis 2.10.12) The choices involved in (ii) can be made to satisfy the following: For any Levi subgroup $\mathrm{M} \subset \mathrm{G}$, sending $\varphi \in \Phi_2(\mathrm{M})/\bar{\mathcal{O}}_{\mathrm{M}}$ to the fiber, over φ, of the map $\mathrm{Irr}_2(\mathrm{M}) \to \Phi_2(\mathrm{M})/\bar{\mathcal{O}}_{\mathrm{M}}$ given by $\sigma \mapsto \varphi_\sigma$, is a bijection $\Phi_2(\mathrm{M})/\bar{\mathcal{O}}_{\mathrm{M}} \to \mathit{\Phi}_2(\mathrm{M})$.

Now we can state the generalization of Theorem 7.3.3 to inner forms of quasi-split classical or odd general spin groups that may not be symmetric:

Theorem 7.3.10. *Let* G *be an inner form of a quasi-split classical or odd general spin group (see Notation 7.3.1(ii)), and let* $\{\bar{\mathcal{O}}_{\mathrm{M}}\}_{\mathrm{M}}$ *be as in Notation 7.3.1(vii). Then* $\{\bar{\mathcal{O}}_{\mathrm{M}}\}_{\mathrm{M}}$ *is nice in the sense of Notation 7.3.7, and with respect to this collection,* G *satisfies Hypothesis 7.3.9.*

We will now make some preparations to prove this theorem, via transfer from the quasi-split form.

Lemma 7.3.11. *We allow* G *to be arbitrary, and do not consider* $\{\mathcal{O}_{\mathrm{M}}\}_{\mathrm{M}}$. *Instead, recalling* G^* *and the fixed inner twist* ψ_{G^*} *from Notation 7.3.1(i), we assume given a collection* $\{\mathcal{O}_{\mathrm{M}^*}\}_{\mathrm{M}^*}$ *of automorphisms of Levi subgroups of* G^*, *analogous to the one in Notation 2.6.1, and for each* M^* *write* $\bar{\mathcal{O}}_{\mathrm{M}^*}$ *for the image of* $\mathcal{O}_{\mathrm{M}^*}$ *in* $\mathrm{Out}(\mathrm{M}^*)$.

(i) For a Levi subgroup $\mathrm{M} \subset \mathrm{G}$, *make a choice of Levi subgroup matching data* $(\mathrm{M}^*, \psi_{\mathrm{M}^*})$ *as in Remark 3.2.2(vi), and use it to define* $\bar{\mathcal{O}}_{\mathrm{M}} \subset \mathrm{Out}(\mathrm{M})$ *as the image* $\bar{\mathcal{O}}_{\mathrm{M}^*}$ *of* $\mathcal{O}_{\mathrm{M}^*}$ *in* $\mathrm{Out}(\mathrm{M}^*) = \mathrm{Out}(\mathrm{M})$. *Then* $\bar{\mathcal{O}}_{\mathrm{M}}$ *is independent of the choice of the Levi subgroup matching data* $(\mathrm{M}^*, \psi_{\mathrm{M}^*})$.

(ii) Let $\mathrm{L} \subset \mathrm{M} \subset \mathrm{G}$ *be Levi subgroups. Then Levi subgroup matching data* $(\mathrm{L}^*, \psi_{\mathrm{L}^*})$ *and* $(\mathrm{M}^*, \psi_{\mathrm{M}^*})$ *can be chosen for* L *and* M, *such that the resulting identifications* ${}^L\mathrm{L} = {}^L\mathrm{L}^*$ *and* ${}^L\mathrm{M} = {}^L\mathrm{M}^*$ *transport some choice for* $\iota_{\mathrm{M},\mathrm{L}}$ *to some choice for* $\iota_{\mathrm{M}^*,\mathrm{L}^*}$.

(iii) The collection $\{\bar{\mathcal{O}}_{\mathrm{M}}\}_{\mathrm{M}}$ *obtained from (i) above is nice.*

Proof. Let us prove (i). Suppose $(\mathrm{M}^*, \psi_{\mathrm{M}^*})$ and $((\mathrm{M}^*)', \psi_{(\mathrm{M}^*)'})$ are two different choices of Levi subgroup matching data. We know from the discussion of Notation 3.2.1(vii) that $\psi_{(\mathrm{M}^*)'} = \psi_{\mathrm{M}^*} \circ \mathrm{Int}(m^* w)$ for some $w \in \mathrm{G}^*(F)$ transporting $(\mathrm{M}^*)'$ to M^*, and some $m^* \in \mathrm{M}^*(\bar{F})$. Since w transports $\mathcal{O}^+_{(\mathrm{M}^*)'} = \mathcal{O}_{(\mathrm{M}^*)'} \circ \mathrm{Int}(\mathrm{M}^*)'(F)$ to $\mathcal{O}^+_{\mathrm{M}^*} = \mathcal{O}_{\mathrm{M}^*} \circ \mathrm{Int}\,\mathrm{M}^*(F)$ (by the condition in (iv)b of Notation 2.6.1), and since $\mathrm{Int}\, m^*$ fixes $\mathrm{Out}(\mathrm{M}^*_{\bar{F}})$, (i) is immediate.
Now let us show (ii). It is easy to choose parabolic subgroups $\mathrm{P}, \mathrm{Q} \subset \mathrm{G}$ with Levi subgroups M, L respectively, such that $\mathrm{P} \supset \mathrm{Q}$. Choose $\psi^* \in \psi_{\mathrm{G}^*} \circ \mathrm{Int}\,\mathrm{G}^*(\bar{F})$ such that $(\psi^*)^{-1}$ takes $(\mathrm{Q}_{\bar{F}}, \mathrm{L}_{\bar{F}})$ to $(\mathrm{Q}^*_{\bar{F}}, \mathrm{L}^*_{\bar{F}})$ for some parabolic-Levi pair $(\mathrm{Q}^*, \mathrm{L}^*)$ in G. Since $(\psi^*)^{-1}(\mathrm{P}_{\bar{F}}, \mathrm{M}_{\bar{F}})$ contains $(\mathrm{Q}^*_{\bar{F}}, \mathrm{L}^*_{\bar{F}})$, and since its $\mathrm{G}^*(\bar{F})$-conjugacy class is defined over F (since ${}^a(\psi^*)^{-1} \in \mathrm{Int}\,\mathrm{L}^*(\bar{F}) \circ (\psi^*)^{-1}$ for all

$a \in \mathrm{Gal}(\bar{F}/F)$), it follows that $(\psi^*)^{-1}(\mathrm{P}_{\bar{F}}, \mathrm{M}_{\bar{F}}) = (\mathrm{P}^*_{\bar{F}}, \mathrm{M}^*_{\bar{F}})$ for some parabolic-Levi pair $(\mathrm{P}^*, \mathrm{M}^*)$ in G^* containing $(\mathrm{Q}^*, \mathrm{L}^*)$. Now note that we have Levi subgroup matching data $(\mathrm{L}^*, \psi_{\mathrm{L}^*})$ and $(\mathrm{M}^*, \psi_{\mathrm{M}^*})$, where ψ_{L^*} and ψ_{M^*} are respectively the restrictions of ψ^* to $\mathrm{L}^*_{\bar{F}}$ and $\mathrm{M}^*_{\bar{F}}$. For these choices, using that ψ_{M^*} takes the parabolic-Levi pair $((\mathrm{Q}^* \cap \mathrm{M}^*)_{\bar{F}}, \mathrm{L}^*_{\bar{F}})$ of $\mathrm{M}^*_{\bar{F}}$ to the parabolic-Levi pair $((\mathrm{Q} \cap \mathrm{M})_{\bar{F}}, \mathrm{L}_{\bar{F}})$ of $\mathrm{M}_{\bar{F}}$, it is easy to see that the identifications ${}^L\mathrm{L} = {}^L\mathrm{L}^*$ and ${}^L\mathrm{M} = {}^L\mathrm{M}^*$ transport a choice for $\iota_{\mathrm{M},\mathrm{L}}$ to one for $\iota_{\mathrm{M}^*,\mathrm{L}^*}$ as desired, giving (ii) (we encountered this in the "second way to describe the resulting identification ${}^L\mathrm{M}_1 = {}^L\mathrm{M}_1^*$" in Remark 3.2.2(vi)).

Now we come to (iii), for which the only nontrivial conditions to verify are the ones in (b) of Notation 7.3.7. Let us prove the first condition. Choose $\beta^* \in \mathcal{O}^*_{\mathrm{G}^*}$ such that β and β^* have the same image in $\mathrm{Out}(\mathrm{G}) = \mathrm{Out}(\mathrm{G}^*)$. This translates to saying that $\beta \circ \psi_{\mathrm{G}^*} \circ (\beta^*)^{-1} \in \mathrm{Int}\, \mathrm{G}(\bar{F}) \circ \psi_{\mathrm{G}^*}$. Using this, it is easy to see that, given Levi subgroup matching data $(\mathrm{M}^*, \psi_{\mathrm{M}^*})$ for M, we can choose Levi subgroup matching data for $\beta(\mathrm{M})$ to be $(\beta^*(\mathrm{M}^*), \psi_{\beta^*(\mathrm{M})^*})$, where $\psi_{\beta^*(\mathrm{M}^*)} = \beta \circ \psi_{\mathrm{M}^*} \circ (\beta^*)^{-1}$, i.e., $\beta = \psi^{-1}_{\beta^*(\mathrm{M}^*)} \circ \beta^* \circ \psi^{-1}_{\mathrm{M}^*}$. Then the map $\mathrm{Out}(\mathrm{M}) \to \mathrm{Out}(\beta(\mathrm{M}))$ induced by transport by β identifies with the map $\mathrm{Out}(\mathrm{M}^*) \to \mathrm{Out}(\beta^*(\mathrm{M}^*))$ induced by transport by β^*. From this and the fact that β^* transports $\mathcal{O}_{\mathrm{M}^*}$ to $\mathcal{O}_{\beta^*(\mathrm{M}^*)}$ (see Notation 2.6.1(iv)b), the first condition of (b) of Notation 7.3.7 is immediate.

Now let us prove the second condition of (b) of Notation 7.3.7. Choose Levi subgroup matching data $(\mathrm{L}^*, \psi_{\mathrm{L}^*})$ and $(\mathrm{M}^*, \psi_{\mathrm{M}^*})$ for L and M, as in (ii). If $\beta^* \in \mathcal{O}_{\mathrm{L}^*}$, then there exists $\tilde{\beta}^* \in \mathcal{O}^+_{\mathrm{M}^*}$ restricting to it (see Notation 2.6.1(iv)b). Lemma 2.4.16(ii) gives an equality that may be written $\iota_{\mathrm{M}^*,\mathrm{L}^*} \circ {}^L\beta^* \in \mathrm{Int}\, \hat{\mathrm{M}}^* \circ \tilde{\beta}^* \circ \iota_{\mathrm{M}^*,\mathrm{L}^*}$. Thus, the second condition of (b) of Notation 7.3.7 follows from the property, ensured by (ii), that our Levi subgroup matching data transports $\iota_{\mathrm{M},\mathrm{L}}$ to $\iota_{\mathrm{M}^*,\mathrm{L}^*}$. This finishes the proof of (iii), and hence of the lemma. □

Corollary 7.3.12. *Let* G *be an inner form of a quasi-split classical or odd general spin group* G^**, and fix an inner twist* ψ_{G^*} *from* G^* *to* G*. Let* $\{\mathcal{O}_{\mathrm{M}^*}\}_{\mathrm{M}^*}$ *be associated to* G^* *as in Notation 7.3.1(vi), and let* $\{\bar{\mathcal{O}}_{\mathrm{M}}\}_{\mathrm{M}}$ *be associated to* G *as in Notation 7.3.1(vii). Then* $\{\bar{\mathcal{O}}_{\mathrm{M}}\}_{\mathrm{M}}$ *is obtained from* $\{\mathcal{O}_{\mathrm{M}^*}\}_{\mathrm{M}^*}$ *as in Lemma 7.3.11, and is nice in the sense of Notation 7.3.7.*

Proof. The first assertion is a straightforward verification, and the second assertion follows from Lemma 7.3.11(iii). □

Proposition 7.3.13. *We allow* G *to be arbitrary, and do not consider* $\{\mathcal{O}_{\mathrm{M}}\}_{\mathrm{M}}$*, instead letting* $\mathrm{G}^*, \psi_{\mathrm{G}^*}, \{\mathcal{O}_{\mathrm{M}^*}\}_{\mathrm{M}^*}$ *and* $\{\bar{\mathcal{O}}_{\mathrm{M}}\}_{\mathrm{M}}$ *be as in Lemma 7.3.11 above, so that the collection* $\{\bar{\mathcal{O}}_{\mathrm{M}}\}_{\mathrm{M}}$ *is nice. Assume that* G^**, with respect to the collection* $\{\mathcal{O}_{\mathrm{M}^*}\}_{\mathrm{M}^*}$*, satisfies Hypotheses 2.7.1, 2.10.3 and 2.10.12. For each Levi subgroup* $\mathrm{M} \subset \mathrm{G}$*, assume that there exists a choice of Levi subgroup matching data as in Notation 3.2.1(vi) (i.e., as in Notation 7.3.1(i)) involving a Levi subgroup* $\mathrm{M}^* \subset \mathrm{G}^*$*, such that for each* $\Sigma^* \in \Phi_2(\mathrm{M}^*)$*, there exists a finite subset* $\Sigma \subset \mathrm{Irr}_2(\mathrm{M})$ *that is a transfer of* Σ^* *in the following sense: under the isomorphism* $SD_{\mathrm{ell}}(\mathrm{M}^*) \to SD_{\mathrm{ell}}(\mathrm{M})$*, some nonzero* $\Theta_{\Sigma^*} \in SD_{\mathrm{ell}}(\mathrm{M}^*)^{\mathcal{O}_{\mathrm{M}^*}}$ *supported on* Σ^* *transfers to some* $\Theta_\Sigma \in SD_{\mathrm{ell}}(\mathrm{M})$ *that is a nonzero multiple of a positive linear combination of the* Θ_σ *as* σ *ranges over* Σ*. Assume that the following additional property is satisfied: if* $\Sigma_1^*, \Sigma_2^* \in \Phi_2(\mathrm{M}^*)$ *transfer this way to* $\Sigma_1, \Sigma_2 \subset \mathrm{Irr}_2(\mathrm{M})$ *and are distinct (and hence disjoint), then* Σ_1 *and* Σ_2 *are also disjoint. For each Levi subgroup* $\mathrm{M} \subset \mathrm{G}$*, choose Levi subgroup matching data* $(\mathrm{M}^*, \psi_{\mathrm{M}^*})$ *such that the* $\Sigma^* \in \Phi_2(\mathrm{M}^*)$ *transfer to* M *in the sense just described, and define the following objects:*

