Algebra 1
HW 1 (Due: 24-08-2023)

1. Let $1 \to H \xrightarrow{\alpha} G \xrightarrow{\beta} K \to 1$ be a short exact sequence of groups. Show that TFAE:

   (a) There exists a homomorphism $\alpha' : G \to H$ such that $\alpha' \circ \alpha = 1$

   (b) There exists isomorphism $\theta : G \to H \times K$ such that TFDC

   $$\begin{array}{ccccccccc} 0 & \longrightarrow & H & \xrightarrow{\alpha} & G & \xrightarrow{\beta} & K & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle \theta} & & \downarrow{\scriptstyle id} & & \\ 0 & \longrightarrow & H & \xrightarrow{i} & H \times K & \xrightarrow{p} & K & \longrightarrow & 0 \end{array}$$

   where id is the identity map, i and p are canonical injection and projection maps respectively.

2. Let $1 \to H \xrightarrow{\alpha} G \xrightarrow{\beta} K \to 1$ be a short exact sequence of groups. Show that TFAE:

   (a) There exists a homomorphism $\beta' : K \to G$ such that $\beta \circ \beta' = 1$

   (b) There exists a homomorphism $\phi : K \to Aut(H)$ and an isomorphism $\theta : G \to H \rtimes_\phi K$ such that TFDC

   $$\begin{array}{ccccccccc} 0 & \longrightarrow & H & \xrightarrow{\alpha} & G & \xrightarrow{\beta} & K & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle \theta} & & \downarrow{\scriptstyle id} & & \\ 0 & \longrightarrow & H & \xrightarrow{i} & H \rtimes_\phi K & \xrightarrow{p} & K & \longrightarrow & 0 \end{array}$$

   where id is the identity map, i and p are canonical injection and projection maps respectively.

In the last two problems make sure you show that all the maps θ and ϕ are actually group homomorphisms.