INTRODUCTION TO CHARACTERISTIC CLASSES

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Basic references for these lectures are:

1. J. W. Milnor and J. D. Stasheff, *Characteristic Classes*, Ann. Math. Studies 76, Princeton (1974).
2. D. Husemoller, *Fibre Bundles* (Second Ed.), Grad. Texts in Math. 20, Springer-Verlag (1966).

1. Vector bundles

For simplicity, we will often restrict the category of topological spaces considered to the category of compact Hausdorff spaces, unless stated otherwise. Recall that for such spaces, we have the following results from point-set topology:

Theorem 1.1. (a) (Tietze extension theorem) *If X is a compact Hausdorff space, $A \subset X$ a closed subset, and $f : A \to \mathbb{R}^n$ a continuous function, then f can be extended to a continuous function $\widetilde{f} : X \to \mathbb{R}^n$.*

(b) (Partititons of unity) *Let $\{U_i\}_{i=1}^n$ be an open cover of X. Then there exist continuous functions $u_i : X \to [0,1]$ with the following properties: (i) $\overline{\{u_i(x) \neq 0\}} \subset U_i$ for each $i = 1, \dots, n$ (ii) $u_1(x) + \cdots + u_n(x) = 1$ for all $x \in X$.*

A collection of functions $\{u_i\}$ as in (b) above is called a *continuous partition of unity* on X subordinate to the open covering $\{U_i\}$. If $V_i = \{x \in X \mid u_i(x) > 0\}$, then $\{V_i\}$ is also an open covering of X, such that $\overline{V_i} \subset U_i$; we call such an open cover $\{V_i\}$ a *shrinking* of $\{U_i\}$.

There are analogous results available for topological and C^∞ manifolds (recall that, by definition, these are paracompact spaces). We will follow the convention that the term "manifold" means "manifold without boundary", unless explicitly stated otherwise. We state the results in the C^∞ case; in the topological case, very similar results hold, except that instead of C^∞ extensions or partitions of unity, one obtains analogous continuous functions. We leave the precise formulation to the reader.

Theorem 1.2. (a) (C^∞ extension theorem) *If M is a C^∞ manifold, $A \subset M$ a closed subset, and $f : U \to \mathbb{R}^n$ a C^∞ function defined on some open neighbourhood of A, then there exists a C^∞ function $\widetilde{f} : M \to \mathbb{R}^n$ which agrees with f on some neighbourhood V of A in U.*

(b) (C^∞ partititons of unity) *Let $\{U_i\}_{i \in I}$ be an open cover of M. Then there exist C^∞ functions $u_i : M \to \mathbb{R}$ with the following properties: (i) $u_i(M) \subset$*

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$[0,1]$, and $\operatorname{supp}(u_i) = \overline{\{u_i(x) \neq 0\}} \subset U_i$ for each $i = 1, \ldots, n$ (ii) the collection of closed sets $\{\operatorname{supp}(u_i)\}_{i\in I}$ is locally finite in M, and $\sum_{i\in I} u_i(x) = 1$ for all $x \in M$ (the sum is well-defined, since locally on M, only a given finite set of terms is non-zero).

Definition 1.3. Let $k = \mathbb{R}$ or $\mathbb{C}$. A *k-vector bundle* of rank n on a topological space X is a space E, together with a continuous map $p : E \to X$, such that "locally on X, E is the product space $X \times k^n$", *i.e.*, there is an open cover $\{U_i\}$ of X and homeomorphisms $\varphi : p^{-1}(U_i) \to U_i \times k^n$ such that

- (i) φ_i is compatible with projection to U_i, *i.e.*, $\varphi_i(x) = (p(x), \widetilde{\varphi}_i(x))$ for all $x \in p^{-1}(U_i)$ for some (continuous) function $\widetilde{\varphi}_i : p^{-1}(U_i) \to k^n$
- (ii) for each $x \in U_i \cap U_j$, the composite homeomorphism
$$A_{ij}(x) : k^n \xrightarrow{\widetilde{\varphi}_i^{-1}} p^{-1}(x) \xrightarrow{\widetilde{\varphi}_j} k^n$$
is a *linear isomorphism* $k^n \to k^n$, *i.e.*, $A_{ij}(x) \in \mathrm{GL}_n(k)$.

Here another such collection of data $\{(V_j, \psi_j)\}$ defines the same vector bundle structure on E if $\{(U_i, \varphi_i)\} \cup \{(V_j, \psi_j)\}$ defines a vector bundle structure on E (in future, for expository reasons, we will omit similar statements from our discussion, though they are implicit).

We refer to the space E as the *total space* and X as the *base space* of the vector bundle; the map p is called the *bundle projection* (generalizing the notion of projection for the product space $X \times k^n$).

Note that for any $x \in U_i$, the homeomorphism $\widetilde{\varphi}_i : p^{-1}(x) \to k^n$ may be used to give $p^{-1}(x)$ the structure of a k-vector space of dimension n. Since A_{ij} are linear isomorphisms, we see that for $x \in U_i \cap U_j$, the vector space structures on $p^{-1}(x)$ induced by $\widetilde{\varphi}_i$ and $\widetilde{\varphi}_j$ agree. We call the k-vector space $p^{-1}(x)$ the *fibre* of $p : E \to X$ over x, and may also denote it by E_x.

Remark 1.4. If $p : E \to X$ is a vector bundle, we will often abuse terminology and refer to "the vector bundle E", if the map p and the vector bundle structure are implicit in the discussion. Strictly speaking, referring to $p : E \to X$ as a vector bundle is itself somewhat imprecise, since we also need to be given an equivalence class of local trivializations; equivalently, we must be given the vector space structures on all the fibers, together with the fact that it is possible to realize these vector space structures via some (unspecified) local trivializations. From this last point of view, one might even dispense with the standard notion of local triviality, and replace it by some weaker notion; this is done, for example, in algebraic or complex analytic geometry, especially when considering more general fibre bundles (like projective bundles, for example).

Remark 1.5. (Transition functions) In the definition 1.3, since φ_i are all homeomorphisms, the matrix valued functions $A_{ij} : U_i \cap U_j \to \mathrm{GL}_n(k)$ are continuous, and we verify easily that they satisfy the condtitions

- (a) for all $x \in U_i$, $A_{ii}(x)$ is the identity
- (b) for all $x \in U_i \cap U_j$, we have $A_{ji}(x) = A_{ij}(x)^{-1}$, the matrix inverse

(c) for all $x \in U_i \cap U_j \cap U_k$, we have a matrix identity $A_{ik}(x) = A_{jk}(x)A_{ij}(x)$, where the expression on the right denotes the product of matrices.

These are called the *transition functions* (or transition matrices) associated to the given local trivializations of the bundle.