- *Let* $\Phi_2(\mathrm{M})$ *be the collection consisting of all the* $\Sigma \subset \mathrm{Irr}_2(\mathrm{M})$ *obtained by transferring the* $\Sigma^* \in \Phi_2(\mathrm{M}^*)$ *in the manner mentioned above, with respect to* $(\mathrm{M}^*, \psi_{\mathrm{M}^*})$*. To proceed, we note that* $\Phi_2(\mathrm{M})$ *is a partition of* $\mathrm{Irr}_2(\mathrm{M})$*: since the image of* $SD_{\mathrm{ell}}(\mathrm{M}^*)^{\mathcal{O}_{\mathrm{M}^*}} \hookrightarrow SD_{\mathrm{ell}}(\mathrm{M}^*) \to SD_{\mathrm{ell}}(\mathrm{M})$ *equals* $SD_{\mathrm{ell}}(\mathrm{M})^{\bar{\mathcal{O}}_{\mathrm{M}}}$ *(use Remark 3.2.2(i) and Proposition 7.3.5(i)), this follows from Proposition 7.3.5(ii).*
- *Using the resulting identifications* ${}^L\mathrm{M} = {}^L\mathrm{M}^*$*, identify* $\Phi(\mathrm{M}) = \Phi(\mathrm{M}^*), \Phi_2(\mathrm{M}) = \Phi_2(\mathrm{M}^*), \Omega(\mathrm{M}) = \Omega(\mathrm{M}^*)$*, etc., and recall that the identification* $\mathrm{Out}(\mathrm{M}) = \mathrm{Out}(\mathrm{M}^*)$ *maps* $\bar{\mathcal{O}}_{\mathrm{M}}$ *to the image* $\bar{\mathcal{O}}_{\mathrm{M}^*}$ *of* $\mathcal{O}_{\mathrm{M}^*}$ *in* $\mathrm{Out}(\mathrm{M}^*)$*. Define a map* $\mathrm{Irr}_2(\mathrm{M}) \to \Phi_2(\mathrm{M})/\bar{\mathcal{O}}_{\mathrm{M}}$ *by* $\sigma \mapsto \varphi_\sigma$*, where one lets* $\Sigma \in \Phi_2(\mathrm{M})$ *be the packet containing* σ*, lets* $\Sigma^* \in \Phi_2(\mathrm{M}^*)$ *be the packet mapping to* Σ *under the bijection* $\Phi_2(\mathrm{M}^*) \to \Phi_2(\mathrm{M})$*, and lets* $\varphi_\sigma \in \Phi_2(\mathrm{M})/\bar{\mathcal{O}}_{\mathrm{M}} = \Phi_2(\mathrm{M}^*)/\mathcal{O}_{\mathrm{M}^*}$ *be*

the element that identifies with φ_{σ^*}, for any $\sigma^* \in \Sigma^*$ (this does not depend on σ^*, since Hypothesis 2.10.12 is being assumed for G^**).*

(i) *For each Levi subgroup $\mathrm{M} \subset \mathrm{G}$, $\Phi_2(\mathrm{M})$ and the map $\mathrm{Irr}_2(\mathrm{M}) \to \Phi_2(\mathrm{M})/\bar{\mathcal{O}}_\mathrm{M}$ can be defined as above starting from any choice of the matching data $(\mathrm{M}^*, \psi_{\mathrm{M}^*})$, and is independent of this choice.*

(ii) *With respect to the collection $\{\bar{\mathcal{O}}_\mathrm{M}\}_\mathrm{M}$ and the objects $\Phi_2(\mathrm{M})$ and the maps $\mathrm{Irr}_2(\mathrm{M}) \to \Phi_2(\mathrm{M})/\bar{\mathcal{O}}_\mathrm{M}$ defined above, Hypothesis 7.3.9 is satisfied.*

Proof. Let us prove (i). Let $\mathrm{M} \subset \mathrm{G}$ be a Levi subgroup, and consider Levi subgroup matching data $(\mathrm{M}^*, \psi_{\mathrm{M}^*})$ and $((\mathrm{M}')^*, \psi_{(\mathrm{M}')^*})$ for M, with $(\mathrm{M}^*, \psi_{\mathrm{M}^*})$ being used to define $\Phi_2(\mathrm{M})$, i.e., $\Phi_2(\mathrm{M}^*)$ transfers to $\Phi_2(\mathrm{M})$ via $(\mathrm{M}^*, \psi_{\mathrm{M}^*})$ in the manner described in the proposition. As in Notation 3.2.1(vii), we have an equality $\psi_{(\mathrm{M}^*)'} = \psi_{\mathrm{M}^*} \circ \mathrm{Int}(m^* w)$, with $m^* \in \mathrm{M}^*(\bar{F})$, and with $w \in \mathrm{G}^*(F)$ transporting $(\mathrm{M}')^*$ to M^*. Let $\Sigma \in \Phi_2(\mathrm{M})$, and assume that Σ is a transfer of $\Sigma^* \in \Phi_2(\mathrm{M}^*)$ via $(\mathrm{M}^*, \psi_{\mathrm{M}^*})$. Then it is easy to see, using Remark 3.2.2(i), that the map $SD_{\mathrm{ell}}(\mathrm{M}^*) \to SD_{\mathrm{ell}}(\mathrm{M})$ is the composite of $SD_{\mathrm{ell}}((\mathrm{M}^*)') \to SD_{\mathrm{ell}}(\mathrm{M})$ and the map $SD_{\mathrm{ell}}(\mathrm{M}^*) \to SD_{\mathrm{ell}}((\mathrm{M}^*)')$ induced by pulling back under $\mathrm{Int}\, w$, which gives that Σ is the transfer of $(\Sigma^*)' := \Sigma^* \circ \mathrm{Int}\, w$ via $((\mathrm{M}^*)', \psi_{(\mathrm{M}^*)'})$. Now it is easy to see that $\Phi_2((\mathrm{M}^*)')$ transfers via $((\mathrm{M}^*)', \psi_{(\mathrm{M}^*)'})$ as well, and gives the same $\Phi_2(\mathrm{M})$ as the set of transfers.

To see that the map $\mathrm{Irr}_2(\mathrm{M}) \to \Phi_2(\mathrm{M})/\bar{\mathcal{O}}_\mathrm{M}$ given by $\sigma \mapsto \varphi_\sigma$ is independent of the Levi subgroup matching data, let $(\mathrm{M}^*, \psi_{\mathrm{M}^*}), ((\mathrm{M}')^*, \psi_{(\mathrm{M}')^*}), \Sigma, \Sigma^*, m^*$ and w be as above, and assume without loss of generality that $\sigma \in \Sigma$. We saw that via $((\mathrm{M}')^*, \psi_{(\mathrm{M}')^*})$, Σ is the transfer of $(\Sigma')^* := \Sigma^* \circ \mathrm{Int}\, w$. If $\sigma^* \in \Sigma^*$, then $\sigma^* \circ \mathrm{Int}\, w \in (\Sigma^*)'$, and $\varphi_{\sigma^* \circ \mathrm{Int}\, w} = {}^L(\mathrm{Int}\, w) \circ \varphi_{\sigma^*}$ by (iii) of Hypothesis 2.10.3 as assumed for G^*, where ${}^L(\mathrm{Int}\, w) : {}^L\mathrm{M}^* \to {}^L(\mathrm{M}^*)'$ is dual to $\mathrm{Int}\, w : (\mathrm{M}')^* \to \mathrm{M}^*$. Therefore, the rest of (i) follows from the easy observation that, because $\psi_{(\mathrm{M}^*)'} = \psi_{\mathrm{M}^*} \circ \mathrm{Int}(m^* w)$, the identification ${}^L\mathrm{M} = {}^L\mathrm{M}^*$, when composed with ${}^L(\mathrm{Int}\, w)$, gives the identification ${}^L\mathrm{M} = {}^L(\mathrm{M}^*)'$ up to an inner automorphism.

Now we come to (ii). It is easy to see, using Remark 3.2.2(i) and Proposition 7.3.5(i), that the image of $SD_{\mathrm{ell}}(\mathrm{M}^*)^{\mathcal{O}_{\mathrm{M}^*}} \hookrightarrow SD_{\mathrm{ell}}(\mathrm{M}^*) \to SD_{\mathrm{ell}}(\mathrm{M})$ equals $SD_{\mathrm{ell}}(\mathrm{M})^{\bar{\mathcal{O}}_\mathrm{M}}$. Proposition 7.3.5(ii) now gives the condition (i) of Hypothesis 7.3.9 (and also another proof of the fact that $\Phi_2(\mathrm{M})$, when defined, is independent of the choice of the Levi subgroup matching data).

For the condition (ii)(a) of Hypothesis 7.3.9, choose Levi subgroup matching data $(\mathrm{M}^*, \psi_{\mathrm{M}^*})$ for M; then the condition is an immediate consequence of the definitions, the compatibility of the relevant endoscopic transfer with unramified twisting (Lemma 3.2.3(i)), and the equality $\varphi_{\sigma^* \otimes \chi_{\alpha^*}} = \alpha^* \cdot \varphi_{\sigma^*}$, where α^* is the image of α under the identification $H^1(W_F/I_F, \mathrm{Z}^{\Gamma,0}_{\hat{\mathrm{M}}}) \to H^1(W_F/I_F, \mathrm{Z}^{\Gamma,0}_{\hat{\mathrm{M}}^*})$; use that χ_α is unitary if and only if χ_{α^*} is unitary, as these identify via the isomorphism $\mathrm{S}_{\mathrm{M}^*} = \mathrm{S}_\mathrm{M}$.

Now we come to the condition (ii)(b) of Hypothesis 7.3.9. Choose parabolic subgroups $\mathrm{P} \supset \mathrm{Q}$ in G, parabolic-Levi pairs $(\mathrm{P}^*, \mathrm{M}^*)$ and $(\mathrm{Q}^*, \mathrm{L}^*)$ in G^*, and inner twists ψ_{L^*} and ψ_{M^*} as in the proof of (ii) of Lemma 7.3.11, so that up to $\hat{\mathrm{M}}$-conjugacy, the embedding $\iota_{\mathrm{M},\mathrm{L}} : {}^L\mathrm{L} \hookrightarrow {}^L\mathrm{M}$ identifies with the embedding $\iota_{\mathrm{M}^*,\mathrm{L}^*} : {}^L\mathrm{L}^* \to {}^L\mathrm{M}^*$. Let $\Sigma \in \Phi_2(\mathrm{M})$ and $\Upsilon \in \Phi_2(\mathrm{L})$ be the 'packets' that contain σ and υ, and assume that these are transferred from $\Sigma^* \in \Phi_2(\mathrm{M}^*)$ and $\Upsilon^* \in \Phi_2(\mathrm{L}^*)$ via $(\mathrm{M}^*, \psi_{\mathrm{M}^*})$ and $(\mathrm{L}^*, \psi_{\mathrm{L}^*})$, respectively. Now we will use the part of the proof of Proposition 6.3.1 starting with (90). Writing $r^\mathrm{M}_\mathrm{Q} \Theta_\Sigma = \Theta_1 + \Theta_2$ and $r^{\mathrm{M}^*}_{\mathrm{Q}^*} \Theta_{\Sigma^*} = \Theta_1^* + \Theta_2^*$ as in that proof, we note as in that proof that $\Theta_\upsilon \chi_\alpha$ contributes nontrivially to Θ_1: this uses the cuspidality of υ and the fact that Θ_Σ is a nonzero multiple of a positive linear combination of the Θ_σ, with σ ranging over Σ. Therefore as in that argument we have $\Theta_1^* \neq 0$, and again (using Lemma 3.2.3(i) in addition), we get that there exists $\upsilon^* \in \Upsilon^*$ such that $\upsilon^* \otimes \chi_{\alpha^*}$, where $\alpha^* \in \mathrm{Z}^{\Gamma,0}_{\hat{\mathrm{L}}^*}$ is the transfer of $\alpha \in \mathrm{Z}^{\Gamma,0}_{\hat{\mathrm{L}}}$, has the same cuspidal support as some $\sigma^* \in \Sigma^*$ (note that this relies on Lemma 6.3.3). Although υ^* may not be supercuspidal, we can apply (ii) of Theorem 2.10.10 and the equality $\varphi_{\upsilon^* \otimes \chi_{\alpha^*}} = \alpha^* \cdot \varphi_{\upsilon^*}$ to conclude that $\iota_{\mathrm{M}^*,\mathrm{L}^*} \circ \lambda(\alpha^* \cdot \varphi_{\upsilon^*}) = \lambda(\varphi_{\sigma^*})$. Since the identifications ${}^L\mathrm{L} = {}^L\mathrm{L}^*$ and ${}^L\mathrm{M} = {}^L\mathrm{M}^*$ identify some $\hat{\mathrm{M}}$-conjugate of $\iota_{\mathrm{M},\mathrm{L}}$ with $\iota_{\mathrm{M}^*,\mathrm{L}^*}$, the equality $\iota_{\mathrm{M},\mathrm{L}} \circ \lambda(\alpha \cdot \varphi_\upsilon) = \lambda(\varphi_\sigma) \in \Omega({}^L\mathrm{M})/\bar{\mathcal{O}}_\mathrm{M}$ follows, giving the condition (ii)(b) of Hypothesis 7.3.9.

