ON CONGRUENT ISOMORPHISMS FOR TORI

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Abstract. Let F and F' be two l-close nonarchimedean local fields, where l is a positive integer, and let T and T' be two tori over F and F', respectively, such that their cocharacter lattices can be identified as modules over the "at most l-ramified" absolute Galois group $\Gamma_F/I_F^l \cong \Gamma_{F'}/I_{F'}^l$. In the spirit of the work of Kazhdan and Ganapathy, for every positive integer m relative to which l is large, we construct a congruent isomorphism $\mathrm{T}(F)/\mathrm{T}(F)_m \cong \mathrm{T}'(F')/\mathrm{T}'(F')_m$, where $\mathrm{T}(F)_m$ and $\mathrm{T}(F')_m$ are the minimal congruent filtration subgroups of $\mathrm{T}(F)$ and $\mathrm{T}(F')$, respectively, defined by J.-K. Yu. We prove that this isomorphism is functorial and compatible with both the isomorphism constructed by Chai and Yu and the Kottwitz homomorphism for tori. We show that, when l is even larger relative to m, it moreover respects the local Langlands correspondence for tori.

1. Introduction

1.1. **A crude version of the main result.** Two nonarchimedean local fields F and F' are said to be l-close, where l is a positive integer, if $\mathfrak{O}_F/\mathfrak{p}_F^l \cong \mathfrak{O}_{F'}/\mathfrak{p}_{F'}^l$, where $\mathfrak{O}_?$ stands for the ring of integers of ?, and and $\mathfrak{p}_?$ for the maximal ideal of $\mathfrak{O}_?$.

If F and F' are l-close, then P. Deligne ([Del84]) constructs an isomorphism $\Gamma_F/I_F^l \to \Gamma_{F'}/I_{F'}^l$ now known as a Deligne isomorphism, where $\Gamma_?$ denotes the Galois group of a chosen separable closure over ?, and $I_?^l$ stands for the l-th upper ramification filtration subgroup of the inertia subgroup $I_? \subset \Gamma_?$. If further F and F' have finite residue fields, Kazhdan isomorphisms (see [Kaz86]), pioneered by D. Kazhdan and studied by various others, notably by R. Ganapathy (see, e.g., [Gan15] and [Gan22]), allow us to relate harmonic analysis on reductive groups over F to that on reductive groups over F'. Thus, for instance, one could hope to study local Langlands correspondence for a group over F' by using local Langlands correspondence for a group over F, if the latter is known.

One has a good understanding of Kazhdan isomorphisms for split groups, by [Kaz86] and [Gan15]. For reductive groups that may not be split, Kazhdan isomorphisms have been constructed by Ganapathy in [Gan22]. However, a lot of the properties of these isomorphisms remain to be studied, and such a study is being pursued by Ganapathy and her collaborators.

In the present paper, we will stick to tori, and investigate questions related to Kazhdan isomorphisms $\mathrm{T}(F)/\mathrm{T}(F)_m \cong \mathrm{T}'(F')/\mathrm{T}'(F')_m$ for tori, when F and F' are l-close and T'/F' is a *transfer* of T/F, that is, we have an identification $X^*(\mathrm{T}) = X^*(\mathrm{T}')$ of character lattices, or equivalently an identification $X_*(\mathrm{T}) = X_*(\mathrm{T}')$ of cocharacter lattices, as modules over Γ_F/I_F^l, identified via [Del84] with $\Gamma_{F'}/I_{F'}^l$, under the implicitly imposed assumption that I_F^l acts trivially on $X^*(\mathrm{T})$ and $I_{F'}^l$ on $X^*(\mathrm{T}')$. Here the filtrations $\{\mathrm{T}(F)_m\}_{m\geq 0}$ and $\{\mathrm{T}(F')_m\}_{m\geq 0}$ are the minimal congruent filtrations of $\mathrm{T}(F)$ and $\mathrm{T}(F')$, respectively, as defined in [Yu15].

A crude version of our main result, which we will state in greater detail in Theorem 1.2.1 below, is as follows:

Theorem 1.1.1. *Suppose a local field F is l-close to a local field F', and a torus T'/F' is a transfer of a torus T/F. Then:*

(i) If l is large relative to m, then there exists a (necessarily unique) "congruent" isomorphism $\mathrm{T}(F)/\mathrm{T}(F)_m \to \mathrm{T}'(F')/\mathrm{T}'(F')_m$.
(ii) These isomorphisms are suitably functorial.

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(iii) *They are compatible with the isomorphisms constructed by Chai and Yu (see* [CY01]*) and with Kottwitz homomorphisms for tori (see* [Kot97, Section 7] *and* [KP23, Section 11.1]*).*
(iv) *If l is even larger relative to m, these isomorphisms respect the local Langlands correspondence for tori.*

1.2. **Statement of the main result — more precise version.** Now we state our main result, Theorem 1.2.1 below, in terms of objects and notation defined in later sections, especially Section 2; however, let us give an introduction to the main objects involved:

(i) If we say $(F, \mathrm{T}) \leftrightarrow_l (F', \mathrm{T}')$ (see Notation 2.3.1(viii)), we roughly mean that F and F' are discretely valued Henselian fields with perfect residue field that are l-close to each other ($F \leftrightarrow_l F'$), and that the torus T' over F' is a transfer of the torus T over F.
(ii) For (F, T) as above, $h(F, \mathrm{T})$ is a positive integer from [CY01, Section 8.1], sort of upper-bounding the nontriviality of the smoothening process required to arrive at the Néron model of T.
(iii) We will be interested in "congruent isomorphisms" $\mathrm{T}(F)/\mathrm{T}(F)_m \to \mathrm{T}'(F')/\mathrm{T}'(F')_m$ (Definition 3.1.3(ii)). These are isomorphisms of abelian groups.
(iv) Interpolating the $\mathrm{T}(F)_m$, with m varying over nonnegative integers, are the "minimal congruent filtration subgroups" $\mathrm{T}(F)_r$ of J.-K. Yu, with r varying over nonnegative real numbers.
(v) Each torus T determines a relation "$m \ll_{\mathrm{T}} l$" meaning that l is sufficiently large relative to m and the Herbrand function of a minimal splitting field for T (see Notation 2.3.1(vii)).

Our more precise version of Theorem 1.1.1 is as follows; note that it has individual assertions that are more precise versions of the corresponding assertions of Theorem 1.1.1.

Theorem 1.2.1. *(i) Suppose $(F, \mathrm{T}) \leftrightarrow_l (F', \mathrm{T}')$, set $h = h(F, \mathrm{T})$, and suppose m is a positive integer, with $0 < m + 3h(F, \mathrm{T}) \ll_{\mathrm{T}} l$. Then there exists a (unique) congruent isomorphism*

$$\mathrm{T}(F)/\mathrm{T}(F)_m \to \mathrm{T}'(F')/\mathrm{T}'(F')_m.$$

Moreover, if $m + 3h(F, \mathrm{T}) + 1 \ll_{\mathrm{T}} l$, then this isomorphism respects the minimal congruent filtration, i.e., takes the image of $\mathrm{T}(F)_r$ to that of $\mathrm{T}'(F')_r$, for $0 \le r \le m$.

(ii) The isomorphisms of (i) *satisfy the following functoriality. Whenever $(F, \mathrm{T}_i) \leftrightarrow_l (F', \mathrm{T}'_i)$ for $i = 1, 2$, with the same underlying $F \leftrightarrow_l F'$, and $0 < m + 3h(F, \mathrm{T}_i) \ll_{\mathrm{T}_j} l$ for $i, j \in \{1, 2\}$, and we are given homomorphisms $\mathrm{T}_1 \to \mathrm{T}_2$ and $\mathrm{T}'_1 \to \mathrm{T}'_2$ inducing the same homomorphism $X^*(\mathrm{T}_2) = X^*(\mathrm{T}'_2) \to X^*(\mathrm{T}'_1) = X^*(\mathrm{T}_1)$, the following diagram is commutative:*

$$\begin{tikzcd}
\mathrm{T}_1(F)/\mathrm{T}_1(F)_m \arrow[r] \arrow[d] & \mathrm{T}_2(F)/\mathrm{T}_2(F)_m \arrow[d] \\
\mathrm{T}'_1(F')/\mathrm{T}'_1(F')_m \arrow[r] & \mathrm{T}'_2(F')/\mathrm{T}'_2(F')_m
\end{tikzcd},$$

where the vertical arrows are as in (i)*, and the horizontal arrows are induced by the homomorphisms $\mathrm{T}_1 \to \mathrm{T}_2$ and $\mathrm{T}'_1 \to \mathrm{T}'_2$.*

(iii) In the setting of (i)*, we have the following compatibility with the Chai-Yu isomorphisms and Kottwitz homomorphisms, in the sense that the following diagram is commutative:*

$$\begin{tikzcd}
\mathrm{T}(F)_b/\mathrm{T}(F)_m \arrow[r] \arrow[d] & \mathrm{T}(F)/\mathrm{T}(F)_m \arrow[r] \arrow[d] & (X_*(\mathrm{T})_{I_F})^{\Gamma_{\kappa_F}} \arrow[d] \\
\mathrm{T}'(F')_b/\mathrm{T}'(F')_m \arrow[r] & \mathrm{T}'(F')/\mathrm{T}'(F')_m \arrow[r] & (X_*(\mathrm{T}')_{I_{F'}})^{\Gamma_{\kappa_{F'}}}
\end{tikzcd}, \tag{1}$$

where $\mathrm{T}(F)_b$ (resp., $\mathrm{T}(F')_b$) denotes the maximal bounded subgroup of $\mathrm{T}(F)$ (resp., $\mathrm{T}(F')$), the left vertical arrow is induced by the Chai-Yu isomorphism of [CY01, Theorem 8.5]*, the middle vertical arrow is as in* (i)*, the right vertical arrow is induced by the $\Gamma_F/I_F^l = \Gamma_{F'}/I_{F'}^l$-equivariant identification $X_*(\mathrm{T}) = X_*(\mathrm{T}')$, and the second horizontal arrow of either row is the Kottwitz homomorphism.*

(iv) In the setting of (i)*, if F and F' are complete and their residue field $\kappa_F = \kappa_{F'}$ is finite, and we assume the stronger inequality $0 < m + 4h(F, \mathrm{T}) \underline{\ll}_{\mathrm{T}} l$, we have the following compatibility with the local Langlands correspondence for tori. We have a commutative diagram*

$$\begin{tikzcd} \mathrm{Hom}(\mathrm{T}(F)/\mathrm{T}(F)_m, \mathbb{C}^\times) \arrow[r, hook] \arrow[d] & H^1(W_F/I_F^l, \hat{\mathrm{T}}) \arrow[d, "\mathrm{Del}_l", "\cong"'] \\ \mathrm{Hom}(\mathrm{T}'(F')/\mathrm{T}'(F')_m, \mathbb{C}^\times) \arrow[r, hook] & H^1(W_{F'}/I_{F'}^l, \hat{\mathrm{T}}') \end{tikzcd} \tag{2}$$

where the horizontal arrows are given by the local Langlands correspondence for tori, the left vertical arrow is induced by the isomorphism $\mathrm{T}(F)/\mathrm{T}(F)_m \to \mathrm{T}'(F')/\mathrm{T}'(F')_m$ of (i)*, and the right vertical arrow is obtained by combining the Deligne isomorphism $W_F/I_F^l \cong W_{F'}/I_{F'}^l$ together with the $\Gamma_F/I_F^l \cong \Gamma_{F'}/I_{F'}^l$-equivariant identification $\hat{\mathrm{T}} = \mathrm{Hom}(X_*(\mathrm{T}), \mathbb{C}^\times) = \mathrm{Hom}(X_*(\mathrm{T}'), \mathbb{C}^\times) = \hat{\mathrm{T}}'$. Here, to make sense of the top row (to which the bottom row is analogous), part of the assertion, implicitly, is that the image of the subset*

$$\mathrm{Hom}(\mathrm{T}(F)/\mathrm{T}(F)_m, \mathbb{C}^\times) \subset \mathrm{Hom}_{\mathrm{cts}}(\mathrm{T}(F), \mathbb{C}^\times)$$

under the local Langlands correspondence is contained in the subset of $H^1(W_F, \hat{\mathrm{T}})$ obtained by inflation from $H^1(W_F/I_F^l, (\hat{\mathrm{T}})^{I_F^l}) = H^1(W_F/I_F^l, \hat{\mathrm{T}})$.

If further T *is weakly induced in the sense of* [KP23]*, i.e., satisfies the condition* (T) *of* [Yu15]*, i.e., becomes an induced torus after base-change to some tamely ramified extension, one can replace $h(F, \mathrm{T}), h(F, \mathrm{T}_1)$ and $h(F, \mathrm{T}_2)$ by 0 in the above statements.*

1.3. **The case of split tori.** The case of split tori is an obvious extension of the GL_1-case covered by [Del84], and is a very special case of [Kaz86].

1.3.1. *Deligne's triples.* We first digress to remark on some fine print. For all these considerations, choosing isomorphisms is important to ensure that various constructions are well-defined. Thus, Deligne works not with isomorphisms $\mathfrak{O}_F/\mathfrak{p}_F^l \cong \mathfrak{O}_{F'}/\mathfrak{p}_{F'}^l$ of truncated discrete valuation rings, but rather with slightly more rigidified data in the form of isomorphisms $(\mathfrak{O}_F/\mathfrak{p}_F^l, \mathfrak{p}_F/\mathfrak{p}_F^{l+1}, \epsilon) \to (\mathfrak{O}_{F'}/\mathfrak{p}_{F'}^l, \mathfrak{p}_{F'}/\mathfrak{p}_{F'}^{l+1}, \epsilon')$ of triples, where ϵ is the obvious map $\mathfrak{p}_F/\mathfrak{p}_F^{l+1} \to \mathfrak{O}_F/\mathfrak{p}_F^l$, and ϵ' is analogous. Fixing such an isomorphism is what lets one construct the Deligne isomorphism $\Gamma_F/I_F^l \cong \Gamma_{F'}/I_{F'}^l$ of [Del84] and show that it is well-defined up to an inner conjugation.

1.3.2. *The case of* GL_1*, from* [Del84]. First, if $\mathrm{T} = \mathrm{GL}_1/F$ and $\mathrm{T}' = \mathrm{GL}_1/F'$ compatibly, the required isomorphism, say when $m = l$, is the isomorphism

$$F^\times/(1 + \mathfrak{p}_F^l) \to {F'}^\times/(1 + \mathfrak{p}_{F'}^l) \tag{3}$$

constructed by Deligne from the realization $F \leftrightarrow_l F'$, in a canonical manner starting from the truncated data, in [Del84, Section 1.2]. A more concrete but slightly less obviously canonical description for (3) requires it to

- restrict to $(\mathfrak{O}_F/\mathfrak{p}_F^l)^\times \cong (\mathfrak{O}_{F'}/\mathfrak{p}_{F'}^l)^\times$ on $\mathfrak{O}_F^\times/(1 + \mathfrak{p}_F^l) = (\mathfrak{O}_F/\mathfrak{p}_F^l)^\times$, and
- send the image of a uniformizer ϖ_F to that of a uniformizer $\varpi_{F'}$ whenever ϖ_F and $\varpi_{F'}$ are compatible under $\mathfrak{p}_F/\mathfrak{p}_F^{l+1} \to \mathfrak{p}_{F'}/\mathfrak{p}_{F'}^{l+1}$.

1.3.3. *General split tori.* When T and T′ are split but otherwise general, the datum relating T′ to T amounts to just an isomorphism $X^*(\mathrm{T}) \to X^*(\mathrm{T}')$ of abelian groups, say $\chi \mapsto \chi'$. Then our isomorphism $\mathrm{T}(F)/\mathrm{T}(F)_m \to \mathrm{T}'(F')/\mathrm{T}'(F')_m$ is defined so as to match the images of $t \in \mathrm{T}(F)$ and $t' \in \mathrm{T}'(F')$ precisely when for each $\chi \in X^*(\mathrm{T})$ identifying with $\chi' \in X^*(\mathrm{T}')$, the images of $\chi(t)$ and $\chi'(t')$ correspond under (3).

For general tori, in the spirit of the above discussion, we find it convenient to specify the isomorphism $\mathrm{T}(F)/\mathrm{T}(F)_m \to \mathrm{T}'(F')/\mathrm{T}'(F')_m$ by forcing compatibilities that characterize it.

1.4. **Standard and congruent isomorphisms.** In addition to the congruent filtration subgroups $\mathrm{T}(F)_m$, we will need the "naive" filtration subgroups $\mathrm{T}(F)_r^{\mathrm{naive}}$ $(r \geq 0)$:

$$\mathrm{T}(F)_r^{\mathrm{naive}} = \{t \in \mathrm{T}(F)_b \mid \mathrm{val}_F(\chi(t) - 1) \geq r, \ \forall\, \chi \in X^*(\mathrm{T})\}, \tag{4}$$

for a suitable extension val_F of the normalized discrete valuation on F.

When $(F, \mathrm{T}) \leftrightarrow_l (F', \mathrm{T}')$, one defines:

(i) for $0 < r \ll_{\mathrm{T}} l$, a standard isomorphism $\mathrm{T}(F)/\mathrm{T}(F)_r^{\mathrm{naive}} \to \mathrm{T}'(F')/\mathrm{T}'(F')_r^{\mathrm{naive}}$ to be one that matches the images of $t \in \mathrm{T}(F)$ and $t' \in \mathrm{T}'(F')$ whenever $\chi(t)$ and $\chi'(t')$ have images that match under a suitable extension of (3) (see Definition 3.1.3(i) for more details).

(ii) a congruent isomorphism $\mathrm{T}(F)/\mathrm{T}(F)_m \to \mathrm{T}'(F')/\mathrm{T}'(F')_m$ to be an isomorphism induced by a standard isomorphism after passage to maximal unramified extensions (see Definition 3.1.3(ii) for more details).

Standard isomorphisms are unique when they exist, and have good functoriality properties, compatibility with the Kottwitz homomorphism, and (in the case of complete fields with finite residue field) compatibility with the local Langlands correspondence. Congruent isomorphisms inherit the first three of these four properties, and under stronger assumptions the fourth too.

Like with [Gan22], we too make use of an argument following the construction of the Kottwitz homomorphism for tori (see [KP23, Proposition 11.1.1]): the following simple yet not entirely obvious fact gets us started (Proposition 3.3.1).

Proposition 1.4.1. *If $(F, \mathrm{T}) \leftrightarrow_l (F', \mathrm{T}')$, F is strictly Henselian, and $0 < r \ll_{\mathrm{T}} l$, then a standard isomorphism $\mathrm{T}(F)/\mathrm{T}(F)_r^{\mathrm{naive}} \to \mathrm{T}'(F')/\mathrm{T}'(F')_r^{\mathrm{naive}}$ exists.*

The difficulty is that we cannot see an obvious way to descend this to the non-strictly-Henselian case, without going through congruent isomorphisms. This difficulty is what motivates congruent isomorphisms for us, notwithstanding their unpleasantness: indeed, if $\tilde{F}/F$ is a maximal unramified extension and $\tilde{F} \leftrightarrow_l \tilde{F}'$ "lies over" $F \leftrightarrow_l F'$, then a "Galois invariant" isomorphism $\mathrm{T}(\tilde{F})/\mathrm{T}(\tilde{F})_m \to \mathrm{T}'(\tilde{F}')/\mathrm{T}'(\tilde{F}')_m$, on taking Galois invariants, gives us an isomorphism $\mathrm{T}(F)/\mathrm{T}(F)_m \to \mathrm{T}'(F')/\mathrm{T}'(F')_m$: $\mathrm{T}(\tilde{F})_m$ has trivial $\mathrm{Gal}(\tilde{F}/F)$-cohomology, being sort of pro-unipotent over the (perfect) residue field of $\mathfrak{O}_F$ ([KP23, Proposition 13.8.1]).

1.5. **Using the work of Chai and Yu.** Thus, the main question now becomes: when can a standard isomorphism $\mathrm{T}(\tilde{F})/\mathrm{T}(\tilde{F})_r^{\mathrm{naive}} \to \mathrm{T}'(\tilde{F}')/\mathrm{T}'(\tilde{F}')_r^{\mathrm{naive}}$ induce an isomorphism $\mathrm{T}(\tilde{F})/\mathrm{T}(\tilde{F})_m \to \mathrm{T}'(\tilde{F}')/\mathrm{T}'(\tilde{F}')_m$? This seems to require a much deeper ingredient: the spectacular work of Chai and Yu ([CY01]).

Under the assumption $m + 3h(F, \mathrm{T}) \ll_{\mathrm{T}} l$ of Theorem 1.2.1, Chai and Yu construct a canonical isomorphism $\mathcal{T}^{\mathrm{ft}} \times_{\mathfrak{O}_F} \mathfrak{O}_F/\mathfrak{p}_F^m \to \mathcal{T}'^{\mathrm{ft}} \times_{\mathfrak{O}_{F'}} \mathfrak{O}_{F'}/\mathfrak{p}_{F'}^m$, where $\mathcal{T}^{\mathrm{ft}}$ is the finite type Néron model of T, and $\mathcal{T}'^{\mathrm{ft}}$ that of T'. The properties that characterize their isomorphism are implicit in their construction. A careful examination of their construction, together with a few arguments that are tedious but not difficult (see Proposition 3.5.1), tells us that their isomorphism $\mathcal{T}^{\mathrm{ft}} \times_{\mathfrak{O}_F} \mathfrak{O}_F/\mathfrak{p}_F^m \to \mathcal{T}'^{\mathrm{ft}} \times_{\mathfrak{O}_{F'}} \mathfrak{O}_{F'}/\mathfrak{p}_{F'}^m$ is characterized by the fact that, upon evaluating on $\mathfrak{O}_{\tilde{F}}/\mathfrak{p}_{\tilde{F}}^m = \mathfrak{O}_{\tilde{F}'}/\mathfrak{p}_{\tilde{F}'}^m$, it is induced by a restriction of a standard isomorphism $\mathrm{T}(\tilde{F})/\mathrm{T}(\tilde{F})_{m+h(F,\mathrm{T})}^{\mathrm{naive}} \to \mathrm{T}'(\tilde{F}')/\mathrm{T}'(\tilde{F}')_{m+h(F,\mathrm{T})}^{\mathrm{naive}}$. This is the key observation that lets us construct congruent (and hence also standard) isomorphisms outside the strictly Henselian and "weakly induced" cases.

Thus, more generally, we define a Chai-Yu isomorphism to be an isomorphism $\mathcal{T}^{\mathrm{ft}} \times_{\mathfrak{O}_F} \mathfrak{O}_F/\mathfrak{p}_F^m \to \mathcal{T}'^{\mathrm{ft}} \times_{\mathfrak{O}_{F'}} \mathfrak{O}_{F'}/\mathfrak{p}_{F'}^m$ that, when evaluated at $\mathfrak{O}_{\tilde{F}}/\mathfrak{p}_{\tilde{F}}^m = \mathfrak{O}_{\tilde{F}'}/\mathfrak{p}_{\tilde{F}'}^m$, is induced by a standard isomorphism $\mathrm{T}(\tilde{F})/\mathrm{T}(\tilde{F})_r^{\mathrm{naive}} \to \mathrm{T}'(\tilde{F}')/\mathrm{T}'(\tilde{F}')_r^{\mathrm{naive}}$ for some $0 < r \ll_{\mathrm{T}} l$ such that $\mathrm{T}(\tilde{F})_r^{\mathrm{naive}} \subset \mathrm{T}(\tilde{F})_m$ and $\mathrm{T}'(\tilde{F}')_r^{\mathrm{naive}} \subset \mathrm{T}'(\tilde{F}')_m$. The existence of a Chai-Yu isomorphism easily gives the existence of a congruent isomorphism that is tautologically compatible with the Chai-Yu isomorphism.

1.6. **The organization of the paper.** In Section 2, we define notation and recall some material, as well as provide some simple arguments such as an explanation as to why various results of [Del84], though stated for "local fields" (quotient fields of complete discrete valuation rings with perfect residue field) automatically extend to the case of Henselian discretely valued fields with perfect residue fields. This extension is convenient because our arguments involve passage to maximal unramified extensions, which preserves Henselian-ness but not completeness. We also review the result of Chai and Yu of interest to us, stating their isomorphism and articulating the characterization implicit in their work, in Theorem 2.5.3.

In Section 3, we study standard isomorphisms, congruent isomorphisms and Chai-Yu isomorphisms. Much of the content of this section has been summarized above.

In Section 4, we restrict to the case of tori that are weakly induced in the sense of [KP23]. These tori are much simpler than general tori. In this case, one can construct a Chai-Yu isomorphism under the milder, natural, assumption $0 < m \lll_{\mathrm{T}} l$ (Proposition 4.2.2), which is what lets us replace $h(F,\mathrm{T})$ by 0 in the statement of Theorem 1.2.1 for weakly induced tori.

Finally, in Section 5, we put things together and prove Theorem 1.2.1.

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2. Notation and review.

2.1. **Discretely valued Henselian fields.** We will abbreviate "discretely valued Henselian field" to "DVHF". We will be interested in DVHFs with perfect (but not necessarily finite) residue fields, unlike [Del84], which additionally imposes completeness. This is because we will need to pass to maximal unramified extensions $\tilde{F}$ of fields F of interest (see [Ber93, just before Corollary 2.4.6]; valued Henselian fields are the quasi-complete fields of [Ber93, Definition 2.3.1 and Proposition 2.4.3]), and doing so preserves only Henselian-ness, not completeness.

2.1.1. *Objects associated to a DVHF.* For a DVHF F, we will denote by $\mathfrak{O}_F$ its ring of integers, $\mathfrak{p}_F \subset \mathfrak{O}_F$ the maximal ideal of $\mathfrak{O}_F$, $\kappa_F = \mathfrak{O}_F/\mathfrak{p}_F$ its residue field, and val_F the normalized discrete valuation of F as well as its own extension to any algebraic extension of F. Given a field F, it will often be implicitly understood that a separable closure F^{sep} has been chosen. For a DVHF F, we will write Γ_F and I_F respectively for the absolute Galois group $\mathrm{Gal}(F^{\mathrm{sep}}/F)$ and the inertia group of F, and $\Gamma_{E/F}$ for the Galois group of any Galois extension E/F. Let $\Gamma_{\kappa_F} = \Gamma_F/I_F$; it is isomorphic to the absolute Galois group of κ_F. In case κ_F is finite, Γ_{κ_F} is topologically generated by a Frobenius element, and we will write $W_F \subset \Gamma_F$ for the Weil group, namely the inverse image in Γ_F of the subgroup of Γ_{κ_F} abstractly generated the Frobenius element. Given a DVHF F with perfect residue field, we will often be interested in separable finitely ramified extensions E/F, which may not be finite or Galois. In such a situation, we will use without further comment that E is also a DVHF with perfect residue field (thus, κ_E is algebraically closed if E contains a maximal unramified extension of F), and denote by $e(E/F)$ the associated ramification degree.