Conversely, given an open cover $\{U_i\}$ of X and a collection of continuous matrix-valued functions $A_{ij} : U_i \cap U_j \to \mathrm{GL}_r(k)$ satisfying (a), (b), (c) one can define a vector bundle E as follows: let $\sim$ be the equivalence relation on the disjoint union $\coprod_i U_i \times k^n$ generated by $(x, v) \sim (x, A_{ij}(x)v)$ for all $x \in U_i \cap U_j$, where $(x, v) \in U_i \times k^n$, and $(x, A_{ij}(x)v) \in U_j \times k^n$. Let $p : E \to X$ be induced by the projections $U_i \times k^n \to U_i$. One verifies at once that the natural map $U_i \times k^n \to E$ is a homeomorphism onto its image $p^{-1}(U_i)$, whose inverse may be taken as φ_i.

Definition 1.6. If $p : E \to X$, $q : F \to Y$ are vector bundles on X and Y respectively, a *morphism of vector bundles* $f : E \to F$ is a continuous map f such that

(i) there is a (necessarily unique) continuous map $f_0 : X \to Y$ such that $f_0 \circ p = q \circ f : E \to Y$ (*i.e.*, , f maps fibres of E into fibers of F), and
(ii) for each x, if $f_0(x) = y$, the induced map on fibres $E_x \to F_y$ is a k-linear transformation.

If f is also a homeomorphism, then we say that it is an *isomorphism* of vector bundles. If $p : E \to X$ is a vector bundle, $F \subset E$ is a *subbundle* if $p\,|_F : F \to X$ is a vector bundle, for which the inclusion $F \subset E$ is a morphism of vector bundles (over the identity map on X). Similarly one can define the notion of a quotient bundle of $p : E \to X$.

Example 1.7. (*The trivial bundle*) $E = X \times k^n$, $p : E \to X$ is the projection.

Thus, in the definition of a vector bundle, the map φ_i (or $\widetilde{\varphi}_i$) is called a *trivialization* of the bundle $p : E \to X$ over the open set U_i.

Example 1.8. (*Möbius band*) Let $k = \mathbb{R}$, $X = S^1$ (the unit circle in $\mathbb{R}^2$), $M =$ Möbius band (without boundary), $p : M \to S^1$ is the projection onto the "equator" of M. We may regard S^1 as the identification space of the unit interval $[0, 1]$ modulo the identification of its end points $0, 1$; the identification map $[0, 1] \twoheadrightarrow S^1$ can be taken to be $t \mapsto (\cos 2\pi t, \sin 2\pi t)$. Then M is the identification space of $[0, 1] \times \mathbb{R}$, modulo the identification of $\{0\} \times \mathbb{R}$ with $\{1\} \times \mathbb{R}$ given by $(0, s) \sim (1, -s)$. Since this identification is via a linear isomorphism $\mathbb{R} \to \mathbb{R}$, we see that $p : M \to S^1$ (induced by the first projection $[0, 1] \times \mathbb{R} \to [0, 1]$) is an $\mathbb{R}$-vector bundle of rank 1.

The Möbius band $p : M \to S^1$ of Example 1.8 can be seen to be *not* isomorphic to the trivial bundle $S^1 \times \mathbb{R}$. Indeed, for any k-vector bundle $p : E \to X$, there is a continuous mapping $0_E : X \to E$ given by $0_E(x) = 0_{E_x}$, where $0_{E_x} \in E_x$ is the 0-element of the fibre vector space E_x. This map satisfies $p \circ 0_E = 1_X$, the identity map of X. Now if $f : S^1 \times \mathbb{R} \to M$ is an isomorphism of vector bundles, it maps the image of $0_{S^1 \times \mathbb{R}}$ homeomorphically to the image of 0_M. Hence it induces a homeomorphism between the complements of these images. But

$S^1 \times \mathbb{R} - \mathrm{im}(0_{S^1\times\mathbb{R}}) = S^1 \times (\mathbb{R} - \{0\})$ is disconnected, while $M - \mathrm{im}(0_M)$ is connected (verify!).

Definition 1.9. If $p : E \to X$ is a vector bundle, and $A \subset X$, then $p : p^{-1}(A) \to A$ is a vector bundle, such that the inclusion $p^{-1}(A) \hookrightarrow E$ defines a morphism of vector bundles (over the inclusion map of A into X); we call $p^{-1}(A) \to A$ the *restriction* of E to A, and write $E\mid_A$ in place of $p^{-1}(A)$.

More generally, if $p : E \to X$ is a k-vector bundle of rank r, and $f : Y \to X$ a continuous map, then the *pullback* $f^*E = Y \times_X E \to Y$ is also a k-vector bundle of rank r. Here $Y \times_X E = \{(y, z) \in Y \times E \mid f(y) = p(z)\}$.

Remark 1.10. Let $f : X \to Y$ be a continuous map, $p : E \to X$, $q : F \to Y$ vector bundles, and $g : F \to E$ a morphism of vector bundles (over f). Then it is easy to see that g induces a morphism $\widetilde{g} : F \to f^*E$ of bundles over the identity map of Y. Conversely any morphism of bundles $\widetilde{g} : F \to f^*E$ arises from a unique morphism of bundles $g : F \to E$.

Example 1.11. (Underlying real vector bunde of a complex bundle) If $p : E \to X$ is a complex vector bundle of rank n, we may also regard $p : E \to X$ as defining a real vector bundle of rank $2n$ in a natural way, such that the real vector space structure on $E_x = p^{-1}(x)$ is that underlying its complex vector space structure. Then we have an isomorphism of real vector bundles $J : E \to E$ given by multiplication by $\sqrt{-1}$ on the fibers E_x; this satisfies $J \circ J = (-1)_E$, where $(-1)_E : E \to E$ is multiplication by -1 on the fibers E_x.

Conversely, given a real vector bundle $p : E \to X$ of even rank $2n$, together with an isomorphism $J : E \to E$ of real vector bundles with $J \circ J = (-1)_E$, we obtain a structure on $p : E \to X$ of a complex vector bundle in a natural way, where the $\mathbb{C}$-vector space structure on E_x is such that scalar multiplication by $\sqrt{-1}$ is defined to be the real linear automorphism $J_x : E_x \to E_x$. We call J a *complex structure* on the real vector bundle $p : E \to X$.

In particular, given a complex vector bundle $p : E \to X$, we can form its *complex conjugate* bundle, usually denoted by $\overline{E}$, which has the same underlying real vector bundle, and complex structure $-J$, where $J : E \to E$ is the complex structure on $p : E \to X$.

Example 1.12. (Tangent bundle for embedded C^∞ manifolds) Let $M \subset \mathbb{R}^N$ be an n-dimensional C^∞ differentiable submanifold. For each $x \in M$, we have

$$\begin{aligned} T_xM &= \text{tangent space to } M \text{ at } x \\ &= \{v \in \mathbb{R}^N \mid \text{ the line } \{x + tv \mid t \in \mathbb{R}\} \text{ is tangent to } M \text{ at } x\}, \\ TM &= \{(x, v) \in M \times \mathbb{R}^N \mid v \in T_xM\}. \end{aligned}$$

Then we have the following facts:

(i) T_xM is a real vector subspace of $\mathbb{R}^N$ of dimension n, for each $x \in M$

(ii) $p : TM \to M$, $p(x, v) = x$, gives TM the structure of a vector bundle of rank n on M, such that the vector space structure on the fibre $(TM)_x = \{x\} \times T_xM$ is that given on $T_xM \subset \mathbb{R}^N$; thus by construction, TM is given as a subbundle of the trivial bundle $M \times \mathbb{R}^N$.