Now we come to the condition (ii)(c) of Hypothesis 7.3.9. Choose $\beta^* \in \mathcal{O}_{\mathrm{G}^*}$ whose image in $\mathrm{Out}(\mathrm{G}^*) = \mathrm{Out}(\mathrm{G})$ identifies with that of β. As in the proof of (iii) of Lemma 7.3.11, given

Levi subgroup matching data $(\mathrm{M}^*, \psi_{\mathrm{M}^*})$ for M, we can choose Levi subgroup matching data for $\beta(\mathrm{M})$ to be $(\beta^*(\mathrm{M}^*), \psi_{\beta^*(\mathrm{M})^*})$, where $\psi_{\beta^*(\mathrm{M}^*)} = \beta \circ \psi_{\mathrm{M}^*} \circ (\beta^*)^{-1}$. By (i), we may work with this data for $\beta(\mathrm{M})$. It is easy to see from Remark 3.2.2(i) that if $\Sigma^* \in \Phi_2(\mathrm{M}^*)$ transfers to $\Sigma \in \Phi_2(\mathrm{M})$ via $(\mathrm{M}^*, \psi_{\mathrm{M}^*})$, then $\Sigma^* \circ (\beta^*)^{-1} \in \Phi_2(\beta^*(\mathrm{M}^*))$ transfers to $\Sigma \circ \beta^{-1} \in \Phi_2(\beta(\mathrm{M}))$ via $(\beta^*(\mathrm{M}^*), \psi_{\beta^*(\mathrm{M}^*)})$ (this argument includes a proof of the inclusion $\Sigma \circ \beta^{-1} \in \Phi_2(\beta(\mathrm{M}))$). Thus, for $\sigma \in \Sigma$ and $\sigma^* \in \Sigma^*$, $\varphi_{\sigma\circ\beta^{-1}}$ is the composite of $\varphi_{\sigma^*\circ(\beta^*)^{-1}}$ and the identification ${}^L\beta^*(\mathrm{M}^*) \to {}^L\beta(\mathrm{M})$ dictated by $\psi_{\beta^*(\mathrm{M}^*)}$, which is easily seen to be a composite of an identification ${}^L\beta^*(\mathrm{M}^*) \to {}^L\mathrm{M}^*$ dual to β^* with the identification ${}^L\mathrm{M}^* \to {}^L\mathrm{M}$ determined by ψ_{M^*}, followed by an identification ${}^L\mathrm{M} \to {}^L\beta(\mathrm{M})$ dual to β^{-1}. The composite of $\varphi_{\sigma^*\circ(\beta^*)^{-1}}$ with ${}^L\beta^*(\mathrm{M}^*) \to {}^L\mathrm{M}^*$ equals φ_{σ^*} by Theorem 2.10.10(iii), and the image of φ_{σ^*} under ${}^L\mathrm{M}^* \to {}^L\mathrm{M}$ equals φ_σ by the definition of φ_σ. Thus, $\varphi_{\sigma\circ\beta^{-1}}$ equals the composite of φ_σ and the map ${}^L\mathrm{M} \to {}^L\beta(\mathrm{M})$ dual to β^{-1} (or more precisely, to $\beta^{-1}|_{\beta(\mathrm{M})}$), as desired.
Finally, the condition (iii) of Hypothesis 7.3.9 is obvious from the definitions and the fact that we are assuming that $(\mathrm{G}^*, \{\mathcal{O}_{\mathrm{M}^*}\}_{\mathrm{M}^*})$ satisfies Hypothesis 2.10.12. □

Theorem 7.3.10 will be proved by transferring the analogous results in the quasi-split cases, discussed in Proposition 7.2.4, using Proposition 7.3.13. We now discuss a few results that will be used to show that the hypotheses of Proposition 7.3.13 are satisfied in the situations we are interested in.

Lemma 7.3.14. *Let* M *be a connected reductive group over F, let $\zeta : \mathrm{A}_\mathrm{M}(F) \to \mathbb{C}^\times$ be a smooth unitary character, and let $\hat{D}_{\zeta,\mathrm{ell}}(\mathrm{M})$ (resp., $\widehat{SD}_{\zeta,\mathrm{ell}}(\mathrm{M})$) denote the complex vector space of invariant (resp., stable) distributions of the form $\sum a_\sigma\Theta_\sigma$ as σ ranges over $T_{\zeta,\mathrm{ell}}(\mathrm{M})$ (see Notation 5.2.1(iv)), where for each σ we implicitly choose a lift $\tilde\sigma \in \tilde T_{\zeta,\mathrm{ell}}(\mathrm{M})$ and let Θ_σ stand for $\Theta_{\tilde\sigma}$. Let R be a commutative unital $\mathbb{C}$-algebra acting on $\hat{D}_{\zeta,\mathrm{ell}}(\mathrm{M})$. Suppose we are given a $\mathbb{C}$-algebra homomorphism $\lambda_\sigma : R \to \mathbb{C}$ for each $\sigma \in T_{\zeta,\mathrm{ell}}(\mathrm{M})$, such that the action of each $z \in R$ on $\hat{D}_{\zeta,\mathrm{ell}}(\mathrm{M})$ is given by $z \cdot (\sum a_\sigma\Theta_\sigma) = \sum \lambda_\sigma(z) a_\sigma\Theta_\sigma$. Suppose $\Theta = \sum a_\sigma\Theta_\sigma \in \widehat{SD}_{\zeta,\mathrm{ell}}(\mathrm{M})$ has the property that $R\cdot\Theta \subset \widehat{SD}_{\zeta,\mathrm{ell}}(\mathrm{M})$.*

(i) For any homomorphism $\lambda : R \to \mathbb{C}$, we have

$$\Theta_\lambda := \sum_{\lambda_\sigma = \lambda} a_\sigma\Theta_\sigma \in \widehat{SD}_{\zeta,\mathrm{ell}}(\mathrm{M}).$$

(ii) Fix an endoscopic datum $\underline{\mathrm{M}}^*$ for M *with underlying group a quasi-split form* M^* *of* M*, as in Notation 3.2.1(i). Note that ζ is a character of $\mathrm{A}_{\mathrm{M}^*}(F) = \mathrm{A}_\mathrm{M}(F)$ as well. Adapting notation from (i), assume that R acts on $\hat{D}_{\zeta,\mathrm{ell}}(\mathrm{M}^*)$ as well, by a similar prescription $z \cdot (\sum a_{\sigma^*}\Theta_{\sigma^*}) = \sum \lambda_{\sigma^*}(z) a_{\sigma^*}\Theta_{\sigma^*}$. Assume also that there exists $\Theta^* = \sum a_{\sigma^*}\Theta_{\sigma^*} \in \widehat{SD}_{\zeta,\mathrm{ell}}(\mathrm{M}^*)$ such that for all $x \in R$, $x\cdot\Theta^*$ transfers to $x \cdot \Theta$. Then for any $\mathbb{C}$-algebra homomorphism $\lambda : R \to \mathbb{C}$, Θ_λ as in (i) is a transfer of the analogously defined Θ^*_λ; i.e.,*

$$\Theta^*_\lambda = \sum_{\lambda_{\sigma^*} = \lambda} a_{\sigma^*}\Theta_{\sigma^*}.$$

In particular, $\Theta_\lambda \in \widehat{SD}_{\zeta,\mathrm{ell}}(\mathrm{M})$ and $\Theta^*_\lambda \in \widehat{SD}_{\zeta,\mathrm{ell}}(\mathrm{M}^*)$.

Proof. The proof of (i) is an easier variant of that of (ii), so we will only prove (ii). Let $f \in C^\infty_\zeta(\mathrm{M}(F))$ and $f^* \in C^\infty_\zeta(\mathrm{M}^*(F))$ have matching orbital integrals. It suffices to show that $\Theta_\lambda(f) = \Theta_\lambda(f^*)$. There exist finite sets $\Sigma \subset T_{\zeta,\mathrm{ell}}(\mathrm{M})$ and $\Sigma^* \subset T_{\zeta,\mathrm{ell}}(\mathrm{M}^*)$ such that $\Theta_\sigma(f) = 0 = \Theta_{\sigma^*}(f^*)$ for all $\sigma \in T_{\zeta,\mathrm{ell}}(\mathrm{M}) \setminus \Sigma$ and $\sigma^* \in T_{\zeta,\mathrm{ell}}(\mathrm{M}^*) \setminus \Sigma^*$.
If $A \subset \hat{D}_{\zeta,\mathrm{ell}}(\mathrm{M})$ and $A^* \subset \hat{D}_{\zeta,\mathrm{ell}}(\mathrm{M}^*)$ are the subspaces supported on $T_{\zeta,\mathrm{ell}}(\mathrm{M})\setminus\Sigma$ and $T_{\zeta,\mathrm{ell}}(\mathrm{M}^*)\setminus\Sigma^*$, respectively, then these subspaces are clearly R-stable and of finite codimension, and vanish respectively on f and f^*. Denote by p the projection

$$\hat{D}_{\zeta,\mathrm{ell}}(\mathrm{M}) \times \hat{D}_{\zeta,\mathrm{ell}}(\mathrm{M}^*) \to \hat{D}_{\zeta,\mathrm{ell}}(\mathrm{M})/A \times \hat{D}_{\zeta,\mathrm{ell}}(\mathrm{M}^*)/A^* =: V.$$

p induces an action of R on V. Since A and A^* vanish on f and f^*, the map $\hat{D}_{\zeta,\mathrm{ell}}(\mathrm{M}) \times \hat{D}_{\zeta,\mathrm{ell}}(\mathrm{M}^*) \to \mathbb{C}$ given by $(\Theta', (\Theta^*)') \mapsto \Theta'(f) - (\Theta^*)'(f^*)$ factors as the composite of p and some map $V \to \mathbb{C}$, whose kernel we denote by $W \subset V$. It then suffices to show that $p(\Theta_\lambda, \Theta^*_\lambda) \in W$.
On the other hand, by the hypothesis that $x \cdot \Theta^*$ transfers to $x \cdot \Theta$ for any $x \in R$, W contains $x \cdot p(\Theta, \Theta^*)$ for any $x \in R$, and hence also the span W' of the $x \cdot p(\Theta, \Theta^*)$ as x varies over R. Thus, it suffices to show that $p(\Theta_\lambda, \Theta^*_\lambda) \in W'$.
Noting that W' is R-stable, this in turn follows from the following observations, each of which is immediate from the description of the R-actions on $\hat{D}_{\zeta,\mathrm{ell}}(\mathrm{M})$ and $\hat{D}_{\zeta,\mathrm{ell}}(\mathrm{M}^*)$:

- The action of R on V is semisimple, and hence so is the restriction of this action to W';
- $p(\Theta_\lambda, \Theta^*_\lambda)$ is simply the projection of $p(\Theta, \Theta^*) \in W' \subset V$ to the λ-eigenspace of the action of R on V, and hence also the projection of $p(\Theta, \Theta^*) \in W'$ to the λ-eigenspace of the action of R on W'.

□

Lemma 7.3.15. *Fix a smooth additive character $\psi : F \to \mathbb{C}^\times$. For this lemma and its proof, given two meromorphic functions f_1, f_2 on $\mathbb{C}$, we write $f_1 \sim f_2$ if there exists a nowhere vanishing entire function f such that $f_1 = f f_2$. Let a be a fixed positive integer. Suppose $\varphi_1, \varphi_2 : W_F \times SL_2(\mathbb{C}) \to \mathrm{GL}_N(\mathbb{C})$ are two admissible bounded Weil-Deligne representations with the property that for every irreducible admissible bounded representation τ of W_F, denoting by S_a the a-dimensional algebraic representation of $SL_2(\mathbb{C})$, we have:*

$$\gamma(s, \varphi_1 \otimes (\tau \otimes S_a), \psi)\gamma(-s, \varphi_1^\vee \otimes (\tau^\vee \otimes S_a), \psi) \sim \gamma(s, \varphi_2 \otimes (\tau \otimes S_a), \psi)\gamma(-s, \varphi_2^\vee \otimes (\tau^\vee \otimes S_a), \psi).$$

Then φ_1 and φ_2 are equivalent (i.e., define isomorphic representations of W_F).