2.1.2. *Passage to completion.* Write $\hat{F}$ for the completion of any DVHF F; it is a complete DVHF. Let F be a DVHF. For the following, we refer to [KP23, Proposition 2.1.6] and [Ber93, Proposition 2.4.1]. If E/F is a finite separable extension, noting that E is also a DVHF, the obvious map $E \otimes_F \hat{F} \to \hat{E}$ of rings is an isomorphism. We have an equivalence of categories between the category $\mathrm{ext}\, F$ of finite separable extensions of F and the analogous category $\mathrm{ext}\, \hat{F}$, given by $E \mapsto \hat{E}$, so that choosing an embedding of F^{sep} into $\hat{F}^{\mathrm{sep}} := (\hat{F})^{\mathrm{sep}}$ gives a canonical identification $\Gamma_F \to \Gamma_{\hat{F}}$.

2.1.3. *Ramification theory.* We assume the setting and notation from Subsubsection 2.1.2 above, but assume also that κ_F is perfect. Let E/F be a finitely ramified separable extension. For each $r \in [-1, \infty)$, we have the "lower ramification (equivalence) relation" Ξ_r on $\mathrm{Hom}_{F\text{-alg}}(E, F^{\mathrm{sep}})$ as in [Del84, (A.3.3)] (whose R_u is our Ξ_r), under which σ and τ are equivalent if and only if for some (or equivalently, any) finitely ramified Galois extension M/F contained in F^{sep} and containing a normal closure of E, we have $\mathrm{val}_M(\sigma(x) - \tau(x)) \geq e(M/E)(r+1)$ for all $x \in \mathfrak{O}_E$. When $E = M$ is Galois, the lower ramification subgroup $\mathrm{Gal}(E/F)_r \subset \mathrm{Gal}(E/F)$ is the equivalence class of the identity element under Ξ_r, which is then just "lies in the same $\mathrm{Gal}(E/F)_r$-coset".

Choosing any M as above, $\mathrm{Gal}(M/F)$ has a well-defined transitive action on the set of Ξ_r-equivalence classes, so that they all have the same cardinality. This cardinality is easily seen to be bounded above by $e(E/F)$ for $r > -1$: if $\sigma, \tau \in \mathrm{Hom}_{F-\mathrm{alg}}(E, F^{\mathrm{sep}})$ belong to the same class under Ξ_r with $r > -1$, it is an easy exercise to see that they agree on the maximal unramified subextension of E/F. For $r > -1$, let $1 \leq g_r \leq e(E/F)$ be the cardinality of each Ξ_r-equivalence class (our g_r is the r_u of [Del84]). This also lets us define the Herbrand function associated to each finitely ramified separable extension E/F by the familiar integral as in [Del84, (A.4.3)], for $r \in [0, \infty)$:

$$e(E/F)^{-1} r \leq \varphi_{E/F}(r) = \int_0^r dt/(g_0/g_t) \leq r. \tag{5}$$

$\varphi_{E/F}$ is a piecewise linear self-homeomorphism of $[0, \infty)$. In fact, one can show that this also defines a self-homeomorphism of $[-1, \infty)$, but we will only be interested in its values on $[0, \infty)$. Let its inverse be $\psi_{E/F}$, another self-homeomorphism of $[0, \infty)$.

The following allow us to reduce the study of $\varphi_{E/F}$ and $\psi_{E/F}$ to the case where E/F is finite and F is complete:

- For any subextension $E_\circ/F$ of E/F with $e(E_\circ/F) = e(E/F)$ — note that there exist finite such $E_\circ/F$ — we have $\varphi_{E/F} = \varphi_{E_\circ/F}$ and $\psi_{E/F} = \psi_{E_\circ/F}$. To see this, make use of the same argument that was used above to prove that $g_r \leq e(E/F)$ for $r > -1$, to see that the value of g_r associated to E/F equals that associated to $E_\circ/F$.
- If E/F is finite, it is easy to see that the identification
$$\mathrm{Hom}_{F-\mathrm{alg}}(E, F^{\mathrm{sep}}) \to \mathrm{Hom}_{\hat{F}-\mathrm{alg}}(\hat{E}, (\hat{F})^{\mathrm{sep}})$$
respects each of the equivalence relations Ξ_r, so that $\varphi_{E/F} = \varphi_{\hat{E}/\hat{F}}$ and $\psi_{E/F} = \psi_{\hat{E}/\hat{F}}$ as functions $[0, \infty) \to [0, \infty)$.

We claim that for finitely ramified separable extensions $E_2/E_1/F$, we have $\varphi_{E_2/F} = \varphi_{E_1/F} \circ \varphi_{E_2/E_1}$, or equivalently $\psi_{E_2/F} = \psi_{E_2/E_1} \circ \psi_{E_1/F}$. To see this, reduce using the arguments above to the case where E_2/F is finite, and then to the case where F, E_1 and E_2 are complete, then to the case where E_2/F is Galois, and use [Del84, (A.4.1) and Proposition A.4.2].

For E/F finitely ramified separable, define the "upper ramification relations" $\Xi^r := \Xi_{\psi_{E/F}(r)}$. If E/F is Galois, this is the "belongs to the same coset" relation for the upper ramification subgroup $\mathrm{Gal}(E/F)^r := \mathrm{Gal}(E/F)_{\psi_{E/F}(r)} \subset \mathrm{Gal}(E/F)$.

When E/F is finite, it is easy to check that $\mathrm{Hom}_{F-\mathrm{alg}}(E, F^{\mathrm{sep}}) \to \mathrm{Hom}_{\hat{F}-\mathrm{alg}}(\hat{E}, (\hat{F})^{\mathrm{sep}})$ preserves the "lower ramification relations" Ξ_r, and hence (using $\psi_{E/F} = \psi_{\hat{E}/\hat{F}}$) also the "upper ramification relations" Ξ^r. The Ξ^r have the following advantage: if E/F is a finite extension and M/F is a finite Galois extension in F^{sep} containing a normal closure of E, then Ξ^r is the same as "lies in the same $\mathrm{Gal}(M/F)^r$-orbit": to see this, pass to completion and use [Del84, (A.3.2) and the last sentence of A.4]. This nice behavior under quotients lets us give Γ_F an upper ramification filtration $\{I_F^r\}_{r \geq 0}$: by definition, the upper ramification filtration subgroup $I_F^r \subset \Gamma_F$ is the subgroup of elements that map to $\mathrm{Gal}(M/F)^r$ for each finite Galois subextension M/F of F^{sep}/F. Each such map $I_F^r \to \mathrm{Gal}(M/F)^r$ is then seen to be surjective. The isomorphism $\Gamma_F \to \Gamma_{\hat{F}}$ maps I_F^r to $I_{\hat{F}}^r$ and quotients to an isomorphism $\Gamma_F/I_F^r \to \Gamma_{\hat{F}}/I_{\hat{F}}^r$, for each $r \geq 0$. The objects associated to F that we have defined above ($\psi_{E/F}, I_F^r$ etc.) are intrinsic, and their definitions did not make use of the embedding $F^{\mathrm{sep}} \hookrightarrow \hat{F}^{\mathrm{sep}}$.

2.1.4. *At most l-ramified extensions.* Let F be a DVHF with perfect residue field, and l a nonnegative integer. A separable (algebraic) extension E/F will be called at most l-ramified if for every finite subextension $E_\circ/F$ of it, the relation Ξ^l associated to $E_\circ/F$ is trivial. Using [Del84, Proposition A.6.1], this can be shown to be equivalent to requiring that I_F^l fixes any F-algebra embedding $E \hookrightarrow F^{\mathrm{sep}}$. Let $(\mathrm{ext}\, F)^l$ denote the category of finite at most l-ramified (and hence separable by definition) extensions of F. The discussion in the previous subsubsection implies that the functor $E \mapsto \hat{E}$ from Subsubsection 2.1.2 induces an equivalence of categories $(\mathrm{ext}\, F)^l \to (\mathrm{ext}\, \hat{F})^l$. The category of the ind-objects of $(\operatorname{ext} F)^l$ is equivalent to the category of the algebraic (not necessarily finite or finitely ramified) at most l-ramified extensions of F.

Remark 2.1.1. Here are some properties of a finitely ramified at most l-ramified extension E/F, that will be used without further comment in what follows:

(i) $l(1) := \psi_{E/F}(l)$ is an integer, equal to $e(E/F)(l+1) - \mathrm{val}_F$(the different of E/F) -1 if E/F is finite (reduce to the case of E/F finite, and see around [Del84, Proposition A.6.1]; this uses that E/F is at most l-ramified). Note also that $l \le l(1) \le le(E/F)$, with the latter equality holding if and only if E/F is tamely ramified (use, e.g., (5) and the previous sentence).

(ii) $I_F^l = I_E^{l(1)}$: reduce to the case where E/F is finite, and note that for any finite Galois extension M/E we have $\mathrm{Gal}(M/F)^l \subset \mathrm{Gal}(M/E)$, and then:

$$\mathrm{Gal}(M/F)^l = \mathrm{Gal}(M/E) \cap \mathrm{Gal}(M/F)_{\psi_{M/F}(l)} = \mathrm{Gal}(M/E)_{\psi_{M/E}\circ\psi_{E/F}(l)} = \mathrm{Gal}(M/E)^{\psi_{E/F}(l)}.$$

2.2. The Krasner-Deligne theory ([Del84]).

2.2.1. *Deligne's triples.* In Subsubsections 2.2.2 and 2.2.3 below, we will write $\mathfrak{T}$ for the category of triples $(A, \mathfrak{m}, \epsilon)$ as in [Del84, Sections 1.1 and 1.4]: A is a truncated discrete valuation ring with perfect residue field, $\mathfrak{m}$ is a free A-module of rank 1, and $\epsilon\colon \mathfrak{m} \to A$ is an A-module epimorphism from $\mathfrak{m}$ to the maximal ideal of A. For an object $S = (A, \mathfrak{m}, \epsilon)$ in this category, $l(S)$ will denote the length of A as an A-module.

For each object $S = (A, \mathfrak{m}, \epsilon)$ of $\mathfrak{T}$ of length say l, Deligne has defined a category $(\operatorname{ext} S)^l$ in [Del84, Definition 2.7], which we will refer to in this subsection. Its objects are the "finite flat" objects over S in $\mathfrak{T}$ satisfying an "at most l-ramified" condition, but its morphisms are only certain equivalence classes of morphisms in $\mathfrak{T}$.

2.2.2. *Deligne's triple* $\mathrm{Tr}_l F$ *associated to* F. If F is a DVHF with perfect residue field, and l a positive integer, we will write $\mathrm{Tr}_l F$ for the object $(\mathfrak{O}_F/\mathfrak{p}_F^l, \mathfrak{p}_F/\mathfrak{p}_F^{l+1}, \epsilon)$ in $\mathfrak{T}$, where $\epsilon\colon \mathfrak{p}_F/\mathfrak{p}_F^{l+1} \to \mathfrak{O}_F/\mathfrak{p}_F^l$ is induced by the inclusion $\mathfrak{p}_F \subset \mathfrak{O}_F$.

[Del84, Theorem 2.8] says that if F is complete, then Tr extends to a well-defined functor from $(\operatorname{ext} F)^l$ to the category $(\operatorname{ext}\mathrm{Tr}_l F)^l$ (see Subsubsection 2.2.1), which is in fact an equivalence of categories. However, even when F is not complete, using the equivalence of categories $(\operatorname{ext} F)^l \to (\operatorname{ext} \hat{F})^l$ (Subsubsection 2.1.4) and the tautological isomorphism $\mathrm{Tr}_l F \to \mathrm{Tr}_l \hat{F}$, we formally get an equivalence of categories $(\operatorname{ext} F)^l \to (\operatorname{ext}\mathrm{Tr}_l F)^l$. Moreover, it is immediate that this equivalence of categories has an intrinsic description independent of $\hat{F}$, exactly as the one used for $\hat{F}$ in [Del84, Theorem 2.8]. In particular, any isomorphism $\mathrm{Tr}_l F \to \mathrm{Tr}_l F'$ determines an equivalence of categories $(\operatorname{ext} F)^l \to (\operatorname{ext} F')^l$; this takes each E in $(\operatorname{ext} F)^l$ to some E' in $(\operatorname{ext} F')^l$ such that we have an isomorphism $\mathrm{Tr}_{el} E = \mathrm{Tr}_{el} E'$ in the category $(\operatorname{ext}\mathrm{Tr}_l F)^l = (\operatorname{ext}\mathrm{Tr}_l F')^l$, where $e = e(E/F) = e(E'/F')$.

2.2.3. *Close local fields.* The notation $F \leftrightarrow_l F'$ will mean not only that F and F' are DVHFs with perfect residue fields that are l-close in the sense that we have an isomorphism $\mathfrak{O}_F/\mathfrak{p}_F^l \cong \mathfrak{O}_{F'}/\mathfrak{p}_{F'}^l$ of rings, but also that the following additional data have been chosen:

- An identification $\mathrm{Tr}_l F = \mathrm{Tr}_l F'$ has been chosen in $\mathfrak{T}$, as also an equivalence of categories $U\colon (\operatorname{ext} F)^l \to (\operatorname{ext} F')^l$ as in Subsubsection 2.2.2 above, which then as in [Del84, Section 3.5], determines the inner class of an isomorphism $\Gamma_F/I_F^l \cong \Gamma_{F'}/I_{F'}^l$.
- An identification $\Gamma_F/I_F^l = \Gamma_{F'}/I_{F'}^l$ from the inner class mentioned above has been chosen by means of a choice of a fixed isomorphism $U((F^{\mathrm{sep}})^{I_F^l}) \to (F'^{\mathrm{sep}})^{I_{F'}^l}$ over F', as follows:

$$\Gamma_F/I_F^l = \mathrm{Aut}_{F\text{-alg}}((F^{\mathrm{sep}})^{I_F^l}) \xrightarrow{U} \mathrm{Aut}_{F'\text{-alg}}(U((F^{\mathrm{sep}})^{I_F^l})) = \mathrm{Aut}_{F'\text{-alg}}((F'^{\mathrm{sep}})^{I_{F'}^l}) = \Gamma_{F'}/I_{F'}^l,$$

  where $U\colon (\operatorname{ext} F)^l \cong (\operatorname{ext} F')^l$ is now extended to the level of the ind-objects.

  We refer to [Del84, Section 3.5, especially Section 3.5(c)] for some of the details, which do not need the assumption that F is complete.

This involves choices of F^{sep} and F'^{sep} among other things, changing which will change associated objects in an appropriate sense, e.g., up to an inner automorphism for Galois groups.

2.2.4. *A variation.* Given $F \leftrightarrow_l F'$, it will be helpful to consider the following variant of U. Let $(\operatorname{ext} F)^{l,+}$ be the category of embeddings $E \hookrightarrow F^{\mathrm{sep}}$, where E/F is finitely ramified and at most l-ramified. Similarly, we have $(\operatorname{ext} F')^{l,+}$. These categories have the following advantage which is important for us: between any two objects in them is either a unique morphism, or none at all, depending on whether the stabilizer of the former in Γ_F or $\Gamma_{F'}$ contains that of the latter. Clearly, U, considered at the level of ind-objects, defines an equivalence of categories $U^+ \colon (\operatorname{ext} F)^{l,+} \to (\operatorname{ext} F')^{l,+}$; it sends $E \hookrightarrow F^{\mathrm{sep}}$ to $U(E) \hookrightarrow U((F^{\mathrm{sep}})^{I_F^l}) \to (F'^{\mathrm{sep}})^{I_{F'}^l} \hookrightarrow F'^{\mathrm{sep}}$, where $U((F^{\mathrm{sep}})^{I_F^l}) \to (F'^{\mathrm{sep}})^{I_{F'}^l}$ is part of the datum $F \leftrightarrow_l F'$. Moreover, it is an easy exercise to describe when $E' \hookrightarrow F'^{\mathrm{sep}}$ is isomorphic in $(\operatorname{ext} F')^{l,+}$ to $U^+(E \hookrightarrow F^{\mathrm{sep}})$: this is so if and only if $E \hookrightarrow F^{\mathrm{sep}}$ and $E' \hookrightarrow F'^{\mathrm{sep}}$ have the same stabilizer in $\Gamma_F/I_F^l = \Gamma_{F'}/I_{F'}^l$.
Upshot: the datum of an extension E'/F and an isomorphism $U(E) \to E'$, is the same as that of an embedding $E' \hookrightarrow F'^{\mathrm{sep}}$ with the same stabilizer as $E \hookrightarrow F^{\mathrm{sep}}$ in $\Gamma_F/I_F^l = \Gamma_{F'}/I_{F'}^l$.

2.2.5. *Close extensions of close local fields.* Given $F \leftrightarrow_l F'$, we will typically need to work with realizations $E \leftrightarrow_{l(1)} E'$ involving finitely ramified at most l-ramified extensions E and E' of F and F' with $U(E) \cong E'$, where $l(1) = \psi_{E/F}(l)$ is an integer. We will use without further mention that, for any such E', we have $\psi_{E/F}(l) = \psi_{E'/F'}(l)$: use the discussion in [Del84, Section 1.5.3] (expressing $\psi_{E/F}(l)$ in terms of truncated data).

Let us describe how a choice of an isomorphism $U(E) \to E'$ gives a realization $E \leftrightarrow_{l(1)} E'$. As in [Del84, Construction 3.4.1] (and using a direct limit argument to reduce to the case of finite extensions), this choice determines an isomorphism $\operatorname{Tr}_{l(1)} E \to \operatorname{Tr}_{l(1)} E'$; by Subsubsection 2.2.2 above this does not need the assumption that F is complete. Moreover, we have obvious choices of the additional data needed to upgrade this to a realization $E \to_{l(1)} E'$, as follows. An analogue U_E of U for this isomorphism can be obtained by restricting U to extensions of E and thinking of $U(E) \to E'$ as an identification, which we can also use to choose F^{sep} and F'^{sep} as algebraic closures of E and E', and get an identification $U_E((F^{\mathrm{sep}})^{I_E^{l(1)}}) = U((F^{\mathrm{sep}})^{I_F^l}) \to (F'^{\mathrm{sep}})^{I_{F'}^l} = (F'^{\mathrm{sep}})^{I_{E'}^{l(1)}}$.

Remark 2.2.1. The above construction of $E \leftrightarrow_{l(1)} E'$ has the following properties, which will be used without further mention in what follows:

(i) Γ_E and $\Gamma_{E'}$, as realized in $E \leftrightarrow_{l(1)} E'$, identify with the stabilizers of $E \hookrightarrow F^{\mathrm{sep}}$ in Γ_F and $E' \hookrightarrow F'^{\mathrm{sep}}$ in $\Gamma_{F'}$, and the resulting isomorphism $\Gamma_E/I_E^{l(1)} \to \Gamma_{E'}/I_{E'}^{l(1)}$ is simply the restriction of $\Gamma_F/I_F^l \to \Gamma_{F'}/I_{F'}^l$.

(ii) We get a bijection

$$\Gamma_{E/F} \to (\Gamma_F/I_F^l)/(\Gamma_E/I_E^{l(1)}) \to (\Gamma_{F'}/I_{F'}^l)/(\Gamma_{E'}/I_{E'}^{l(1)}) \to \Gamma_{E'/F'}, \tag{6}$$

which is an isomorphism of groups if E/F is Galois, in which case these groups act compatibly on $\operatorname{Tr}_{l(1)} E = \operatorname{Tr}_{l(1)} E'$ over $\operatorname{Tr}_l F = \operatorname{Tr}_l F'$.

(iii) Suppose L/F is a finitely ramified at most l-ramified extension 'containing' E/F. Then any extension L'/F' together with an isomorphism $U(L) \to L'$ determines an extension L'/E' (via $E' \to U(E) \to U(L) \to L'$), together with an isomorphism $U_E(L) = U(L) \to L'$, and vice versa. Given any such L' and $U(L) \to L'$, it is easily verified that the realization $L \leftrightarrow_{\psi_{L/F}(l)} L'$, obtained using the above construction starting from $F \leftrightarrow_l F'$ and $U(L) \to L'$, is the same as the realization $L \leftrightarrow_{\psi_{L/E}(\psi_{E/F}(l))} L'$, obtained using the above construction starting from $E \leftrightarrow_{\psi_{E/F}(l)} E'$ and $U_E(L) = U(L) \to L'$.

However, it seems inconvenient to keep track of fixed isomorphisms $U(E) \to E'$, or even to keep referring to U. This is why we had Subsubsection 2.2.4: the discussion there shows that, given $E \hookrightarrow F^{\mathrm{sep}}$ in $(\operatorname{ext} F)^{l,+}$, any $E' \hookrightarrow F'^{\mathrm{sep}}$ with the same stabilizer as it in $\Gamma_F/I_F^l = \Gamma_{F'}/I_{F'}^l$ determines a unique isomorphism $U(E) \to E'$, and hence a realization $E \leftrightarrow_{l(1)} E'$.

Notation 2.2.2. If a realization $F \leftrightarrow_l F'$ is understood, and we talk of compatible embeddings $E \hookrightarrow F^{\mathrm{sep}}$ and $E' \hookrightarrow F'^{\mathrm{sep}}$, we will mean that E/F and E'/F' are finitely ramified at most l-ramified extensions, and that $E \hookrightarrow F^{\mathrm{sep}}$ and $E' \hookrightarrow F'^{\mathrm{sep}}$ have the same stabilizer in $\Gamma_F/I_F^l = \Gamma_{F'}/I_{F'}^l$. Thus, the compatible embeddings $E \hookrightarrow F^{\mathrm{sep}}$ and $E' \hookrightarrow F'^{\mathrm{sep}}$ give a realization $E \leftrightarrow_{l(1)} E'$, where $l(1) = \psi_{E/F}(l)$, "lying over" $F \leftrightarrow_l F'$.

Remark 2.2.3. By the above discussion above, Remark 2.2.1 can be stated in terms of compatible embeddings, with, in particular, Remark 2.2.1(iii) taking the following shape (with a slight change of notation): if $L_1/L_2/F$ are finitely ramified at most l-ramified extensions, and $L_i \hookrightarrow F^{\mathrm{sep}}$ and $L'_i \hookrightarrow F'^{\mathrm{sep}}$ are compatible embeddings for $i = 1, 2$, then $L'_1 \hookrightarrow F'^{\mathrm{sep}}$ is the composite of $L'_2 \hookrightarrow F'^{\mathrm{sep}}$ with an embedding $L'_1 \hookrightarrow L'_2$. Moreover, with $l_1 = \psi_{L_1/F}(l)$ and $l_2 = \psi_{L_2/F}(l) = \psi_{L_2/L_1}(l_1)$, the realization $L_2 \leftrightarrow_{l_2} L'_2$ produced from $F \leftrightarrow_l F'$ is the same as the $L_2 \leftrightarrow_{\psi_{L_2/L_1}(l_1)} L'_2$ produced from the $L_1 \leftrightarrow_{l_1} L'_1$ in turn produced from $F \leftrightarrow_l F'$.

2.2.6. *Relating the multiplicative groups of close local fields.* Suppose that $F \leftrightarrow_l F'$, and that we have embeddings $L_1 \hookrightarrow L_2 \hookrightarrow F^{\mathrm{sep}}$, with L_1/F and L_2/F finitely ramified and at most l-ramified. Assume that for $i = 1, 2$, $L_i \hookrightarrow F^{\mathrm{sep}}$ and $L'_i \hookrightarrow F'^{\mathrm{sep}}$ are compatible embeddings, so that $L'_1 \hookrightarrow F'^{\mathrm{sep}}$ factors through $L'_2 \hookrightarrow F'^{\mathrm{sep}}$ (Remark 2.2.3). Set $l_i = \psi_{L_i/F}(l)$ for $i = 1, 2$. The inclusion $L_1^\times \hookrightarrow L_2^\times$ induces $L_1^\times/(1 + \mathfrak{p}_{L_1}^{l_1}) \to L_2^\times/(1 + \mathfrak{p}_{L_2}^{l_2})$, because $l_2 = \psi_{L_2/L_1}(l_1) \le l_1 e(L_2/L_1)$, by Remark 2.1.1(i). Part of the datum defining the map $\mathrm{Tr}_{l_1} L_1 \to \mathrm{Tr}_{l_2} L_2$ (as in [Del84, Section 1.4]) is an $\mathfrak{O}_{L_1}/\mathfrak{p}_{L_1}^{l_1}$-linear map $\mathfrak{p}_{L_1}/\mathfrak{p}_{L_1}^{l_1+1} \to (\mathfrak{p}_{L_2}/\mathfrak{p}_{L_2}^{l_2+1})^{\otimes e(L_2/L_1)}$ that sends a generator of the source to a generator of the target as an $\mathfrak{O}_{L_2}/\mathfrak{p}_{L_2}^{l_2}$-module. The description of $L_i^\times/(1 + \mathfrak{p}_{L_i}^{l_i})$ $(i = 1, 2)$ in terms of $\mathrm{Tr}_{l_i} L_i$, given in [Del84, Section 1.2] as the group of homogeneous units of the graded $\mathfrak{O}_{L_i}/\mathfrak{p}_{L_i}^{l_i}$-algebra $\bigoplus_{n\in\mathbb{Z}}(\mathfrak{p}_{L_i}/\mathfrak{p}_{L_i}^{l_i+1})^{\otimes n}$, implies that the map $L_1^\times/(1 + \mathfrak{p}_{L_1}^{l_1}) \to L_2^\times/(1 + \mathfrak{p}_{L_2}^{l_2})$ can be described in terms of the extension $\mathrm{Tr}_{l_1} L_1 \to \mathrm{Tr}_{l_2} L_2$, as obtained by putting together the various $(\mathfrak{p}_{L_1}/\mathfrak{p}_{L_1}^{l_1+1})^{\otimes n} \to (\mathfrak{p}_{L_2}/\mathfrak{p}_{L_2}^{l_2+1})^{\otimes e(L_2/L_1)n}$ (this does not use the completeness of F or of the L_i). Using the isomorphism $\mathrm{Tr}_{l_i} L_i \to \mathrm{Tr}_{l_i} L'_i$, we get an isomorphism $L_i^\times/(1 + \mathfrak{p}_{L_i}^{l_i}) \to L_i'^\times/(1 + \mathfrak{p}_{L'_i}^{l_i})$. Now it is clear that, whenever $F \leftrightarrow_l F'$ and we have $L_1 \leftrightarrow_{l_1} L'_1$ and $L_2 \leftrightarrow_{l_2} L'_2$ as above, we have an obvious commutative diagram:

$$\begin{tikzcd}
F^\times/(1 + \mathfrak{p}_F^l) \arrow[r] \arrow[d] & L_1^\times/(1 + \mathfrak{p}_{L_1}^{l_1}) \arrow[r] \arrow[d] & L_2^\times/(1 + \mathfrak{p}_{L_2}^{l_2}) \arrow[d] \\
F'^\times/(1 + \mathfrak{p}_{F'}^l) \arrow[r] & L_1'^\times/(1 + \mathfrak{p}_{L'_1}^{l_1}) \arrow[r] & L_2'^\times/(1 + \mathfrak{p}_{L'_2}^{l_2})
\end{tikzcd} .$$

By the discussion at the end of Subsubsection 2.2.5, if some L_i/F is Galois, then the vertical arrow in the above diagram involving L_i is invariant under $\Gamma_{L_i/F} = \Gamma_{L'_i/F'}$.

Notation 2.2.4. Let F be a DVHF with perfect residue field, and L/F be an algebraic extension. For $r, l > 0$, we say that $r \underline{\ll}_L l$ (or $l \underline{\gg}_L r$), if L/F is finitely ramified, at most l-ramified, and satisfies that $r \le \psi_{L/F}(l)/e(L/F)$ (usually l will be an integer for our purposes).