The idea of the proof is as follows. For each $x \in M$, there is a neighbourhood U of x in $\mathbb{R}^N$, and C^∞ functions $f_1, \ldots, f_{N-n}$ on U such that

(a) $U \cap M = \{y \in U \mid f_1(y) = \cdots = f_{N-n}(y) = 0\}$, and

(b) for any $x \in M \cap U$, the Jacobian matrix $J(f)(x) = \left[\frac{\partial f_j}{\partial x_i}(x)\right]$ has maximal rank $N - n$. Then one sees that $T_xM = \ker J(f)(x)$.

Now after permuting the coordinates, if we write $J(f)(x) = [A, B]$ where the submatrix $A = \left[\frac{\partial f_j}{\partial x_i}(x)\right]_{1\leq i,j\leq N-n}$ has rank $N - n$, then $\mathbb{R}^n \to \ker J(f)(x)$, $v \mapsto \begin{pmatrix} A^{-1}Bv \\ v \end{pmatrix}$ is a linear isomorphism; the inverse isomorphism gives a trivialization of TM over $U \cap M$.

Example 1.13. (Tangent bundle for "abstract" C^∞ manifolds) There is a also a construction for the tangent bundle of an "abstract" C^∞ n-manifold M, independent of any chosen embedding in Euclidean space $\mathbb{R}^N$. By definition, we are given a collection of coordinate charts (*i.e.*, an "atlas") $\{(U_i, \varphi_i)\}_{i\in I}$, where $\varphi_i : U_i \to V_i \subset \mathbb{R}^n$ is a homeomorphism of U_i with an open subset V_i of $\mathbb{R}^n$, such that the homeomorphism $\varphi_{ij} = \varphi_j \circ \varphi_i^{-1} : \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$ is a diffeomorphism (C^∞ homeomorphism with C^∞ inverse) between open subsets of $\mathbb{R}^n$. The Jacobian matrix $J_{ij} = J(\varphi_{ij})$ of partial derivatives yields a continuous (in fact, C^∞) function $\widetilde{J}_{ij} : U_i \cap U_j \to \mathrm{GL}_n(\mathbb{R})$, given by $\widetilde{J}_{ij}(x) = J_{ij}(\varphi_i(x))$. Then $\widetilde{J}_{ij}$ are transition functions for an $\mathbb{R}$-vector bundle of rank n, which we define to be the tangent bundle TM. One checks that this is independent of the choice of coordinate charts defining the C^∞ structure on M. Also, the local trivializations make TM into a C^∞ manifold of dimension $2n$ in a natural way, such that the bundle projection $TM \to M$ is a C^∞ submersion.

It is easy to show that if $f : M \to N$ is a C^∞ mapping, there is an induced morphism of bundles $Df : TM \to TN$, which is also a C^∞ mapping, and which is compatible with composition of C^∞ maps. One verifies also (exercise!) that, for an embedded C^∞ n-manifold M, this definition of TM agrees with the previous definition.

Remark 1.14. (Manifolds with boundary) The definition of tangent bundle $\pi : TM \to M$ can be extended to the case of a C^∞ n-manifold M with boundary ∂M, such that $TM\mid_{M-\partial M}$ is the tangent bundle as defined earlier, and $TM\mid_{\partial M} = T\partial M \oplus (\partial M \times \mathbb{R})$ (here $\oplus$ denotes the direct sum of bundles, defined in the next section). We leave the necessary modifications of our earlier discussion to the reader.

Example 1.15. A manifold M for which $TM \to M$ is the trivial bundle is called *parallelizable*. The unit $2n$-sphere $S^{2n} \subset \mathbb{R}^{2n+1}$ is an example of a non-parallelizable manifold (see Exercise 1.25 below).

Remark 1.16. In the above examples, one saw that for an open cover $\{U_i\}$ of a C^∞ n-manifold M determined by an atlas of coordinate charts, the transition functions for the tangent bundle $U_i \cap U_j \to \mathrm{GL}_n(\mathbb{R})$ are C^∞ functions. We express this by saying that $p : TM \to M$ is a C^∞ vector bundle, as defined below.

Definition 1.17. A C^∞ *k-vector bundle* of rank n on a C^∞ manifold M of dimension d is a C^∞ manifold E (of dimension $n+d$, or $2n+d$, depending on whether $k=\mathbb{R}$ or $k=\mathbb{C}$), together with a C^∞ map $p:E\to M$, such that $p:E\to M$ has the structure of a k-vector bundle of rank n, whose local trivializations are C^∞, *i.e.*, there is an open cover $\{U_i\}$ of M and *diffeomorphisms* $\varphi : p^{-1}(U_i)\to U_i\times k^n$ such that the conditions (i),(ii) of definition (1.3) are satisfied.

Example 1.18. Let $M\subset\mathbb{R}^N$ be a C^∞ submanifold of dimension n. We may similarly define the *normal bundle* $q:E\to M$ to M in $\mathbb{R}^N$ by setting

$$E=\{(x,v)\in M\times\mathbb{R}^N \mid v\in (T_xM)^\perp\},$$

where p is induced by the first projection. We leave it as an exercise to show that this does define a real C^∞ vector bundle on M of rank $N-n=N-\dim M$.

Example 1.19. Similarly, we can define the normal bundle of any C^∞ submanifold M of a given C^∞ manifold N. If $i:M\to N$ is the inclusion, then we have a morphism of vector bundles $Di:TM\to TN$, giving rise to an *inclusion* $TM\to i^*TN$ of vector bundles on M. The normal bundle of M in N is defined to be the quotient vector bundle i^*TN/TM, whose fiber over any point $x\in M\subset N$ is the quotient vector space T_xN/T_xM.

If we choose a positive definite inner product (in the sense of definition 2.2 below) on the bundle i^*TN (*e.g.*, by choosing such an inner product on TN, that is to say, a Riemannian metric on N), then the normal bundle is isomorphic to the subbundle $TM^\perp\subset i^*TN$.

Finally, if $i:M\to N$ is merely an immersion, so that $Di:TM\to i^*TN$ is an inclusion, then one can still define a“normal bundle to i” as the quotient i^*TN/TM.