Proof. It suffices to show that φ_1 and φ_2 have an irreducible factor φ' in common: for then, writing $\varphi_i = \varphi' \oplus \varphi_i^-$ and noting that $\gamma(s, \varphi_i \otimes (\tau \otimes S_a), \psi) = \gamma(s, \varphi_i^- \otimes (\tau \otimes S_a), \psi)\gamma(s, \varphi' \otimes (\tau \otimes S_a), \psi)$ and similarly with $\gamma(-s, \varphi_i^\vee \otimes (\tau^\vee \otimes S_a), \psi)$ for $i = 1, 2$ and any irreducible admissible bounded representation τ of W_F, we get

$$\gamma(s, \varphi_1^- \otimes (\tau \otimes S_a), \psi)\gamma(-s, \varphi_1^{-,\vee} \otimes (\tau^\vee \otimes S_a), \psi) \sim \gamma(s, \varphi_2^- \otimes (\tau \otimes S_a), \psi)\gamma(-s, \varphi_2^{-,\vee} \otimes (\tau^\vee \otimes S_a), \psi),$$

applying an appropriate induction hypothesis to which yields the lemma.
Exchanging φ_1 and φ_2 if necessary, we may and do fix an irreducible admissible bounded representation ρ of W_F and $r > 0$ such that $\rho \otimes S_r$ is an irreducible constituent of φ_1, and such that if $r' > r$, then $\rho \otimes S_{r'}$ is not an irreducible constituent of either of φ_1 or φ_2. It now suffices to show that $\rho \otimes S_r$ is also an irreducible constituent of φ_2.
This in turn follows if we prove the following: given any admissible bounded representation φ of $W_F \times \mathrm{SL}_2(\mathbb{C})$ with the property that $\rho \otimes S_{r'}$ is not an irreducible constituent of φ for any $r' > r$, $\gamma(s, \varphi \otimes (\rho^\vee \otimes S_a), \psi)\gamma(-s, \varphi^\vee \otimes (\rho \otimes S_a), \psi)$ has a pole at $s = (a + r)/2$ if and only if $\rho \otimes S_r$ is an irreducible constituent of φ.
For this, it suffices to assume that φ is irreducible and not of the form $\rho \otimes S_{r'}$ with $r' > r$, and prove the following two assertions about $\gamma(s, \varphi \otimes (\rho^\vee \otimes S_a), \psi)\gamma(-s, \varphi^\vee \otimes (\rho \otimes S_a), \psi)$: that it has a pole at $s = (a + r)/2$ if $\varphi \cong \rho \otimes S_r$, and that it has neither a pole nor a zero at $s = (a + r)/2$ otherwise.
For this, note:

$$\gamma(s, \varphi \otimes (\rho^\vee \otimes S_a), \psi)\gamma(-s, \varphi^\vee \otimes (\rho \otimes S_a), \psi) \sim \frac{L(1 - s, \varphi^\vee \otimes (\rho \otimes S_a))L(1 + s, \varphi \otimes (\rho^\vee \otimes S_a))}{L(s, \varphi \otimes (\rho^\vee \otimes S_a))L(-s, \varphi^\vee \otimes (\rho \otimes S_a))}.$$

Writing $\varphi = \rho' \otimes S_l$ and noting that $S_l \otimes S_a$ is the direct sum of the $S_{a+l-1-2t}$ as t ranges from 0 to $\min(a, l) - 1$, this expression equals

$$\prod_{t=0}^{\min(a,l)-1} \frac{L(1 - s, ({\rho'}^\vee \otimes \rho) \otimes S_{a+l-1-2t})L(1 + s, (\rho' \otimes \rho^\vee) \otimes S_{a+l-1-2t})}{L(s, (\rho' \otimes \rho^\vee) \otimes S_{a+l-1-2t})L(-s, ({\rho'}^\vee \otimes \rho) \otimes S_{a+l-1-2t})}.$$

Using the formula $L(s, \rho'' \otimes S_{t'}) = L(s + (t'-1)/2, \rho'')$, this equals

$$\prod_{t=0}^{\min(a,l)-1} \frac{L(-s + (a+l-2t)/2, \rho'^{\vee} \otimes \rho) L(s + (a+l-2t)/2, \rho' \otimes \rho^{\vee})}{L(s + (a+l-2-2t)/2, \rho' \otimes \rho^{\vee}) L(-s + (a+l-2-2t)/2, \rho'^{\vee} \otimes \rho)}.$$

Let us study the poles and zeroes of the above expression at $s = (a+r)/2$. In what follows, we will use the following easy consequence of the definitions: if an admissible representation ρ'' of W_F is bounded, then $L(s, \rho'')$ has a pole on the real line if and only if ρ'' contains the trivial representation as a constituent, in which case the only pole that it has on the real line is at $s = 0$. None of the four factors above has a zero (as they are local L-functions). Neither does any of them have a pole on the real line unless $\rho' \cong \rho$. Thus, assume that $\rho' = \rho$. Then $l \leq r$ by the hypothesis on φ. Now it is clear that none of the factors in the above expression other than possibly $L(-s + (a+l-2t)/2, \rho'^{\vee} \otimes \rho)$ with $t = 0$ has a pole at $s = (a+r)/2$, while the factor $L(-s + (a+l-2t)/2, \rho'^{\vee} \otimes \rho)$ with $t = 0$ has a pole at $s = (a+r)/2$ if and only if $l = r$, i.e., $\varphi = \rho \otimes S_r$. This proves the two assertions above, and the lemma follows. ☐

Lemma 7.3.15 has the following corollary, which allows us to get the Langlands parameter of a representation of a quasi-split classical group from certain Plancherel μ-function values associated to it.

Corollary 7.3.16. *Assume that* G *is a quasi-split symplectic, special orthogonal, unitary or odd general spin group, and let* $\Phi_2(\mathrm{G})$ *be as defined in the proof of Proposition 7.2.4 (i.e., as in the proof of Proposition 7.2.2, except that, in the even special orthogonal case, we only consider packets up to the nontrivial collection* $\{\mathcal{O}_{\mathrm{M}}\}$ *as in Proposition 7.2.4). Let* ζ *be a smooth unitary character of* $\mathrm{A_G}(F)$ *(which is trivial unless* G *is general spin). For* $\sigma, \sigma' \in \mathrm{Irr}_{\zeta,2}(\mathrm{G}) := \mathrm{Irr}_2(\mathrm{G})_\zeta$*, let us say that* $\sigma \sim \sigma'$ *if* $\mu'''(\upsilon \otimes \sigma) = \mu'''(\upsilon \otimes \sigma')$ *for every* $\upsilon \in \mathrm{Irr}_2(\mathrm{H}_m)$*, where* m *is any nonnegative even integer (resp., any nonnegative integer) if* G *is symplectic, special orthogonal or unitary (resp.,* G *is odd general spin). Then* $\sigma \sim \sigma'$ *if and only if they belong to the same element of* $\Phi_2(\mathrm{G})$.

Proof. The 'if' part is immediate from Corollary 5.2.12(i), so it suffices to prove the 'only if' part. We write $\mathrm{G} = \mathrm{G}_n$ (as in Notation 7.3.1(iv)).
First we consider the case where G is symplectic, special orthogonal or unitary. Let $E = F$ except in the unitary case, where we let E/F be the quadratic extension that splits G. Choosing one of the embeddings of ${}^L\mathrm{G}$ into the L-group of a group GL_W of the form $\mathrm{Res}_{E/F}\,\mathrm{GL}(W)$ or $\mathrm{GL}(W)$ as in the proof of Proposition 7.2.4, and using Shapiro's lemma if $E \neq F$, we identify φ_σ and φ_υ, for any $\sigma \in \mathrm{Irr}_2(\mathrm{G})$ and $\upsilon \in \mathrm{Irr}_2(\mathrm{H}_d)$ where $d \in \mathbb{N}$, with a representation of $W_E \times \mathrm{SL}_2(\mathbb{C})$. In the cases under consideration, by the work of Arthur ([Art13]) or Mok ([Mok15]), intertwining operators can be normalized using Artin L and ε-factors, as follows from [Art13, Proposition 2.3.1] and [Mok15, Proposition 3.3.1]. This has the consequence that for any $\sigma \in \mathrm{Irr}_2(\mathrm{G})$ and $\upsilon \in \mathrm{Irr}_2(\mathrm{H}_d)$ with $d \in \mathbb{N}$, we have, adapting notation from Lemma 7.3.15:

$$\mu(\upsilon |\det|_E^s \otimes \sigma) \sim \gamma(s, \varphi_\sigma^\vee \otimes \varphi_\upsilon, \psi_E)\gamma(-s, \varphi_\sigma \otimes \varphi_\upsilon^\vee, \psi_E) f,$$

where f is a nonzero meromorphic function that depends only on φ_υ (see, e.g., the expressions in [GL18, the proof of Proposition 7.3] and [CZ20, Theorem 2.5.1(4)]; changing ψ_E does not affect '$\sim$'). Now the corollary (except in the odd general spin case) is an easy consequence of Lemma 7.3.15, applied with $a = 2$ and with E in place of F, using the fact that for any irreducible admissible bounded representation τ of W_E, say of dimension m, $\tau \otimes S_2$ in the notation of that lemma is the Langlands parameter of some element of $\mathrm{Irr}_2(\mathrm{H}_{2m})$.
It remains to prove the odd general spin case. We will instead allow G to be any of the groups mentioned in the corollary, but only prove the weaker assertion that has been claimed for odd general spin groups, i.e., we will make the stronger assumption that the equality $\mu'''(\upsilon \otimes \sigma) = \mu'''(\upsilon \otimes \sigma')$ holds for all positive integers m and $\upsilon \in \mathrm{Irr}_2(\mathrm{H}_m)$, and not just when m is even.
We first prove:
Claim. Under the assumption that $\mu'''(\upsilon \otimes \sigma) = \mu'''(\upsilon \otimes \sigma')$ for all $m \geq 0$ and $\upsilon \in \mathrm{Irr}_2(\mathrm{H}_m)$, given an irreducible supercuspidal representation ρ of $\mathrm{H}_d(F)$ and a positive even integer a, the representation $\mathrm{St}(\rho, a) \rtimes \sigma$ of $\mathrm{G}_{2ad+n}(F)$ parabolically induced from $\mathrm{St}(\rho, a) \otimes \sigma$ is irreducible if and

only if the analogously defined representation $\mathrm{St}(\rho, a) \rtimes \sigma'$ of $\mathrm{G}_{2ad+n}(F)$ is irreducible (here and in what follows, $\mathrm{St}(\rho, a)$ denotes the generalized Steinberg representation of $\mathrm{GL}_{ad}(F)$ associated to ρ and a, as in [Mg14]).
Let us prove this claim. First, given any $a \geq 1$ and $d \in \mathbb{N}$, and given any irreducible supercuspidal representation ρ of $\mathrm{H}_d(F)$, it is easy to see that the parabolic induction from $\mathrm{H}_{ad} \times \mathrm{G}$ to G_{2ad+n} realizing the induced representation $\mathrm{St}(\rho, a) \rtimes \sigma$ is unramified if and only if the parabolic induction giving rise to $\mathrm{St}(\rho, a) \rtimes \sigma'$ is (here, 'unramified' means that the inducing representation $\mathrm{St}(\rho, a) \otimes \sigma$ or $\mathrm{St}(\rho, a) \otimes \sigma'$ is not fixed by any element of $\mathrm{G}_{2ad+n}(F)$ taking some parabolic subgroup containing $\mathrm{H}_{ad} \times \mathrm{G}$ to an opposite) ; in these cases, both $\mathrm{St}(\rho, a) \rtimes \sigma$ and $\mathrm{St}(\rho, a) \rtimes \sigma'$ are irreducible by Harish-Chandra theory. Now consider ρ, a such that these parabolic inductions are both ramified. In this situation, we know from Harish-Chandra theory that $\mathrm{St}(\rho, a) \rtimes \sigma$ (resp., $\mathrm{St}(\rho, a) \rtimes \sigma'$) is reducible unless, and exactly unless, the associated μ-function $\mu(\mathrm{St}(\rho, a) \otimes \sigma)$ (resp., $\mu(\mathrm{St}(\rho, a) \otimes \sigma')$) vanishes (e.g., note from [Luo20, around Lemma 2.1] that, adapting notation from that paper, the kernel $W_0(\mathrm{St}(\rho, a) \otimes \sigma)$ of the map from $W(\mathrm{St}(\rho, a) \otimes \sigma)$ to the R-group $R(\mathrm{St}(\rho, a) \otimes \sigma)$ is nontrivial if and only if $\mu(\mathrm{St}(\rho, a) \otimes \sigma) = 0$, and similarly with σ'). Thus, the claim follows from the hypothesis $\sigma \sim \sigma'$, which gives us that $\mu(\mathrm{St}(\rho, a) \otimes \sigma) = \mu(\mathrm{St}(\rho, a) \otimes \sigma')$.
Now that the claim is proved, the corollary is an immediate consequence of [Mg14, Remark 7.3], which asserts that the set "Jord(σ)" of Jordan blocks of σ as defined in [Mg14], which reads off the Langlands parameter $\varphi_\sigma \in \Phi_2(\mathrm{M})/\mathcal{O}_{\mathrm{M}}$ of σ, agrees with the similarly notated notion found in earlier works of Mœglin and Tadic, defined in terms of reducibility of parabolic induction. □