Note that, if $F \leftrightarrow_l F'$, $r \underline{\ll}_L l$, and $L \hookrightarrow F^{\mathrm{sep}}$ and $L' \hookrightarrow F'^{\mathrm{sep}}$ are compatible extensions, then with $l(1) = \psi_{E/F}(l)$, the isomorphisms $\mathfrak{O}_L/\mathfrak{p}_L^{l(1)} \to \mathfrak{O}_{L'}/\mathfrak{p}_{L'}^{l(1)}$ and $L^\times/(1 + \mathfrak{p}_L^{l(1)}) \to L'^\times/(1 + \mathfrak{p}_{L'}^{l(1)})$ induce isomorphisms $\mathfrak{O}_L/\mathfrak{p}_L^{\lceil e(L/F)r\rceil}\mathfrak{O}_L \to \mathfrak{O}_{L'}/\mathfrak{p}_{L'}^{\lceil e(L/F)r\rceil}\mathfrak{O}_{L'}$ and $L^\times/(1+\mathfrak{p}_L^{\lceil e(L/F)r\rceil}) \to L'^\times/(1+\mathfrak{p}_{L'}^{\lceil e(L/F)r\rceil})$.

2.3. Notation related to tori over DVHFs.

Notation 2.3.1. (i) Henceforth we will subscripting to indicate base-change: the base-change of a scheme X/S to S' will be denoted by $\mathrm{X}_{S'}$.

(ii) For any torus T over a field F with a chosen separable closure F^{sep}, we will denote by $X^*(\mathrm{T})$ and $X_*(\mathrm{T})$ respectively the character lattice and the cocharacter lattice of the base-change $\mathrm{T}_{F^{\mathrm{sep}}}$ of T to F^{sep}, viewed with the obvious action of $\mathrm{Gal}(F^{\mathrm{sep}}/F)$ on these. If an embedding $E \hookrightarrow F^{\mathrm{sep}}$ is understood from the context, where E/F a separable extension splitting T, we may use it to view each $\chi \in X^*(\mathrm{T})$ as a homomorphism $\mathrm{T}_E \to \mathbb{G}_m/E$, $\chi(t)$ as an element of $E^\times$ for $\chi \in X^*(\mathrm{T})$ and $t \in \mathrm{T}(E)$, etc.

(iii) If T is a torus over a DVHF F with perfect residue field, its ft-Néron model and connected Néron model (see [KP23, Definition B.8.9]) will be denoted by $\mathcal{T}^{\mathrm{ft}}$ and $\mathcal{T}$, respectively (thus, we follow [Gan22] in writing $\mathcal{T}$ for the $\mathcal{T}^0$ of [KP23]).

(iv) If T is a torus over a DVHF F with perfect residue field, then $\mathrm{T}(F)_b \subset \mathrm{T}(F)$ will denote its maximal bounded subgroup; thus, $\mathcal{T}^{\mathrm{ft}}(\mathfrak{O}_F)$ identifies with $\mathrm{T}(F)_b$.

(v) Let T be a torus over a DVHF F with perfect residue field. We will consider three filtrations of $\mathrm{T}(F)$:

$$\{\mathrm{T}(F)_r^{\mathrm{naive}}\}_{r\geq 0}, \quad \{\mathrm{T}(F)_r^{\mathrm{std}}\}_{r\geq 0} \quad \text{and} \quad \{\mathrm{T}(F)_r\}_{r\geq 0}.$$

For $\{\mathrm{T}(F)_r^{\mathrm{naive}}\}_{r\geq 0}$ (defined in (4)), we have for each $r \geq 0$:

$$\mathrm{T}(F)_r^{\mathrm{naive}} = \{t \in \mathrm{T}(F)_b \mid \mathrm{val}_F(\chi(t) - 1) \geq r,\ \forall\, \chi \in X^*(\mathrm{T})\} \tag{7}$$

(here, each $\chi(t)$ is valued in F^{sep}, and we recall that the normalized valuation val_F is canonically extended to algebraic extensions of F). The filtration $\{\mathrm{T}(F)_r^{\mathrm{std}}\}_{r\geq 0}$, is the standard or Moy-Prasad filtration of $\mathrm{T}(F)$ (see [KP23, Definition B.5.1]):

$$\mathrm{T}(F)_r^{\mathrm{std}} = \{t \in \mathrm{T}(F)^0 \mid \mathrm{val}_F(\chi(t) - 1) \geq r,\ \forall\, \chi \in X^*(\mathrm{T})\}, \tag{8}$$

where $\mathrm{T}(F)^0$ is the Iwahori subgroup of $\mathrm{T}(F)$ as defined in [KP23, Definition 2.5.13], a subgroup of finite index of $\mathrm{T}(F)_b$ (which can be strictly contained in $\mathrm{T}(F)_b$). The filtration $\{\mathrm{T}(F)_r\}_{r\geq 0}$ is the minimal congruent filtration of $\mathrm{T}(F)$, originally introduced by Yu in [Yu15], but interpreted as in [KP23, Section B.10]. We have $\mathrm{T}(F)_r^{\mathrm{std}} = \mathrm{T}(F)_0 \cap \mathrm{T}(F)_r^{\mathrm{naive}}$ for each $r \geq 0$, and one can check that $\mathrm{T}(F)_0^{\mathrm{naive}} = \mathrm{T}(F)_b$. If $r = m$ is an integer, then $\mathrm{T}(F)_m$ is the group of $\mathfrak{O}_F$-points of what is defined in [KP23, Definition A.5.12] as the m-th congruence subgroup scheme of $\mathcal{T}$ (this is what the "congruent" of "minimal congruent filtration" refers to). By [KP23, Remark A.5.14], it also equals $\ker(\mathcal{T}(\mathfrak{O}_F) \to \mathcal{T}(\mathfrak{O}_F/\mathfrak{p}_F^m))$, and hence is what [Gan22] denotes as T_m.

(vi) In the setting of (v) above, if E/F is a finitely ramified separable algebraic extension, we set $\mathrm{T}(E)_r^{\mathrm{naive}} \coloneqq \mathrm{T}_E(E)_r^{\mathrm{naive}}$. In slight contrast, if E/F is unramified (algebraic), we let $\mathrm{T}(E)_r$ be $\mathcal{T}_r(\mathfrak{O}_E)$, where $\mathcal{T}_r$ is the r-th minimal congruence filtration group scheme of T from [KP23, Definition B.10.8(3)] (thus, $\mathrm{T}(E)_r \cap \mathrm{T}(F) = \mathrm{T}(F)_r$, but we do not know if $\mathrm{T}(E)_r$ equals $\mathrm{T}_E(E)_r$).

(vii) Let T be a torus over a DVHF F with perfect residue field, and let $r, l > 0$. We say that $r \ll_{\mathrm{T}} l$ (or $l \gg_{\mathrm{T}} r$), if there exists a (finitely ramified at most l-ramified) extension L/F that splits T, such that $r \ll_L l$ (see Notation 2.2.4; usually l will be an integer for our purposes).

(viii) If we write $(F, \mathrm{T}) \leftrightarrow_l (F', \mathrm{T}')$, we will mean that $F \leftrightarrow_l F'$, and that T, T' are tori over F, F' such that the actions of I_F^l on $X^*(\mathrm{T})$ and $I_{F'}^l$ on $X^*(\mathrm{T}')$ are trivial (in other words, T and T' are "at most l-ramified"), and that an isomorphism $X^*(\mathrm{T}) \to X^*(\mathrm{T}')$ has been chosen that is equivariant for the actions of $\Gamma_F/I_F^l = \Gamma_{F'}/I_{F'}^l$ (recall that the identification $\Gamma_F/I_F^l = \Gamma_{F'}/I_{F'}^l$ is part of the datum defining $F \leftrightarrow_l F'$).

(ix) Suppose $F \leftrightarrow_l F'$, and that $E \hookrightarrow F^{\mathrm{sep}}$ and $E' \hookrightarrow F'^{\mathrm{sep}}$ are compatible embeddings, so (see Subsubsection 2.2.5) we have $E \leftrightarrow_{l(1)} E'$, where $l(1) = \psi_{E/F}(l)$. Suppose further that $(F, \mathrm{T}) \leftrightarrow_l (F', \mathrm{T}')$ and $(E, \mathrm{S}) \leftrightarrow_{l(1)} (E', \mathrm{S}')$ extend our $F \leftrightarrow_l F'$ and $E \leftrightarrow_{l(1)} E'$, respectively. Then we will implicitly work with realizations $(E, \mathrm{T}_E) \leftrightarrow_{l(1)} (E', \mathrm{T}'_{E'})$ and $(F, \mathrm{R} \coloneqq \mathrm{Res}_{E/F}\,\mathrm{S}) \leftrightarrow_l (F', \mathrm{R}' \coloneqq \mathrm{Res}_{E'/F'}\,\mathrm{S}')$, extending $E \leftrightarrow_{l(1)} E'$ and $F \leftrightarrow_l F'$, obtained from the following identifications:

(a) $X^*(\mathrm{T}_E) = X^*(\mathrm{T}) \to X^*(\mathrm{T}') = X^*(\mathrm{T}'_{E'})$, which is equivariant for $\Gamma_F/I_F^l = \Gamma_{F'}/I_{F'}^l$ and hence for $\Gamma_E/I_E^{l(1)} = \Gamma_{E'}/I_{E'}^{l(1)}$; and

(b) $X^*(\mathrm{R}) = \mathrm{Ind}_{\Gamma_E}^{\Gamma_F} X^*(\mathrm{S}) = \mathrm{Ind}_{\Gamma_E/I_E^{l(1)}}^{\Gamma_F/I_F^l} X^*(\mathrm{S}) = \mathrm{Ind}_{\Gamma_{E'}/I_{E'}^{l(1)}}^{\Gamma_{F'}/I_{F'}^l} X^*(\mathrm{S}') = \mathrm{Ind}_{\Gamma_{E'}}^{\Gamma_{F'}} X^*(\mathrm{S}') = X^*(\mathrm{R}')$, where we recall that S and S' are at most $l(1)$-ramified, use the discussion of (i) and (ii) of Remark 2.2.1, and use the canonical identifications $X^*(\mathrm{R}) = \mathrm{Ind}_{\Gamma_E}^{\Gamma_F} X^*(\mathrm{S}) = \mathbb{Z}[\Gamma_F] \otimes_{\mathbb{Z}[\Gamma_E]} X^*(\mathrm{S})$ and $X^*(\mathrm{R}') = \mathrm{Ind}_{\Gamma_{E'}}^{\Gamma_{F'}} X^*(\mathrm{S})$ reviewed in Remark 2.3.2 below.

As usual, the above notation will be adapted in obvious ways: for any torus T_1 over any DVHF $F^\flat$, we will make sense of $\mathcal{T}_1^{\mathrm{ft}}$ and $\mathrm{T}_1(F^\flat)_m$, etc.

Remark 2.3.2. Let E/F be a finite separable field extension, S/E a torus, and $\mathrm{R} \coloneqq \mathrm{Res}_{E/F}\,\mathrm{S}$. Let us recall the canonical realization $X^*(\mathrm{R}) = \mathrm{Ind}_{\Gamma_E}^{\Gamma_F} X^*(\mathrm{S})$, where $\Gamma_F = \mathrm{Gal}(F^{\mathrm{sep}}/F)$ and $\Gamma_E = \mathrm{Gal}(F^{\mathrm{sep}}/E)$. We have a "universal", surjective, homomorphism $\mathrm{R}_E \to \mathrm{S}_E$, which, at the level of A-valued points for an E-algebra A, is the map $\mathrm{R}(A) = \mathrm{S}(E \otimes_F A) \to \mathrm{S}(A)$, obtained by applying the functor S to the multiplication map $E \otimes_F A \to A$ of E-algebras (where $E \otimes_F A$ is an E-algebra via the first factor). This map has a well-known universal property: for any multiplicative type group scheme T over F, base-changing to E followed by composition with $\mathrm{R}_E \to \mathrm{S}_E$ gives us a functorial bijection between homomorphisms $\mathrm{T} \to \mathrm{R}$ and homomorphisms $\mathrm{T}_E \to \mathrm{S}_E$. Hence, composition with the injection $X^*(\mathrm{S}) \to X^*(\mathrm{R})$ dual to $\mathrm{R}_E \to \mathrm{S}_E$ gives a functorial identification $\mathrm{Hom}_{\Gamma_E}(X^*(\mathrm{S}), X^*(\mathrm{T})) \to \mathrm{Hom}_{\Gamma_F}(X^*(\mathrm{R}), X^*(\mathrm{T}))$ (the notation $X^*(\mathrm{T})$ extends to the case where T is a multiplicative type group scheme). Hence Frobenius reciprocity gives an identification

$$X^*(\mathrm{R}) = \mathrm{Ind}_{\Gamma_E}^{\Gamma_F} X^*(\mathrm{S}) = \mathbb{Z}[\Gamma_F] \otimes_{\mathbb{Z}[\Gamma_E]} X^*(\mathrm{S}).$$

Remark 2.3.3. We will often use without further comment the following nice property of the naive filtration (Notation 2.3.1(v)): for any injective homomorphism $\mathrm{T}_1 \hookrightarrow \mathrm{T}_2$ of tori over a DVHF F with perfect residue field, $r \geq 0$, and a finitely ramified separable extension E/F, $\mathrm{T}_1(F)_r^{\mathrm{naive}} = \mathrm{T}_1(F) \cap \mathrm{T}_2(F)_r^{\mathrm{naive}} = \mathrm{T}_1(F) \cap \mathrm{T}_2(E)_{e(E/F)r}^{\mathrm{naive}}$. For the first equality, use that $X^*(\mathrm{T}_2) \to X^*(\mathrm{T}_1)$ is surjective. For the second, recall that by the convention in Notation 2.3.1(vi), each $\mathrm{T}_i(E)_s$ is defined using the normalized discrete valuation on E.

2.4. **Weil restriction for tori across close local fields.** The following lemma is implicit in the last sentence of [CY01, Section 3.6].

Lemma 2.4.1. *Suppose $(F, \mathrm{T}) \leftrightarrow_l (F', \mathrm{T}')$. Let $L \hookrightarrow F^{\mathrm{sep}}$ and $L' \hookrightarrow {F'}^{\mathrm{sep}}$ be compatible embeddings, with L/F finite, so that by Notation 2.3.1*(ix)*, we have a realization $(F, \mathrm{R}) \leftrightarrow_l (F', \mathrm{R}')$, where $\mathrm{R} = \mathrm{Res}_{L/F} \mathrm{T}_L$ and $\mathrm{R}' = \mathrm{Res}_{L'/F'} \mathrm{T}'_{L'}$. Then the "diagonal" inclusions $\mathrm{T} \hookrightarrow \mathrm{R}$ and $\mathrm{T}' \hookrightarrow \mathrm{R}'$ induce the same (necessarily surjective) homomorphisms $X^*(\mathrm{R}) = X^*(\mathrm{R}') \to X^*(\mathrm{T}') = X^*(\mathrm{T})$.*

Proof. L'/F' is also finite, so the statement makes sense. By the universal property of $\mathrm{Res}_{L/F}$, the "diagonal" map $\mathrm{T} \hookrightarrow \mathrm{R}$ is the unique homomorphism that, when base-changed to L and composed with $\mathrm{R}_L \to \mathrm{T}_L$, yields the identity map $\mathrm{T}_L \to \mathrm{T}_L$. Dually, $X^*(\mathrm{R}) \to X^*(\mathrm{T})$ is the unique homomorphism of Γ_F-modules that, when viewed as a homomorphism of Γ_L-modules and precomposed with the "universal" Γ_L-module homomorphism $X^*(\mathrm{T}) \hookrightarrow X^*(\mathrm{R})$, yields the identity. In the previous sentence, we can replace Γ_F by $\Gamma_F/I_F^l = \Gamma_{F'}/I_{F'}^l$ and Γ_E by $\Gamma_E/I_E^{l(1)} = \Gamma_{E'}/I_{E'}^{l(1)}$, where $l(1) = \psi_{L/F}(l)$. Since an analogous assertion applies for (F', T'), we are done. □

2.5. **A review of some results of Chai and Yu.** Unfortunately, we will need to quote from the proofs, and not just the lemmas, of [CY01]. Therefore, we summarize what we will need from that paper in this subsection.

Notation 2.5.1. (i) If T is a torus over a DVHF F with perfect residue field, and L/F is a finite Galois extension splitting T, we will denote by $h(F, \mathrm{T}, L)$ the nonnegative integer $h(\mathfrak{O}_F, \mathfrak{O}_L, \Gamma_F, X_*(\mathrm{T}))$ defined as in [CY01, Section 8.1, just before the lemma]. If L/F is a minimal splitting extension for T, i.e., isomorphic to the fixed field of the kernel of $\Gamma_F \to \mathrm{Aut}(X^*(\mathrm{T}))$, we write $h(F, \mathrm{T}) = h(F, \mathrm{T}, L)$. By [CY01, Lemma 8.1], whenever $(F, \mathrm{T}) \leftrightarrow_l (F', \mathrm{T}')$, $L \hookrightarrow F^{\mathrm{sep}}$ and $L' \hookrightarrow {F'}^{\mathrm{sep}}$ are compatible extensions, and $h(F, \mathrm{T}, L) < \lfloor \psi_{L/F}(l)/e(L/F) \rfloor$, we have $h(F, \mathrm{T}, L) = h(F', \mathrm{T}', L')$ and $h(F, \mathrm{T}) = h(F, \mathrm{T}')$: indeed, if we set $h = h(F, \mathrm{T}, L)$ and $e = e(L/F)$, then L'/F' splits T' (minimally if L does), and we have an identification $\mathrm{Tr}_{e(h+1)} L \cong \mathrm{Tr}_{e(h+1)} L'$ over $\mathrm{Tr}_{h+1} F \cong \mathrm{Tr}_{h+1} F'$, feeding into the hypothesis of [CY01, Lemma 8.1].

(ii) Suppose $(F, \mathrm{T}) \leftrightarrow_l (F', \mathrm{T}')$. Let L/F be an at most l-ramified finite Galois extension splitting T, and assume that $L \hookrightarrow F^{\mathrm{sep}}$ and $L' \hookrightarrow {F'}^{\mathrm{sep}}$ are compatible embeddings. Recall that we have $(F, \mathrm{R} := \mathrm{Res}_{L/F} \mathrm{T}_L) \leftrightarrow_l (F', \mathrm{R}' := \mathrm{Res}_{L'/F'} \mathrm{T}_{L'})$ (see Notation 2.3.1(ix)). Whenever $0 < m \underline{\ll}_L l$, [CY01, the proof of Proposition 8.4(ii)] gives us an isomorphism

$$\mathcal{R}^{\mathrm{ft}} \times_{\mathfrak{O}_F} \mathfrak{O}_F/\mathfrak{p}_F^m \to {\mathcal{R}'}^{\mathrm{ft}} \times_{\mathfrak{O}_{F'}} \mathfrak{O}_{F'}/\mathfrak{p}_{F'}^m, \tag{9}$$

(we recall that $\mathcal{R}^{\mathrm{ft}}$ (resp., ${\mathcal{R}'}^{\mathrm{ft}}$) is the finite type Néron model of R (resp., R')). Explicitly, the identification $\mathfrak{O}_L/\mathfrak{p}_F^m\mathfrak{O}_L = \mathfrak{O}_{L'}/\mathfrak{p}_{F'}^m\mathfrak{O}_{L'}$ (from $L \leftrightarrow_{\psi_{L/F}(l)} L'$) allows us to identify both sides, at the level of A-valued points for an $\mathfrak{O}_F/\mathfrak{p}_F^m = \mathfrak{O}_{F'}/\mathfrak{p}_{F'}^m$-algebra A, with

$$A \mapsto \mathrm{Hom}(X^*(\mathrm{T}), ((\mathfrak{O}_L/\mathfrak{p}_F^m\mathfrak{O}_L) \otimes_{\mathfrak{O}_F} A)^\times) = \mathrm{Hom}(X^*(\mathrm{T}'), ((\mathfrak{O}_{L'}/\mathfrak{p}_{F'}^m\mathfrak{O}_{L'}) \otimes_{\mathfrak{O}_{F'}} A)^\times).$$

Remark 2.5.2. Assume the setting of Notation 2.5.1(ii). The following is from [CY01, Section 8.1].

(i) Note that $\mathrm{Res}_{\mathfrak{O}_L/\mathfrak{O}_F}\mathbb{G}_m$ has an obvious realization as a closed subscheme of the affine space $\mathbb{A}^{[L:F]+1}/\mathfrak{O}_F$ associated to the free $\mathfrak{O}_F$-module $\mathfrak{O}_L\oplus\mathfrak{O}_F$ of rank $[L:F]+1$ (with some chosen basis). Using a *compatible* basis of $\mathfrak{O}_{L'}\oplus\mathfrak{O}_{F'}$, we have $\mathrm{Res}_{\mathfrak{O}_{L'}/\mathfrak{O}_{F'}}\mathbb{G}_m\hookrightarrow\mathbb{A}^{[L:F]+1}/\mathfrak{O}_{F'}$. Note that the obvious isomorphism $(\mathbb{A}^{[L:F]+1}/\mathfrak{O}_F)\times_{\mathfrak{O}_F}\mathfrak{O}_F/\mathfrak{p}_F^m\to(\mathbb{A}^{[L:F]+1}/\mathfrak{O}_{F'})\times_{\mathfrak{O}_{F'}}\mathfrak{O}_{F'}/\mathfrak{p}_{F'}^m$ restricts to the isomorphism $\mathrm{Res}_{\mathfrak{O}_L/\mathfrak{O}_F}\mathbb{G}_m\times_{\mathfrak{O}_F}\mathfrak{O}_F/\mathfrak{p}_F^{m+1}\to\mathrm{Res}_{\mathfrak{O}_{L'}/\mathfrak{O}_{F'}}\mathbb{G}_m\times_{\mathfrak{O}_{F'}}\mathfrak{O}_{F'}/\mathfrak{p}_{F'}^{m+1}$ defined as in (in fact as a special case of) Notation 2.5.1(ii).

(ii) Choosing bases $\{\chi_i=\chi_i'\}$ of $X^*(\mathrm{T})=X^*(\mathrm{T}')$, we can realize $\mathcal{R}^{\mathrm{ft}}$, which sends an $\mathfrak{O}_F$-algebra A to $\mathrm{Hom}(X^*(\mathrm{T}),(\mathfrak{O}_L\otimes_{\mathfrak{O}_F}A)^\times)$, as a product of copies of $\mathrm{Res}_{\mathfrak{O}_L/\mathfrak{O}_F}\mathbb{G}_m$ indexed by $\{\chi_i=\chi_i'\}$, giving using (i) an embedding $\mathcal{R}^{\mathrm{ft}}\hookrightarrow\mathbb{A}^{\dim\mathrm{T}([L:F]+1)}/\mathfrak{O}_F$, as a closed subscheme. Similarly, we get $\mathcal{R}'^{\mathrm{ft}}\hookrightarrow\mathbb{A}^{\dim\mathrm{T}([L:F]+1)}/\mathfrak{O}_{F'}$. It is immediate that the obvious isomorphism $(\mathbb{A}^{[L:F]+1}/\mathfrak{O}_F)\times_{\mathfrak{O}_F}\mathfrak{O}_F/\mathfrak{p}_F^m\to(\mathbb{A}^{[L:F]+1}/\mathfrak{O}_{F'})\times_{\mathfrak{O}_{F'}}\mathfrak{O}_{F'}/\mathfrak{p}_{F'}^m$ restricts to the isomorphism $\mathcal{R}^{\mathrm{ft}}\times_{\mathfrak{O}_F}\mathfrak{O}_F/\mathfrak{p}_F^m\to\mathcal{R}'^{\mathrm{ft}}\times_{\mathfrak{O}_{F'}}\mathfrak{O}_{F'}/\mathfrak{p}_{F'}^m$ of Notation 2.5.1(ii).

The following is one of the main results of [CY01]:

Theorem 2.5.3 (Chai and Yu). *Let $(F,\mathrm{T})\leftrightarrow_l(F',\mathrm{T}')$. Suppose m is a positive integer such that $m+3h(F,\mathrm{T},L)\underline{\ll}_L l$, for a fixed at most l-ramified finite Galois extension L/F splitting* T *(which exists since $(F,\mathrm{T})\leftrightarrow_l(F',\mathrm{T}')$). Let $L\hookrightarrow F^{\mathrm{sep}}$ and $L'\hookrightarrow F'^{\mathrm{sep}}$ be compatible embeddings, so that $(F,\mathrm{R}:=\mathrm{Res}_{L/F}\mathrm{T}_L)\leftrightarrow_l(F',\mathrm{R}':=\mathrm{Res}_{L'/F'}\mathrm{T}_{L'})$. Set $h=h(F,\mathrm{T},L)$. Then there exists a unique isomorphism $\mathcal{T}^{\mathrm{ft}}\times_{\mathfrak{O}_F}\mathfrak{O}_F/\mathfrak{p}_F^m\to\mathcal{T}'^{\mathrm{ft}}\times_{\mathfrak{O}_{F'}}\mathfrak{O}_{F'}/\mathfrak{p}_{F'}^m$ satisfying the following property: for some (or equivalently, any) compatible embeddings $\tilde F\hookrightarrow F^{\mathrm{sep}}$ and $\tilde F'\hookrightarrow F'^{\mathrm{sep}}$, where $\tilde F/F$ is a maximal unramified extension, evaluating this isomorphism (resp., (9)) at $\mathfrak{O}_{\tilde F}/\mathfrak{p}_{\tilde F}^m=\mathfrak{O}_{\tilde F'}/\mathfrak{p}_{\tilde F'}^m$ (resp., $\mathfrak{O}_{\tilde F}/\mathfrak{p}_{\tilde F}^{m+h}=\mathfrak{O}_{\tilde F'}/\mathfrak{p}_{\tilde F'}^{m+h}$) gives us the right-most (resp., the left-most) vertical arrow of an obvious commutative diagram*

$$\begin{tikzcd}
\mathrm{R}(\tilde F)_b/\mathrm{R}(\tilde F)_{m+h} \arrow[d] & \mathrm{T}(\tilde F)_b/\mathrm{T}(\tilde F)^{\mathrm{naive}}_{m+h} \arrow[l, hook'] \arrow[r] \arrow[d] & \mathrm{T}(\tilde F)_b/\mathrm{T}(\tilde F)_m \arrow[d] \\
\mathrm{R}'(\tilde F')_b/\mathrm{R}'(\tilde F')_{m+h} & \mathrm{T}'(\tilde F')_b/\mathrm{T}'(\tilde F')^{\mathrm{naive}}_{m+h} \arrow[l, hook'] \arrow[r] & \mathrm{T}'(\tilde F')_b/\mathrm{T}'(\tilde F')_m
\end{tikzcd} \quad . \tag{10}$$

Here, we have identified $\mathcal{T}^{\mathrm{ft}}(\mathfrak{O}_{\tilde F})$ with $\mathrm{T}(\tilde F)_b$, and hence $\mathcal{T}^{\mathrm{ft}}(\mathfrak{O}_{\tilde F}/\mathfrak{p}_{\tilde F}^m)$ with $\mathrm{T}(\tilde F)_b/\mathrm{T}(\tilde F)_m$, etc. That the left vertical arrow induces the middle vertical arrow, and the inclusions $\mathrm{T}(\tilde F)^{\mathrm{naive}}_{m+h}\subset\mathrm{T}(\tilde F)_m$ and $\mathrm{T}'(\tilde F')^{\mathrm{naive}}_{m+h}\subset\mathrm{T}'(\tilde F')_m$ needed to make sense of the right horizontal arrows in the two rows, are part of the assertion.