Example 1.20. (Complex manifolds) Recall that a *complex manifold* is a C^∞ manifold M of even dimension $2n$, for which there is an admissible atlas of coordinate charts $\{(U_i,\phi_i)\}_{i\in I}$, such that, identifying $\mathbb{R}^{2n}$ with the real vector space underlying $\mathbb{C}^n$, the transition functions

$$\phi_{ij}=\phi_j\circ\phi_i^{-1}:\phi_i(U_i\cap U_j)\to\phi_j(U_i\cap U_j),$$

which are given to be C^∞ diffeomorphisms, are actually complex analytic functions (*i.e.*, holomorphic functions) on open subsets of $\mathbb{C}^n$, as well. This implies that the transition matrix valued functions $J_{ij}:U_i\cap U_j\to \mathrm{GL}_{2n}(\mathbb{R})$ of Example 1.13, which are used to define the tangent bundle TM, can in fact be viewed as (holomorphic) maps $U_i\to U_j\to\mathrm{GL}_n(\mathbb{C})$, where we view $\mathrm{GL}_n(\mathbb{C})\subset\mathrm{GL}_{2n}(\mathbb{R})$ in the standard way (by associating to a complex linear automorphism of $\mathbb{C}^n$ the underlying real linear automorphism of $\mathbb{R}^{2n}$). In particular, we may view TM as a complex vector bundle of rank n, whose underlying real vector bundle is the tangent bundle of the underlying C^∞ manifold M, as defined earlier in Example 1.13.

Definition 1.21. A *section* of a k-vector bundle $p:E\to X$ is a continuous map $s:X\to E$ such that $p\circ s:X\to X$ is the identity map. Let $\Gamma(X,E)$ denote the set of sections of E. Similarly, let $\Gamma_\infty(M, E)$ denote the space of C^∞ sections of a C^∞ vector bundle $p : E \to M$; note that $\Gamma_\infty(M, E) \subset \Gamma(M, E)$ is then a k-subspace.

Notations 1.22. The map $0_E : X \to E$ described earlier, given by $0_E(x) = 0_{E_x}$, is a section, called the *zero section* (or 0-section) of $p : E \to X$. We will sometimes denote its image, which is a homeomorphic copy of the base space X, by 0_X.

Remark 1.23. Note that a section of the trivial bundle $s : X \times k^r \to X$ is essentially just a vector valued continuous function $f : X \to k^r$; here f corresponds to the section $x \mapsto (x, f(x))$. Thus a section of an arbitrary vector bundle may be viewed, locally on X, as a vector valued function; though this depends on the local trivialization chosen, the notions of a section vanishing, or of a collection of sections being linearly independent at each point, *do not* depend on the local trivialization.

Remark 1.24. The vector space structures on the fibres E_x of a vector bundle $p : E \to X$ determine on $\Gamma(X, E)$ a natural structure of a module over the ring $C_k(X)$ of k-valued continuous functions on X, given by $(a \cdot s)(x) = a(x) \cdot s(x)$ for any $a \in C_k(X)$, $s \in \Gamma(X, E)$; here $a(x) \in k$, $s(x) \in E_x$, and $a(x) \cdot s(x)$ denotes the scalar multiplication for the k-vector space structure on E_x. If $p : E \to X$, $q : F \to X$ are k-vector bundles, and $f : E \to F$ is a morphism of vector bundles, then $s \mapsto f \circ s$ induces a homomorphism of $C_k(X)$-modules $f_* : \Gamma(X, E) \to \Gamma(X, F)$. Note that a k-vector bundle $p : E \to X$ of rank n is a trivial bundle precisely when $\Gamma(X, E)$ is a free $C_k(X)$-module (also of rank n); equivalently, E has n sections $s_1, \ldots , s_n$ which are each nowehere vanishing, and are linearly independent at each point of X (that is to say, $s_1(x), \ldots , s_n(x) \in E_x$ are linearly independent elements, for each $x \in X$).

If $M \subset \mathbb{R}^N$ is a C^∞ manifold, then a section of its tangent bundle $TM \to M$ is called a *vector field* on M; a C^∞ section of TM is called a C^∞ (or *smooth*) vector field. It is a standard convention that the term "vector field", without any further qualification, referes to a smooth vector field. In particular, note that an n-dimensional C^∞ manifold M is parallelizable precisely when it has n continuous vector fields $v_1, \ldots , v_n$ which are each nowhere vanishing, and linearly independent at each point. It is then a fact that there exists a C^∞ trivialization of TM; this is a particular case of general results on approximation of continuous maps between C^∞ manifolds by C^∞ maps, which implies that the classifications of topological and C^∞ vector bundles are equivalent.

Example 1.25. If $S^m \subset \mathbb{R}^{m+1}$ has a vector field v which is nowhere 0, then one shows easily that the identity map on S^m is homotopic to the *antipodal map* (the map $x \mapsto -x$). From a standard result in topology, this implies that m is odd. Conversely, if m is odd, then S^m does support a C^∞ vector field v which is nowhere zero.

Remark 1.26. We will see later (see Remark 5.9) that if M is an oriented compact C^∞ manifold of dimension n, and v is a C^∞ vector field on M with isolated zeroes, then the "number of zeroes" (called the *index*) of v equals the topological

Euler characteristic of M,

$$\chi(M) = \sum_{i\geq 0}(-1)^i \operatorname{rank} H^i(M,\mathbb{Z}).$$

Here the zeroes have to be counted "with multiplicity", and taking the orientation into account (*i.e.*, zeroes may possibly be counted with negative multiplicity). This is called the *Poincaré-Hopf theorem.*

Theorem 1.27. (Swan) *There is an (anti-)equivalence of categories between k-vector bundles on a "good" topological space X and finitely generated projective modules over the ring $C_k(X)$, given by $(p : E \to X) \mapsto \Gamma(X,E)$.* (Here "good" includes the case when X is a compact Hausdorff space.)

Proof. (Sketch) We will assume X is a compact Hausdorff space. Let $p : E \to X$ be a k-vector bundle of rank r, and let $\{U_i\}_{i=1}^n$ be an open cover on which there are trivializations $\varphi_i : p^{-1}(U_i) \to U_i \times k^r$. Let $s_{ij} : U_i \to p^{-1}(U_i)$ be such that $\varphi_i \circ s_{ij} : U_i \to U_i \times k^r$ corresponds to the j-th coordinate function on k^r. Choose a partition of unity (see Theorem 1.1(b)) u_i subordinate to the open cover $\{U_i\}$. Define

$$\widetilde{s}_{ij}(x) = \begin{cases} u_i(x)s_{ij}(x) & \text{if } x \in U_i \\ 0_{E_x} & \text{if } x \notin U_i \end{cases}$$

One checks at once that $\widetilde{s}_{ij}$ is continuous, so that $\widetilde{s}_{ij} \in \Gamma(X,E)$ for all i,j. Now define a morphism of vector bundles

$$\Phi : X \times k^{rn} \to E,$$

$$\Phi(x, \{a_{ij}\}_{1\leq i\leq n, 1\leq j\leq r}) = \sum a_{ij}\widetilde{s}_{ij}(x).$$

If $V_i = \{x \in X \mid u_i(x) > 0\}$, so then $\{V_i\}$ is also an open covering of X, shrinking $\{U_i\}$; for $x \in V_i$, the map $k^r \to E_x$, $(a_1, \dots, a_r) \mapsto \sum_{j=1}^r a_j\widetilde{s}_{ij}(x) = u_i(x)\sum_j a_j s_{ij}(x)$, which is an isomorphism of k-vector spaces, since $s_{ij}(x)$ is a basis for E_x, and $u_i(x) \in k$ is a non-zero scalar. Hence $\Phi : X \times k^{rn} \to E$ is a *surjection.*