Proof of Theorem 7.3.10. The case where G is quasi-split has been handled in Proposition 7.2.4. However, for a while we will allow G to be quasi-split as well, since the strategy of our proof is to transfer the result in the quasi-split case to the non-quasi-split case. Let $E = F$ unless G^* or equivalently G is a unitary group, in which case we assume that E/F is a quadratic extension and that G is associated to an E/F-Hermitian space.
We will assume the notation from Notation 7.3.1 and Lemma 7.3.14, including the set $T_{\zeta,\mathrm{ell}}(\mathrm{G})$ and the choices fixing Θ_σ for any $\sigma \in T_{\zeta,\mathrm{ell}}(\mathrm{G})$, where $\zeta : \mathrm{A}_{\mathrm{G}}(F) \to \mathbb{C}^\times$ is a smooth unitary character. Fix such a character ζ for now (A_{G} and ζ are trivial unless G is general spin). Let

$$R = \mathbb{C}[\upsilon \mid \upsilon \in \mathrm{Irr}_2(\mathrm{H}_d) \text{ for some } d \in \mathbb{N}_{\mathrm{G}}]$$

be the polynomial ring whose variables are the elements of the $\mathrm{Irr}_2(\mathrm{H}_d)$ as d ranges over $\mathbb{N}_{\mathrm{G}}$ (including $d = 0$); thus, there are uncountably many variables. We now let R act on $\hat{D}_{\zeta,\mathrm{ell}}(\mathrm{G})$ as follows (compare with the expressions in Lemma 7.3.14). It suffices to specify the endomorphism T_υ with which $\upsilon \in \mathrm{Irr}_2(\mathrm{H}_d)$ acts on $\hat{D}_{\zeta,\mathrm{ell}}(\mathrm{G})$, which we stipulate to be given by:

$$T_\upsilon\Big(\sum_{\sigma \in T_{\zeta,\mathrm{ell}}(\mathrm{G})} a_\sigma \Theta_\sigma\Big) = \sum_{\sigma \in T_{\zeta,\mathrm{ell}}(\mathrm{G})} a_\sigma \mu'''(\upsilon \otimes \sigma)\Theta_\sigma, \tag{101}$$

where we artificially set $\mu'''(\upsilon \otimes \sigma) = 0$ if $\sigma \in T_{\zeta,\mathrm{ell}}(\mathrm{G}) \setminus \mathrm{Irr}_{\zeta,2}(\mathrm{G})$, and set $\mu'''(\upsilon \otimes \sigma) = \mu(\upsilon \otimes \sigma) = 1$ if $d = 0$ and $\sigma \in \mathrm{Irr}_{\zeta,2}(\mathrm{G})$. Similarly, we define an action of R on $\hat{D}_{\zeta,\mathrm{ell}}(\mathrm{G}^*)$ by requiring that any $\upsilon \in \mathrm{Irr}_2(\mathrm{H}_d)$, whose Jacquet-Langlands transfer to $\mathrm{H}_d^* = \mathrm{Res}_{E/F}\,\mathrm{GL}_d$ we denote by υ^*, act via the endomorphism T_{υ^*} of $\hat{D}_{\zeta,\mathrm{ell}}(\mathrm{G}^*)$ given by:

$$T_{\upsilon^*}\Big(\sum_{\sigma^* \in T_{\zeta,\mathrm{ell}}(\mathrm{G}^*)} a_{\sigma^*} \Theta_{\sigma^*}\Big) = \sum_{\sigma^* \in T_{\zeta,\mathrm{ell}}(\mathrm{G}^*)} a_{\sigma^*} \mu'''(\upsilon^* \otimes \sigma^*)\Theta_{\sigma^*}, \tag{102}$$

again following an analogous convention as above concerning $\mu'''(\upsilon^* \otimes \sigma^*)$ and $\mu(\upsilon^* \otimes \sigma^*)$.
We give $\mathrm{G}(F)$ and $\mathrm{G}^*F)$ ψ_{G^*}-compatible Haar measures, and choose a Haar measure on $\mathrm{A}_{\mathrm{G}}(F) = \mathrm{A}_{\mathrm{G}^*}(F)$. Let $\Theta = e(\mathrm{G}) \sum d(\sigma)\Theta_\sigma \in \hat{D}_{\zeta,\mathrm{ell}}(\mathrm{G})$ and $\Theta^* = \sum d(\sigma^*)\Theta_{\sigma^*} \in \hat{D}_{\zeta,\mathrm{ell}}(\mathrm{G}^*)$, where the sums run over $\sigma \in \mathrm{Irr}_{\zeta,2}(\mathrm{G})$ and $\sigma^* \in \mathrm{Irr}_{\zeta,2}(\mathrm{G}^*)$, and we recall that $e(\cdot)$ stands for the Kottwitz sign.
Claim. For all $x \in R$, $x \cdot \Theta^* \in \hat{D}_{\zeta,\mathrm{ell}}(\mathrm{G}^*)$ is stable, and transfers to $x \cdot \Theta \in \hat{D}_{\zeta,\mathrm{ell}}(\mathrm{G})$ via $\underline{\mathrm{G}}^*$.
Let us prove this claim. Write $\mathbf{T}_{\underline{\mathrm{G}}^*}$ for the transfer of stable distributions from G^* to G. To prove this claim, it suffices to consider the case where $x = \upsilon_1 \ldots \upsilon_r$, for some r and $\upsilon_i \in \mathrm{Irr}_2(\mathrm{H}_{d_i})$,

$1 \leq i \leq r$, with each $d_i \in \mathbb{N}_{\mathrm{G}}$, and show:
(103)

$$\mathbf{T}_{\underline{\mathrm{G}}^*}\left(\sum_{\sigma^*\in \mathrm{Irr}_{\zeta,2}(\mathrm{G}^*)} d(\sigma^*)\left(\prod_{i=1}^{r}\mu'''(\upsilon_i^*\otimes\sigma^*)\right)\Theta_{\sigma^*}\right) = e(\mathrm{G})\sum_{\sigma\in\mathrm{Irr}_{\zeta,2}(\mathrm{G})} d(\sigma)\left(\prod_{i=1}^{r}\mu'''(\upsilon_i\otimes\sigma)\right)\Theta_\sigma$$

(this includes showing that the parenthetical expression on the left-hand side is stable).
For lightness of notation, we do not fix the υ_i yet. We consider the Levi subgroups $\mathrm{M}^+ := \mathrm{M}(d_1,\dots,d_r;n) \subset \mathrm{G}^+ := \mathrm{G}_{2(d_1+\cdots+d_r)+n}$ and $(\mathrm{M}^+)^* := \mathrm{H}^*_{d_1}\times\cdots\times\mathrm{H}^*_{d_r}\times\mathrm{G}^* \subset (\mathrm{G}^+)^* = \mathrm{G}^*_{2(d_1+\cdots+d_r)+n}$ as in Notation 7.3.1(iv). For $1 \leq i \leq r$, we choose compatible Haar measures on $\mathrm{H}_{d_i}(F)$ and $\mathrm{H}^*_{d_i}(F)$, and a Haar measure on $\mathrm{A}_{\mathrm{H}_{d_i}}(F) = \mathrm{A}_{\mathrm{H}^*_{d_i}}(F)$. The resulting product measures on $\mathrm{M}^+(F)$ and $(\mathrm{M}^+)^*(F)$ are then compatible as well (see the conventions in Notation 7.3.1(iv)), and we get a Haar measure on $\mathrm{A}_{\mathrm{M}^+}(F) = \mathrm{A}_{(\mathrm{M}^+)^*}(F)$. Using Corollary 5.2.11(ii), applied considering it is easy to see that for fixed smooth unitary characters ζ_i of $\mathrm{A}_{\mathrm{H}_{d_i}}(F) = \mathrm{A}_{\mathrm{H}^*_{d_i}}(F)$ for $1 \leq i \leq r$, we have that

$$\sum_{\sigma^*\in\mathrm{Irr}_{\zeta,2}(\mathrm{G}^*)}\ \sum_{(\upsilon_i^*)_i\in\prod_{1\leq i\leq r}\mathrm{Irr}_{\zeta_i,2}(\mathrm{H}^*_{d_i})} d(\upsilon_1^*)\dots d(\upsilon_r^*)d(\sigma^*)\mu'''(\upsilon_1^*\otimes\cdots\otimes\upsilon_r^*\otimes\sigma^*)\Theta_{\upsilon_1^*\otimes\cdots\otimes\upsilon_r^*\otimes\sigma^*} \tag{104}$$

is a stable distribution on $(\mathrm{M}^+)^*$, and transfers via $\underline{(\mathrm{G}^+)^*}$ (see Notation 7.3.1(iv)) to the distribution

$$e(\mathrm{G}^+)\sum_{\sigma\in\mathrm{Irr}_{\zeta,2}(\mathrm{G})}\ \sum_{(\upsilon_i)_i\in\prod_{1\leq i\leq r}\mathrm{Irr}_{\zeta_i,2}(\mathrm{H}_{d_i})} d(\upsilon_1)\dots d(\upsilon_r)d(\sigma)\mu'''(\upsilon_1\otimes\cdots\otimes\upsilon_r\otimes\sigma)\Theta_{\upsilon_1\otimes\cdots\otimes\upsilon_r\otimes\sigma} \tag{105}$$

on $\mathrm{M}^+ = \mathrm{M}(d_1,\dots,d_r;n)$.
We apply this in the following setting. We now fix $\upsilon_i \in \mathrm{Irr}_2(\mathrm{H}_{d_i})$ for $1 \leq i \leq r$, denote by ζ_i the central character of υ_i or equivalently of its Deligne-Kazhdan-Vigneras transfer υ_i^*, and choose $f_i \in C^\infty_{\zeta_i}(\mathrm{H}_{d_i}(F))$ and $f_i^* \in C^\infty_{\zeta_i}(\mathrm{H}^*_{d_i}(F))$ that are pseudocoefficients for $\upsilon_i \in \mathrm{Irr}_{\zeta_i,2}(\mathrm{H}_{d_i})$ and $\upsilon_i^* \in \mathrm{Irr}_{\zeta_i,2}(\mathrm{H}^*_{d_i})$, respectively. One knows that $e(\mathrm{H}_{d_i})^{-1}d(\upsilon_i)^{-1}f_i \in C^\infty_{\zeta_i}(\mathrm{H}_{d_i}(F))$ and $d(\upsilon_i^*)^{-1}f_i^* \in C^\infty_{\zeta_i}(\mathrm{H}^*_{d_i}(F))$ have matching orbital integrals (e.g., combine Claim 1 in the proof of Proposition 3.3.7, together with the equality $a = e(\mathrm{M})$ seen later in that proof). Thus, whenever $f \in C^\infty_\zeta(\mathrm{G}(F))$ and $f^* \in C^\infty_\zeta(\mathrm{G}^*(F))$ have matching orbital integrals for $\underline{\mathrm{G}}^*$, so do $d(\upsilon_1)^{-1}\dots d(\upsilon_r)^{-1}f_1\otimes\cdots\otimes f_r\otimes f \in C^\infty_{\zeta^+}(\mathrm{M}^+(F))$ and $d(\upsilon_1^*)^{-1}\dots d(\upsilon_r^*)^{-1}f_1^*\otimes\cdots\otimes f_r^*\otimes f^* \in C^\infty_{\zeta^+}((\mathrm{M}^+)^*(F))$ for $\underline{(\mathrm{G}^+)^*}$, where $\zeta^+ = \zeta_1\otimes\dots\zeta_r\otimes\zeta$. Applying these observations and the equality $e(\mathrm{G}^+) = e(\mathrm{M}^+) = e(\mathrm{H}_1)\dots e(\mathrm{H}_r)e(\mathrm{G})$, to the fact that (104) transfers to (105), we conclude that:
(106)

$$\mathbf{T}_{\underline{\mathrm{G}}^*}\left(\sum_{\sigma^*\in\mathrm{Irr}_{\zeta,2}(\mathrm{G}^*)} d(\sigma^*)\mu'''(\upsilon_1^*\otimes\cdots\otimes\upsilon_r^*\otimes\sigma^*)\Theta_{\sigma^*}\right) = e(\mathrm{G})\sum_{\sigma\in\mathrm{Irr}_{\zeta,2}(\mathrm{G})} d(\sigma)\mu'''(\upsilon_1\otimes\cdots\otimes\upsilon_r\otimes\sigma)\Theta_\sigma.$$

(and in particular the parenthetical expression on the left-hand side is stable). Thus, we would like to deduce (103) from (106).
Combining (106) with the 'multiplicativity' expressions (99) and (98) for $\mu'''(\upsilon_1^*\otimes\cdots\otimes\upsilon_r^*\otimes\sigma^*)$ and $\mu'''(\upsilon_1\otimes\cdots\otimes\upsilon_r\otimes\sigma)$, and combining with (100), (103) follows whenever each $\mu'''_\alpha(\upsilon_1,\dots,\upsilon_r)$ that contributes to the right-most term in (98) is nonzero. Then (103) follows in general by analytic continuation, finishing the proof of the claim.
Now that the claim is proved, we no longer fix ζ. Let $\Phi_2(\mathrm{G})$ be the partition of $\mathrm{Irr}_2(\mathrm{G})$ determined by the equivalence relation $\sim$ according to which $\sigma \sim \sigma'$ if and only if the central characters of σ and σ' have the same restriction, say ζ, to $\mathrm{A}_{\mathrm{G}}(F)$, and moreover $\mu'''(\upsilon\otimes\sigma) = \mu'''(\upsilon\otimes\sigma')$ whenever $\upsilon \in \mathrm{Irr}_2(\mathrm{H}_d)$ for some $d \in \mathbb{N}_{\mathrm{G}}$. Similarly, we define a partition $\Phi_2(\mathrm{G}^*)$ of $\mathrm{Irr}_2(\mathrm{G}^*)$.
We let the collection $\{\mathcal{O}_{\mathrm{M}^*}\}_{\mathrm{M}^*}$ to be as in Proposition 7.2.4, but with G replaced by G^*. The collection $\{\bar{\mathcal{O}}_{\mathrm{M}}\}_{\mathrm{M}}$, which we recall is as in Notation 7.3.1(vii), is nice by Corollary 7.3.12, which also tells us that it is obtained from $\{\mathcal{O}_{\mathrm{M}^*}\}_{\mathrm{M}^*}$ as in Lemma 7.3.11. Note that in such a situation,