The above description is not present right at the point of statement of [CY01, Theorem 8.5], but can be assembled from various parts of [CY01]. To help the reader do so, we will give more references later in this subsection. To this end, we now make some preparation.

The following assertion is a special case of [CY01, Proposition 4.2], understood using [CY01, the proof of Proposition 8.4(iii)]:

Proposition 2.5.4 (Chai and Yu). *Let $F\leftrightarrow_l F'$. Let X,X' be smooth algebraic schemes over $\mathfrak{O}_F,\mathfrak{O}_{F'}$, $\mathrm{W}\subset\mathrm{X}_{\kappa_F}$ and $\mathrm{W}'\subset\mathrm{X}'_{\kappa_{F'}}$ closed smooth subschemes, and Y,Y' the dilatations of W,W' on X,X'. Assume that, for some $0\le m<l$, we are given an isomorphism $\mathrm{X}\times_{\mathfrak{O}_F}\mathfrak{O}_F/\mathfrak{p}_F^{m+1}\to\mathrm{X}'\times_{\mathfrak{O}_{F'}}\mathfrak{O}_{F'}/\mathfrak{p}_{F'}^{m+1}$ over $\mathfrak{O}_F/\mathfrak{p}_F^{m+1}=\mathfrak{O}_{F'}/\mathfrak{p}_{F'}^{m+1}$, that on tensoring with $\kappa_F=\kappa_{F'}$ identifies W with W'. Then there is a unique isomorphism $\mathrm{Y}\times_{\mathfrak{O}_F}\mathfrak{O}_F/\mathfrak{p}_F^m\to\mathrm{Y}'\times_{\mathfrak{O}_{F'}}\mathfrak{O}_{F'}/\mathfrak{p}_{F'}^m$ over $\mathfrak{O}_F/\mathfrak{p}_F^m=\mathfrak{O}_{F'}/\mathfrak{p}_{F'}^m$ with the following property: for some (or equivalently, any) compatible embeddings $\tilde F\hookrightarrow F^{\mathrm{sep}}$ and $\tilde F'\hookrightarrow F'^{\mathrm{sep}}$, where $\tilde F/F$ is a maximal unramified extension, it maps the image of $y\in\mathrm{Y}(\mathfrak{O}_{\tilde F})\subset\mathrm{X}(\mathfrak{O}_{\tilde F})$ in $\mathrm{Y}\times_{\mathfrak{O}_F}\mathfrak{O}_F/\mathfrak{p}_F^m$ (by which we mean its image in $\mathrm{Y}(\mathfrak{O}_{\tilde F}/\mathfrak{p}_{\tilde F}^m)$) to that of $y'\in\mathrm{Y}'(\mathfrak{O}_{\tilde F'})\subset\mathrm{X}(\mathfrak{O}_{\tilde F'})$ in $\mathrm{Y}'\times_{\mathfrak{O}_{\tilde F'}}\mathfrak{O}_{\tilde F'}/\mathfrak{p}_{\tilde F'}^m$ whenever $\mathrm{X}\times_{\mathfrak{O}_F}\mathfrak{O}_F/\mathfrak{p}_F^{m+1}\to\mathrm{X}'\times_{\mathfrak{O}_{F'}}\mathfrak{O}_{F'}/\mathfrak{p}_{F'}^{m+1}$ does so.*

References for the proof. First we address the uniqueness, assuming the existence. It is immediately seen that $\tilde{F}'/F'$ is automatically a maximal unramified extension. Given $y \in \mathrm{Y}(\mathfrak{O}_{\tilde{F}}) \subset \mathrm{X}(\mathfrak{O}_{\tilde{F}})$, its image in $\mathrm{X}(\mathfrak{O}_{\tilde{F}}/\mathfrak{p}_{\tilde{F}}^{m+1}) = \mathrm{X}'(\mathfrak{O}_{\tilde{F}'}/\mathfrak{p}_{\tilde{F}'}^{m+1})$ has image in $\mathrm{X}(\kappa_{\tilde{F}}) = \mathrm{X}'(\kappa_{\tilde{F}'})$ that belongs to $\mathrm{W} = \mathrm{W}'$. Thus, any $y' \in \mathrm{X}'(\mathfrak{O}_{\tilde{F}'})$ that lifts this image (such y' exist as X' is smooth) belongs to $\mathrm{Y}'(\mathfrak{O}_{\tilde{F}'}) \subset \mathrm{X}'(\mathfrak{O}_{\tilde{F}'})$. Thus, $\mathrm{Y} \times_{\mathfrak{O}_F} \mathfrak{O}_F/\mathfrak{p}_F^m \to \mathrm{Y}' \times_{\mathfrak{O}_{F'}} \mathfrak{O}_{F'}/\mathfrak{p}_{F'}^m$ is pinned down on the image of $\mathrm{Y}(\mathfrak{O}_{\tilde{F}})$ in $\mathrm{Y}(\mathfrak{O}_{\tilde{F}}/\mathfrak{p}_{\tilde{F}}^m)$. Now the uniqueness follows by the schematic density of this set of points, as asserted in [CY01, Lemma 8.5.1], which applies since Y is smooth (see [BLR90, Section 3.2, Proposition 3], or [KP23, Lemma A.5.10] or [MRR23, Proposition 2.16]). Note that this argument does not use the completeness assumption of [CY01, Section 8].

Now we discuss the existence. The non-dependence on $\tilde{F} \hookrightarrow F^{\mathrm{sep}}$ and $\tilde{F}' \hookrightarrow F'^{\mathrm{sep}}$ is easy. Write $\mathrm{X} = \operatorname{Spec} C$ and $\mathrm{X}' = \operatorname{Spec} C'$, for an $\mathfrak{O}_F$-algebra C and an $\mathfrak{O}_{F'}$-algebra C'. Then $\mathrm{X} \times_{\mathfrak{O}_F} \mathfrak{O}_F/\mathfrak{p}_F^{m+1} \to \mathrm{X}' \times_{\mathfrak{O}_{F'}} \mathfrak{O}_{F'}/\mathfrak{p}_{F'}^{m+1}$ is dual to the inverse of an isomorphism $C/\mathfrak{p}_F^{m+1}C \to C'/\mathfrak{p}_{F'}^{m+1}C'$. Let the subscheme $\mathrm{W} \subset \mathrm{X}_{\kappa_F} \subset \mathrm{X}$ be defined by an ideal I_W generated by $\varpi_F, g_1, \ldots, g_s$, where $\varpi_F \in \mathfrak{O}_F$ is a uniformizer, and $g_1, \ldots, g_s \in C$. Then the subscheme $\mathrm{W}' \subset \mathrm{X}'$ is defined by the ideal $I_{W'}$ generated by $\varpi_{F'}, g_1', \ldots, g_s'$, where $\varpi_{F'} \in \mathfrak{O}_{F'}$ is a uniformizer matching ϖ_F under $\mathfrak{O}_F/\mathfrak{p}_F^{m+1} = \mathfrak{O}_{F'}/\mathfrak{p}_{F'}^{m+1}$, and where g_i' matches g_i under $C/\mathfrak{p}_F^{m+1}C \to C'/\mathfrak{p}_{F'}^{m+1}C'$ for $1 \le i \le s$.

Over the isomorphism $C/\mathfrak{p}_F^{m+1}C = C'/\mathfrak{p}_{F'}^{m+1}C'$ of $\mathfrak{O}_F/\mathfrak{p}_F^{m+1} = \mathfrak{O}_{F'}/\mathfrak{p}_{F'}^{m+1}$-algebras lies an isomorphism

$$\frac{C[x_1, \ldots, x_s]}{(\varpi_F x_1 - g_1, \ldots, \varpi_F x_s - g_s)} \otimes_{\mathfrak{O}_F} \mathfrak{O}_F/\mathfrak{p}_F^{m+1} \to \frac{C'[x_1, \ldots, x_s]}{(\varpi_{F'} x_1 - g_1', \ldots, \varpi_{F'} x_s - g_s)} \otimes_{\mathfrak{O}_{F'}} \mathfrak{O}_{F'}/\mathfrak{p}_{F'}^{m+1}, \tag{11}$$

induced by sending x_i to x_i for each i. The ring $C[x_1, \ldots, x_s]/(\varpi_F x_1 - g_1, \ldots, \varpi_F x_s - g_s)$, *modulo its* ϖ_F^∞*-torsion*, is the coordinate ring $\mathfrak{O}_F[\mathrm{Y}]$ of Y (see, e.g., [KP23, the discussion of Remark A.5.9]). A similar assertion applies with F' in place of F. While $\mathfrak{O}_F[\mathrm{Y}] \otimes_{\mathfrak{O}_F} \mathfrak{O}_F/\mathfrak{p}_F^m$ and $\mathfrak{O}_{F'}[\mathrm{Y}'] \otimes_{\mathfrak{O}_{F'}} \mathfrak{O}_{F'}/\mathfrak{p}_{F'}^m$ are quotients of the left-hand side and the right-hand side of the isomorphism (11), it is not obvious that (11) induces an isomorphism between these quotients. Nevertheless, it does so, by [CY01, Proposition 4.2], as we now explain.

We will superscript with $[m]$ to denote base-change to $\mathfrak{O}_F/\mathfrak{p}_F^m$ or $\mathfrak{O}_{F'}/\mathfrak{p}_{F'}^m$. In particular, $I_W^{[m]} = I_W \otimes_{\mathfrak{O}_F} \mathfrak{O}_F/\mathfrak{p}_F^m$. Since $I_W \supset \varpi_F \mathfrak{O}_F[X]$, we get $\varpi_F^m I_W \supset \varpi_F^{m+1} \mathfrak{O}_F[X]$. This allows us to identify $I_W^{[m]} = I_W/\varpi_F^m I_W$ with $I_{W'}^{[m]} = I_{W'}/\varpi_{F'}^m I_{W'}$. One considers the following diagram:

$$\begin{tikzcd}
(C[x_1, \ldots, x_s]/(\varpi_F x_1 - g_1, \ldots, \varpi_F x_s - g_s))^{[m]} \arrow[r] \arrow[d] & \left(\bigoplus_{t \ge 0} \operatorname{Sym}^t_{C^{[m]}} I_W^{[m]}\right)_{(\varpi_F)} \arrow[r] \arrow[d] & (\mathfrak{O}_F[\mathrm{Y}])^{[m]} \arrow[d] \\
(C'[x_1, \ldots, x_s]/(\varpi_{F'} x_1 - g_1', \ldots, \varpi_{F'} x_s - g_s'))^{[m]} \arrow[r] & \left(\bigoplus_{t \ge 0} \operatorname{Sym}^t_{C'^{[m]}} I_{W'}^{[m]}\right)_{(\varpi_{F'})} \arrow[r] & (\mathfrak{O}_{F'}[\mathrm{Y}'])^{[m]}
\end{tikzcd}$$

where the top middle term refers to the homogeneous localization of $\bigoplus_{t \ge 0} \operatorname{Sym}^t_{C^{[m]}} I_W^{[m]}$ at the homogeneous element of degree 1 given by the image of $\varpi_F \in I_W$, so that its spectrum is an open subset of the scheme $\mathrm{Bl}'(\mathrm{X}, \mathrm{W}) \times_{\mathfrak{O}_F} \mathfrak{O}_F/\mathfrak{p}_F^m$ defined as in [CY01, Section 4.2.1]. Thus, the right square and its commutativity, and the fact that the right vertical arrow is an isomorphism, follow from the canonicity description of [CY01, Section 4.2.1]. The top left arrow maps the image of x_i to g_i/ϖ_F for each i, and the bottom left arrow is similar, so that the commutativity of the left square is clear.

Thus, (11) indeed quotients to an isomorphism $\mathfrak{O}_F[\mathrm{Y}] \otimes_{\mathfrak{O}_F} \mathfrak{O}_F/\mathfrak{p}_F^m \to \mathfrak{O}_{F'}[\mathrm{Y}'] \otimes_{\mathfrak{O}_{F'}} \mathfrak{O}_{F'}/\mathfrak{p}_{F'}^m$, i.e., an isomorphism $\mathrm{Y} \times_{\mathfrak{O}_F} \mathfrak{O}_F/\mathfrak{p}_F^m \to \mathrm{Y}' \times_{\mathfrak{O}_{F'}} \mathfrak{O}_{F'}/\mathfrak{p}_{F'}^m$. It remains to show that this morphism is as described in the proposition, so let y, y' be as in it. It suffices to show that $f(y) = f'(y')$ in $\mathfrak{O}_{\tilde{F}}/\mathfrak{p}_{\tilde{F}}^m = \mathfrak{O}_{\tilde{F}'}/\mathfrak{p}_{\tilde{F}'}^m$ whenever f and f' match under (11). Without loss of generality, f and f' are both represented by x_i for some $1 \le i \le s$. But since we have $\varpi_F f = g_i$ and $\varpi_{F'} f' = g_i'$, this follows from the fact that g_i and g_i' match each other under $C/\mathfrak{p}_F^{m+1}C \to C'/\mathfrak{p}_{F'}^{m+1}C'$, as do ϖ_F and $\varpi_{F'}$ under $\mathfrak{O}_F/\mathfrak{p}_F^{m+1} \to \mathfrak{O}_{F'}/\mathfrak{p}_{F'}^{m+1}$, so that $g_i(y)$ and $g_i'(y')$ match under $\mathfrak{O}_{\tilde{F}}/\mathfrak{p}_{\tilde{F}}^{m+1} \to \mathfrak{O}_{\tilde{F}'}/\mathfrak{p}_{\tilde{F}'}^{m+1}$. $\square$

References for the proof of Theorem 2.5.3. The uniqueness follows as in the proof of Proposition 2.5.4. Using the latter argument of [CY01, Remark 8.6] ($\mathfrak{O}_F/\mathfrak{p}_F^m$, $\mathcal{T}^{\mathrm{ft}} \times_{\mathfrak{O}_F} \mathfrak{O}_F/\mathfrak{p}_F^m$ and $\mathcal{R}^{\mathrm{ft}} \times_{\mathfrak{O}_F} \mathfrak{O}_F/\mathfrak{p}_F^m$ remain unchanged when F is replaced by its completion), together with [KP23, Proposition 2.3.4(2)], from which it follows that $\mathcal{T}^{\mathrm{ft}}(\mathfrak{O}_{\tilde{F}}) = \mathrm{T}(\tilde{F})_b$ is dense in the analogous group associated to the completion of F (so that the source and target of middle vertical arrow are unchanged when we replace F by its completion), we may and shall assume that F and F' are complete. This is being done so that we may use results from [CY01, Section 8].

We will denote by $\underline{\mathrm{T}}^i$ and $\underline{\mathrm{R}}^i$ the integral models $\underline{T}^i$ for T and $\underline{R}^i$ for R from [CY01, Sections 3.2, 3.4, 3.6]. By definition, $\underline{\mathrm{R}}^0 \coloneqq \mathcal{R}^{\mathrm{ft}}$, where $\mathrm{R} \coloneqq \mathrm{Res}_{L/F}\mathrm{T}_L$, and $\underline{\mathrm{T}}^0$ (see [CY01, Section 3.6]) is the schematic closure of T in $\underline{\mathrm{R}}^0$. Thus, $\underline{\mathrm{T}}^0$ is the standard model of T in the sense of [KP23, Section B.4]. If $\underline{\mathrm{T}}^i$ and $\underline{\mathrm{R}}^i$ are defined, then $\underline{\mathrm{T}}^{i+1}$ (resp., $\underline{\mathrm{R}}^{i+1}$) is the dilatation of a smooth subscheme $\mathrm{Z}^i \subset \underline{\mathrm{T}}^i \times_{\mathfrak{O}_F} \kappa_F$ on $\underline{\mathrm{T}}^i$ (resp., $\mathrm{W}^i \subset \underline{\mathrm{R}}^i \times_{\mathfrak{O}_F} \kappa_F$ on $\underline{\mathrm{R}}^i$). Further, $\underline{\mathrm{T}}^i$ can be realized as the schematic closure of T in $\underline{\mathrm{R}}^i$ ([CY01, Lemma 3.5]). We have similar objects $(\underline{\mathrm{T}}')^i$ and $(\underline{\mathrm{R}}')^i$ associated to T'/F'.

By [CY01, Corollary 8.2.4], or rather its proof, thanks to the inequalities $\psi_{L/F}(l)/e(L/F) > 2h$ and $\psi_{L/F}(l)/e(L/F) - h \geq m+h$, the isomorphism $\underline{\mathrm{R}}^0 \times_{\mathfrak{O}_F} \mathfrak{O}_F/\mathfrak{p}_F^{m+h} \to (\underline{\mathrm{R}}')^0 \times_{\mathfrak{O}_{F'}} \mathfrak{O}_{F'}/\mathfrak{p}_{F'}^{m+h}$ given by (ii) of Notation 2.5.1 restricts to an isomorphism $\underline{\mathrm{T}}^0 \times_{\mathfrak{O}_F} \mathfrak{O}_F/\mathfrak{p}_F^{m+h} \to (\underline{\mathrm{T}}')^0 \times_{\mathfrak{O}_{F'}} \mathfrak{O}_{F'}/\mathfrak{p}_{F'}^{m+h}$, as the following two sentences explain. Recall the chains of inclusions of closed subschemes $\underline{\mathrm{T}}^0 \subset \underline{\mathrm{R}}^0 \subset \mathbb{A}^{\dim \mathrm{T}\cdot([L:F]+1)}/\mathfrak{O}_F$ and $(\underline{\mathrm{T}}')^0 \subset (\underline{\mathrm{R}}')^0 \subset \mathbb{A}^{\dim \mathrm{T}\cdot([L:F]+1)}/\mathfrak{O}_{F'}$ from [CY01, Section 8.1 and the beginning of Section 8.3], reviewed in Remark 2.5.2. The isomorphism $\underline{\mathrm{R}}^0 \times_{\mathfrak{O}_F} \mathfrak{O}_F/\mathfrak{p}_F^{m+h} \to (\underline{\mathrm{R}}')^0 \times_{\mathfrak{O}_{F'}} \mathfrak{O}_{F'}/\mathfrak{p}_{F'}^{m+h}$ from (ii) of Notation 2.5.1 has been observed to be a restriction of the obvious identification of affine spaces $\mathbb{A}^{\dim \mathrm{T}\cdot([L:F]+1)}/(\mathfrak{O}_F/\mathfrak{p}_F^{m+h}) = \mathbb{A}^{\dim \mathrm{T}\cdot([L:F]+1)}/(\mathfrak{O}_{F'}/\mathfrak{p}_{F'}^{m+h})$, and the isomorphism $\underline{\mathrm{T}}^0 \times_{\mathfrak{O}_F} \mathfrak{O}_F/\mathfrak{p}_F^{m+h} \to (\underline{\mathrm{T}}')^0 \times_{\mathfrak{O}_{F'}} \mathfrak{O}_{F'}/\mathfrak{p}_{F'}^{m+h}$ is constructed in [CY01, Corollary 8.2.4] to also satisfy this property.

Since $\underline{\mathrm{T}}^0(\mathfrak{O}_{\tilde{F}}) \subset \underline{\mathrm{R}}^0(\mathfrak{O}_{\tilde{F}})$ identifies with the inclusion $\mathrm{T}(\tilde{F})_b \subset \mathrm{R}(\tilde{F})_b$, and $\mathrm{T}(\tilde{F})_{m+h}^{\mathrm{naive}} = \mathrm{T}(\tilde{F}) \cap \mathrm{R}(\tilde{F})_{m+h}$, and similar assertions hold for F', the above paragraph implies that we indeed have a well-defined middle vertical arrow of (10) making the left square of that diagram commute.

A positive integer δ is introduced in [CY01, Section 8.5], and it is observed that $\delta \leq h$. As observed there (the invocation of [CY01, Lemma 5.5]) and using our assumption that $m + 3h \leq l$, working with F' instead of F does not change δ. Thus, as observed at [CY01, the beginning of Section 8.5.2], $\mathcal{T}^{\mathrm{ft}}$ equals $\underline{\mathrm{T}}^h$ and $\mathcal{T}'^{\mathrm{ft}}$ equals $(\underline{\mathrm{T}}')^h$.

$\underline{\mathrm{R}}^h$ and $(\underline{\mathrm{R}}')^h$ are obtained from $\underline{\mathrm{R}}^0$ and $(\underline{\mathrm{R}}')^0$ by a series of h dilatations. At the i-th step, one inductively assumes given an identification $\underline{\mathrm{R}}^{i-1} \times_{\mathfrak{O}_F} \mathfrak{O}_F/\mathfrak{p}_F^{m+h+1-i} \to (\underline{\mathrm{R}}')^{i-1} \times_{\mathfrak{O}_{F'}} \mathfrak{O}_{F'}/\mathfrak{p}_{F'}^{m+h+1-i}$, and dilatates $\underline{\mathrm{R}}^{i-1}$ and $(\underline{\mathrm{R}}')^{i-1}$ with respect to the same subscheme of $\underline{\mathrm{R}}^{i-1} \times_{\mathfrak{O}_F} \kappa_F = (\underline{\mathrm{R}}')^{i-1} \times_{\mathfrak{O}_{F'}} \kappa_{F'}$ to get $\underline{\mathrm{R}}^i$ and $(\underline{\mathrm{R}}')^i$, yielding by Proposition 2.5.4 an isomorphism $\underline{\mathrm{R}}^i \times_{\mathfrak{O}_F} \mathfrak{O}_F/\mathfrak{p}_F^{m+h-i} \to (\underline{\mathrm{R}}')^i \times_{\mathfrak{O}_{F'}} \mathfrak{O}_{F'}/\mathfrak{p}_{F'}^{m+h-i}$ (see [CY01, Proposition 8.4]). It therefore follows from h-many applications of Proposition 2.5.4 that there is a unique isomorphism $\underline{\mathrm{R}}^h \times_{\mathfrak{O}_F} \mathfrak{O}_F/\mathfrak{p}_F^m \to (\underline{\mathrm{R}}')^h \times_{\mathfrak{O}_{F'}} \mathfrak{O}_{F'}/\mathfrak{p}_{F'}^m$ that maps the image of $y \in \underline{\mathrm{R}}^h(\mathfrak{O}_{\tilde{F}}) \subset \mathcal{R}^{\mathrm{ft}}(\mathfrak{O}_{\tilde{F}})$ to that of $y' \in (\underline{\mathrm{R}}')^h(\mathfrak{O}_{\tilde{F}'}) \subset \mathcal{R}'^{\mathrm{ft}}(\mathfrak{O}_{\tilde{F}'})$ whenever the image of y maps to that of y' under $\underline{\mathrm{R}}^0 \times_{\mathfrak{O}_F} \mathfrak{O}_F/\mathfrak{p}_F^{m+h} \to (\underline{\mathrm{R}}')^0 \times_{\mathfrak{O}_{F'}} \mathfrak{O}_{F'}/\mathfrak{p}_{F'}^{m+h}$ (these applications are justified by the fact that the $\underline{\mathrm{R}}^i$ and the $(\underline{\mathrm{R}}')^i$ are smooth, unlike the $\underline{\mathrm{T}}^i$ and the $(\underline{\mathrm{T}}')^i$).

It is argued in [CY01, Section 8.5.2] that $\underline{\mathrm{R}}^h \times_{\mathfrak{O}_F} \mathfrak{O}_F/\mathfrak{p}_F^m \to (\underline{\mathrm{R}}')^h \times_{\mathfrak{O}_{F'}} \mathfrak{O}_{F'}/\mathfrak{p}_{F'}^m$ restricts to an isomorphism $\mathcal{T}^{\mathrm{ft}} \times_{\mathfrak{O}_F} \mathfrak{O}_F/\mathfrak{p}_F^m = \underline{\mathrm{T}}^h \times_{\mathfrak{O}_F} \mathfrak{O}_F/\mathfrak{p}_F^m \to (\underline{\mathrm{T}}')^h \times_{\mathfrak{O}_{F'}} \mathfrak{O}_{F'}/\mathfrak{p}_{F'}^m = \mathcal{T}'^{\mathrm{ft}} \times_{\mathfrak{O}_{F'}} \mathfrak{O}_{F'}/\mathfrak{p}_{F'}^m$. Hence, under this isomorphism, the image of $y \in \mathcal{T}^{\mathrm{ft}}(\mathfrak{O}_{\tilde{F}}) \subset \mathcal{R}^{\mathrm{ft}}(\mathfrak{O}_{\tilde{F}})$ maps to that of $y' \in \mathcal{T}'^{\mathrm{ft}}(\mathfrak{O}_{\tilde{F}'}) \subset \mathcal{R}'^{\mathrm{ft}}(\mathfrak{O}_{\tilde{F}'})$ whenever, under the middle vertical arrow of (10), the image of y maps to that of y'. This makes sense of the right square of (10) and gives its commutativity. □

Note that in the above proof, the implicit assertions such as that $\mathrm{T}(\tilde{F})_{m+h}^{\mathrm{naive}} \subset \mathrm{T}(\tilde{F})_m$ have been implicitly taken care of. This could be compared with the easier containment $\mathrm{T}(\tilde{F})_r^{\mathrm{naive}} \supset \mathrm{T}(\tilde{F})_r$ (use [KP23, Propositions B.10.4 and B.10.13]), which can be proper for tori that are not "weakly induced". In any case, let us record the containment $\mathrm{T}(\tilde{F})_{m+h}^{\mathrm{naive}} \subset \mathrm{T}(\tilde{F})_m$, since it applies in greater generality (without assuming $m + 3h(F, \mathrm{T}) \ll_{\mathrm{T}} l$), and does not need the strength of [CY01, Proposition 4.2] (i.e., of Proposition 2.5.4):

Lemma 2.5.5. *Let* T *be a torus over a DVHF* F *with perfect residue field. Assume that* $h := h(F, \mathrm{T}) < \lfloor \psi_{L/F}(l)/e(L/F) \rfloor$ *for some at most* l*-ramified finite Galois extension* L/F *splitting* T*. Then for any positive integer* m*, we have* $\mathrm{T}(\tilde{F})^{\mathrm{naive}}_{m+h} = \mathrm{T}(\tilde{F})^{\mathrm{std}}_{m+h} \subset \mathrm{T}(\tilde{F})_m$*.*

Remark 2.5.6. In the situation of the above lemma, we claim that whenever $F \leftrightarrow_{m+h} F'$, and $\tilde{F} \hookrightarrow F^{\mathrm{sep}}$ and $\tilde{F}' \hookrightarrow F'^{\mathrm{sep}}$ are compatible embeddings, where $\tilde{F}/F$ is a maximal unramified extension, we also have an analogous containment $\mathrm{T}'(\tilde{F}')^{\mathrm{naive}}_{m+h} = \mathrm{T}'(\tilde{F}')^{\mathrm{std}}_{m+h} \subset \mathrm{T}'(\tilde{F}')_m$: this is because $h(F, \mathrm{T}) = h(F', \mathrm{T}')$ by the discussion of Notation 2.5.1(i).