We claim the surjective bundle morphism Φ is split. Equivalently, there is an injective bundle map $\Psi : E \to X \times k^{rn}$ such that the composition $\Phi \circ \Psi : E \to E$ is the identity. We prove this as follows. If $k = \mathbb{R}$, let $<\ ,\ >$ denote a positive definite inner product on k^{rn}; if $k = \mathbb{C}$, let $<\ ,\ >$ denote a positive definite Hermitian inner product on k^{rn}. For each $x \in X$, the k-linear surjection $\Phi_x : k^{rn} \cong \{x\} \times k^{rn} \twoheadrightarrow E_x$ induces an isomorphism $\alpha_x : \ker(\Psi_x)^{\perp} \xrightarrow{\cong} E_x$. Let $\Psi_x : E_x \to \{x\} \times k^{rn}$ be the inverse isomorphism α_x^{-1}, composed with the inclusion $\ker(\Psi_x)^{\perp} \hookrightarrow \{x\} \times k^{rn}$. We leave it to the reader to check that $\Psi : E \to X \times k^{rn}$, $\Psi(y) = \Psi_{p(y)}(y)$ is continuous, and defines the desired splitting of Φ.

This implies that $\Gamma(X,E)$ is a direct summand of $\Gamma(X, X \times k^{rn}) = C_k(X)^{rn}$ as a $C_k(X)$-module; in particular, $\Gamma(X,E)$ is a finitely generated projective $C_k(X)$-module.

We leave the (easy) proof of the converse statement to the reader. □

One consequence of the above proof is worth making explicit.

Corollary 1.28. *Let X be a compact Hausdorff space, E a k-vector on X. Then E is a direct summand of a trivial vector bundle on X (of some rank, depending on E).*

Using Theorem 1.2 instead of Theorem 1.1, one can obtain similar results on C^∞ manifolds. Similarly, one has analogous results on topological manifolds (using the ring of continuous, rather than C^∞, functions).

Theorem 1.29. *There is an (anti-)equivalence between the category of C^∞ k-vector bundles on a C^∞ manifold M and the category of finitely generated projective modules over the ring $C^\infty_k(M)$ of C^∞ functions $M \to k$. In particular, any C^∞ vector bundle $p : E \to M$ is a direct summand of a trivial bundle $M \times k^N$, for some N (depending on E).*

The proof for compact smooth manifolds is rather similar to that of Swan's result given above. However, the proof for non-compact (C^∞ or topological) manifolds M requires more care; we omit the details here.

2. Operations on vector bundles

The usual operations on finite dimensional vector spaces which are familiar from linear algebra carry over to similar operations on vector bundles. Examples are given by the direct sum, tensor product, dual, Hom, and the exterior and symmetric powers. These are defined on the trivial bundle $X \times k^n$ through the standard operation on k^n, and may be defined for an arbitrary bundle using local trivializations. We then recover the standard operations on the fibres. Equivalently, applying the equivalence $\Gamma(X, -)$, these correspond to the standard operations on finitely generated projective $C_k(X)$-modules.

Example 2.1. (i) If $E \to X$, $F \to X$ are vector bundles, then $(E \oplus F)_x = E_x \oplus F_x$, where the right side denotes the vector space direct sum of the k-vctor spaces E_x and F_x.

(ii) $\mathrm{Hom}(E, F)_x = \mathrm{Hom}_k(E_x, F_x)$, and $\mathrm{Hom}(E, F)$ is the vector bundle whose module of sections $\Gamma(X, \mathrm{Hom}(E, F))$ is the $C_k(X)$-module of $C_k(X)$-linear homomorphisms $\Gamma(X, E) \to \Gamma(X, F)$. If $E^\vee$ is the dual k-vector bundle, so that $(E^\vee)_x = (E_x)^\vee$, then for any vector bundle F, we have a natural isomorphism of vector bundles $E^\vee \otimes F \cong \mathrm{Hom}(E, F)$.

(iii) $(E^\vee)^\vee \cong E$ for any vector bundle E.

(iv) $\overset{n}{\wedge} E = 0$ for $n > \mathrm{rank}\, E$.

Definition 2.2. An *inner product* on E is a symmetric bundle morphism $E \otimes E \to X \times k$. If $k = \mathbb{R}$, the inner product is called *positive definite* if the induced inner product $E_x \otimes_{\mathbb{R}} E_x \to \mathbb{R}$ on each fibre is positive definite; we also refer to the positive definite inner product as a *Euclidean structure* on the vector bundle E. In a similar way, we can define the notion of a positive definite *Hermitian inner product* on a $\mathbb{C}$-vector bundle E; its real part is a positive definite inner product on the underlying real vector bundle, such that the complex structure J (given by multiplication by i on each fiber) is an isometry.

Remark 2.3. Since every k-vector bundle on a compact Hausdorff space is a subbundle of a trivial bundle of finite rank (corollary 1.28), we see that every real vector bundle on such a space carries a positive definite inner product. Similarly every complex vector bundle carries a positive definite Hermitian inner product. If E is any vector bundle supporting a positive definite inner product, then for any subbundle $F \subset E$, we can form the orthogonal complement subbundle $F^{\perp}$ with $(F^{\perp})_x = F_x^{\perp} \subset E_x$. Clearly we have $F \oplus F^{\perp} = E$. Hence, any injective or surjective bundle morphism $E \to F$ on such a space X is *split* injective or surjective, respectively. This is of course consistent with Theorem 1.27.

Remark 2.4. The operation of pullback preserves the above operations on vector bundles (direct sums, Hom, duals, tensor, exterior and symmetric products, inner products, etc.). On the level of modules, there is a corresponding homomorphism $f^* : \Gamma(X, E) \to \Gamma(Y, f^*E)$; if $f^* : C_k(X) \to C_k(Y)$ is the ring homomorphism given by $g \mapsto g \circ f$, then $\Gamma(X, E) \to \Gamma(Y, E)$ is $C_k(X)$-linear, and the induced $C_k(Y)$-linear map $\Gamma(X, E) \otimes_{C_k(X)} C_k(Y) \to \Gamma(Y, f^*E)$ is an *isomorphism.* Notice that if $f : Y \hookrightarrow X$ is the inclusion of a subset, so that $f^*E \cong p^{-1}(Y)$, then the homomorphism $f^* : \Gamma(X, E) \to \Gamma(Y, f^*E)$ is given by restriction of functions.

Proposition 2.5. *If $f, g : Y \to X$ are homotopic maps, with Y paracompact, then for any bundle $p : E \to X$, the bundles $f^*E \to Y$ and $g^*E \to Y$ are isomorphic.*

Proof. We will give a proof when Y is a compact Hausdorff space. The idea of the proof in general is similar, but technically more complicated. We make use of a simple lemma.