the isomorphism $SD_{\mathrm{ell}}(\mathrm{M}^*) \to SD_{\mathrm{ell}}(\mathrm{M})$ restricts, by Proposition 7.3.5(i), to an isomorphism $SD_{\mathrm{ell}}(\mathrm{M}^*)^{\mathcal{O}_{\mathrm{M}^*}} \to SD_{\mathrm{ell}}(\mathrm{M})^{\bar{\mathcal{O}}_{\mathrm{M}}}$.
By Corollary 7.3.16 (applied with G^* in place of G), and recalling the action of R on $\hat{D}_{\zeta,\mathrm{ell}}(\mathrm{G}^*)$ and using that the Deligne-Kazhdan-Vigneras correspondence gives bijections $\mathrm{Irr}_2(\mathrm{H}_d) \to \mathrm{Irr}_2(\mathrm{H}_d^*)$ for each $d \in \mathbb{N}_{\mathrm{G}}$, we conclude that $\Phi_2(\mathrm{G}^*)$ is also what it should denote according to the proof of Proposition 7.2.4 (thus, it agrees with the $\Phi_2(\mathrm{G}^*)$ as in Proposition 7.2.2 except in the even special orthogonal case, in which case it is coarser due to the outer automorphism). In particular, each element of $\Phi_2(\mathrm{G}^*)$ is finite, and $\Phi_2(\mathrm{G}^*)$ indexes a basis for $SD_{\mathrm{ell}}(\mathrm{G}^*)^{\bar{\mathcal{O}}_{\mathrm{G}^*}}$.
For each unitary character ζ of $\mathrm{A}_{\mathrm{G}}(F) = \mathrm{A}_{\mathrm{G}^*}(F)$, recalling the actions of R on $\hat{D}_{\zeta,\mathrm{ell}}(\mathrm{G})$ and $\hat{D}_{\zeta,\mathrm{ell}}(\mathrm{G}^*)$, the claim above allows us to apply Lemma 7.3.14, which in turn shows that for each $\Sigma^* \in \Phi_2(\mathrm{G}^*)$, the virtual character $\Theta_{\Sigma^*} := \sum d(\sigma^*)\Theta_{\sigma^*}$, where the sum runs over $\sigma^* \in \Sigma^*$, is stable and transfers to $\Theta_\Sigma := e(\mathrm{G}) \sum d(\sigma)\Theta_\sigma$, where the sum runs over the elements σ of a uniquely determined $\Sigma \in \Phi_2(\mathrm{G})$. Since endoscopic transfer takes $SD_{\mathrm{ell}}(\mathrm{G}^*)$ to $SD_{\mathrm{ell}}(\mathrm{G})$ (which ultimately relies on [Art96], as explicated in [MW16, Chapter XI]), and since Σ^* is finite as observed above, it follows that Σ is finite (here, we use the easy observation that two elements $\sum a_\sigma \Theta_\sigma, \sum a'_\sigma \Theta_\sigma \in \hat{D}_{\zeta,\mathrm{ell}}(\mathrm{G})$, where each sum runs over $T_{\zeta,\mathrm{ell}}(\mathrm{G})$, are distinct as distributions unless $a_\sigma = a'_\sigma$ for all σ: use pseudocoefficients). Moreover, since $\{\Theta_{\Sigma^*} \mid \Sigma^* \in \Phi_2(\mathrm{G}^*)\}$ is a basis for $SD_{\mathrm{ell}}(\mathrm{G}^*)^{\mathcal{O}_{\mathrm{G}^*}}$, and since $SD_{\mathrm{ell}}(\mathrm{G}^*)^{\mathcal{O}_{\mathrm{G}^*}} \to SD_{\mathrm{ell}}(\mathrm{G})^{\bar{\mathcal{O}}_{\mathrm{G}}}$ is an isomorphism, it follows that $\{\Theta_\Sigma \mid \Sigma^* \in \Phi_2(\mathrm{G}^*)\}$ is a basis for $SD_{\mathrm{ell}}(\mathrm{G})^{\bar{\mathcal{O}}_{\mathrm{G}}}$. By Proposition 7.3.5(ii), it follows that $\{\Sigma \mid \Sigma^* \in \Phi_2(\mathrm{G}^*)\} \subset \Phi_2(\mathrm{G})$ is a partition of $\mathrm{Irr}_2(\mathrm{G})$, forcing that the map $\Phi_2(\mathrm{G}^*) \to \Phi_2(\mathrm{G})$ given by $\Sigma^* \mapsto \Sigma$ is a bijection. Therefore, $\Phi_2(\mathrm{G})$ is a partition of $\mathrm{Irr}_2(\mathrm{G})$ into finite subsets.
Given a Levi subgroup $\mathrm{M} \subset \mathrm{G}$, choosing Levi subgroup matching data $(\mathrm{M}^*, \psi_{\mathrm{M}^*})$, and applying the above considerations to the 'inner type classical' part G_{M} of M and the Deligne-Kazhdan-Vigneras transfer to the 'inner type GL-type' part of M, we can make analogous statments involving Levi subgroups: we get partitions $\Phi_2(\mathrm{M})$ and $\Phi_2(\mathrm{M}^*)$ of $\mathrm{Irr}_2(\mathrm{M})$ and $\mathrm{Irr}_2(\mathrm{M}^*)$ into finite subsets, where $\Phi_2(\mathrm{M}^*)$ agrees with what it denotes according to the proof of Proposition 7.2.4, and a bijection $\Phi_2(\mathrm{M}^*) \to \Phi_2(\mathrm{M})$, denoted $\Sigma^* \mapsto \Sigma$ and defined by the requirement that $\Theta_{\Sigma^*} := \sum d(\sigma^*)\Theta_{\sigma^*}$ transfers to $\Theta_\Sigma := e(\mathrm{M}) \sum d(\sigma)\Theta_\sigma$.
Thus, all the hypotheses of Proposition 7.3.13 are met, where we note that by Proposition 7.2.4, G^*, with respect to the $\{\mathcal{O}_{\mathrm{M}^*}\}$ as in that proposition, satisfies Hypotheses 2.7.1, 2.10.3 and 2.10.12. It follows from Proposition 7.3.13 that G, together with the partitions $\Phi_2(\mathrm{M})$ as described above, satisfies the weak versions of these hypotheses as described in Hypothesis 7.3.9. □

Proof of Theorem 7.3.3. By Theorem 7.3.10, Hypothesis 7.3.9 is satisfied. On the other hand, since G is symmetric, it follows from the discussion in Notation 7.3.1(vii) that the image of each $\mathcal{O}_{\mathrm{M}}$ (which we recall is as in Notation 7.3.1(vi)) in $\mathrm{Out}(\mathrm{M})$ equals $\bar{\mathcal{O}}_{\mathrm{M}}$. Using this, one can check that any construction satisfying Hypothesis 7.3.9 also satisfies Hypotheses 2.7.1, 2.10.3 and 2.10.12, which are therefore satisfied. □

Remark 7.3.17. For simplicity, assume that G is an inner form of a quasi-split symplectic or odd special orthogonal group over F, so that $\mathcal{O}_{\mathrm{G}}$ (as in Notation 7.3.1(vi)) is trivial. As in Remark 3.4.13, it follows from Theorem 7.3.3, Corollary 3.4.12 and Lemma 3.3.8(iii) that, if $p \gg 0$, regular supercuspidal packets on $\mathrm{G}(F)$ as defined by Kaletha are also packets for the unrefined local Langlands correspondence of Theorem 7.3.3. An analogous assertion, but involving the local Langlands correspondence of Theorem 7.3.10 and accounting for an outer automorphism, should apply to inner forms of quasi-split even special orthogonal groups over F, but we have not worked out the details.

7.4. **Some consequences for classical groups and their inner forms.**

Corollary 7.4.1. *If* G *is a quasi-split symplectic, special orthogonal, unitary, general symplectic or odd general spin group (*$\mathrm{SO}_n, \mathrm{Sp}_{2n}, \mathrm{U}_n, \mathrm{GSp}_{2n}$ *or* GSpin_{2n+1}*), or an inner form of a quasi-split symplectic, odd special orthogonal or unitary group, then* G *satisfies the stable center conjecture, i.e.,* $\mathcal{Z}_1(\mathrm{G}) = \mathcal{Z}_2\mathrm{G})$*. If* G *is a quasi-split even general special orthogonal group* GSO_{2n} *(resp., a* *symmetric inner form of an even special orthogonal group* SO_{2n}*), then we have a weaker equality* $\mathcal{Z}_1(\mathrm{G})^{\mathcal{O}} = \mathcal{Z}_{2,\mathcal{O}}(\mathrm{G})$*, where* $\mathcal{O} = \mathcal{O}_{\mathrm{G}}$ *is as in Proposition 7.2.2 (resp., as in Notation 7.3.1(vi)).*

Proof. This follows from combining Theorem 5.4.2 with either Proposition 7.2.2 (in the quasi-split cases) or Theorem 7.3.3 (in the non-quasi-split cases). □

Now we address questions related to depth preservation:

Proposition 7.4.2. *(i) Let* G *be as in Proposition 7.2.2, i.e.,* G *is quasi-split and is of the form* $\mathrm{Sp}_{2n}, \mathrm{SO}_n, \mathrm{U}_n, \mathrm{GSp}_{2n}, \mathrm{GSO}_{2n}$ *or* GSpin_{2n+1}*. Let* $\Sigma \in \Phi_2(\mathrm{G})$ *be an* $\mathcal{O}_{\mathrm{G}}$*-stable discrete series packet as in Proposition 7.2.2, with* $\mathcal{O}_{\mathrm{G}}$ *nontrivial in the* GSO_{2n}*-case, but not in any of the other cases including the* SO_{2n}*-case. Assume that* $p > 2$ *and that, in the unitary case,* p *is greater than the rank of* G*. Then for each* $\sigma_1, \sigma_2 \in \Sigma$*, we have* $\mathrm{depth}(\sigma_1) = \mathrm{depth}(\sigma_2)$*.*

(ii) Let G *be an inner form of a quasi-split classical or odd general spin group, as in Notation 7.3.1(ii), and let* $\mathrm{G}^*, \psi_{\mathrm{G}^*}$ *be as in Notation 7.3.1(i). This time, we let* $\{\mathcal{O}_{\mathrm{M}^*}\}_{\mathrm{M}^*}$ *and* $\{\bar{\mathcal{O}}_{\mathrm{M}}\}_{\mathrm{M}}$ *be as in Notation 7.3.1(vi) and Notation 7.3.1(vii), respectively, and let* $\Phi_2(\mathrm{G}^*)$ *and* $\Phi_2(\mathrm{G})$ *be defined as in Proposition 7.2.4 and the proof of Theorem 7.3.10, respectively. Thus,* $\Phi_2(\mathrm{G})$ *is also obtained from* $\Phi_2(\mathrm{G}^*)$ *as in the proof of Proposition 7.3.13. Let* $\Sigma_0^* \in \Phi_2(\mathrm{G}^*)$*, and let* $\Sigma_0 \in \Phi_2(\mathrm{G})$ *be a transfer of* $\Sigma_0^* \in \Phi_2(\mathrm{G}^*)$ *in the sense of Proposition 7.3.13. Assume that* $p > 2$ *and that, in the unitary case,* p *is greater than the rank of* G*. Then for each* $\sigma_0 \in \Sigma_0$ *and* $\sigma_0^* \in \Sigma_0^*$*, we have* $\mathrm{depth}(\sigma_0) = \mathrm{depth}(\sigma_0^*)$*.*

Proof. In each case, the assumptions on p imply that p is a very good prime for G in the sense of [BKV16, Section 8.10]. Now (i) follows from Corollary 5.3.4(i), since its hypotheses are satisfied by Proposition 7.2.4.

Coming to (ii), it is easy to reduce (as in the proof of Corollary 5.3.4(ii)) to showing that for each $r \geq 0$, denoting by E_r and E_r^* the depth r projectors associated to G and G^*, we have $\hat{E}_r(\sigma_0) = \hat{E}_r^*(\sigma_0^*)$. In this case, we do have by [AR00, Proposition 4.1] a nice bilinear form (in the sense of Definition 5.3.6(iii)) on $\mathfrak{g}$. If G is a symmetric inner form, then the proposition is now an immediate consequence of Corollary 5.3.4(ii). We now briefly indicate how this can be adapted to deal with the case where G may not be symmetric.

We give $\mathrm{G}(F)$ and $\mathrm{G}^*(F)$ Haar measures that are compatible under ψ_{G^*}, and we give $\mathrm{A}_{\mathrm{G}}(F) = \mathrm{A}_{\mathrm{G}^*}(F)$ a ("shared") Haar measure. Then by Proposition 5.3.5, the distribution $e(\mathrm{G})E_r$ on $\mathrm{G}(F)$ is a transfer of the distribution E_r^* on $\mathrm{G}^*(F)$. The restriction of the central character of either of σ_0 or σ_0^* to $\mathrm{A}_{\mathrm{G}}(F) = \mathrm{A}_{\mathrm{G}^*}(F)$ is the same, say ζ. Let $\Phi_{\zeta,2}(\mathrm{G}^*)_{\leq r} \subset \Phi_2(\mathrm{G}^*)$ be the subset of packets the central character of one (or equivalently, each) of whose representations restricts to ζ, and the depth of one (or equivalently by (i), each) of whose representations is at most r. Let $\Phi_{\zeta,2}(\mathrm{G})_{\leq r}$ be the set of elements of $\Phi_{\zeta,2}(\mathrm{G})$ obtained by transferring elements of $\Phi_{\zeta,2}(\mathrm{G}^*)_{\leq r}$. It suffices to show that given $\sigma_0 \in \mathrm{Irr}_{\zeta,2}(\mathrm{G})$, we have $\mathrm{depth}\, \sigma_0 \leq r$ if and only if $\sigma_0 \in \Sigma_0$ for some $\Sigma_0 \in \Phi_{\zeta,2}(\mathrm{G})_{\leq r}$.