Proof of Lemma 2.5.5. It suffices to show that $\mathrm{T}(\tilde{F})^{\mathrm{naive}}_{m+h} \subset \mathrm{T}(\tilde{F})_m$, since it will then follow that

$$\mathrm{T}(\tilde{F})^{\mathrm{std}}_{m+h} \subset \mathrm{T}(\tilde{F})^{\mathrm{naive}}_{m+h} = \mathrm{T}(\tilde{F})^{\mathrm{naive}}_{m+h} \cap \mathrm{T}(\tilde{F})_m \subset \mathrm{T}(\tilde{F})^{\mathrm{naive}}_{m+h} \cap \mathrm{T}(\tilde{F})_0 = \mathrm{T}(\tilde{F})^{\mathrm{std}}_{m+h}.$$

Let $\mathrm{R} = \mathrm{Res}_{L/F}\mathrm{T}$. Since the connected Néron model $\mathcal{T}$ is obtained from $\mathcal{T}^{\mathrm{ft}}$ by dilatating with respect to the identity component of the special fiber, it follows that the group $\mathrm{T}(\tilde{F})_m = \ker(\mathcal{T}(\mathfrak{O}_{\tilde{F}}) \to \mathcal{T}(\mathfrak{O}_{\tilde{F}}/\mathfrak{p}_{\tilde{F}}^m))$ also equals $\ker(\mathcal{T}^{\mathrm{ft}}(\mathfrak{O}_{\tilde{F}}) \to \mathcal{T}^{\mathrm{ft}}(\mathfrak{O}_{\tilde{F}}/\mathfrak{p}_{\tilde{F}}^m))$. The kernel of $\mathcal{R}^{\mathrm{ft}}(\mathfrak{O}_{\tilde{F}}) \to \mathcal{R}^{\mathrm{ft}}(\mathfrak{O}_{\tilde{F}}/\mathfrak{p}_{\tilde{F}}^{m+h})$ equals $\mathrm{R}(\tilde{F})_{m+h}$, which equals $\mathrm{R}(\tilde{F})^{\mathrm{naive}}_{m+h}$ by [KP23, Corollary B.10.13] (and the equality $\mathcal{R} = \mathcal{R}^{\mathrm{ft}}$).

We will use notation from the above (outline of) proof. It suffices to show that, if $t \in \underline{\mathrm{T}}^0(\mathfrak{O}_{\tilde{F}})$ has trivial image in $\mathcal{R}^{\mathrm{ft}}(\mathfrak{O}_{\tilde{F}}/\mathfrak{p}_{\tilde{F}}^{m+h}) = \underline{\mathrm{R}}^0(\mathfrak{O}_{\tilde{F}}/\mathfrak{p}_{\tilde{F}}^{m+h})$, then t has trivial image in $\underline{\mathrm{R}}^h(\mathfrak{O}_{\tilde{F}}/\mathfrak{p}_{\tilde{F}}^m)$ (and hence belongs to $\ker(\mathcal{T}^{\mathrm{ft}}(\mathfrak{O}_{\tilde{F}}) \to \mathcal{T}^{\mathrm{ft}}(\mathfrak{O}_{\tilde{F}}/\mathfrak{p}_{\tilde{F}}^m)))$. Indeed, one shows by induction on $0 \le i \le h$ that t has trivial image in $\underline{\mathrm{R}}^i(\mathfrak{O}_{\tilde{F}}/\mathfrak{p}_{\tilde{F}}^{m+h-i})$. The induction step is as in the arguments around and below (11) in the proof of Proposition 2.5.4: if the coordinate ring of R^i is C, write the coordinate ring of R^{i+1} as a quotient of $C[x_1, \dots, x_s]/(\varpi_F x_1 - g_1, \dots, \varpi_F x_s - g_s)$, and note that the induction hypothesis that $f(t) \in \varpi_F^{m+h-i}\mathfrak{O}_{\tilde{F}}$ for all $f \in C$ implies $f(t) \in \varpi_F^{m+h-i-1}\mathfrak{O}_{\tilde{F}}$ for all $f \in C[x_1, \dots, x_s]/(\varpi_F x_1 - g_1, \dots, \varpi_F x_s - g_s)$. □

Note that the proof of the above lemma shows that we can replace $h = h(F, \mathrm{T})$ by any i such that $\underline{\mathrm{T}}^i = \mathcal{T}^{\mathrm{ft}}$, with $\underline{\mathrm{T}}^i$ as in the proof of Theorem 2.5.3 (e.g., i could be the δ of that proof).

3. Standard, congruent and Chai-Yu isomorphisms

3.1. The definition of standard and congruent isomorphisms.

Notation 3.1.1. Let $(F, \mathrm{T}) \leftrightarrow_l (F', \mathrm{T}')$, and let $r > 0$. Elements $t\mathrm{T}(F)^{\mathrm{naive}}_r \in \mathrm{T}(F)/\mathrm{T}(F)^{\mathrm{naive}}_r$ and $t'\mathrm{T}'(F')^{\mathrm{naive}}_r \in \mathrm{T}'(F')/\mathrm{T}'(F')^{\mathrm{naive}}_r$ (or by abuse of notation, $t \in \mathrm{T}(F)$ and $t' \in \mathrm{T}'(F')$) are said to be standard correspondents of each other if for some (and hence by Lemma 3.2.1 below, any) compatible embeddings $L \hookrightarrow F^{\mathrm{sep}}$ and $L' \hookrightarrow F'^{\mathrm{sep}}$, such that L splits T and $r \ll_L l$ (see Notation 2.2.4), the following holds: for every $\chi = \chi' \in X^*(\mathrm{T}') = X^*(\mathrm{T})$, $\chi(t)$ and $\chi'(t')$ have images that match under the isomorphism $L^\times/(1 + \mathfrak{p}_L^{\lceil e(L/F)r \rceil}) \to L'^\times/(1 + \mathfrak{p}_{L'}^{\lceil e(L/F)r \rceil})$ (described below Notation 2.2.4).

Remark 3.1.2. (i) Note that the condition in Notation 3.1.1 does not change if t or t' is replaced by another element of $t\mathrm{T}(F)^{\mathrm{naive}}_r$ or $t'\mathrm{T}'(F')^{\mathrm{naive}}_r$ (this follows from the definition of the naive filtration subgroups $\mathrm{T}(F)^{\mathrm{naive}}_r$ and $\mathrm{T}'(F')^{\mathrm{naive}}_r$).
(ii) We will see in Lemma 3.2.1 below that, in the setting of Notation 3.1.1, every element of $\mathrm{T}(F)/\mathrm{T}(F)^{\mathrm{naive}}_r$ has either a unique standard correspondent in $\mathrm{T}'(F')/\mathrm{T}'(F')^{\mathrm{naive}}_r$, or none at all (and vice versa). This will sometimes be used without further comment in what follows.

Definition 3.1.3. Let $(F, \mathrm{T}) \leftrightarrow_l (F', \mathrm{T}')$.

(i) Let $r > 0$ be a positive real number. An isomorphism $\mathrm{T}(F)/\mathrm{T}(F)^{\mathrm{naive}}_r \to \mathrm{T}'(F')/\mathrm{T}'(F')^{\mathrm{naive}}_r$ of abelian groups is said to be a standard isomorphism if it maps every element of its source to a standard correspondent of it.

(ii) Let m be a positive integer. An isomorphism $\mathrm{T}(F)/\mathrm{T}(F)_m \to \mathrm{T}'(F')/\mathrm{T}'(F')_m$ is said to be a congruent isomorphism if for some (or equivalently by Lemma 3.4.1(i) below, any) compatible embeddings $\tilde{F} \hookrightarrow F^{\mathrm{sep}}$ and $\tilde{F}' \hookrightarrow F'^{\mathrm{sep}}$, where $\tilde{F}/F$ is a maximal unramified extension, and some $r > 0$ such that $\mathrm{T}(\tilde{F})_r^{\mathrm{naive}} \subset \mathrm{T}(\tilde{F})_m$ and $\mathrm{T}'(\tilde{F}')_r^{\mathrm{naive}} \subset \mathrm{T}'(\tilde{F}')_m$, the given isomorphism is a restriction of an isomorphism $\mathrm{T}(\tilde{F})/\mathrm{T}(\tilde{F})_m \to \mathrm{T}'(\tilde{F}')/\mathrm{T}'(\tilde{F}')_m$, which in turn is induced by a standard isomorphism $\mathrm{T}(\tilde{F})/\mathrm{T}(\tilde{F})_r^{\mathrm{naive}} \to \mathrm{T}'(\tilde{F}')/\mathrm{T}'(\tilde{F}')_r^{\mathrm{naive}}$ associated to $(\tilde{F}, \mathrm{T}_{\tilde{F}}) \leftrightarrow_l (\tilde{F}', \mathrm{T}'_{\tilde{F}'})$.

(iii) Let m be a positive integer. An isomorphism $\mathcal{T}^{\mathrm{ft}} \times_{\mathfrak{O}_F} \mathfrak{O}_F/\mathfrak{p}_F^m \to \mathcal{T}'^{\mathrm{ft}} \times_{\mathfrak{O}_{F'}} \mathfrak{O}_{F'}/\mathfrak{p}_{F'}^m$ is said to be a Chai-Yu isomorphism if for some (or equivalently by Lemma 3.4.1(i) below, any) compatible embeddings $\tilde{F} \hookrightarrow F^{\mathrm{sep}}$ and $\tilde{F}' \hookrightarrow F'^{\mathrm{sep}}$, where $\tilde{F}/F$ is a maximal unramified extension, and some $r > 0$ such that $\mathrm{T}(\tilde{F})_r^{\mathrm{naive}} \subset \mathrm{T}(\tilde{F})_m$ and $\mathrm{T}'(\tilde{F}')_r^{\mathrm{naive}} \subset \mathrm{T}'(\tilde{F}')_m$, the isomorphism $\mathrm{T}(\tilde{F})_b/\mathrm{T}(\tilde{F})_m \to \mathrm{T}'(\tilde{F}')_b/\mathrm{T}'(\tilde{F}')_m$ obtained by evaluating the given isomorphism at $\mathfrak{O}_{\tilde{F}}/\mathfrak{p}_{\tilde{F}}^m = \mathfrak{O}_{\tilde{F}'}/\mathfrak{p}_{\tilde{F}'}^m$ is induced by a "restricted standard isomorphism" $\mathrm{T}(\tilde{F})_b/\mathrm{T}(\tilde{F})_r^{\mathrm{naive}} \to \mathrm{T}'(\tilde{F}')_b/\mathrm{T}'(\tilde{F}')_r^{\mathrm{naive}}$, namely, one that maps each element of its source to a standard correspondent of it, for the realization $(\tilde{F}, \mathrm{T}_{\tilde{F}}) \leftrightarrow_l (\tilde{F}', \mathrm{T}'_{\tilde{F}'})$.

The following remark helps make sense of the above definition.

Remark 3.1.4.

(i) Note that, by definition, a standard isomorphism $\mathrm{T}(F)/\mathrm{T}(F)_r^{\mathrm{naive}} \to \mathrm{T}'(F')/\mathrm{T}'(F')_r^{\mathrm{naive}}$ does not exist unless $0 \le r \underline{\ll}_{\mathrm{T}} l$, i.e., $r \le \psi_{L/F}(l)/e(L/F)$ for some finitely ramified at most l-ramified separable extension L/F splitting T. Similarly, a congruent isomorphism $\mathrm{T}(\tilde{F})/\mathrm{T}(\tilde{F})_m \to \mathrm{T}'(\tilde{F}')/\mathrm{T}'(\tilde{F}')_m$ or a Chai-Yu isomorphism $\mathcal{T}^{\mathrm{ft}} \times_{\mathfrak{O}_F} \mathfrak{O}_F/\mathfrak{p}_F^m \to \mathcal{T}'^{\mathrm{ft}} \times_{\mathfrak{O}_{F'}} \mathfrak{O}_{F'}/\mathfrak{p}_{F'}^m$ does not exist unless there exist $r \ge 0$ such that $l \underline{\gg}_{\mathrm{T}} r$, $\mathrm{T}(\tilde{F})_r^{\mathrm{naive}} \subset \mathrm{T}(\tilde{F})_m$ and $\mathrm{T}'(\tilde{F}')_r^{\mathrm{naive}} \subset \mathrm{T}'(\tilde{F}')_m$ (these latter conditions force $r \ge m$, since $\mathrm{T}(F)_r \subset \mathrm{T}(F)_r^{\mathrm{naive}}$).

(ii) In Definition 3.1.3(ii), the notion of restriction from $\mathrm{T}(\tilde{F})/\mathrm{T}(\tilde{F})_m$ to $\mathrm{T}(F)/\mathrm{T}(F)_m$ makes sense, because $\mathrm{T}(\tilde{F})_m \cap \mathrm{T}(F) = \mathrm{T}(F)_m$ (see Notation 2.3.1(vi)). For L/F finitely ramified separable with $e = e(L/F)$, since $\mathrm{T}(L)_{er}^{\mathrm{naive}} \cap \mathrm{T}(F) = \mathrm{T}(F)_r^{\mathrm{naive}}$ (Remark 2.3.3), we may restrict from $\mathrm{T}(L)/\mathrm{T}(L)_{er}^{\mathrm{naive}}$ to $\mathrm{T}(F)/\mathrm{T}(F)_r^{\mathrm{naive}}$.

(iii) To relate the definition of a Chai-Yu isomorphism to isomorphisms constructed in [CY01], see Proposition 3.5.1 below.

3.2. Some first properties of standard isomorphisms.

Lemma 3.2.1. *Let $(F, \mathrm{T}) \leftrightarrow_l (F', \mathrm{T}')$, and let $r > 0$. If $t \in \mathrm{T}(F)$ and $t' \in \mathrm{T}'(F')$ satisfy the conditions in the definition of a standard correspondent (Notation 3.1.1) with respect to some choice of $L \hookrightarrow F^{\mathrm{sep}}$ and $L' \hookrightarrow F'^{\mathrm{sep}}$ (with L/F finitely ramified, such that L splits* T *and $r \underline{\ll}_L l$), then it satisfies those conditions with respect to any other such choice, say $L_1 \hookrightarrow F^{\mathrm{sep}}$ and $L'_1 \hookrightarrow F'^{\mathrm{sep}}$. Moreover, every element of $\mathrm{T}(F)/\mathrm{T}(F)_r^{\mathrm{naive}}$ has either a unique standard corresondent in $\mathrm{T}'(F')/\mathrm{T}'(F')_r^{\mathrm{naive}}$, or none at all.*

Proof. We first prove the former assertion. Assume first that L/F is minimal, i.e., isomorphic to the fixed field of $\ker(\Gamma_F \to \mathrm{Aut}(X^*(\mathrm{T})))$. In this case, $L \hookrightarrow F^{\mathrm{sep}}$ factors as the composite of $L_1 \hookrightarrow F^{\mathrm{sep}}$ and some F-algebra embedding $L \hookrightarrow L_1$. Moreover, L' is automatically a minimal splitting extension for T', and we similarly get $L' \hookrightarrow L'_1 \hookrightarrow F'^{\mathrm{sep}}$. In this case, the lemma follows from the commutative diagram of Subsubsection 2.2.6. A similar argument, with the roles of L and L_1 swapped, reduces the case of general L to that of minimal L: note that replacing L with a smaller splitting extension increases $\psi_{L/F}(l)/e(L/F)$, and hence preserves the relation $r \underline{\ll}_L l$.

To see the uniqueness assertion, setting $e = e(L/F) = e(L'/F')$ and recalling that $\mathrm{T}(F)/\mathrm{T}(F)_r^{\mathrm{naive}} \subset \mathrm{T}(L)/\mathrm{T}(L)_{er}^{\mathrm{naive}}$, use the identifications

$$\mathrm{T}(L)/\mathrm{T}(L)_{er}^{\mathrm{naive}} \xrightarrow{\cong} \mathrm{Hom}(X^*(\mathrm{T}), L^\times/(1+\mathfrak{p}_L^{\lceil er \rceil})) \xrightarrow{\cong} \mathrm{Hom}(X^*(\mathrm{T}'), L'^\times/(1+\mathfrak{p}_{L'}^{\lceil er \rceil})) \xrightarrow{\cong} \mathrm{T}'(L')/\mathrm{T}'(L')_{er}^{\mathrm{naive}},$$

where the first map is $t \mapsto (\chi \mapsto \chi(t))$, and note that t and t' are standard correspondents if and only if they define the same element of this identified object. $\square$

Remark 3.2.2. Assume the setting of the above lemma.

(i) If T/F is split, a unique standard isomorphism $\mathrm{T}(F)/\mathrm{T}(F)_r^{\mathrm{naive}} \to \mathrm{T}'(F')/\mathrm{T}'(F')_r^{\mathrm{naive}}$ exists:

$$\mathrm{T}(F)/\mathrm{T}(F)_r^{\mathrm{naive}} \to \mathrm{Hom}(X^*(\mathrm{T}), F^\times/1+\mathfrak{p}_F^{\lceil r\rceil}) = \mathrm{Hom}(X^*(\mathrm{T}'), F'^\times/1+\mathfrak{p}_{F'}^{\lceil r\rceil}) \to \mathrm{T}'(F')/\mathrm{T}'(F')_r^{\mathrm{naive}}.$$

(ii) For general T, there is either a unique standard isomorphism

$$\mathrm{T}(F)/\mathrm{T}(F)_r^{\mathrm{naive}} \to \mathrm{T}'(F')/\mathrm{T}'(F')_r^{\mathrm{naive}},$$

or none at all. Indeed, choose compatible embeddings $L \hookrightarrow F^{\mathrm{sep}}$ and $L' \hookrightarrow F'^{\mathrm{sep}}$, with L/F a splitting extension for T such that $r \ll_L l$. Set $e = e(L/F) = e(L'/F')$. If the standard isomorphism $\mathrm{T}(L)/\mathrm{T}(L)_{er}^{\mathrm{naive}} \to \mathrm{T}'(L')/\mathrm{T}'(L')_{er}^{\mathrm{naive}}$ associated to $(L, \mathrm{T}_L) \leftrightarrow_{\psi_{L/F}(l)} (L', \mathrm{T}'_{L'})$ (made sense of using Notation 2.3.1(ix)) restricts to an isomorphism $\mathrm{T}(F)/\mathrm{T}(F)_r^{\mathrm{naive}} \to \mathrm{T}'(F')/\mathrm{T}'(F')_r^{\mathrm{naive}}$, then this restriction defines a standard isomorphism. If not, there is no standard isomorphism $\mathrm{T}(F)/\mathrm{T}(F)_r^{\mathrm{naive}} \to \mathrm{T}'(F')/\mathrm{T}'(F')_r^{\mathrm{naive}}$.

Lemma 3.2.3. *Standard isomorphisms have the following functoriality. Let $(F, \mathrm{T}_i) \leftrightarrow_l (F', \mathrm{T}'_i)$ for $i = 1, 2$, with the same underlying $F \leftrightarrow_l F'$, and let $r > 0$. Let $f : \mathrm{T}_1 \to \mathrm{T}_2$ and $f' : \mathrm{T}'_1 \to \mathrm{T}'_2$ be homomorphisms inducing the same homomorphism $X^*(\mathrm{T}'_2) = X^*(\mathrm{T}_2) \to X^*(\mathrm{T}_1) = X^*(\mathrm{T}'_1)$ at the level of character lattices. Then, if standard isomorphisms $\mathrm{T}_i(F)/\mathrm{T}_i(F)_r^{\mathrm{naive}} \to \mathrm{T}'_i(F')/\mathrm{T}'_i(F')_r^{\mathrm{naive}}$ exist for $i = 1, 2$, they are the vertical arrows of the following commutative diagram:*

$$\begin{tikzcd} \mathrm{T}_1(F)/\mathrm{T}_1(F)_r^{\mathrm{naive}} \arrow[r, "f"] \arrow[d] & \mathrm{T}_2(F)/\mathrm{T}_2(F)_r^{\mathrm{naive}} \arrow[d] \\ \mathrm{T}'_1(F')/\mathrm{T}'_1(F')_r^{\mathrm{naive}} \arrow[r, "f'"] & \mathrm{T}'_2(F')/\mathrm{T}'_2(F')_r^{\mathrm{naive}} \end{tikzcd}.$$

Remark 3.2.4. The lemma would be immediate if we had $r \ll_L l$ for some L/F splitting both T_1 and T_2. We are not making this assumption, hence the longer proof.

Proof of Lemma 3.2.3. The assertion of the lemma is equivalent to the following statement: if $t_1 \in \mathrm{T}_1(F)$ is a standard correspondent of $t'_1 \in \mathrm{T}'_1(F')$, then $t_2 := f(t_1) \in \mathrm{T}_2(F)$ is a standard correspondent of $t'_2 := f'(t'_1) \in \mathrm{T}'_2(F')$.

First, we consider a slightly different situation. We make the stronger assumption that there exists a finite separable extension L/F, splitting both T_1 and T_2, such that $r \ll_L l$. However, we do not impose the assumption that either of the standard isomorphisms is well-defined. Choose compatible embeddings $L \hookrightarrow F^{\mathrm{sep}}$ and $L' \hookrightarrow F'^{\mathrm{sep}}$.

Under these assumptions, the following claim is formal: if $t_1 \in \mathrm{T}_1(F)$ is a standard correspondent of $t'_1 \in \mathrm{T}'_1(F')$, then $f(t_1) \in \mathrm{T}_2(F)$ is a standard correspondent of $f'(t'_1) \in \mathrm{T}'_2(F')$.

Now consider the general case. Let $\mathrm{T}_3 \subset \mathrm{T}_2$ be the image of f, and $\mathrm{T}'_3 \subset \mathrm{T}'_2$ that of f'. Thus, $X^*(\mathrm{T}_3) \subset X^*(\mathrm{T}_1)$ is the image of $X^*(\mathrm{T}_2) \to X^*(\mathrm{T}_1)$, and similarly with $X^*(\mathrm{T}'_3)$. We have an obvious realization $(F, \mathrm{T}_3) \leftrightarrow_l (F', \mathrm{T}'_3)$.

We have $r \ll_{\mathrm{T}_3} l$, since any splitting field of either of T_1 or T_2 also splits T_3. However, we cannot assume that there exists a standard isomorphism $\mathrm{T}_3(F)/\mathrm{T}_3(F)_r^{\mathrm{naive}} \to \mathrm{T}'_3(F')/\mathrm{T}'_3(F')_r^{\mathrm{naive}}$.

Nevertheless, if $t_1 \in \mathrm{T}_1(F)$ is a standard correspondent of $t'_1 \in \mathrm{T}'_1(F')$, then letting t_2, t_3 be the images of t_1 in $\mathrm{T}_2(F)$ and $\mathrm{T}_3(F)$, and t'_2 and t'_3 those of t'_1 in $\mathrm{T}'_2(F')$ and $\mathrm{T}'_3(F')$:

- t_3 is a standard correspondent of t'_3: apply the above claim with $\mathrm{T}_1 \to \mathrm{T}_2$ and $\mathrm{T}'_1 \to \mathrm{T}'_2$ replaced by $\mathrm{T}_1 \to \mathrm{T}_3$ and $\mathrm{T}'_1 \to \mathrm{T}'_3$; and
- hence t_2 is a standard correspondent of t'_2: apply the above claim with $\mathrm{T}_1 \to \mathrm{T}_2$ and $\mathrm{T}'_1 \to \mathrm{T}'_2$ replaced by $\mathrm{T}_3 \to \mathrm{T}_2$ and $\mathrm{T}'_3 \to \mathrm{T}'_2$.

As observed earlier, this implies the lemma. □

Lemma 3.2.5. *Let $(F, \mathrm{T}) \leftrightarrow_l (F', \mathrm{T}')$, and let $r > 0$. Assume that there is a standard isomorphism $\mathrm{T}(F)/\mathrm{T}(F)_r^{\mathrm{naive}} \to \mathrm{T}'(F')/\mathrm{T}'(F')_r^{\mathrm{naive}}$. Then for all $0 < s \le r$, this standard isomorphism induces a standard isomorphism $\mathrm{T}(F)/\mathrm{T}(F)_s^{\mathrm{naive}} \to \mathrm{T}'(F')/\mathrm{T}'(F')_s^{\mathrm{naive}}$, which further restricts to a "restricted standard isomorphism" $\mathrm{T}(F)_b/\mathrm{T}(F)_s^{\mathrm{naive}} \to \mathrm{T}'(F')_b/\mathrm{T}'(F')_s^{\mathrm{naive}}$, uniquely characterized by the fact that it sends each element of its source to a standard correspondent of it.*

Proof. Easy, using the following two facts. First, whenever $L \leftrightarrow_{l_1} L'$, the resulting isomorphism $L^\times/(1+\mathfrak{p}_L^{l_1}) \to L'^\times/(1+\mathfrak{p}_{L'}^{l_1})$ induces, for all $0 < s \le l_1$, an isomorphism $L^\times/(1+\mathfrak{p}_L^{\lceil s\rceil}) \to L'^\times/(1+\mathfrak{p}_{L'}^{\lceil s\rceil})$. Secondly, given $t \in \mathrm{T}(F)$ and a finitely ramified separable extension $L \hookrightarrow F^{\mathrm{sep}}$ that splits T, we have $t \in \mathrm{T}(F)_b$ if and only if for all $\chi \in X^*(\mathrm{T})$, the element $\chi(t)$ of $L^\times$ belongs to $\mathfrak{O}_L^\times$. □

Lemma 3.2.6. *Standard isomorphisms also have the following functoriality. Let $(F,\mathrm{T}) \leftrightarrow_l (F',\mathrm{T}')$. For $i = 1,2$, let $E_i \hookrightarrow F^{\mathrm{sep}}$ and $E'_i \hookrightarrow F'^{\mathrm{sep}}$ be compatible embeddings, and assume that there is a factorization $E_1 \hookrightarrow F^{\mathrm{sep}} = (E_2 \hookrightarrow F^{\mathrm{sep}}) \circ (E_1 \hookrightarrow E_2)$, giving an analogous factorization $E'_1 \hookrightarrow F'^{\mathrm{sep}} = (E'_2 \hookrightarrow F'^{\mathrm{sep}}) \circ (E'_1 \hookrightarrow E'_2)$. Then, if for $i = 1,2$, standard isomorphisms $\mathrm{T}(E_i)/\mathrm{T}(E_i)_{r_i}^{\mathrm{naive}} \to \mathrm{T}'(E'_i)/\mathrm{T}'(E'_i)_{r_i}^{\mathrm{naive}}$ exist (as before, using Notation 2.3.1*(ix) *to make sense of the $(E_i,\mathrm{T}_{E_i}) \leftrightarrow_{\psi_{E_i/F}(l)} (E'_i,\mathrm{T}'_{E'_i})$), and $r_2 \le e(E_2/E_1)r_1$, then these are the vertical arrows of the following commutative diagram, whose horizontal arrows are induced by $\mathrm{T}(E_1) \to \mathrm{T}(E_2)$ and $\mathrm{T}'(E'_1) \to \mathrm{T}'(E'_2)$:*

$$\begin{tikzcd}
\mathrm{T}(E_1)/\mathrm{T}(E_1)_{r_1}^{\mathrm{naive}} \arrow[r] \arrow[d] & \mathrm{T}(E_2)/\mathrm{T}(E_2)_{r_2}^{\mathrm{naive}} \arrow[d] \\
\mathrm{T}'(E'_1)/\mathrm{T}'(E'_1)_{r_1}^{\mathrm{naive}} \arrow[r] & \mathrm{T}'(E'_2)/\mathrm{T}'(E'_2)_{r_2}^{\mathrm{naive}}
\end{tikzcd}.$$

Note that we do not assume the existence of any standard isomorphism

$$\mathrm{T}(F)/\mathrm{T}(F)_r^{\mathrm{naive}} \to \mathrm{T}'(F')/\mathrm{T}'(F')_r^{\mathrm{naive}}.$$

Proof. Since $r_2 \le e(E_2/E_1)r_1$, the horizontal arrows are well-defined. By Lemma 3.2.5, we may decrease r_1 if necessary, to assume that $r_2 = e(E_2/E_1)r_1$.