Lemma 2.6. *Let X be a compact Hausdorff space, $i : A \to X$ be the inclusion of a closed subset. Then for any vector bundle $p : E \to X$, the restriction map $i^* : \Gamma(X, E) \to \Gamma(A, i^*E)$ is surjective.*

Proof. For the trivial bundle, this follows from the Tietze extension theorem (Theorem 1.1(a)). Since any vector bundle on X is a direct summand of a trivial bundle, we reduce immediately to the special case. □

Now let $H : Y \times I \to X$ be a homotopy between f and g, where $I = [0, 1]$ is the unit interval. Let $p_1 : Y \times I \to Y$ be the projection. For $t \in I$, let $f_t : Y \to X$ be the map $f_t(y) = H(y, t)$; then $f_0 = f$, and $f_1 = g$.

We claim that for each $t \in I$, there is a neighbourhood V_t of $t \in I$ such that for all $t' \in T$, we have an isomorphism of vector bundles $f_t^*E \cong f_{t'}^*E$. If we grant the claim, a finite number of these open subsets cover I, and it is then clearly possible to find a sequence $t_0 = 0 < t_1 < \cdots < t_n = 1$ in I such that we have isomorphisms $f_{t_i}^*E \cong f_{t_{i+1}}^*E$ for $0 \leq i < n$; the comnposition of these isomorphisms is the desired one.

To prove the claim, consider the vector bundle $F = \mathrm{Hom}(H^*E, p_1^*f_t^*E)$ on $Y \times I$. If $i_t : Y \cong Y \times \{t\} \hookrightarrow Y \times I$ is the inclusion, then $i_t^*F \cong \mathrm{Hom}(f_t^*E, f_t^*E)$. By lemma 2.6, the identity endomorphism of f_t^*E extends to a global section $s \in \Gamma(Y, F)$. The subset $\mathrm{Iso}\,(H^*E, p_1^*f_t^*E) \subset \mathrm{Hom}(H^*E, p_1^*f_t^*E) = F$ is an *open* subset, where the fibre over $z \in Y \times I$ of $\mathrm{Iso}\,(H^*E, p_1^*f_t^*E)$ is the set

Iso $(H^*E, p_1^*f_t^*E)_z$ of vector space isomorphisms of $(H^*E)_z$ with $(p_1^*f_t^*E)_z$. Hence $s^{-1}(\mathrm{Iso}\,(H^*E, p_1^*f_t^*E))$ is an open subset of $Y \times I$ containing $Y \times \{t\}$; hence it also contains an open subset of the form $Y \times V_t$ for some (relatively) open interval $V_t \subset I$ containing t. Now for $t' \in V_t$, the restriction of s to $Y \times \{t'\}$ gives the desired isomorphism, proving the claim. □

3. Classifying maps to Grassmannians

For $n \leq m$, let $\mathbb{G}_k(n,m)$ denote the Grassmannian of n-dimensional subspaces of k^m. For $k = \mathbb{R}$, fix the standard Euclidean inner product on $\mathbb{R}^m$; for $k = \mathbb{C}$, fix the standard positive definite Hermitian inner product on $\mathbb{C}^m$. In each case, the standard basis vectors form an orthonormal basis. We can then make identifications

$$\mathbb{G}_{\mathbb{R}}(n,m) = \mathrm{O}\,(m)/\mathrm{O}\,(n) \times \mathrm{O}\,(m-n), \quad \mathbb{G}_{\mathbb{C}}(n,m) = \mathrm{U}\,(m)/\mathrm{U}\,(n) \times \mathrm{U}\,(m-n)$$

as homogeneous spaces for the orthogonal group $\mathrm{O}\,(m)$ and the unitray group $\mathrm{U}\,(m)$, respectively. We have a tautological k-vector bundle $\gamma_{n,m} \to \mathbb{G}_k(n,m)$, whose fibre $(\gamma_{n,m})_x$ is $\{x\} \times V$, where $V \subset k^m$ is the subspace of dimension n corresponding to the point $x \in \mathbb{G}_k(n,m)$. The orthogonal projection of $\mathbb{G}_k(n,m) \times k^m$ onto the subbundle $\gamma_{n,m}$ gives us m tautological sections $\mathbf{s}_1, \ldots, \mathbf{s}_m$ of $\gamma_{n,m}$.

Example 3.1. (C^∞ and complex structures on Grassmannians) The Grassmannians can be regarded as C^∞ manifolds in a standard way, and the complex Grassmannians are in fact complex manifolds. We recall one description of these structures (other equivalent descriptions are possible) on $\mathbb{G}_k(n,m)$. Let $I = \{i_1, \ldots, i_n\} \subset \{1, \ldots, m\}$ be any ordered subset of cardinality n, and let $U_I \subset \mathbb{G}_k(n,m)$ be the (open) subset on which the sections $\{\mathbf{s}_i \mid i \in I\}$ generate $\gamma_{n,m}$. Since $\gamma_{n,m}$ is a vector bundle of rank n, and the set $\{\mathbf{s}_i \mid 1 \leq i \leq m\}$ generate $\gamma_{n,m}$ everywhere, the open sets U_i cover $\mathbb{G}_k(n,m)$. Let $\{j_1, \ldots, j_{m-n}\}$ be the ordered set $\{1, \ldots, m\} - I$. By the definition of U_I, $\mathbf{s}_{i_1}, \ldots, \mathbf{s}_{i_n}$ trivialize $\gamma_{n,m}\,|_{U_I}$ (in particular, $\gamma_{n,m} \to \mathbb{G}_k(n,m)$ is indeed locally trivial!). Hence we can find continuous functions a_{pq}, $1 \leq p \leq m-n, 1 \leq q \leq n$ on U_I such that

$$\mathbf{s}_{j_p}\,|_{U_I} = \sum_q a_{pq}\mathbf{s}_{i_q} \ \ \forall\, 1 \leq p \leq m-n.$$

Now one can show (exercise for the reader!) that

(i) the map $U_I \to k^{n(m-n)}$ given by the a_{pq} is in fact a homeomorphism; in particular, this gives $\mathbb{G}_k(n,m)$ the structure of a topological manifold

(ii) if I, I' are two such sets of n indices, the corresponding transition homeomorphisms, determined by the intersections $U_I \cap U_{I'}$, are in fact rational functions; hence they are C^∞, and when $k = \mathbb{C}$, are also analytic functions.

In particular, we see that $\mathbb{G}_{\mathbb{R}}(n,m)$ is a C^∞ manifold, and $\mathbb{G}_{\mathbb{C}}(n,m)$ is a complex manifold; the tautological bundles are also C^∞.

If $p : E \to X$ is a k-vector bundle of rank n, and $s_1, \ldots, s_m$ are sections which give rise to a surjective bundle morphism $\psi : X \times k^m \to E$, we say that the sections s_j *generate* E. In this situation, the map

$$f_\psi : X \to \mathbb{G}_k(n,m),$$

$$x \mapsto [\ker(\psi_x)^\perp],$$

is easily seen to be *continuous*. Further, by construction, we see that there is a natural identification $f_\psi^*\gamma_{n,m} \cong E$ with $f_\psi^*(\mathbf{s}_i) = s_i$.