By Corollary 5.2.11(ii), applied with $\mathrm{M} = \mathrm{G}$, we get an equality

$$\cdot\, \mathbf{T}_{\underline{\mathrm{G}}^*} \left(\sum_{\sigma^* \in \mathrm{Irr}_2(\mathrm{G}^*)_\zeta} d(\sigma^*) \hat{E}_r^*(\sigma^*) \Theta_{\sigma^*} \right) = e(\mathrm{G}) \left(\sum_{\sigma \in \mathrm{Irr}_2(\mathrm{G})_\zeta} d(\sigma) \hat{E}_r(\sigma) \Theta_\sigma \right). \tag{107}$$

Each side of (107) features a finite sum. Since the parenthetical expression on the left-hand side of (107) is stable and invariant under $\mathcal{O}_{\mathrm{G}^*}$, it can be written as a linear combination, with each coefficient nonzero, of the $\Theta_{\Sigma^*} = \sum d(\sigma^*)\Theta_{\sigma^*}$ as Σ^* ranges over elements of $\Phi_2(\mathrm{G}^*)_{\leq r}$. Therefore, the right-hand side of (107) is a linear combination, with each coefficient nonzero, of the $\Theta_\Sigma = \sum d(\sigma)\Theta_\sigma$ as Σ ranges over the elements of $\Phi_{\zeta,2}(\mathrm{G})_{\leq r}$. But since the right-hand side of (107) is manifestly a linear combination, with each coefficient nonzero, of characters Θ_σ, where σ ranges precisely over representations of $\mathrm{G}(F)$ of depth at most r, it follows that these σ are precisely those that belong to some element of $\Phi_{\zeta,2}(\mathrm{G})_{\leq r}$, as desired. □

The work of M. Oi ([Oi22]) allows us to deduce the following corollary:

Corollary 7.4.3. *Let* G *be an inner form of a quasi-split symplectic, special orthogonal or unitary group. There exists a constant* $N_{\mathrm{G}} > 2$*, depending only on the absolute root datum of* G*, such that the following holds if* $p > N_{\mathrm{G}}$*. Let* σ *be a discrete series representation of* $\mathrm{G}(F)$*, and let* φ_σ *be its Langlands parameter in a set* $\Phi_2(\mathrm{G})/\mathcal{O}_{\mathrm{G}}$ *or* $\Phi_2(\mathrm{G})/\bar{\mathcal{O}}_{\mathrm{G}}$ *as in Proposition 7.2.4 or Theorem 7.3.10. Then:*

$$\inf\{r \geq 0 \mid \dot{\varphi}_\sigma|_{I_F^{r+}} = s|_{I_F^{r+}} \text{ for a preferred section } s : W_F \to {}^L\mathrm{G}\} =: \operatorname{depth}\varphi_\sigma = \operatorname{depth}\sigma,$$

where $\dot{\varphi}_\sigma : W_F \to {}^L\mathrm{G}$ *is a representative for* φ_σ*.*

Proof. If G is quasi-split, and Σ is the packet in $\Phi_2(\mathrm{G})$ (in the sense of Proposition 7.2.4) containing σ, then [Oi22, Theorem 1.2] gives:

$$\max\{\operatorname{depth}\sigma' \mid \sigma' \in \Sigma\} = \operatorname{depth}\varphi_\sigma.$$

Therefore, in this case, the corollary follows from Proposition 7.4.2(i). Given the construction of φ_σ in the non-quasi-split case, the non-quasi-split case now follows by combining with Proposition 7.4.2(ii). We also remark that if G is a possibly non-quasi-split unitary group, the corollary follows from [Oi22, Theorem 1.4] and [Oi21, Theorem 1.3] (without any need for Proposition 7.4.2), and that the precise bounds for N_{G} are given in [Oi21] and [Oi22]. □

Now we address [Hai14, Remark 5.5.4] and [SS13, Conjecture 6.3 and Remark 6.4] for quasi-split classical groups, but up to an outer automorphism in the even special orthogonal case.

Proposition 7.4.4. *Let* G *be a quasi-split symplectic, special orthogonal or unitary group, and let* $\{\mathcal{O}_{\mathrm{M}}\}_{\mathrm{M}}$ *be as in Proposition 7.2.4. Then the maps* $p_1 : \Omega(\mathrm{G}) \to \underline{\Omega}({}^L\mathrm{G})$ *and* $p_2 : \Omega(\mathrm{G}) \to \underline{\Omega}^{\mathrm{st}}(\mathrm{G})$ *(see Definitions 4.3.1 and 4.3.3) are well-defined, and satisfy that* $p_1^*(\mathbb{C}[\underline{\Omega}({}^L\mathrm{G})]) = p_2^*(\mathbb{C}[\underline{\Omega}^{\mathrm{st}}(\mathrm{G})]) = \mathcal{Z}_{2,\mathcal{O}}(\mathrm{G}) = \mathcal{Z}_1(\mathrm{G})^{\mathcal{O}} \subset \mathcal{Z}(\mathrm{G}) = \mathbb{C}[\Omega(\mathrm{G})]$*. In particular, in the symplectic, odd special orthogonal and unitary cases we have* $p_1^*(\mathbb{C}[\underline{\Omega}({}^L\mathrm{G})]) = p_2^*(\mathbb{C}[\underline{\Omega}^{\mathrm{st}}(\mathrm{G})]) = \mathcal{Z}_2(\mathrm{G}) = \mathcal{Z}_1(\mathrm{G})$*.*

Proof. This follows from Proposition 5.5.1 and Corollary 5.5.2, since their hypotheses are satisfied by Proposition 7.2.4; the equality $\mathcal{Z}_{2,\mathcal{O}}(\mathrm{G}) = \mathcal{Z}_1(\mathrm{G})^{\mathcal{O}}$ is from Theorem 5.4.2. □

Remark 7.4.5. Suppose that $F = \mathbb{Q}_p$, and that G is a unitary group in an odd number of variables, associated to an unramified extension E/F. In this case, the work [MHN22] of Bertoloni Meli, Hamann and Nguyen proves that the local Langlands correspondence for G constructed in [Mok15] and [KMSW14] agrees with the local Langlands correspondence constructed by Fargues and Scholze in [FS21]. It seems to us that for such a G, combining [MHN22] with Proposition 7.2.2 and Theorem 5.4.2 should show that the elements of $\mathcal{Z}(\mathrm{G})$ constructed by Fargues and Scholze using excursion operators belong to the stable Bernstein center, i.e., to $\mathcal{Z}_2(\mathrm{G})$. Given the work of Hamann in [Ham21], it might also be interesting to ask a similar question when G is an inner form of GSp_4.

To finish this subsection, we will describe how considerations related to the stable Bernstein center give an easy proof of Mœglin's result that, given any two irreducible representations σ_1, σ_2 of a quasi-split symplectic, special orthogonal or unitary group belonging to the same Arthur packet Σ, their Langlands parameters φ_{σ_1} and φ_{σ_2} satisfy that $\lambda(\varphi_{\sigma_1}) = \lambda(\varphi_{\sigma_2})$. To be sure, we cannot prove any of the much deeper results that Mœglin has proved concerning Arthur parameters.

Notation 7.4.6.
(i) Assume that G is quasi-split. Recall that an Arthur parameter ψ for G refers to the $\hat{\mathrm{G}}$-conjugacy class of a homomorphism $\dot{\psi} : W_F \times \mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C}) \to {}^L\mathrm{G}$ such that its restriction to the product $W_F \times \mathrm{SL}_2(\mathbb{C})$ of the first two factors is a Langlands parameter, its restriction to the second $\mathrm{SL}_2(\mathbb{C})$-factor is a homomorphism $\mathrm{SL}_2(\mathbb{C}) \to \hat{\mathrm{G}}$ of algebraic groups, and such that $\dot{\psi}(W_F)$ is contained in $Cs(W_F)$ for some preferred section $s : W_F \to {}^L\mathrm{G}$ and a bounded subset $C \subset \hat{\mathrm{G}}$.
(ii) Assume that G is a quasi-split symplectic, special orthogonal or unitary group over F. In the even special orthogonal case, we will consider Arthur parameters ψ only up to outer automorphism (i.e., up to the action of $\mathcal{O}_{\mathrm{G}}$ as in Proposition 7.2.4). To each Arthur parameter ψ for G, Arthur has associated, in [Art13, Chapter 7], a finite multiset $\Sigma(\psi)$

over $\operatorname{Irr}(\mathrm{G})$ consisting of unitary representations, called the Arthur packet attached to ψ. Mœglin has shown that $\Sigma(\psi)$ is multiplicity-free. Thus, $\Sigma(\psi)$ may and shall be thought of as a set. If G is even orthogonal, then $\Sigma(\psi)$ could be coarser than what an actual Arthur packet (without the involvement of the outer automorphism) should be.

(iii) Associated to $\Sigma(\psi) \subset \operatorname{Irr}(\mathrm{G})$ is a stable virtual character $\Theta(\psi)$ supported in $\Sigma(\psi)$, and with the following property. Suppose G underlies an elliptic twisted endoscopic datum $\mathcal{E}$ for the usual twisted space $\tilde{\mathrm{GL}}$ associated to a GL-type group $\mathrm{GL} = \mathrm{GL}_N/F$ or $\operatorname{Res}_{E/F}\mathrm{GL}_N$ and an outer automorphism θ that preserves a pinning, as in [Art13] or [Mok15]. Denote by $\iota : {}^L\mathrm{G} \hookrightarrow {}^L\mathrm{GL}_N$ the embedding that is part of the data of $\mathcal{E}$. Let $\pi(\iota\circ\psi)$ denote the representation of $\mathrm{GL}(F)$ constituting the Arthur packet associated to the Arthur parameter $\iota\circ\psi$ of $\mathrm{GL}(F)$. $\pi(\iota\circ\psi)$ is known to be self-dual, so there exists a representation $\tilde{\pi}(\iota\circ\psi)$ of $\tilde{\mathrm{GL}}(F)$ with underlying $\mathrm{GL}(F)$-representation $\pi(\iota\circ\psi)$. Then the endoscopic transfer of $\Theta(\psi)$ to $\tilde{\mathrm{GL}}$ is, up to a nonzero scalar, the twisted character $\Theta_{\tilde{\pi}(\iota\circ\psi)}$ of $\tilde{\pi}(\iota\circ\psi)$. In the even special orthogonal case, $\Theta(\psi)$ is invariant under the group $\mathcal{O}_{\mathrm{G}}$ as in Proposition 7.2.4.

Proposition 7.4.7. *Suppose* G *is a quasi-split symplectic, special orthogonal or unitary group, and let* ψ *be an Arthur parameter for* G. *Then for all* $\sigma_1, \sigma_2 \in \Sigma(\psi)$ *with Langlands parameters* $\varphi_{\sigma_1}, \varphi_{\sigma_2} \in \Phi(\mathrm{G})/\mathcal{O}_{\mathrm{G}}$ *(as assigned by Proposition 7.2.4 together with Theorem 2.10.10, and with* $\mathcal{O} = \mathcal{O}_{\mathrm{G}}$ *as in Proposition 7.2.4), we have* $\lambda(\varphi_{\sigma_1}) = \lambda(\varphi_{\sigma_2}) \in \Omega({}^L\mathrm{G})/\mathcal{O}_{\mathrm{G}}$.

Proof. Write $\mathcal{O} = \mathcal{O}_{\mathrm{G}}$. We let $\mathrm{GL} = \mathrm{GL}_N$ or $\operatorname{Res}_{E/F}\mathrm{GL}_N$, $\tilde{\mathrm{GL}}, \mathcal{E}, \iota, \pi(\iota\circ\psi), \theta$ and $\tilde{\pi}(\iota\circ\psi)$ be as in Notation 7.4.6(iii). We assume that $\mathcal{E}$ is the 'simplest possible endoscopic datum' — as in [Wal10, Section 1.8], but such that the 'H^-' of that reference is a trivial group.

One knows that the maps $\Phi(\mathrm{G})/\mathcal{O} \to \Phi(\mathrm{GL})$ and $\underline{\Omega}({}^L\mathrm{G}) := \Omega({}^L\mathrm{G})/\mathcal{O} \to \Omega({}^L\mathrm{G})$ induced by ι are injective; we will use the injectivity of the latter map. Every element $f^{\mathrm{G}} \in C_c^\infty(\mathrm{G}(F))$ whose image in the vector space $\mathcal{I}(\mathrm{G})$ of $\operatorname{Int}\mathrm{G}(F)$-coinvariants of $C_c^\infty(\mathrm{G}(F))$ is fixed by $\mathcal{O}_{\mathrm{G}}$, matches some $f \in C_c^\infty(\tilde{\mathrm{GL}}(F))$, by [Art13, Corollary 2.1.2] and [Mok15, Proposition 3.1.1(b)]. Since $\mathcal{O}_{\mathrm{G}}$-invariant distributions are determined by their values on $\mathcal{O}_{\mathrm{G}}$-invariant functions ($\mathcal{O}_{\mathrm{G}}$ is of order at most 2), it follows that the dual map from the space of $\mathcal{O}_{\mathrm{G}}$-invariant stable distributions on $\mathrm{G}(F)$ to the space of distributions on the twisted space $\tilde{\mathrm{GL}}(F)$ is injective, as is its restriction to the space of stable (not necessarily tempered) virtual characters on $\mathrm{G}(F)$.