Choose compatible embeddings $L \hookrightarrow F^{\mathrm{sep}}$ and $L' \hookrightarrow F'^{\mathrm{sep}}$ for the realization $E_2 \leftrightarrow_{l_2} E'_2$ (see Subsubsection 2.2.5), splitting T, and such that $r_2 \underline{\ll}_{\mathrm{T}_{E_2}} l_2$. Thus, we have

$$r_1 = r_2/e(E_2/E_1) \le \psi_{L/E_2}(l_2)/e(L/E_1) = \psi_{L/E_2}(\psi_{E_2/F}(l))/e(L/E_1) = \psi_{L/E_1}(l_1)/e(L/E_1).$$

Thus, we also have $r_1 \underline{\ll}_{\mathrm{T}_{E_1}} l_1$. Note that $L \hookrightarrow F^{\mathrm{sep}}$ and $L' \hookrightarrow F'^{\mathrm{sep}}$ are also compatible embeddings for $F \leftrightarrow_l F'$ (since their stabilizers in $\Gamma_F/I_F^l = \Gamma_{F'}/I_{F'}^l$ are contained in $\Gamma_{E_2}/I_{E_2}^{l_2} = \Gamma_{E'_2}/I_{E'_2}^{l_2}$), and hence also for $E_1 \leftrightarrow_{l_1} E'_1$. All these descriptions give the same realization $L \leftrightarrow_{l^\circ} L'$, where $l^\circ = \psi_{L/E_2}(l_2) = \psi_{L/E_1}(l_1)$ (see Remark 2.2.3).

Thus, by Remark 3.2.2, both the vertical arrows are obtained by restriction from the standard isomorphism $\mathrm{T}(L)/\mathrm{T}(L)_{e(L/E_i)r_i}^{\mathrm{naive}} \to \mathrm{T}'(L')/\mathrm{T}'(L')_{e(L/E_i)r_i}^{\mathrm{naive}}$ (with $e(L/E_i)r_i$ independent of i), and the lemma follows. □

Corollary 3.2.7. *Let $(F,\mathrm{T}) \leftrightarrow_l (F',\mathrm{T}')$. Let $E \hookrightarrow F^{\mathrm{sep}}$ and $E' \hookrightarrow F'^{\mathrm{sep}}$ be compatible embeddings, and let $r > 0$. Assume that E/F is Galois (and hence so is E'/F'), and that a standard isomorphism $\mathrm{T}(E)/\mathrm{T}(E)_r^{\mathrm{naive}} \to \mathrm{T}'(E')/\mathrm{T}'(E')_r^{\mathrm{naive}}$, associated to $(E,\mathrm{T}_E) \leftrightarrow_{\psi_{E/F}(l)} (E',\mathrm{T}'_{E'})$, exists. Then this isomorphism is equivariant for the action of $\Gamma_{E/F} = \Gamma_{E'/F'}$.*

Proof. This is a special case of Lemma 3.2.6. □

3.3. **Further properties of standard isomorphisms.** In this subsection, we prove less obvious properties of standard isomorphisms: their existence when F is strictly Henselian, and compatibility with Kottwitz homomorphisms and the local Langlands correspondence.

Proposition 3.3.1. *Let $(F,\mathrm{T}) \leftrightarrow_l (F',\mathrm{T}')$, and let $0 < r \underline{\ll}_{\mathrm{T}} l$. If further F is strictly Henselian (and hence so is F'), a standard isomorphism $\mathrm{T}(F)/\mathrm{T}(F)_r^{\mathrm{naive}} \to \mathrm{T}'(F')/\mathrm{T}'(F')_r^{\mathrm{naive}}$ exists.*

Proof. Choose a finite separable extension L/F, splitting T, such that $r \underline{\ll}_L l$. Without loss of generality, L/F is minimal such (making L smaller increases $\psi_{L/F}(l)/e(L/F)$), and hence Galois. Let $L \hookrightarrow F^{\mathrm{sep}}$ and $L' \hookrightarrow F'^{\mathrm{sep}}$ be compatible embeddings, so $L \leftrightarrow_{\psi_{L/F}(l)} L'$. Abbreviate $e := e(L/F)$.

We have a standard isomorphism $\mathrm{T}(L)/\mathrm{T}(L)_{er}^{\mathrm{naive}} \to \mathrm{T}'(L')/\mathrm{T}'(L')_{er}^{\mathrm{naive}}$, since L splits T and $er \le \psi_{L/F}(l)$. It suffices to show that this isomorphism restricts to an isomorphism $\mathrm{T}(F)/\mathrm{T}(F)_r^{\mathrm{naive}} \to \mathrm{T}'(F')/\mathrm{T}'(F')_r^{\mathrm{naive}}$.

The isomorphism $\mathrm{T}(L)/\mathrm{T}(L)^{\mathrm{naive}}_{er} \to \mathrm{T}'(L')/\mathrm{T}'(L')^{\mathrm{naive}}_{er}$ is equivariant for $\Gamma_{L/F} = \Gamma_{L'/F'}$, by Corollary 3.2.7, and hence for the action of the product $N_{L/F} = N_{L'/F'}$ of the elements of $\Gamma_{L/F} = \Gamma_{L'/F'}$. Therefore, it suffices to show that $N_{L/F} : \mathrm{T}(L) \to \mathrm{T}(F)$ and $N_{L'/F'} : \mathrm{T}'(L') \to \mathrm{T}'(F')$ are surjective, or equivalently, that the analogous maps $\mathrm{Res}_{L/F}\,\mathrm{T}_L \to \mathrm{T}$ and $\mathrm{Res}_{L'/F'}\,\mathrm{T}'_{L'} \to \mathrm{T}'$ are surjective respectively at the levels of F-rational points and F'-rational points. This is well-known: the kernel T_0 of $N_{L/F} : \mathrm{Res}_{L/F}\,\mathrm{T}_L \to \mathrm{T}$ is connected, and $H^1(F, \mathrm{T}_0) = 0$ by [KP23, Corollary 2.3.7 and Lemma 2.5.4], since F is strictly Henselian and κ_F is perfect. $\square$

Proposition 3.3.2. *Standard isomorphisms have the following compatibility with Kottwitz homomorphisms. Let $(F, \mathrm{T}) \leftrightarrow_l (F', \mathrm{T}')$, and assume that a standard isomorphism $\mathrm{T}(F)/\mathrm{T}(F)^{\mathrm{naive}}_r \to \mathrm{T}'(F')/\mathrm{T}'(F')^{\mathrm{naive}}_r$ exists. Then it is the left vertical arrow of the following commutative diagram, whose horizontal arrows are the relevant Kottwitz homomorphisms, and whose right vertical arrow is an isomorphism induced by the $\Gamma_F/I^l_F = \Gamma_{F'}/I^l_{F'}$-equivariant identification $X_*(\mathrm{T}) = X_*(\mathrm{T}')$.*

$$\begin{tikzcd}
\mathrm{T}(F)/\mathrm{T}(F)^{\mathrm{naive}}_r \arrow[r] \arrow[d] & (X_*(\mathrm{T})_{I_F})^{\Gamma_{\kappa_F}} \arrow[d] \\
\mathrm{T}'(F')/\mathrm{T}'(F')^{\mathrm{naive}}_r \arrow[r] & (X_*(\mathrm{T}')_{I_{F'}})^{\Gamma_{\kappa_{F'}}}
\end{tikzcd}$$

(subscripting with I_F or $I_{F'}$ stands for taking the group of I_F-coinvariants or $I_{F'}$-coinvariants).

Proof. To make sense of the right vertical arrow, use that the identification $\Gamma_F/I^l_F = \Gamma_{F'}/I^l_{F'}$ restricts to an identification $I_F/I^l_F = I_{F'}/I^l_{F'}$ and induces an identification $\Gamma_{\kappa_F} = \Gamma_{\kappa_{F'}}$.

If T is split, the lemma follows from Remark 3.2.2(i) and the following factorization of the Kottwitz homomorphism:

$$\mathrm{T}(F) = \mathrm{Hom}(X^*(\mathrm{T}), F^\times) \overset{\mathrm{val}}{\to} \mathrm{Hom}(X^*(\mathrm{T}), \mathbb{Z}) = X_*(\mathrm{T}).$$

Choose compatible embeddings $\tilde{F} \hookrightarrow F^{\mathrm{sep}}$ and $\tilde{F}' \hookrightarrow F'^{\mathrm{sep}}$, with $\tilde{F}/F$ a maximal unramified extension. By Proposition 3.3.1, we have a standard isomorphism

$$\mathrm{T}(\tilde{F})/\mathrm{T}(\tilde{F})^{\mathrm{naive}}_r \to \mathrm{T}'(\tilde{F}')/\mathrm{T}'(\tilde{F}')^{\mathrm{naive}}_r$$

associated to $(\tilde{F}, \mathrm{T}_{\tilde{F}}) \leftrightarrow_l (\tilde{F}', \mathrm{T}'_{\tilde{F}'})$, which by Lemma 3.2.6 restricts to the standard isomorphism $\mathrm{T}(F)/\mathrm{T}(F)^{\mathrm{naive}}_r \to \mathrm{T}'(F')/\mathrm{T}'(F')^{\mathrm{naive}}_r$. Since the Kottwitz homomorphism too is defined by restricting from the maximal unramified extension, we may now replace $(F, \mathrm{T}) \leftrightarrow_l (F', \mathrm{T}')$ by $(\tilde{F}, \mathrm{T}_{\tilde{F}}) \leftrightarrow_l (\tilde{F}', \mathrm{T}'_{\tilde{F}'})$, and assume that F is strictly Henselian.

Now we are in the setting of Proposition 3.3.1. Let $L \hookrightarrow F^{\mathrm{sep}}$ and $L' \hookrightarrow F'^{\mathrm{sep}}$ be as in the proof of that proposition, and set $e = e(L/F)$. We have a diagram

$$\begin{tikzcd}
 & \mathrm{T}(L)/\mathrm{T}(L)^{\mathrm{naive}}_{er} \arrow[dl, "N_{L/F}"'] \arrow[rr] \arrow[dd, dotted] & & \mathrm{T}'(L')/\mathrm{T}'(L')^{\mathrm{naive}}_{er}, \arrow[dl, "N_{L'/F'}"'] \arrow[dd] \\
\mathrm{T}(F)/\mathrm{T}(F)^{\mathrm{naive}}_r \arrow[rr] \arrow[dd] & & \mathrm{T}'(F')/\mathrm{T}'(F')^{\mathrm{naive}}_r \arrow[dd] & \\
 & X_*(\mathrm{T}_L) \arrow[rr, dotted] \arrow[dl, dotted] & & X_*(\mathrm{T}'_{L'}) \arrow[dl] \\
X_*(\mathrm{T})_{I_F} \arrow[rr] & & X_*(\mathrm{T}')_{I_{F'}} &
\end{tikzcd}$$

whose 'top face' is given by the proof of Proposition 3.3.1, vertical arrows are the appropriate Kottwitz homomorphisms, and the 'bottom' face consists of obvious maps.

The proof of Proposition 3.3.1 also gives the commutativity of the 'top face'. The two 'side faces' are commutative by [KP23, Lemma 11.1.4]. The 'hind face' (the four terms involving L) is commutative since the split case of the lemma is known. The 'bottom face' is clearly commutative.

Since $N_{L/F}$ is surjective, as we saw in the proof of Proposition 3.3.1, it is now easy to see that the 'front face' is commutative as well, which is exactly the commutative diagram the lemma seeks to prove. □

Proposition 3.3.3. *Standard isomorphisms have the following compatibility with the* LLC. *Let* $(F, \mathrm{T}) \leftrightarrow_l (F', \mathrm{T}')$, *and assume that a standard isomorphism* $\mathrm{T}(F)/\mathrm{T}(F)_r^{\mathrm{naive}} \to \mathrm{T}'(F')/\mathrm{T}'(F')_r^{\mathrm{naive}}$ *exists (in particular,* $r \underline{\ll}_{\mathrm{T}} l$*). Assume further that* F *and* F' *are complete with finite residue field. Then we have the following commutative diagram analogous to* (2)*, whose left vertical arrow is induced by the given isomorphism, and the right vertical arrow is induced by the isomorphism* $\hat{\mathrm{T}} = X^*(\mathrm{T}) \otimes \mathbb{C}^\times = X^*(\mathrm{T}') \otimes \mathbb{C}^\times = \hat{\mathrm{T}}'$ *of modules over* $W_F/I_F^l = W_{F'}/I_{F'}^l$:

(12)
$$\begin{tikzcd} \mathrm{Hom}(\mathrm{T}(F)/\mathrm{T}(F)_r^{\mathrm{naive}}, \mathbb{C}^\times) \arrow[r, hook, "\mathrm{LLC}"] \arrow[d] & H^1(W_F/I_F^l, \hat{\mathrm{T}}) \arrow[d, "\cong"', "\mathrm{Del}_l"] \\ \mathrm{Hom}(\mathrm{T}'(F')/\mathrm{T}'(F')_r^{\mathrm{naive}}, \mathbb{C}^\times) \arrow[r, hook, "\mathrm{LLC}"] & H^1(W_{F'}/I_{F'}^l, \hat{\mathrm{T}}') \end{tikzcd} .$$

Proof. First suppose $\mathrm{T} = \mathbb{G}_m/F$, and $\mathrm{T}' = \mathbb{G}_m/F'$ compatibly. In this case, the lemma is easy to see: briefly, the local class field theory map $W_F \to F^\times$ sends I_F^l to $1 + \mathfrak{p}_F^l$, making the horizontal arrows well-defined (as $r \le l$), and the commutativity of the square follows from the analogous statement for local class field theory, proved by Deligne (see [Del84, Proposition 3.6.1]). From this, the case where T is split follows, so we will consider the split case as known.

Let L/F be a finite (say minimal, and hence) Galois extension splitting T, with $r \underline{\ll}_L l$. Let χ: $\mathrm{T}(F)/\mathrm{T}(F)_r^{\mathrm{naive}} \to \mathbb{C}^\times$ have image χ': $\mathrm{T}'(F')/\mathrm{T}'(F')_r^{\mathrm{naive}} \to \mathbb{C}^\times$ under the left vertical arrow.

Write $e = e(L/F)$. Since $\mathrm{T}(F)/\mathrm{T}(F)_r^{\mathrm{naive}} \hookrightarrow \mathrm{T}(L)/\mathrm{T}(L)_{er}^{\mathrm{naive}}$, we can extend χ to a homomorphism χ_1: $\mathrm{T}(L)/\mathrm{T}(L)_{er}^{\mathrm{naive}} \to \mathbb{C}^\times$. It is clear that the homomorphism $\chi_1' : \mathrm{T}'(L')/\mathrm{T}'(L')_{er}^{\mathrm{naive}} \to \mathbb{C}^\times$ obtained by transferring χ_1 under the standard isomorphism $\mathrm{T}(L)/\mathrm{T}(L)_{er}^{\mathrm{naive}} \to \mathrm{T}'(L')/\mathrm{T}'(L')_{er}^{\mathrm{naive}}$ (which exists since L splits T and $r \underline{\ll}_L l$) has χ' as its restriction to $\mathrm{T}'(F')/\mathrm{T}'(F')_r^{\mathrm{naive}} \subset \mathrm{T}'(L')/\mathrm{T}'(L')_{er}^{\mathrm{naive}}$ (see Remark 3.2.2).

Let $\varphi_\chi \in H^1(W_F, \hat{\mathrm{T}}), \varphi_{\chi'} \in H^1(W_{F'}, \hat{\mathrm{T}}'), \varphi_{\chi_1} \in H^1(W_L, \hat{\mathrm{T}}_L)$, $\varphi_{\chi_1'} \in H^1(W_{L'}, \hat{\mathrm{T}}'_{L'})$ be the local Langlands parameters of $\chi, \chi', \chi_1, \chi_1'$. Write $l_1 = \psi_{L/F}(l) = \psi_{L'/F'}(l)$.

Since T_L is split, and since $er \le \psi_{L/F}(l) = l_1$, it follows from the split case (discussed at the beginning of this proof) that $\varphi_{\chi_1} \in H^1(W_L/I_L^{l_1}, \hat{\mathrm{T}}) \subset H^1(W_L, \hat{\mathrm{T}})$ (the inflation map), that $\varphi_{\chi_1'} \in H^1(W_{L'}/I_{L'}^{l_1}) \subset H^1(W_{L'}, \hat{\mathrm{T}}')$, and that $\varphi_{\chi_1'}$ is the image of φ_{χ_1} under the obvious isomorphism $H^1(W_L/I_L^{l_1}, \hat{\mathrm{T}}) \to H^1(W_{L'}/I_{L'}^{l_1}, \hat{\mathrm{T}}')$.

We will show that φ_χ is the image of φ_{χ_1} under the corestriction map $H^1(W_L, \hat{\mathrm{T}}) \to H^1(W_F, \hat{\mathrm{T}})$: since L/F is Galois, this follows from the construction of the local Langlands correspondence for tori in [Yu09, Section 7.7] (see especially the definition of φ_T in (c) there). This corestriction map is a composite $H^1(W_L, \hat{\mathrm{T}}) \to H^1(W_F, \mathrm{Ind}_{W_L}^{W_F} \hat{\mathrm{T}}) \to H^1(W_F, \hat{\mathrm{T}})$, where the first map is the isomorphism given by Shapiro's lemma, and the second is induced by an appropriate surjection $\mathrm{Ind}_{W_L}^{W_F} \hat{\mathrm{T}} \to \hat{\mathrm{T}}$, involving a certain sum over representatives for W_F/W_L: see [Ser02, towards the end of Section 2.5] (though this reference treats profinite groups, the same applies in our context; $W_L \subset W_F$ is of finite index). Restricted to $H^1(W_L/I_L^{l_1}, \hat{\mathrm{T}}) \subset H^1(W_L, \hat{\mathrm{T}})$, this map is a similarly defined composite $H^1(W_L/I_L^{l_1}, \hat{\mathrm{T}}) \to H^1(W_F/I_F^l, \mathrm{Ind}_{W_L/I_L^{l_1}}^{W_F/I_F^l} \hat{\mathrm{T}}) \to H^1(W_F/I_F^l, \hat{\mathrm{T}})$, where this time one uses, as one clearly may, the Shapiro's lemma isomorphism $H^1(W_L/I_L^{l_1}, \hat{\mathrm{T}}) \to H^1(W_F/I_F^l, \mathrm{Ind}_{W_L/I_L^{l_1}}^{W_F/I_F^l} \hat{\mathrm{T}})$, and a sum over representatives for $(W_F/I_F^l)/(W_L/I_L^{l_1})$. Similarly, $\varphi_{\chi'}$ is the image of $\varphi_{\chi_1'}$ under a composite $H^1(W_{L'}/I_{L'}^{l_1}, \hat{\mathrm{T}}') \to H^1(W_{F'}/I_{F'}^l, \mathrm{Ind}_{W_{L'}/I_{L'}^{l_1}}^{W_{F'}/I_{F'}^l} \hat{\mathrm{T}}') \to H^1(W_{F'}/I_{F'}^l, \hat{\mathrm{T}}')$.

Now, using the identification $W_F/I_F^l = W_{F'}/I_{F'}^l \supset W_{L'}/I_{L'}^{l_1} = W_L/I_L^{l_1}$, the identification $\hat{\mathrm{T}} = \hat{\mathrm{T}}'$ as modules over $W_F/I_F^l = W_{F'}/I_{F'}^l$, and also using the observation above relating φ_{χ_1} and $\varphi_{\chi_1'}$, we conclude that φ_χ is indeed the image of $\varphi_{\chi'}$ under the isomorphism $H^1(W_F/I_F^l, \hat{\mathrm{T}}) \to H^1(W_{F'}/I_{F'}^l, \hat{\mathrm{T}}')$, finishing the proof of the proposition. □

3.4. **Properties of congruent and Chai-Yu isomorphisms.**

Lemma 3.4.1. *Let $(F, \mathrm{T}) \leftrightarrow_l (F', \mathrm{T}')$, and let m be a positive integer.*

(i) If an isomorphism $\mathrm{T}(F)/\mathrm{T}(F)_m \to \mathrm{T}'(F')/\mathrm{T}'(F')_m$ satisfies the conditions of Definition 3.1.3(ii) *with respect to some choice of $\tilde{F} \hookrightarrow F^{\mathrm{sep}}, \tilde{F}' \hookrightarrow F'^{\mathrm{sep}}$ and $r > 0$, then so does it with respect to any other such choice (as in the definition). Thus, there is either a unique congruent isomorphism $\mathrm{T}(F)/\mathrm{T}(F)_m \to \mathrm{T}'(F')/\mathrm{T}'(F')_m$, or none at all. A similar assertion applies to Chai-Yu isomorphisms (Definition 3.1.3*(iii)*).*

(ii) Congruent isomorphisms have the following functoriality. Let $(F, \mathrm{T}_i) \leftrightarrow_l (F', \mathrm{T}'_i)$ for $i = 1, 2$, with the same underlying $F \leftrightarrow_l F'$, and let m be a positive integer. Let $f : \mathrm{T}_1 \to \mathrm{T}_2$ and $f' : \mathrm{T}'_1 \to \mathrm{T}'_2$ be homomorphisms inducing the same homomorphism $X^*(\mathrm{T}'_2) = X^*(\mathrm{T}_2) \to X^*(\mathrm{T}_1) = X^*(\mathrm{T}'_1)$ at the level of character lattices. Assume that congruent isomorphisms $c_i : \mathrm{T}_i(F)/\mathrm{T}_i(F)_m \to \mathrm{T}'_i(F')/\mathrm{T}'_i(F')_m$ exist for $i = 1, 2$, and form the following diagram:

$$\begin{array}{ccc} \mathrm{T}_1(F)/\mathrm{T}_1(F)_m & \xrightarrow{f} & \mathrm{T}_2(F)/\mathrm{T}_2(F)_m \\ {\scriptstyle c_1}\downarrow & & \downarrow{\scriptstyle c_2} \\ \mathrm{T}'_1(F')/\mathrm{T}'_1(F')_m & \xrightarrow{f'} & \mathrm{T}'_2(F')/\mathrm{T}'_2(F')_m \end{array} .$$

Then this diagram is commutative under the following additional assumption: "the same r can be used to define both the congruent isomorphisms", i.e., there exists $r > 0$ such that for $i = 1, 2$, we have $r \ll_{\mathrm{T}_i} l$, $\mathrm{T}_i(\tilde{F})_r^{\mathrm{naive}} \subset \mathrm{T}_i(\tilde{F})_m$ and $\mathrm{T}'_i(\tilde{F}')_r^{\mathrm{naive}} \subset \mathrm{T}'_i(\tilde{F}')_m$, for some choice of compatible embeddings $\tilde{F} \hookrightarrow F^{\mathrm{sep}}$ and $\tilde{F}' \hookrightarrow F'^{\mathrm{sep}}$, where $\tilde{F}/F$ is a maximal unramified extension.

(iii) Recall the following necessary condition for the existence of a congruent isomorphism

$$\mathrm{T}(F)/\mathrm{T}(F)_m \to \mathrm{T}'(F')/\mathrm{T}'(F')_m :$$

for some compatible embeddings $\tilde{F} \hookrightarrow F^{\mathrm{sep}}$ and $\tilde{F}' \hookrightarrow F'^{\mathrm{sep}}$, where $\tilde{F}/F$ is a maximal unramified extension, and some $r \ll_{\mathrm{T}} l$ such that $\mathrm{T}(\tilde{F})_r^{\mathrm{naive}} \subset \mathrm{T}(\tilde{F})_m$ and $\mathrm{T}'(\tilde{F}')_r^{\mathrm{naive}} \subset \mathrm{T}'(\tilde{F}')_m$, a standard isomorphism $\mathrm{T}(\tilde{F})/\mathrm{T}(\tilde{F})_r^{\mathrm{naive}} \to \mathrm{T}'(\tilde{F}')/\mathrm{T}'(\tilde{F}')_r^{\mathrm{naive}}$ induces an isomorphism $\mathrm{T}(\tilde{F})/\mathrm{T}(\tilde{F})_m \to \mathrm{T}'(\tilde{F}')/\mathrm{T}'(\tilde{F}')_m$. This condition is also sufficient.

(iv) Suppose there exists a congruent isomorphism $\mathrm{T}(F)/\mathrm{T}(F)_m \to \mathrm{T}'(F')/\mathrm{T}'(F')_m$. Then for all $0 < s \leq m$, it induces a standard isomorphism $\mathrm{T}(F)/\mathrm{T}(F)_s^{\mathrm{naive}} \to \mathrm{T}'(F')/\mathrm{T}'(F')_s^{\mathrm{naive}}$.

Proof. To see (i), combine Lemma 3.2.6 (to see the non-dependence on $\tilde{F} \hookrightarrow F^{\mathrm{sep}}$ and $\tilde{F}' \hookrightarrow F'^{\mathrm{sep}}$) with Lemma 3.2.5 (to see the non-dependence on r). For the assertion concerning Chai-Yu isomorphisms, one also uses [CY01, Lemma 8.5.1].

Now we come to (ii). For $i = 1, 2$, the relation $r \ll_{\mathrm{T}_i} l$ easily implies $r \ll_{(\mathrm{T}_i)_{\tilde{F}}} l$, and hence Proposition 3.3.1 gives a standard isomorphism $\mathrm{T}_i(\tilde{F})/\mathrm{T}_i(\tilde{F})_r^{\mathrm{naive}} \to \mathrm{T}'_i(\tilde{F}')/\mathrm{T}'_i(\tilde{F}')_r^{\mathrm{naive}}$. Thus, (ii) follows from Lemma 3.2.3 and the fact that we can work with the given r (by (i)).

Now we come to (iii). It follows from Corollary 3.2.7 that the standard isomorphism

$$\mathrm{T}(\tilde{F})/\mathrm{T}(\tilde{F})_r^{\mathrm{naive}} \to \mathrm{T}'(\tilde{F}')/\mathrm{T}'(\tilde{F}')_r^{\mathrm{naive}},$$

and hence also the isomorphism $\mathrm{T}(\tilde{F})/\mathrm{T}(\tilde{F})_m \to \mathrm{T}'(\tilde{F}')/\mathrm{T}'(\tilde{F}')_m$, is invariant under $\Gamma_{\tilde{F}/F} = \Gamma_{\tilde{F}'/F'}$. Thus, it suffices to show that $(\mathrm{T}(\tilde{F})/\mathrm{T}(\tilde{F})_m)^{\Gamma_{\tilde{F}/F}} = \mathrm{T}(F)/\mathrm{T}(F)_m$ (then the analogous assertion for F' will be true as well). This in turn follows if we show that $H^1(\Gamma_{\tilde{F}/F}, \mathrm{T}(\tilde{F})_m) = 0$, which is a special case of [KP23, Proposition 13.8.1]. This gives (iii).