Let $\mathbb{G}_k(n) = \varinjlim_m \mathbb{G}_k(n,m)$, induced by $i_{n,m} : \mathbb{G}_k(n,m) \hookrightarrow \mathbb{G}_k(n,m+1)$, given by $k^m \hookrightarrow k^m \perp k = k^{m+1}$, $v \mapsto (v,0)$. One gives $\mathbb{G}_k(n)$ the direct limit topology: thus a subset of $\mathbb{G}_k(n)$ is closed if its intersection with each $\mathbb{G}_k(n,m)$ is closed. This implies easily that $\mathbb{G}_k(n)$ is Hausdorff. We claim that any given compact subset of $\mathbb{G}_k(n)$ is contained in $\mathbb{G}_k(n,m)$ for some sufficiently large m; indeed, if $A \subset \mathbb{G}_k(n)$ is any infinite subset which has finite intersection with each $\mathbb{G}_k(n,m)$, then all subsets of A are closed, and so A is an infinite discrete subset of $\mathbb{G}_k(n)$, necessarily non-compact. The space $\mathbb{G}_k(n)$ is called an *infinite Grassmannian*.

The inclusion $k^m \hookrightarrow k^{m+1}$ induces an inclusion of total spaces $\gamma_{n,m} \hookrightarrow \gamma_{n,m+1}$, allowing us to define $\gamma_n = \varinjlim_m \gamma_{n,m}$, together with a natural map $p : \gamma_n \to \mathbb{G}_k(n)$. It can be shown (see Milnor's book, lemma 5.4) that this makes γ_n into a k-vector bundle of rank n on $\mathbb{G}_k(n)$ (the main issue being the local triviality of γ_n); by construction, the restriction $\gamma_n \mid_{\mathbb{G}_k(n,m)}$ is just $\gamma_{n,m}$. Note that by construction, γ_n supports a positive definite inner product, if $k = \mathbb{R}$, and a positive definite Hermitian iner product, if $k = \mathbb{C}$, such that the induced inner products on all $\gamma_{n,m}$ are the standard ones (coming from their descriptions as subbundles of trivial bundles).

Lemma 3.2. *Let $p : E \to X$, ψ and $f_\psi : X \to \mathbb{G}_k(n,m)$ be as above, and let $s \in \Gamma(X,E)$. Suppose $\psi' : X \times k^{m+1} \to E$ is given by $\{s_1, \ldots, s_m, s\}$, and $f_{\psi'} : X \to \mathbb{G}_k(n,m+1)$ is the corresponding map. Then ψ' is homotopic to $i_{n,m} \circ \psi : X \to \mathbb{G}_k(n,m+1)$.*

Proof. If $p_X : X \times I \to X$ is the projection, then the sections has

$$p_X^*(s_1), \ldots, p_X^*(s_m), tp_X^*(s) \in \Gamma(X \times I, p_X^*E)$$

yield a bundle surjection $(X \times I) \times k^{m+1} \twoheadrightarrow p_X^*E$. The corresponding map $H : X \times I \to \mathbb{G}_k(n,m+1)$ yields the desired homotopy. □

Theorem 3.3. *Let $p : E \to X$ be a k-vector bundle which has is generated by a finite set of global sections, i.e., there is a surjection $\psi : k^N \times X \to E$ from a trivial k-bundle on X. The homotopy class of the map $\widetilde{f}_\psi : X \xrightarrow{\psi} \mathbb{G}_k(n,m) \hookrightarrow \mathbb{G}_k(n)$ depends only on the vector bundle E. For compact Hausdorff spaces or topological manifolds X, the association $E \mapsto [\widetilde{f}_\psi]$ gives a bijection between the sets*

$$\mathbf{Vect}_n(X) = \textit{ isomorphism classes of (k-)vector bundles of rank } n \textit{ on } X$$

and

$$[X, \mathbb{G}_k(n)] = \textit{ homotopy classes of maps } X \to \mathbb{G}_k(n).$$

(See also Remark 3.7, below.)

Proof. If $s_1, \dots, s_l$ and $t_1, \dots, t_m$ are two sets of sections generating E, then the corresponding maps $f : X \to \mathbb{G}_k(n,l)$, $g : X \to \mathbb{G}_k(n,m)$ yield homotopic maps into $\mathbb{G}_k(n, l+m+1)$. Indeed, first consider the the maps $\widetilde{f} : X \to \mathbb{G}_k(n, l+m)$, $\widetilde{g} : X \to \mathbb{G}_k(n, l+m)$ arising (respectively) from the sets of sections $s_1, \dots, s_l, 0, \dots, 0$ (with m zeroes) and $t_1, \dots, t_m, 0, \dots, 0$ (with l zeroes); the first map is induced by f, and the second by g. By the above lemma and induction, these two maps $X \to \mathbb{G}_k(n, l+m)$ are respectively homotopic to the maps $F : X \to \mathbb{G}_k(n, l+m)$, $G : X \to \mathbb{G}_k(n, l+m)$ corresponding to the sets of sections $s_1, \dots, s_l, t_1, \dots, t_m$ and $t_1, \dots, t_m, s_1, \dots, s_l$. Now F and G are related by translation by a permutation matrix in $\mathrm{GL}_{l+m}(k)$, for the natural action of $\mathrm{GL}_{l+m}(k)$ on $\mathbb{G}_k(n, l+m)$. For $k = \mathbb{C}$, this permutation matrix is in $\mathrm{U}\,(l+m)$, which is path connected; a path in $\mathrm{U}\,(l+m)$ from the identity element to this permutation yields a homotopy between F and G. If $k = \mathbb{R}$, then the permutation is an element of $\mathrm{O}, (l+m)$, which need not be connected. But then the linear map $k^{l+m+1} \to k^{l+m+1}$ which is the given permutation on the first $l+m$ cordinates, and is multiplication by the sign of the permutation on the $l+m+1$-st coordinate, is an element in the identity component of $\mathrm{O}\,(l+m+1)$, hence again cane be joined to the identity element by a path in $\mathrm{O}\,(l+m+1)$. Thus the maps $X \to \mathbb{G}_k(n, l+m+1)$ induced by f, g are homotopic. Hence there is a well-defined map $\alpha : \mathbf{Vect}_n(X) \to [X, \mathbb{G}_k(n)]$.