By Proposition 7.2.4 and Corollary 5.5.2(ii), we can identify $\mathbb{C}[\underline{\Omega}({}^L\mathrm{G})]$ with $\mathcal{Z}_{2,\mathcal{O}}(\mathrm{G})$. By Corollary 5.5.2(ii) (or use [Coh18]), we can identify $\mathbb{C}[\Omega({}^L\mathrm{GL})]$ with $\mathcal{Z}_2(\mathrm{GL})$. Thus, the map $\mathbb{C}[\Omega({}^L\mathrm{GL})] \to \mathbb{C}[\underline{\Omega}({}^L\mathrm{G})]$ obtained by pulling back the map $\underline{\Omega}({}^L\mathrm{G}) := \Omega({}^L\mathrm{G})/\mathcal{O} \hookrightarrow \Omega({}^L\mathrm{GL})$ induced by ι (defined exactly like the map $\underline{\Omega}(\underline{\mathrm{H}})$ of Notation 6.1.25(iv)), identifies with a map $\mathcal{Z}_2(\mathrm{GL}) \to \mathcal{Z}_{2,\mathcal{O}}(\mathrm{G})$, which we denote by $z \mapsto z_{\mathrm{G}}$. Using Proposition 7.2.4, it is easy to check that the conditions of Theorem 6.2.3, as applied in 'Scenario 2', are met; the condition (I)(c) of that theorem follows from the invariance of the transfer factors under outer automorphism (in our particular situations) as mentioned in [Art13, Section 2.1, page 56], and the condition (II)(d) of that theorem is baked into the definition of the local Langlands correspondence of [Art13] or [Mok15], which is consistent with Proposition 7.2.4. Therefore, whenever $f \in C_c^\infty(\tilde{\mathrm{GL}}(F))$ and $f^{\mathrm{G}} \in C_c^\infty(\mathrm{G}(F))$ whose image in $\mathcal{I}(\mathrm{G})$ is $\mathcal{O}$-invariant have matching orbital integrals, then so do $z * f$ and $z_{\mathrm{G}} * f^{\mathrm{G}}$.

Dually at the level of characters, we conclude that, since the $\mathcal{O}$-invariant stable virtual character $\Theta(\psi)$ on $\mathrm{G}(F)$ transfers to a nonzero scalar multiple of the twisted character $\Theta(\iota,\psi,\theta) := \Theta_{\tilde{\pi}_{\iota\circ\psi}}$ on $\tilde{\mathrm{GL}}(F)$, the distribution $f^{\mathrm{G}} \mapsto \Theta(\psi)(z_{\mathrm{G}} * f^{\mathrm{G}})$ on $\mathrm{G}(F)$ transfers to the distribution $f \mapsto \Theta(\iota,\psi,\theta)(z * f)$ on $\tilde{\mathrm{GL}}(F)$, for all $z \in \mathcal{Z}_2(\mathrm{GL})$ and $z_{\mathrm{G}} \in \mathcal{Z}_{2,\mathcal{O}}(\mathrm{G})$ as above: this uses that $f^{\mathrm{G}} \mapsto \Theta(\psi)(z_{\mathrm{G}} * f^{\mathrm{G}})$ is $\mathcal{O}$-invariant whenever $z_{\mathrm{G}} \in \mathcal{Z}_{2,\mathcal{O}}(\mathrm{G})$ (use, e.g., (57)).

The distribution $f \mapsto \Theta(\iota,\psi,\theta)(z * f)$ equals $\hat{z}(\pi(\iota\circ\psi))\Theta(\iota,\psi,\theta)$. Since the space of $\mathcal{O}$-invariant stable distributions on $\mathrm{G}(F)$ has been observed above to inject via endoscopic transfer into the space of distributions on $\tilde{\mathrm{GL}}(F)$, it follows that for all $z_{\mathrm{G}} \in \mathcal{Z}_{2,\mathcal{O}}(\mathrm{G})$ belonging to the image of $\mathcal{Z}_2(\mathrm{GL}) \to \mathcal{Z}_{2,\mathcal{O}}(\mathrm{G})$, $f^{\mathrm{G}} \mapsto \Theta(\psi)(z_{\mathrm{G}} * f^{\mathrm{G}})$ is a scalar multiple of $\Theta(\psi)$. Thus, every $z_{\mathrm{G}} \in \mathcal{Z}_{2,\mathcal{O}}(\mathrm{G})$ that lies in the image of $\mathbb{C}[\Omega({}^L\mathrm{GL})] = \mathcal{Z}_2(\mathrm{GL}) \to \mathcal{Z}_{2,\mathcal{O}}(\mathrm{G}) = \mathbb{C}[\underline{\Omega}({}^L\mathrm{G})]$ acts by the same scalar on all the elements in $\Sigma(\psi)$. Here, we used that for each $\sigma \in \Sigma(\psi)$, Θ_σ contributes nontrivially to $\Theta(\psi)$, as follows from [Art13, Theorem 2.2.1] and [Mok15, Theorem 3.2.1], and the fact that the $\mathcal{S}_\psi$ of these references is abelian.

Let $\sigma_1, \sigma_2 \in \Sigma(\psi)$ be as in the proposition. The scalar with which $z_{\mathrm{G}} \in \mathbb{C}[\underline{\Omega}({}^L\mathrm{G})] = \mathcal{Z}_{2,\mathcal{O}}(\mathrm{G})$ acts on σ_i is given by $z_{\mathrm{G}}(\lambda(\varphi_{\sigma_i}))$. Thus, we conclude that for all $z \in \mathbb{C}[\Omega({}^L\mathrm{GL})] = \mathcal{Z}_2(\mathrm{GL})$, z takes the same value on the images of $\lambda(\varphi_{\sigma_1})$ and $\lambda(\varphi_{\sigma_2})$ in $\Omega({}^L\mathrm{GL})$. Since $\mathbb{C}[\Omega({}^L\mathrm{GL})]$ separates points on $\Omega({}^L\mathrm{GL})$, the images of $\lambda(\varphi_{\sigma_1})$ and $\lambda(\varphi_{\sigma_2})$ in $\Omega({}^L\mathrm{GL})$ are equal. Since $\underline{\Omega}({}^L\mathrm{G}) \to \Omega({}^L\mathrm{GL})$ is an injection, it follows that $\lambda(\varphi_{\sigma_1}) = \lambda(\varphi_{\sigma_2})$, as desired. □

Remark 7.4.8. It might be more satisfying to replace the last paragraph of the proof of the above proposition with a justification that the map $\mathcal{Z}_2(\mathrm{GL}) \to \mathcal{Z}_{2,\mathcal{O}}(\mathrm{G})$ is surjective, followed by the observation that $\mathcal{Z}_{2,\mathcal{O}}$ separates the points on $\underline{\Omega}({}^L\mathrm{G})$. But we have not attempted to prove that.

Remark 7.4.9. The local Langlands correspondence, even when combined with a description of stable packets and character identities, does not tell us how to relate the adjoint gamma factor to harmonic analysis. Therefore, it is not sufficient to yield the formal degree conjecture of Ichino, Ikeda and Hiraga ([HII08a, Conjecture 1.4], but we will follow [Oha22, Conjecture 3.2], because we have not worked out the relation between the group $\tilde{\mathcal{S}}_\varphi$ of [Art06] and the group $\mathcal{S}_\varphi$ of [HII08a]). However, it could help reduce its proof to that for the quasi-split inner form, as is explained in [ILM17, Section 5, before Remark 5.2]. We will now recall this argument, in a form that involves Proposition 3.3.7 and illustrates that a weaker conclusion can be drawn with a weaker hypothesis.

(i) Assume that Σ and Σ^* are atomically stable discrete series packets on $\mathrm{G}(F)$ and its quasi-split inner form $\mathrm{G}^*(F)$, respectively, and that Σ^* transfers to Σ in the manner described in Proposition 3.3.7(ii), with both $\mathcal{O}_{\mathrm{G}}$ and its analogue $\mathcal{O}_{\mathrm{G}^*}$ trivial. Assume also that a Langlands parameter $\varphi = \varphi_\Sigma = \varphi_{\Sigma^*} \in \Phi_2(\mathrm{G}) = \Phi_2(\mathrm{G}^*)$ is assigned to them. Associated to φ are three finite groups $\tilde{\mathcal{S}}_\varphi, \mathcal{S}_\varphi$ and $\mathcal{S}_\varphi^\natural$, where $\tilde{\mathcal{S}}_\varphi$ and $\mathcal{S}_\varphi$ are as in [Art06, (3.2) and (1.1)], and $\mathcal{S}_\varphi^\natural$ is as in [HII08a, page 287] or [Oha22, Conjecture 3.2]. If $\hat{Z}_{\mathrm{sc}}$ is as in [Art06], we have a homomorphism $\hat{Z}_{\mathrm{sc}} \to \tilde{\mathcal{S}}_\varphi$, whose image is a central subgroup, and whose cokernel is $\mathcal{S}_\varphi$ (see [Art06, shortly below (3.2)]).

(ii) By [Art06, page 209, Conjecture], one expects that, for some character $\hat{\zeta}_\varrho$ of $\hat{Z}_{\mathrm{sc}}$ (we write ϱ for the ρ of [Art06]), there is a bijection $\rho \mapsto \sigma_\rho$ onto Σ from the set $\hat{\tilde{\mathcal{S}}}_\varphi(\hat{\zeta}_\varrho)$ of irreducible characters on $\tilde{\mathcal{S}}_\varphi$ whose central character pulls back to $\hat{\zeta}_\varrho$ on $\hat{Z}_{\mathrm{sc}}$. Assume that this expectation is satisfied. [Oha22, Conjecture 3.2] states that, for an appropriate choice of measures spelled out in [HII08b], we have an equality of the form:

$$d(\sigma_\rho) = \frac{\dim \rho}{\#\mathcal{S}_\varphi^\natural} |\gamma(0, \mathrm{Ad} \circ \varphi, \psi)|.$$

(iii) Under our assumptions, it is easy to see that the formal degree conjecture is equivalent to the combination of the following two assertions:

- If $\rho_1, \rho_2 \in \hat{\tilde{\mathcal{S}}}_\varphi(\hat{\zeta}_\varrho)$, then:

$$d(\sigma_{\rho_1})(\dim \rho_1)^{-1} = d(\sigma_{\rho_2}) \dim(\rho_2)^{-1}. \tag{108}$$

- The following weakened version of the formal degree conjecture holds:

$$\sum_{\sigma \in \Sigma} d(\sigma)^2 = \frac{\#\mathcal{S}_\varphi}{(\#\mathcal{S}_\varphi^\natural)^2} \cdot |\gamma(0, \mathrm{Ad} \circ \varphi, \psi)|^2, \tag{109}$$

where we used that the sum of the $(\dim \rho)^2$ as ρ ranges over $\hat{\tilde{\mathcal{S}}}_\varphi(\hat{\zeta}_\varrho)$ equals $\#\mathcal{S}_\varphi$: this is because the regular representation of $\tilde{\mathcal{S}}_\varphi \times \tilde{\mathcal{S}}_\varphi$ on the $(\#\mathcal{S}_\varphi)$-dimensional vector space of $(\hat{\mathrm{Z}}_{\mathrm{sc}}, \hat{\zeta}_\varrho)$-equivariant complex valued functions on $\tilde{\mathcal{S}}_\varphi$ decomposes as the sum of the $\rho^\vee \otimes \rho$, with ρ ranging over $\hat{\tilde{\mathcal{S}}}_\varphi(\hat{\zeta}_\varrho)$.

(iv) Proposition 3.3.7(ii) gives the equality

$$\sum_{\sigma \in \Sigma} d(\sigma)^2 = \sum_{\sigma^* \in \Sigma^*} d(\sigma^*)^2,$$

so that (109) for G is equivalent to that for G^* (the compatibility of measures imposed in Proposition 3.3.7 is consistent with that in [HII08b]).

(v) On the other hand, we claim that (108) follows if the following weak form of the character identities as in [Art06, page 209, Conjecture] is satisifed: for each $s \in \tilde{\mathcal{S}}_\varphi$ that does not belong to the image of $\hat{Z}_{\mathrm{sc}}$, $\sum(\operatorname{tr}\rho(\tilde{s})) \cdot \Theta_{\sigma_\rho}$ is obtained by endoscopic transfer from a 'non-principal' endoscopic datum for G, where here and in what follows, the sum ranges over $\rho \in \hat{\tilde{\mathcal{S}}}_\varphi(\hat{\zeta}_\varrho)$. This has the consequence that $\sum_\rho d(\sigma_\rho) \operatorname{tr}\rho(\tilde{s}) = 0$ for all such s: the argument of [Sha90, Corollary 9.10] applies without needing G to be quasi-split. Thus, the $(\hat{Z}_{\mathrm{sc}}, \hat{\zeta}_\varrho)$-equivariant class function $\sum_\rho d(\sigma_\rho) \operatorname{tr}\rho$ on $\tilde{\mathcal{S}}_\varphi$ is supported in the image of $\hat{Z}_{\mathrm{sc}} \to \tilde{\mathcal{S}}_\varphi$, which forces the virtual representation $\sum_\rho d(\sigma_\rho)\rho$ of $\tilde{\mathcal{S}}_\varphi$ to be a multiple of $\sum_\rho (\dim\rho)\rho$. Thus, (108) follows.

(vi) Another way to look at all this is that, if character identities as in (v) above are satisfied, then the formal degree conjecture for the elements of $\Sigma \cup \Sigma^*$ needs to be checked only for a single element of $\Sigma \cup \Sigma^*$ (for example, [ILM17], in their context, used the generic element).

(vii) None of this yields anything new for (possibly non-quasi-split) odd special orthogonal or unitary groups, since the formal degree conjecture is well-known in these cases from the work of Ichino, Lapid and Mao ([ILM17]) and Beuzart-Plessis ([BP21]); there is also the work of Morimoto ([Mor22]). However, e.g., for inner forms of symplectic groups, the above considerations do reduce the proof of the weakened form (109) to the split case.

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