It remains to prove (iv). Choose compatible embeddings $\tilde{F} \hookrightarrow F^{\mathrm{sep}}$ and $\tilde{F}' \hookrightarrow F'^{\mathrm{sep}}$, where $\tilde{F}/F$ is a maximal unramified extension. Choose r such that a standard isomorphism $\mathrm{T}(\tilde{F})/\mathrm{T}(\tilde{F})_r^{\mathrm{naive}} \to \mathrm{T}'(\tilde{F}')/\mathrm{T}'(\tilde{F}')_r^{\mathrm{naive}}$ induces an isomorphism $\mathrm{T}(\tilde{F})/\mathrm{T}(\tilde{F})_m \to \mathrm{T}'(\tilde{F}')/\mathrm{T}'(\tilde{F}')_m$ that restricts to $\mathrm{T}(F)/\mathrm{T}(F)_m \to \mathrm{T}'(F')/\mathrm{T}'(F')_m$. Recall that $r \geq m$ (since $\mathrm{T}(\tilde{F})_r^{\mathrm{naive}} \subset \mathrm{T}(\tilde{F})_m \subset \mathrm{T}(\tilde{F})_m^{\mathrm{naive}}$). It suffices to show that, whenever $t \in \mathrm{T}(F)$ and $t' \in \mathrm{T}'(F')$ have images that match under $\mathrm{T}(F)/\mathrm{T}(F)_m \to \mathrm{T}'(F')/\mathrm{T}'(F')_m$, $t\mathrm{T}(F)_s^{\mathrm{naive}}$ and $t'\mathrm{T}'(F')_s^{\mathrm{naive}}$ are standard correspondents (for "level s"). An easy argument reduces this to showing that t and t' have images that match under the standard isomorphism $\mathrm{T}(\tilde{F})/\mathrm{T}(\tilde{F})_s^{\mathrm{naive}} \to \mathrm{T}'(\tilde{F}')/\mathrm{T}'(\tilde{F}')_s^{\mathrm{naive}}$ (which exists by Lemma 3.2.5 and the fact that $s \le m \le r$). Now we are done by Lemma 3.2.5, since there exist $t_\circ \in \mathrm{T}(\tilde{F})_m \subset \mathrm{T}(\tilde{F})_m^{\mathrm{naive}} \subset \mathrm{T}(\tilde{F})_s^{\mathrm{naive}}$, and similarly $t'_\circ \in \mathrm{T}'(\tilde{F}')_s^{\mathrm{naive}}$, such that $tt_\circ$ and $t't'_\circ$ have images that match under the standard isomorphism $\mathrm{T}(\tilde{F})/\mathrm{T}(\tilde{F})_r^{\mathrm{naive}} \to \mathrm{T}'(\tilde{F}')/\mathrm{T}'(\tilde{F}')_r^{\mathrm{naive}}$. ☐

Lemma 3.4.2. *Let $(F, \mathrm{T}) \leftrightarrow_l (F', \mathrm{T}')$, and let m be a positive integer. Then there exists associated to this data at most one Chai-Yu isomorphism $\mathcal{T}^{\mathrm{ft}} \times_{\mathfrak{O}_F} \mathfrak{O}_F/\mathfrak{p}_F^m \to \mathcal{T}'^{\mathrm{ft}} \times_{\mathfrak{O}_{F'}} \mathfrak{O}_{F'}/\mathfrak{p}_{F'}^m$.*

Proof. Combine the argument for congruent isomorphisms in Lemma 3.4.1(i) with the schematic density of the image of $\mathrm{T}(\tilde{F})_b = \mathcal{T}^{\mathrm{ft}}(\mathfrak{O}_{\tilde{F}})$ in $\mathcal{T}^{\mathrm{ft}} \times_{\mathfrak{O}_F} \mathfrak{O}_F/\mathfrak{p}_F^m$ ([CY01, Lemma 8.5.1]). ☐

Proposition 3.4.3. *Let $(F, \mathrm{T}) \leftrightarrow_l (F', \mathrm{T}')$, and let m be a positive integer. If there exists a Chai-Yu isomorphism $\mathcal{T}^{\mathrm{ft}} \times_{\mathfrak{O}_F} \mathfrak{O}_F/\mathfrak{p}_F^m \to \mathcal{T}'^{\mathrm{ft}} \times_{\mathfrak{O}_{F'}} \mathfrak{O}_{F'}/\mathfrak{p}_{F'}^m$, then there exists a congruent isomorphism $\mathrm{T}(F)/\mathrm{T}(F)_m \to \mathrm{T}'(F')/\mathrm{T}'(F')_m$. Moreover, this congruent isomorphism restricts to an isomorphism $\mathrm{T}(F)_b/\mathrm{T}(F)_m \to \mathrm{T}'(F')_b/\mathrm{T}'(F')_m$ obtained by evaluating the Chai-Yu isomorphism at $\mathfrak{O}_F/\mathfrak{p}_F^m = \mathfrak{O}_{F'}/\mathfrak{p}_{F'}^m$.*

Proof. Choose compatible embeddings $\tilde{F} \hookrightarrow F^{\mathrm{sep}}$ and $\tilde{F}' \hookrightarrow F'^{\mathrm{sep}}$, where $\tilde{F}/F$ is a maximal unramified extension. By the definition of a Chai-Yu isomorphism (Definition 3.1.3(iii)), for some $0 < r \lll_{\mathrm{T}} l$, there exists a "restricted standard isomorphism" $\mathrm{T}(\tilde{F})_b/\mathrm{T}(\tilde{F})_r^{\mathrm{naive}} \to \mathrm{T}'(\tilde{F}')_b/\mathrm{T}'(\tilde{F}')_r^{\mathrm{naive}}$ that induces the isomorphism $\mathrm{T}(\tilde{F})_b/\mathrm{T}(\tilde{F})_m \to \mathrm{T}'(\tilde{F}')_b/\mathrm{T}'(\tilde{F}')_m$ obtained by evaluating the given Chai-Yu isomorphism at $\mathfrak{O}_{\tilde{F}}/\mathfrak{p}_{\tilde{F}}^m = \mathfrak{O}_{\tilde{F}'}/\mathfrak{p}_{\tilde{F}'}^m$. There also exists a standard isomorphism $\mathrm{T}(\tilde{F})/\mathrm{T}(\tilde{F})_r^{\mathrm{naive}} \to \mathrm{T}'(\tilde{F}')/\mathrm{T}'(\tilde{F}')_r^{\mathrm{naive}}$ by Proposition 3.3.1, which restricts to the restricted standard isomorphism $\mathrm{T}(\tilde{F})_b/\mathrm{T}(\tilde{F})_r^{\mathrm{naive}} \to \mathrm{T}'(\tilde{F}')_b/\mathrm{T}'(\tilde{F}')_r^{\mathrm{naive}}$, by Lemma 3.2.5.

The restricted standard isomorphism induces an isomorphism $\mathrm{T}(\tilde{F})_b/\mathrm{T}(\tilde{F})_m \to \mathrm{T}'(\tilde{F}')_b/\mathrm{T}'(\tilde{F}')_m$ and hence takes the image of $\mathrm{T}(\tilde{F})_m$ to that of $\mathrm{T}'(\tilde{F}')_m$. Hence so does the standard isomorphism as well, which therefore induces an isomorphism $\mathrm{T}(\tilde{F})/\mathrm{T}(\tilde{F})_m \to \mathrm{T}'(\tilde{F}')/\mathrm{T}'(\tilde{F}')_m$. As in the proof of Lemma 3.4.1(iii), using that $H^1(\Gamma_{\tilde{F}/F}, \mathrm{T}(\tilde{F})_m) = 0 = H^1(\Gamma_{\tilde{F}'/F'}, \mathrm{T}'(\tilde{F}')_m)$, this isomorphism $\mathrm{T}(\tilde{F})/\mathrm{T}(\tilde{F})_m \to \mathrm{T}'(\tilde{F}')/\mathrm{T}'(\tilde{F}')_m$ restricts to an isomorphism $\mathrm{T}(F)/\mathrm{T}(F)_m \to \mathrm{T}'(F')/\mathrm{T}'(F')_m$, which is clearly a congruent isomorphism that satisfies the latter assertion of the lemma. ☐

Now we study the behavior of Chai-Yu isomorphisms with respect to minimal congruent filtrations.

Proposition 3.4.4. *Let $(F, \mathrm{T}) \leftrightarrow_l (F', \mathrm{T}')$, and let m be a positive integer. Fix compatible embeddings $\tilde{F} \hookrightarrow F^{\mathrm{sep}}$ and $\tilde{F}' \hookrightarrow F'^{\mathrm{sep}}$, where $\tilde{F}/F$ is a maximal unramified extension. Assume that, associated to $(F, \mathrm{T}) \leftrightarrow_l (F', \mathrm{T}')$, there exists a Chai-Yu isomorphism $\mathcal{T}^{\mathrm{ft}} \times_{\mathfrak{O}_F} \mathfrak{O}_F/\mathfrak{p}_F^{m+1} \to \mathcal{T}'^{\mathrm{ft}} \times_{\mathfrak{O}_{F'}} \mathfrak{O}_{F'}/\mathfrak{p}_{F'}^{m+1}$ ("one higher level"), say induced by some isomorphism $\mathrm{T}(\tilde{F})_b/\mathrm{T}(\tilde{F})_r^{\mathrm{naive}} \to \mathrm{T}'(\tilde{F}')_b/\mathrm{T}'(\tilde{F}')_r^{\mathrm{naive}}$ as in Definition 3.1.3*(iii) *(where $m+1 \le r \lll_{\mathrm{T}} l$). Then for all $0 \le s \le m$:*

- *(i) The isomorphism $\mathrm{T}(\tilde{F})_b/\mathrm{T}(\tilde{F})_m \to \mathrm{T}'(\tilde{F}')_b/\mathrm{T}'(\tilde{F}')_m$, obtained by evaluating the given Chai-Yu isomorphism at $\mathfrak{O}_{\tilde{F}}/\mathfrak{p}_{\tilde{F}}^m = \mathfrak{O}_{\tilde{F}'}/\mathfrak{p}_{\tilde{F}'}^m$, sends the image of $\mathrm{T}(\tilde{F})_s$ to that of $\mathrm{T}'(\tilde{F}')_s$.*
- *(ii) At the level of schemes, letting $\mathcal{T}_s$ and $\mathcal{T}'_s$ be the minimal congruent filtration group schemes associated to T and T' of level s, one has a unique isomorphism $\mathcal{T}_s \times_{\mathfrak{O}_F} \mathfrak{O}_F/\mathfrak{p}_F^{\lfloor m+1-s \rfloor} \to \mathcal{T}'_s \times_{\mathfrak{O}_{F'}} \mathfrak{O}_{F'}/\mathfrak{p}_{F'}^{\lfloor m+1-s \rfloor}$ of schemes over $\mathfrak{O}_F/\mathfrak{p}_F^{\lfloor m+1-s \rfloor} = \mathfrak{O}_{F'}/\mathfrak{p}_{F'}^{\lfloor m+1-s \rfloor}$, under which the images of $t \in \mathcal{T}_s(\mathfrak{O}_{\tilde{F}}) = \mathrm{T}(\tilde{F})_s$ and $t' \in \mathcal{T}'_s(\mathfrak{O}_{\tilde{F}'}) = \mathrm{T}'(\tilde{F}')_s$ correspond whenever the Chai-Yu isomorphism being considered sends the image of t in $\mathcal{T}^{\mathrm{ft}}(\mathfrak{O}_{\tilde{F}}/\mathfrak{p}_{\tilde{F}}^{m+1}) = \mathrm{T}(\tilde{F})_b/\mathrm{T}(\tilde{F})_{m+1}$ to that of t' in $\mathcal{T}'^{\mathrm{ft}}(\mathfrak{O}_{\tilde{F}'}/\mathfrak{p}_{\tilde{F}'}^{m+1}) = \mathrm{T}'(\tilde{F}')_b/\mathrm{T}'(\tilde{F}')_{m+1}$.*

Proof. We have $m+1 \lll_{\mathrm{T}} l$ (as is implicit in the existence of the given Chai-Yu isomorphism), i.e., $m+1 \lll_L l$ for some extension L/F splitting T. If $\tilde{L}$ is a compositum of L and $\tilde{F}$, we have $\psi_{\tilde{L}/F}(l) = \psi_{\tilde{L}/L} \circ \psi_{L/F}(l) = \psi_{L/F}(l)$, so $m+1 \lll_{\mathrm{T}_{\tilde{F}}} l$. Fix compatible embeddings $\tilde{L} \hookrightarrow F^{\mathrm{sep}}$ and $\tilde{L}' \hookrightarrow F'^{\mathrm{sep}}$, the former extending $\tilde{F} \hookrightarrow F^{\mathrm{sep}}$. Hence $\tilde{F}' \hookrightarrow F'^{\mathrm{sep}}$ factors through $\tilde{L}' \hookrightarrow F'^{\mathrm{sep}}$ as well. Tautologically, $\tilde{L} \hookrightarrow F^{\mathrm{sep}}$ and $\tilde{L}' \hookrightarrow F'^{\mathrm{sep}}$ are also compatible embeddings for $\tilde{F} \leftrightarrow_{\psi_{\tilde{F}/F}(l)=l} \tilde{F}'$.

Suppose that (i) and the existence assertion of (ii) are known. Then each t as in (ii) has a corresponding t' (by (i)) and vice versa (by symmetry); therefore the uniqueness assertion in (ii) follows from [CY01, Lemma 8.5.1].

The proposition being trivial for $s = 0$, our first aim is to prove just (i) for $0 < s < 1$. By the definitions of $\mathrm{T}(\tilde{F})_s$ and $\mathrm{T}'(\tilde{F}')_s$ (see [KP23, Definition B.10.8(2)], and the description involving dilatation in [KP23, the proof of Lemma B.10.9]), and the fact that $\mathrm{T}(\tilde{F})_b/\mathrm{T}(\tilde{F})_{m+1} \to \mathrm{T}'(\tilde{F}')_b/\mathrm{T}'(\tilde{F}')_{m+1}$ takes the image of $\mathrm{T}(\tilde{F})_1$ to that of $\mathrm{T}'(\tilde{F}')_1$ (because the Chai-Yu isomorphism is a morphism of schemes over $\mathfrak{O}_F/\mathfrak{p}_F^{m+1} = \mathfrak{O}_{F'}/\mathfrak{p}_{F'}^{m+1}$), it suffices to show that for any homomorphism $\mathrm{S} \to \mathrm{T}_{\tilde{F}}$ with S an induced torus over $\tilde{F}$, the isomorphism $\mathrm{T}(\tilde{F})_b/\mathrm{T}(\tilde{F})_{m+1} \to \mathrm{T}'(\tilde{F}')_b/\mathrm{T}'(\tilde{F}')_{m+1}$ takes the image of $\mathrm{S}(\tilde{F})_s$ in the source into that of $\mathrm{S}'(\tilde{F}')_s$ under some homomorphism from an induced torus S' over $\tilde{F}'$ to $\mathrm{T}'_{\tilde{F}'}$ (we thank Kaletha for informing us that in [KP23, Definition B.10.8(2)], R varies over induced K-tori; this is why we take S to be an induced torus over $\tilde{F}$ and not over F).

The map $\mathrm{S} \to \mathrm{T}_{\tilde{F}}$ factors through the maximal $\tilde{L}$-split ($\tilde{F}$-torus) quotient of S, since $X^*(\mathrm{T}) \to X^*(\mathrm{S})$ has image inside $X^*(\mathrm{S})^{\mathrm{Gal}(F^{\mathrm{sep}}/\tilde{L})}$. $\mathrm{Gal}(F^{\mathrm{sep}}/\tilde{F})$ permutes some basis for $X^*(\mathrm{S})$, and hence also the set of $\mathrm{Gal}(F^{\mathrm{sep}}/\tilde{L})$-orbits of elements of this basis, and hence also some basis of the character lattice $X^*(\mathrm{S})^{\mathrm{Gal}(F^{\mathrm{sep}}/\tilde{L})}$ of the maximal $\tilde{L}$-split quotient of S. Thus, the maximal $\tilde{L}$-split quotient of S is an induced torus as well, with which we may now replace S, to assume that S is $\tilde{L}$-split, and in particular at most l-ramified, and satisfying $m + 1 \ll_{\mathrm{S}} l$.

This gives us a torus S' over $\tilde{F}'$, which is clearly induced and splits over $\tilde{L}'$, and a homomorphism $\mathrm{S}' \to \mathrm{T}'_{\tilde{F}'}$ such that $X^*(\mathrm{T}') \to X^*(\mathrm{S}')$ identifies with the homomorphism $X^*(\mathrm{T}) \to X^*(\mathrm{S})$ dual to $\mathrm{S} \to \mathrm{T}_{\tilde{F}}$. Since S' is induced, (i) will follow if we show that $\mathrm{T}(\tilde{F})/\mathrm{T}(\tilde{F})_{m+1} \to \mathrm{T}'(\tilde{F}')/\mathrm{T}'(\tilde{F}')_{m+1}$ takes the image of $\mathrm{S}(\tilde{F})_s$ in the source to that of $\mathrm{S}'(\tilde{F}')_s$ in the target.

Proposition 3.3.1 shows that standard isomorphisms $\mathrm{S}(\tilde{F})/\mathrm{S}(\tilde{F})_r^{\mathrm{naive}} \to \mathrm{S}'(\tilde{F}')/\mathrm{S}'(\tilde{F}')_r^{\mathrm{naive}}$ and $\mathrm{T}(\tilde{F})/\mathrm{T}(\tilde{F})_r^{\mathrm{naive}} \to \mathrm{T}'(\tilde{F}')/\mathrm{T}'(\tilde{F}')_r^{\mathrm{naive}}$ exist, the latter clearly restricting to the isomorphism $\mathrm{T}(\tilde{F})_b/\mathrm{T}(\tilde{F})_r^{\mathrm{naive}} \to \mathrm{T}'(\tilde{F}')_b/\mathrm{T}'(\tilde{F}')_r^{\mathrm{naive}}$ in the statement of the proposition. Applying Lemma 3.2.3 in the context of the homomorphisms $\mathrm{S} \to \mathrm{T}_{\tilde{F}}$ and $\mathrm{S}' \to \mathrm{T}'_{\tilde{F}'}$, and using that the standard isomorphism $\mathrm{S}(\tilde{F})/\mathrm{S}(\tilde{F})_r^{\mathrm{naive}} \to \mathrm{S}'(\tilde{F}')/\mathrm{S}'(\tilde{F}')_r^{\mathrm{naive}}$ identifies the images of $\mathrm{S}(\tilde{F})_s$ and $\mathrm{S}'(\tilde{F}')_s$ (by Lemma 3.2.5), it follows that the images of $\mathrm{S}(\tilde{F})_s$ and $\mathrm{S}'(\tilde{F}')_s$ agree in $\mathrm{T}(\tilde{F})/\mathrm{T}(\tilde{F})_r^{\mathrm{naive}} = \mathrm{T}'(\tilde{F}')/\mathrm{T}'(\tilde{F}')_r^{\mathrm{naive}}$.

By the choice of r, the images of $\mathrm{S}(\tilde{F})_s$ and $\mathrm{S}'(\tilde{F}')_s$ in $\mathrm{T}(\tilde{F})/\mathrm{T}(\tilde{F})_{m+1}$ and $\mathrm{T}'(\tilde{F}')/\mathrm{T}'(\tilde{F}')_{m+1}$, respectively, match under the isomorphism $\mathrm{T}(\tilde{F})_b/\mathrm{T}(\tilde{F})_{m+1} \to \mathrm{T}'(\tilde{F}')_b/\mathrm{T}'(\tilde{F}')_{m+1}$. Thus, (i) follows for $0 < s < 1$, and hence for $0 \le s < 1$.

Now let us prove (ii) for $0 \le s < 1$; this is what necessitated needing a Chai-Yu isomorphism of level $m + 1$. The case of $s = 0$ is immediate: $\mathcal{T}^{\mathrm{ft}} \times_{\mathfrak{O}_F} \mathfrak{O}_F/\mathfrak{p}_F^{m+1} \to \mathcal{T}'^{\mathrm{ft}} \times_{\mathfrak{O}_{F'}} \mathfrak{O}_{F'}/\mathfrak{p}_{F'}^{m+1}$ restricts to an isomorphism $\mathcal{T} \times_{\mathfrak{O}_F} \mathfrak{O}_F/\mathfrak{p}_F^{m+1} \to \mathcal{T}' \times_{\mathfrak{O}_{F'}} \mathfrak{O}_{F'}/\mathfrak{p}_{F'}^{m+1}$, and we have $\mathcal{T} = \mathcal{T}_0$ and $\mathcal{T}' = \mathcal{T}'_0$. Hence we assume $s > 0$. Since (i) is known in this case, with m replaced by $m + 1$, $\mathcal{T}_s$ and $\mathcal{T}'_s$ are respectively the dilatations of $\mathcal{T} = \mathcal{T}_0$ and $\mathcal{T}' = \mathcal{T}'_0$ with respect to the same subgroup W_s of $\mathcal{T} \times_{\mathfrak{O}_F} \kappa_F = \mathcal{T}' \times_{\mathfrak{O}_{\tilde{F}'}} \kappa_{F'}$ (identified using the Chai-Yu isomorphism). Now the required isomorphism $\mathcal{T}_s \times_{\mathfrak{O}_F} \mathfrak{O}_F/\mathfrak{p}_F^m \to \mathcal{T}'_s \times_{\mathfrak{O}_{F'}} \mathfrak{O}_{F'}/\mathfrak{p}_{F'}^m$, described as in (ii), follows from Proposition 2.5.4, which applies since this subgroup is reduced and hence smooth over $\kappa_F = \kappa_{F'}$, and since $\mathcal{T}$ (over $\mathfrak{O}_F$) and $\mathcal{T}'$ (over $\mathfrak{O}_{F'}$) are smooth; note that $m = \lfloor m + 1 - s \rfloor$. For this step, we needed $m + 1$ in place of m. Note that $\mathcal{T}_s$ is not a subgroup scheme of $\mathcal{T} = \mathcal{T}_0$, and [CY01, Proposition 4.2] (summarized in Proposition 2.5.4) is doing much work here.

Now consider general s with $0 \le s < m$.

Let us prove (i). If $t \in \mathrm{T}(\tilde{F})_s$, and if $t' \in \mathcal{T}'(\mathfrak{O}_{\tilde{F}'})$ has the same image as t in $\mathcal{T}(\mathfrak{O}_{\tilde{F}}/\mathfrak{p}_{\tilde{F}}^{m+1}) = \mathcal{T}'(\mathfrak{O}_{\tilde{F}'}/\mathfrak{p}_{\tilde{F}'}^{m+1})$, then since t and t' have the same image in the special fiber $\mathcal{T}(\mathfrak{O}_{\tilde{F}}/\mathfrak{p}_{\tilde{F}}) = \mathcal{T}'(\mathfrak{O}_{\tilde{F}'}/\mathfrak{p}_{\tilde{F}'})$, it follows that $t' \in \mathrm{T}'(\tilde{F}')_{s-\lfloor s \rfloor}$. Thus, by (ii) in the case where $0 \le s < 1$ (applied with $s - \lfloor s \rfloor$

in place of s), t and t' have the same image in $\mathcal{T}_{s-\lfloor s\rfloor}(\mathfrak{O}_{\tilde{F}}/\mathfrak{p}_{\tilde{F}}^m) = \mathcal{T}'_{s-\lfloor s\rfloor}(\mathfrak{O}_{\tilde{F}'}/\mathfrak{p}_{\tilde{F}'}^m)$. The conditions $t \in \mathrm{T}(\tilde{F})_s$ and $t' \in \mathrm{T}'(\tilde{F}')_s$ both translate to this image having trivial further image in $\mathcal{T}_{s-\lfloor s\rfloor}(\mathfrak{O}_{\tilde{F}}/\mathfrak{p}_{\tilde{F}}^{\lfloor s\rfloor}) = \mathcal{T}'_{s-\lfloor s\rfloor}(\mathfrak{O}_{\tilde{F}'}/\mathfrak{p}_{\tilde{F}'}^{\lfloor s\rfloor})$. Thus, (i) follows.

Applying (ii) with $s-\lfloor s\rfloor$ in place of s, and applying Proposition 2.5.4 $\lfloor s\rfloor$ times (ii) follows (use that $\lfloor m+1-(s-\lfloor s\rfloor)\rfloor - \lfloor s\rfloor = \lfloor m+1-s\rfloor$). □

3.5. **Relating to the work of Chai and Yu.**

Proposition 3.5.1. *The isomorphism of Chai and Yu described in Theorem 2.5.3 (the right-most vertical arrow of* (10)*) is a Chai-Yu isomorphism.*

The main input into the proof of the above proposition is the following lemma.

Lemma 3.5.2. *Consider the setting of Notation 2.5.1*(ii)*. Thus,* $(F,\mathrm{T}) \leftrightarrow_l (F',\mathrm{T}')$*, and we consider* $(F, \mathrm{R} := \mathrm{Res}_{L/F}\mathrm{T}_L) \leftrightarrow_l (F', \mathrm{R}' := \mathrm{Res}_{L'/F'}\mathrm{T}'_{L'})$*, where* L/F *is an at most* l*-ramified finite Galois extension splitting* T*, and* $L \hookrightarrow F^{\mathrm{sep}}$ *and* $L' \hookrightarrow F'^{\mathrm{sep}}$ *are compatible embeddings. Let* $0 < m \leq_L l$*. Then the isomorphism* $\mathcal{R} \times_{\mathfrak{O}_F} \mathfrak{O}_F/\mathfrak{p}_F^m \to \mathcal{R}' \times_{\mathfrak{O}_{F'}} \mathfrak{O}_{F'}/\mathfrak{p}_{F'}^m$ *of* (9) *is a Chai-Yu isomorphism.*

Proof. Note that $\mathcal{R}^{\mathrm{ft}} = \mathcal{R}$ and $\mathcal{R}'^{\mathrm{ft}} = \mathcal{R}'$. Some of the proof will be written informally, for lightness of reading.

Let $\{\chi_i = \chi'_i\}_i$ be a basis for $X^*(\mathrm{T}) = X^*(\mathrm{T}')$. It gives an isomorphism $\mathrm{R} = \prod_i \mathrm{Res}_{L/F}\mathbb{G}_m, \mathrm{R}' = \prod_i \mathrm{Res}_{L'/F'}\mathbb{G}_m$. The realization $(F,\mathrm{R}) \leftrightarrow_l (F',\mathrm{R}')$ is then, in an obvious sense, a product of the obvious realizations $\prod_i (F, \mathrm{Res}_{L/F}\mathbb{G}_m) \leftrightarrow_l (F', \mathrm{Res}_{L'/F'}\mathbb{G}_m)$.

