Algebra 1
HW 2(Due: 31-08-2023)

1. Show that subgroups and quotients of solvable groups are solvable. Show the same for nilpotent groups.

2. We have seen in the class that if N is a normal subgroup of G and if N and G/N are solvable, then G is solvable. Prove that this does not hold for nilpotent groups by giving an example of semi-direct product G of two groups that are nilpotent but with G not nilpotent.

3. Show that D_{2n} is solvable but is nilpotent only when $n = 2^i$ for some i.

4. Show that A_4 is not simple and A_5 is simple.

5. Show the following:

   (a) Every finite p-group is nilpotent

   (b) $G \simeq \prod_{1 \leq i \leq n} G_i$ is nilpotent if and only if each G_i is nilpotent

6. Let G be a finite nilpotent group. Show that if $H < G$ is a proper subgroup then $H < N_G(H)$ is also proper. Use this to show that G is a direct product of its Sylow subgroups. Conclude that G contains a subgroup of order m for every m that divides $|G|$
   Note that the above two problems implies that a finite G is nilpotent if and only if it is the direct product of its Sylow subgroups

7. Show that a non-abelian group of order ab where a and b are distinct primes is solvable. Is it nilpotent? (Hint: Use previous exercise).

8. Show that group of upper triangular matrices in $GL_2(\mathbb{F}_p)$ where $\mathbb{F}_p$ is finite field with p elements is solvable but not nilpotent for $p \geq 3$.