Algebra 1
HW 6 (Due: 28-09-2023)

1. For a commutative ring R and let $Nil(R)$ denote the set of nilpotent elements of R. Let $\mathbf{CommRing}$ and $\mathbf{Sets}$ denote respectively the category of commutative rings and sets. Are the following functors representable? If so find the representing object. Else show that it is not representable

   (a)

   $$\begin{aligned} \mathcal{F} : \mathbf{CommRing} &\to \mathbf{Sets} \\ R &\mapsto Nil(R) \end{aligned}$$

   (b) Fix $n \geq 1$,

   $$\begin{aligned} \mathcal{G} : \mathbf{CommRing} &\to \mathbf{Sets} \\ R &\mapsto \{r \in R | r^n = 0\} \end{aligned}$$

2. Let $\mathbf{Cat}$ denote the category of small categories (objects are small categories and morphisms are functors between categories). Let $\mathcal{H}$ be the functor that maps a category to all of the morphisms in the category. Is $\mathcal{H}$ representable? If so find the representing object. Else show that it is not representable.

3. Prove Cayley's theorem using the Yoneda lemma

4. Let $\mathbf{Top}$ be the category of topological spaces with continuous maps as morphisms. Consider the functor

   $$\begin{aligned} \mathcal{O} : \mathbf{Top}^{op} &\to \mathbf{Sets} \\ X &\mapsto \{U \subseteq X | U \text{ is } \text{ open in } \ X\} \end{aligned}$$

   Is $\mathcal{O}$ representable? If so find the representing object. Else show that it is not representable.