Further, the isomorphism $\mathcal{R} \times_{\mathfrak{O}_F} \mathfrak{O}_F/\mathfrak{p}_F^m \to \mathcal{R}' \times_{\mathfrak{O}_{F'}} \mathfrak{O}_{F'}/\mathfrak{p}_{F'}^m$ given by (9) then becomes the product of the isomorphisms $\mathrm{Res}_{\mathfrak{O}_L/\mathfrak{O}_F}\mathbb{G}_m \times_{\mathfrak{O}_F} \mathfrak{O}_F/\mathfrak{p}_F^m \to \mathrm{Res}_{\mathfrak{O}_{L'}/\mathfrak{O}_{F'}}\mathbb{G}_m \times_{\mathfrak{O}_{F'}} \mathfrak{O}_{F'}/\mathfrak{p}_{F'}^m$, each of which is given, at the level of A-points for an algebra A over $\mathfrak{O}_F/\mathfrak{p}_F^m = \mathfrak{O}_{F'}/\mathfrak{p}_{F'}^m$, by the identification $((\mathfrak{O}_L/\mathfrak{p}_F^m\mathfrak{O}_L) \otimes_{\mathfrak{O}_F} A)^\times \to ((\mathfrak{O}_{L'}/\mathfrak{p}_{F'}^m\mathfrak{O}_{L'}) \otimes_{\mathfrak{O}_{F'}} A)^\times$. It is enough to prove that this isomorphism is a Chai-Yu isomorphism for $(F, \mathrm{Res}_{L/F}\mathbb{G}_m) \leftrightarrow_l (F', \mathrm{Res}_{L'/F'}\mathbb{G}_m)$.

In other words, we may assume that $\mathrm{T} = \mathbb{G}_m$, though the chosen splitting extension used to define $\mathrm{R} = \mathrm{Res}_{L/F}\mathrm{T}_L = \mathrm{Res}_{L/F}\mathbb{G}_m$ is still L/F.

Let $\tilde{F} \hookrightarrow F^{\mathrm{sep}}$ and $\tilde{F}' \hookrightarrow F'^{\mathrm{sep}}$ be compatible extensions, with $\tilde{F}/F$ a maximal unramified extension. Since $\mathrm{R} = \mathrm{Res}_{L/F}\mathbb{G}_m$ is an induced torus, it is standard (and easy) that $\mathrm{R}(\tilde{F})_m^{\mathrm{naive}} = \mathrm{R}(\tilde{F})_m$. Therefore, keeping in mind Lemma 3.4.1(i), we may take $r = m$ in the definition of a Chai-Yu isomorphism. It is enough to show that the isomorphism $\mathrm{R}(\tilde{F})_b/\mathrm{R}(\tilde{F})_m \to \mathrm{R}'(\tilde{F}')_b/\mathrm{R}'(\tilde{F}')_m$ obtained by evaluating $\mathcal{R} \times_{\mathfrak{O}_F} \mathfrak{O}_F/\mathfrak{p}_F^m \to \mathcal{R}' \times_{\mathfrak{O}_{F'}} \mathfrak{O}_{F'}/\mathfrak{p}_{F'}^m$ at $\mathfrak{O}_{\tilde{F}}/\mathfrak{p}_{\tilde{F}}^m = \mathfrak{O}_{\tilde{F}'}/\mathfrak{p}_{\tilde{F}'}^m$ is a "restricted standard isomorphism" for $(\tilde{F}, \mathrm{R}_{\tilde{F}}) \leftrightarrow_l (\tilde{F}', \mathrm{R}'_{\tilde{F}'})$.

Let $\tilde{L} \hookrightarrow F^{\mathrm{sep}}$ (resp., $\tilde{L}' \hookrightarrow F'^{\mathrm{sep}}$) be a compositum of $L \hookrightarrow F^{\mathrm{sep}}$ and $\tilde{F} \hookrightarrow F^{\mathrm{sep}}$ (resp., $L' \hookrightarrow F'^{\mathrm{sep}}$ and $\tilde{F}' \hookrightarrow F'^{\mathrm{sep}}$). It is then immediate that $\tilde{L} \hookrightarrow F^{\mathrm{sep}}$ and $\tilde{L}' \hookrightarrow F'^{\mathrm{sep}}$ have the same stabilizer in $\Gamma_F/I_F^l = \Gamma_{F'}/I_{F'}^l$, i.e., are compatible embeddings for $F \leftrightarrow_l F'$, and hence also for for $\tilde{F} \leftrightarrow_l \tilde{F}'$. Note that $m \leq_{\tilde{L}} l$ (since $\psi_{\tilde{L}/F} = \psi_{L/F}$), and that $\tilde{L} \leftrightarrow_{\psi_{\tilde{L}/F}(l)} \tilde{L}'$ lies over both $L \leftrightarrow_{\psi_{L/F}(l)} L'$ and $\tilde{F} \leftrightarrow_l \tilde{F}'$. Set $e = e(L/F) = e(\tilde{L}/\tilde{F})$. We use $\tilde{L}/\tilde{F}$ as a splitting extension for $\mathrm{R}_{\tilde{F}}$.

Thus, if $t \in \mathrm{R}(\tilde{F})_b = \mathcal{R}(\mathfrak{O}_{\tilde{F}})$ and $t' \in \mathrm{R}'(\tilde{F}')_b = \mathcal{R}'(\mathfrak{O}_{\tilde{F}'})$ have the same image in $\mathcal{R}(\mathfrak{O}_{\tilde{F}}/\mathfrak{p}_{\tilde{F}}^m) = \mathcal{R}'(\mathfrak{O}_{\tilde{F}'}/\mathfrak{p}_{\tilde{F}'}^m)$, it is enough to show that for all $\chi = \chi' \in X^*(\mathrm{R}'_{\tilde{F}'}) = X^*(\mathrm{R}_{\tilde{F}})$, $\chi(t) \in \mathfrak{O}_{\tilde{L}}^\times$ and $\chi'(t') \in \mathfrak{O}_{\tilde{L}'}^\times$ have the same image in $(\mathfrak{O}_{\tilde{L}}/\mathfrak{p}_{\tilde{L}}^{em})^\times = (\mathfrak{O}_{\tilde{L}'}/\mathfrak{p}_{\tilde{L}'}^{em})^\times$, i.e., in $(\mathfrak{O}_{\tilde{L}}/\mathfrak{p}_{\tilde{F}}^m\mathfrak{O}_{\tilde{L}})^\times = (\mathfrak{O}_{\tilde{L}'}/\mathfrak{p}_{\tilde{F}'}^m\mathfrak{O}_{\tilde{L}'})^\times$.

It is enough to prove this for $\chi = \chi'$ running over some basis of $X^*(\mathrm{R}_{\tilde{F}}) = X^*(\mathrm{R}) = X^*(\mathrm{R}') = X^*(\mathrm{R}'_{\tilde{F}'})$. We use the basis $\{\chi_\sigma = \chi'_{\sigma'} \mid \sigma = \sigma' \in \Gamma_{L'/F'} = \Gamma_{L/F}\}$, where for each $\tilde{L}$-algebra A, $\chi_\sigma : (\mathrm{Res}_{L/F}\mathbb{G}_m)(A) = (L \otimes_F A)^\times \to A^\times = \mathbb{G}_m(A)$ is a restriction of the map $L \otimes_F A \to A$ that takes $l \otimes a$ to $\sigma(l)a$, and $\chi'_{\sigma'}$ has a similar description. Taking $A = \tilde{L}$ and viewing t as an element of

$$\mathcal{R}(\mathfrak{O}_{\tilde{F}}) \subset \mathcal{R}(\mathfrak{O}_{\tilde{L}}) = (\mathfrak{O}_L \otimes_{\mathfrak{O}_F} \mathfrak{O}_{\tilde{L}})^\times \subset (L \otimes_F \tilde{L})^\times = \mathrm{R}(\tilde{L}),$$

and similarly with t', the lemma follows from the following commutative diagram:

$$\begin{tikzcd}
(\mathfrak{O}_L/\mathfrak{p}_F^m\mathfrak{O}_L)\otimes_{\mathfrak{O}_F}(\mathfrak{O}_{\tilde{L}}/\mathfrak{p}_{\tilde{F}}^m\mathfrak{O}_{\tilde{L}}) \arrow[r] \arrow[d] & \mathfrak{O}_{\tilde{L}}/\mathfrak{p}_{\tilde{F}}^m\mathfrak{O}_{\tilde{L}} \arrow[d] \\
(\mathfrak{O}_{L'}/\mathfrak{p}_{F'}^m\mathfrak{O}_{L'})\otimes_{\mathfrak{O}_{F'}}(\mathfrak{O}_{\tilde{L}'}/\mathfrak{p}_{\tilde{F}'}^m\mathfrak{O}_{\tilde{L}'}) \arrow[r] & \mathfrak{O}_{\tilde{L}'}/\mathfrak{p}_{\tilde{F}'}^m\mathfrak{O}_{\tilde{L}'}
\end{tikzcd},$$

where the top horizontal arrow sends $l\otimes\tilde{l}$ to $\sigma(l)\tilde{l}$, and the bottom horizontal arrow is analogous. □

Proof of Proposition 3.5.1. Since the map (9) is a Chai-Yu isomorphism (Lemma 3.5.2), the left vertical arrow of (10) takes any element of its source to a standard correspondent of it. In other words, it is a restriction of the standard isomorphism $\mathrm{R}(\tilde{F})/\mathrm{R}(\tilde{F})_{m+h} = \mathrm{R}(\tilde{F})/\mathrm{R}(\tilde{F})^{\mathrm{naive}}_{m+h} \to \mathrm{R}'(\tilde{F}')/\mathrm{R}'(\tilde{F}')^{\mathrm{naive}}_{m+h} = \mathrm{R}'(\tilde{F}')/\mathrm{R}'(\tilde{F}')_{m+h}$, which exists by Proposition 3.3.1. Since the maps $X^*(\mathrm{R}) = X^*(\mathrm{R}') \to X^*(\mathrm{T}') = X^*(\mathrm{T})$ that are dual to $\mathrm{T} \hookrightarrow \mathrm{R}$ and $\mathrm{T}' \hookrightarrow \mathrm{R}'$ coincide (Lemma 2.4.1), it follows from Proposition 3.3.1 and Lemma 3.2.3 that the middle vertical arrow of (10) also sends each element of its source to a standard correspondent of it (in fact, this gives an alternate justification for the existence of the middle vertical arrow of (10)). By definition (see Definition 3.1.3(iii)), this implies that $\mathcal{T}^{\mathrm{ft}} \times_{\mathfrak{O}_F} \mathfrak{O}_F/\mathfrak{p}_F^m \to \mathcal{T}'^{\mathrm{ft}} \times_{\mathfrak{O}_{F'}} \mathfrak{O}_{F'}/\mathfrak{p}_{F'}^m$ is a Chai-Yu isomorphism. □

4. The case of weakly induced tori

In this section, we will restrict to a class of tori that includes all induced tori, namely, the class of tori satisfying the beautiful condition (T) identified in [Yu15], which, following [KP23], we will refer to as the class of weakly induced tori. For these tori, the standard and minimal congruent filtrations coincide ([KP23, Corollary B.10.13]). We will show that for weakly induced tori, standard, congruent and Chai-Yu isomorphisms exist in the "best possible" generality. It will follow that for these tori, congruent isomorphisms are a special case of standard isomorphisms.

4.1. **Weakly induced tori.**

Notation 4.1.1. A torus T over a DVHF F is said to be weakly induced if it becomes an induced torus over some finite tamely ramified extension of F. It is easy to see ([KP23, Remark B.6.3]) that T is weakly induced if and only if $X^*(\mathrm{T})$ has a basis that is permuted by the wild inertia group $I_F^{>0} := \bigcup_{r>0} I_F^r \subset I_F \subset \Gamma_F$.

The following lemma is one reason why weakly induced tori are easy to work with.

Lemma 4.1.2. *Let* T *be a weakly induced torus over a DVHF F with perfect residue field. For any $r \geq 0$, we have $\mathrm{T}(F)_r = \mathrm{T}(F)^{\mathrm{std}}_r$, and for any $r > 0$, we have $\mathrm{T}(F)^{\mathrm{std}}_r = \mathrm{T}(F)^{\mathrm{naive}}_r$. Consequently, using Remark 2.3.3, $\mathrm{T}(L_1)_r = (\mathrm{T}(L_2)_{e(L_2/L_1)r})^{\mathrm{Gal}(L_2/L_1)}$ and $\mathrm{T}_1(F)_r = \mathrm{T}_1(F) \cap \mathrm{T}_2(F)_r$ whenever $r > 0$, $L_2/L_1/F$ is a chain of finitely ramified separable field extensions with L_2/L_1 Galois, and $\mathrm{T}_1 \hookrightarrow \mathrm{T}_2$ is an injective homomorphism of weakly induced tori over F.*

Proof. For the equality $\mathrm{T}(F)_r = \mathrm{T}(F)^{\mathrm{std}}_r$, use [KP23, Corollary B.10.13] (and intersect with $\mathrm{T}(F)$). For the equality $\mathrm{T}(F)^{\mathrm{std}}_r = \mathrm{T}(F)^{\mathrm{naive}}_r$ when $r > 0$, see [KP23, Proposition B.6.4(3)]. □

4.2. **Standard, congruent and Chai-Yu isomorphisms for weakly induced tori.**

Proposition 4.2.1. *Let $(F, \mathrm{T}) \leftrightarrow_l (F', \mathrm{T}')$, with* T *assumed to be weakly induced. Suppose $0 < r \leq\!\!\!\!\!\leq_{\mathrm{T}} l$. Then there is a standard isomorphism $\mathrm{T}(F)/\mathrm{T}(F)^{\mathrm{naive}}_r = \mathrm{T}(F)/\mathrm{T}(F)_r \to \mathrm{T}'(F')/\mathrm{T}'(F')_r = \mathrm{T}'(F')/\mathrm{T}'(F')^{\mathrm{naive}}_r$. If $r = m$ is an integer, then this is also a congruent isomorphism.*

Proof. Since T is weakly induced over F, so is T': $I_F^{>0}$, acting through $I_F^{>0}/I_F^l = I_{F'}^{>0}/I_{F'}^l$, permutes a basis of $X^*(\mathrm{T}) = X^*(\mathrm{T}')$. Choose compatible embeddings $\tilde{F} \hookrightarrow F^{\mathrm{sep}}$ and $\tilde{F}' \hookrightarrow F'^{\mathrm{sep}}$, where $\tilde{F}/F$ is a maximal unramified extension. Consider the standard isomorphism $\mathrm{T}(\tilde{F})/\mathrm{T}(\tilde{F})_r \to \mathrm{T}'(\tilde{F}')/\mathrm{T}'(\tilde{F}')_r$ associated to $(\tilde{F}, \mathrm{T}_{\tilde{F}}) \leftrightarrow_l (\tilde{F}', \mathrm{T}'_{\tilde{F}'})$ (Proposition 3.3.1). It is equivariant for $\Gamma_{\tilde{F}/F} = \Gamma_{\tilde{F}'/F'}$ (Corollary 3.2.7). Thus, as in the proof of Lemma 3.4.1(iii), the first assertion follows if we show that $H^1(\Gamma_{\tilde{F}/F}, \mathrm{T}(\tilde{F})_r) = 0 = H^1(\Gamma_{\tilde{F}'/F'}, \mathrm{T}'(\tilde{F}')_r)$. This is a special case of [KP23, Proposition 13.8.1]. The second assertion is immediate. □

Proposition 4.2.2. *Let $(F, \mathrm{T}) \leftrightarrow_l (F', \mathrm{T}')$, with T a weakly induced torus over F. Let m be a positive integer, with $m \lll_{\mathrm{T}} l$. Then there is a unique Chai-Yu isomorphism $\mathcal{T}^{\mathrm{ft}} \times_{\mathfrak{O}_F} \mathfrak{O}_F/\mathfrak{p}_F^m \cong \mathcal{T}'^{\mathrm{ft}} \times_{\mathfrak{O}_{F'}} \mathfrak{O}_{F'}/\mathfrak{p}_{F'}^m$.*

Proof. Let $m \lll_L l$ for some finite Galois extension L/F splitting T, and let $L \hookrightarrow F^{\mathrm{sep}}$ and $L' \hookrightarrow F'^{\mathrm{sep}}$ be compatible embeddings. Form $(F, \mathrm{R} \coloneqq \mathrm{Res}_{L/F} \mathrm{T}_L) \leftrightarrow_l (F', \mathrm{R}' \coloneqq \mathrm{Res}_{L'/F'} \mathrm{T}_{L'})$.

By Lemma 3.5.2, there exists a Chai-Yu isomorphism $\mathcal{R}^{\mathrm{ft}} \times_{\mathfrak{O}_F} \mathfrak{O}_F/\mathfrak{p}_F^m \to \mathcal{R}'^{\mathrm{ft}} \times_{\mathfrak{O}_{F'}} \mathfrak{O}_{F'}/\mathfrak{p}_{F'}^m$. On the other hand, we also know that $\mathcal{T}^{\mathrm{ft}} \hookrightarrow \mathcal{R}^{\mathrm{ft}}$ and $\mathcal{T}'^{\mathrm{ft}} \hookrightarrow \mathcal{R}'^{\mathrm{ft}}$ are closed immersions, as a special case of [KP23, Lemma B.7.11] (this nontrivially uses the fact that T and T' are weakly induced). This allows us to make sense of the following claim: that the Chai-Yu isomorphism $\mathcal{R}^{\mathrm{ft}} \times_{\mathfrak{O}_F} \mathfrak{O}_F/\mathfrak{p}_F^m \to \mathcal{R}'^{\mathrm{ft}} \times_{\mathfrak{O}_{F'}} \mathfrak{O}_{F'}/\mathfrak{p}_{F'}^m$ restricts to an isomorphism $\mathcal{T}^{\mathrm{ft}} \times_{\mathfrak{O}_F} \mathfrak{O}_F/\mathfrak{p}_F^m \to \mathcal{T}'^{\mathrm{ft}} \times_{\mathfrak{O}_{F'}} \mathfrak{O}_{F'}/\mathfrak{p}_{F'}^m$ (which we will show to be the desired Chai-Yu isomorphism).

Since the image of $\mathcal{T}^{\mathrm{ft}}(\mathfrak{O}_{\tilde{F}})$ is schematically dense in $\mathcal{T}^{\mathrm{ft}} \times_{\mathfrak{O}_F} \mathfrak{O}_F/\mathfrak{p}_F^m$ (by [CY01, Lemma 8.5.1]), this claim follows if we show that $\mathcal{R}^{\mathrm{ft}} \times_{\mathfrak{O}_F} \mathfrak{O}_F/\mathfrak{p}_F^m \to \mathcal{R}'^{\mathrm{ft}} \times_{\mathfrak{O}_{F'}} \mathfrak{O}_{F'}/\mathfrak{p}_{F'}^m$ takes the image of $\mathcal{T}^{\mathrm{ft}}(\mathfrak{O}_{\tilde{F}}) = \mathrm{T}(\tilde{F})_b$ isomorphically onto that of $\mathcal{T}'^{\mathrm{ft}}(\mathfrak{O}_{\tilde{F}'}) = \mathrm{T}'(\tilde{F}')_b$. In other words, if we show that the map $\mathrm{R}(\tilde{F})_b/\mathrm{R}(\tilde{F})_m \to \mathrm{R}'(\tilde{F}')_b/\mathrm{R}'(\tilde{F}')_m$, obtained by evaluating $\mathcal{R}^{\mathrm{ft}} \times_{\mathfrak{O}_F} \mathfrak{O}_F/\mathfrak{p}_F^m \to \mathcal{R}'^{\mathrm{ft}} \times_{\mathfrak{O}_{F'}} \mathfrak{O}_{F'}/\mathfrak{p}_{F'}^m$ at $\mathfrak{O}_{\tilde{F}}/\mathfrak{p}_{\tilde{F}}^m = \mathfrak{O}_{\tilde{F}'}/\mathfrak{p}_{\tilde{F}'}^m$, induces an isomorphism $\mathrm{T}(\tilde{F})_b/\mathrm{T}(\tilde{F})_m \to \mathrm{T}'(\tilde{F}')_b/\mathrm{T}'(\tilde{F}')_m$.

But by the definition of a Chai-Yu isomorphism, this map $\mathrm{R}(\tilde{F})_b/\mathrm{R}(\tilde{F})_m \to \mathrm{R}'(\tilde{F}')_b/\mathrm{R}'(\tilde{F}')_m$ is a restriction of a standard isomorphism (use Proposition 3.3.1 and Lemmas 3.2.5 and 4.1.2), and hence restricts to an isomorphism $\mathrm{T}(\tilde{F})_b/\mathrm{T}(\tilde{F})_m \to \mathrm{T}'(\tilde{F}')_b/\mathrm{T}'(\tilde{F}')_m$ that is also a restriction of a standard isomorphism (combine Proposition 3.3.1 with Lemmas 3.2.5, 4.1.2 and 3.2.3).

This not only proves that $\mathcal{R}^{\mathrm{ft}} \times_{\mathfrak{O}_F} \mathfrak{O}_F/\mathfrak{p}_F^m \to \mathcal{R}'^{\mathrm{ft}} \times_{\mathfrak{O}_{F'}} \mathfrak{O}_{F'}/\mathfrak{p}_{F'}^m$ restricts to an isomorphism $\mathcal{T}^{\mathrm{ft}} \times_{\mathfrak{O}_F} \mathfrak{O}_F/\mathfrak{p}_F^m \to \mathcal{T}'^{\mathrm{ft}} \times_{\mathfrak{O}_{F'}} \mathfrak{O}_{F'}/\mathfrak{p}_{F'}^m$, but also that the restricted isomorphism, evaluated on $\mathfrak{O}_{\tilde{F}}/\mathfrak{p}_{\tilde{F}}^m = \mathfrak{O}_{\tilde{F}'}/\mathfrak{p}_{\tilde{F}'}^m$, is the isomorphism $\mathrm{T}(\tilde{F})_b/\mathrm{T}(\tilde{F})_m \to \mathrm{T}'(\tilde{F}')_b/\mathrm{T}'(\tilde{F}')_m$ obtained by restricting a standard isomorphism. Thus, by definition (and Lemma 4.1.2), $\mathcal{T}^{\mathrm{ft}} \times_{\mathfrak{O}_F} \mathfrak{O}_F/\mathfrak{p}_F^m \to \mathcal{T}'^{\mathrm{ft}} \times_{\mathfrak{O}_{F'}} \mathfrak{O}_{F'}/\mathfrak{p}_{F'}^m$ is a Chai-Yu isomorphism. Its uniqueness follows from Lemma 3.4.2. □

5. Putting things together

Proof of Theorem 1.2.1. In the setting of (i) of the theorem, Theorem 2.5.3, interpreted using Proposition 3.5.1, gives us a Chai-Yu isomorphism $\mathcal{T}^{\mathrm{ft}} \times_{\mathfrak{O}_F} \mathfrak{O}_F/\mathfrak{p}_F^m \to \mathcal{T}'^{\mathrm{ft}} \times_{\mathfrak{O}_{F'}} \mathfrak{O}_{F'}/\mathfrak{p}_{F'}^m$. Hence Proposition 3.4.3 provides us with a congruent isomorphism $\mathrm{T}(F)/\mathrm{T}(F)_m \to \mathrm{T}'(F')/\mathrm{T}'(F')_m$. Note that this automatically also gives the compatibility with the Chai-Yu isomorphism (i.e., the commutativity of the left square of (1)). The latter assertion of (i) therefore follows from Proposition 3.4.4. Item (ii) of the theorem follows from Lemma 3.4.1(ii), whose extra condition is satisfied by taking r to be any real number between $m + \max(h(F, \mathrm{T}_1), h(F, \mathrm{T}_2))$ and $\min(\psi_{L_1/F}(l)/e(L_1/F), \psi_{L_2/F}(l)/e(L_2/F))$, where L_1 and L_2 are minimal splitting extensions for T_1 and T_2 (use Lemma 2.5.5).

As for (iii) of the theorem, it remains to prove the compatibility with the Kottwitz homomorphism (the commutativity of the right-square of (1)). By definition (Definition 3.1.3(ii)) we reduce to a similar assertion for a suitable $\mathrm{T}(\tilde{F})/\mathrm{T}(\tilde{F})_m \to \mathrm{T}'(\tilde{F}')/\mathrm{T}'(\tilde{F}')_m$, with $\tilde{F}/F$ a maximal unramified extension, which is induced by a standard isomorphism $\mathrm{T}(\tilde{F})/\mathrm{T}(\tilde{F})_r^{\mathrm{naive}} \to \mathrm{T}'(\tilde{F}')/\mathrm{T}'(\tilde{F}')_r^{\mathrm{naive}}$. Thus, the desired compatibility with the Kottwitz homomorphism follows from the compatibility of the standard isomorphism with the Kottwitz homomorphism (Proposition 3.3.2).

Now we come to (iv). Set $h = h(F, \mathrm{T})$. The assumption $m + 4h \lll_{\mathrm{T}} l$ implies that we have a congruent isomorphism $\mathrm{T}(F)/\mathrm{T}(F)_{m+h} \to \mathrm{T}'(F')/\mathrm{T}'(F')_{m+h}$, which by Lemma 3.4.1(iv) induces a standard isomorphism $\mathrm{T}(F)/\mathrm{T}(F)_{m+h}^{\mathrm{naive}} \to \mathrm{T}'(F')/\mathrm{T}'(F')_{m+h}^{\mathrm{naive}}$, and also induces a congruent isomorphism $\mathrm{T}(F)/\mathrm{T}(F)_m \to \mathrm{T}'(F')/\mathrm{T}'(F')_m$. Since $\mathrm{T}(F)_{m+h}^{\mathrm{naive}} \subset \mathrm{T}(F)_m$ and $\mathrm{T}'(F')_{m+h}^{\mathrm{naive}} \subset \mathrm{T}'(F')_m$ (Remark 2.5.6), $\mathrm{T}(F)/\mathrm{T}(F)_{m+h}^{\mathrm{naive}} \to \mathrm{T}'(F')/\mathrm{T}'(F')_{m+h}^{\mathrm{naive}}$ induces $\mathrm{T}(F)/\mathrm{T}(F)_m \to \mathrm{T}'(F')/\mathrm{T}'(F')_m$ as well. Now (iv) is easy to see from Proposition 3.3.3, applied with $m + h$ in place of r.

Now we address the weakly induced case. Proposition 4.2.1 gives (i) with $h(F,\mathrm{T})$ replaced by 0. Lemma 3.2.3 then gives (ii) with $h(F,\mathrm{T}_1)$ and $h(F,\mathrm{T}_2)$ replaced by 0. For the compatibility with the Chai-Yu isomorphism, first note that a Chai-Yu isomorphism $\mathcal{T}^{\mathrm{ft}} \times_{\mathfrak{O}_F} \mathfrak{O}_F/\mathfrak{p}_F^m \to \mathcal{T}'^{\mathrm{ft}} \times_{\mathfrak{O}_{F'}} \mathfrak{O}_{F'}/\mathfrak{p}_{F'}^m$ exists (Proposition 4.2.2). In this weakly induced case, the middle vertical arrow of (1) is a standard isomorphism (Proposition 4.2.1), so the commutativity of the left square of (1) is automatic from the definition of a Chai-Yu isomorphism. Since the middle vertical arrow of (1) is a standard isomorphism, its compatibility with the Kottwitz homomorphism is immediate from Proposition 3.3.2. For the same reason, the compatibility with the LLC is obvious from Proposition 3.3.3. □

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