Conversely, assume given a continuous map $\psi : X \to \mathbb{G}_k(n)$. When X is compact, we must have that $\psi(X) \subset \mathbb{G}_k(n,m)$ for some m. Then $E = \psi^*\gamma_{n,m}$ is generated by the sections $\psi^*(\mathbf{s}_1), \dots, \psi^*(\mathbf{s}_m)$, and the corresponding map $X \to \mathbb{G}_k(n,m)$ is just ψ itself. Two homotopic maps yield isomorphic bundles on X, by lemma 2.5. This gives a well-defined map $\beta : [X, \mathbb{G}_k(n)] \to \mathbf{Vect}_n(X)$. It is clear from the definitions that the two maps α, β are inverse to each other. In the case when X is a topological manifold, it first needs to be shown that any given map $\psi : X \to \mathbb{G}_k(n)$ is homotopic to a map with image contained in some $\mathbb{G}_k(n,m)$, and that, up to increasing m if necessary, the homotopy class of this map is unique. More precisely, the natural map $\varinjlim_{m} [X, \mathbb{G}_k(n,m)] \to [X, \mathbb{G}_k(n)]$ is bijective. This leads to the definition the map β as before, which is again clearly inverse to α. □

Remark 3.4. In a similar way, Theorem 1.29 implies that for a C^∞ manifold M, there is a natural bijection between isomorphism classes of C^∞ k-vector bundles of rank n and C^∞ homotopy classes of maps $M \to \mathbb{G}_k(n)$, where a map $M \to \mathbb{G}_k(n)$ is defined to be C^∞ if, locally on M, it factors through a C^∞ map into the subspace $\mathbb{G}_k(n,m)$, for some m (note that, since a manifold is locally compact, any continuous map $f : M \to \mathbb{G}_k(n)$ automatically has a factorization through $\mathbb{G}_k(n,m)$, locally on M, where the number m may depend on the chosen neighbourhood in M). One shows that any such map is C^∞ homotopic to a map factoring (on all of M) through some subspace $\mathbb{G}_k(n,m)$. In particular, since any continuous map $M \to \mathbb{G}_k(n,m)$ is homotopic to a C^∞ map, and any two continuously homotopic such C^∞ maps are in fact C^∞ homotopic, we see that the isomorphism classes of topological and C^∞ vector bundles on a C^∞ manifold M coincide.

Remark 3.5. A vector bundle $p : E \to X$ is called *finite* if there exists a finite open covering $\{U_i\}$ such that $E\mid_{U_i}$ is trivial for all i. For any continuous map $f : X \to \mathbb{G}_k(n,m)$, the pull-back bundle $f^*\gamma_{n,m}$ is finite; more generally, the pull-back of a finite vector bundle under any continuous map is again finite. One can show (see Husemoller's book) the following.

(i) Any vector bundle is finite over a paracompact space X which has finite (combinatorial) dimension, *i.e.*, such that for some integer $d > 0$, any open cover of X has a refinement such that all $d+1$-fold intersections are empty; in fact any vector bundle over such a space X is a direct summand of a trivial bundle. This holds, in particular, if X is a topological manifold.

(ii) Any finite vector bundle over a paracompact space supports a positive definite inner product (Euclidean or Hermitian, according as $k = \mathbb{R}$ or $\mathbb{C}$).

(ii) $\gamma_n \to \mathbb{G}_k(n)$ is *not* finite (note that, however, it does support a positive definite inner product, as mentioned earlier).

Remark 3.6. The above bijections α and β are natural (functorial), in the sense that if $f : Y \to X$ is a continuous map between (say) compact Hausdorff spaces, the map

$$f^* : \mathbf{Vect}_n(X) \to \mathbf{Vect}_n(Y),$$

$$[E] \mapsto [f^*E],$$

corresponds to the map

$$[X, \mathbb{G}_k(n)] \to [Y, \mathbb{G}_k(n)],$$

$$(\psi : Y \to \mathbb{G}_k(n)) \mapsto (\psi \circ f : X \to \mathbb{G}_k(n)).$$

In more abstract language, we say that the space $\mathbb{G}_k(n)$ *represents the functor* $\mathbf{Vect}_n(-)$ (on the category of compact Hausdorff spaces). This is actually a slight abuse of terminology, since the "representing object", namely $\mathbb{G}_k(n)$, is not itself a compact Hausdorff space. We have similar statements for the categories of topological or C^∞ manifolds.

Remark 3.7. If X is a paracompact space, the natural transformation $\beta : [X, \mathbb{G}_k(n)] \to \mathbf{Vect}_n(X)$ can be shown to be bijective, even though vector bundles on X need not be finite. See Husemoller's book for the proof. Of course it is not necessarily then true that $[X, \mathbb{G}_k(n)] = \varinjlim_m [X, \mathbb{G}_k(n,m)]$.

Thus if E is any k-vector bundle of rank n on an appropriate space X, so that we have an element $[E] \in \mathbf{Vect}_n(X)$, then we get an associated ring homomorphism

$$[E]^* : H^*(\mathbb{G}_k(n), A) \to H^*(X, A)$$

on cohomology rings, for any coefficient ring A; images in $H^*(X, A)$ of cohomology classes in $H^*(\mathbb{G}_k(n), A)$ are called the *characteristic classes of E in $H^*(X, A)$*. From the above remark 3.6, it follows that if $\theta(E) \in H^*(X, A)$ is a characteristic class for a vector bundle E on (say) a compact Hausdorff space X, and if $f : Y \to X$ is a continuous map, then $\theta(f^*E) = f^*\theta(E)$, where on the right, f^* denotes the ring homomorphism $H^*(X, A) \to H^*(Y, A)$. This latter functoriality property with respect to pull-backs is another common definition of a characteristic class, which is equivalent to the one we have given, at least for bundles on "good enough" spaces (like compact Hausdorff spaces, or manifolds).

In view of the above, it is interesting to compute the cohomology rings of the infinite Grassmannians $H^*(\mathbb{G}_k(n), A)$ for various rings A; each such computation leads to a corresponding "theory of characteristic classes" for vector bundles.

4. Cohomology rings of Grassmannians, and characteristic classes

We first state a result giving a computation of the cohomology ring of $\mathbb{G}_k(n)$ in the most important cases. Other results can be deduced from these, using the universal coefficient theorem.

Theorem 4.1. (a) $H^1(\mathbb{G}_{\mathbb{C}}(n), \mathbb{Z}) = \mathbb{Z}[\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_n]$ *is a graded polynomial algebra in n variables, where* $\mathbf{c}_i \in H^{2i}(\mathbb{G}_k(n), \mathbb{Z})$ *is homogeneous of degree* $2i$*. For any* $\mathbb{C}$*-vector bundle* E *of rank n, we call* $c_i(E) := [E]^*(\mathbf{c}_i)$ *the* i**-th Chern class** *of* E*.*

(b) $H^*(\mathbb{G}_{\mathbb{R}}(n), \mathbb{Z}[\frac{1}{2}]) = \mathbb{Z}[\frac{1}{2}][\mathbf{p}_1, \ldots, \mathbf{p}_{[\frac{n}{2}]}]$ *is a graded polynomial algebra, where* $\mathbf{p}_i$ *is homogeneous of degree* $4i$*. For any* $\mathbb{R}$*-vector bundle of rank n, we call* $p_i(E) := [E]^*(\mathbf{p}_i)$ *the* i**-th Pontryagin class** *of* E*.*

(c) $H^*(\mathbb{G}_{\mathbb{R}}(n), \mathbb{Z}/2\mathbb{Z}) = (\mathbb{Z}/2\mathbb{Z})[\mathbf{w}_1, \ldots, \mathbf{w}_n]$ *is a graded polynomial algebra, where* $\mathbf{w}_i$ *is homogeneous of degree* i*. For any* $\mathbb{R}$*-vector bundle* E *of rank n, we call* $w_i(E) := [E]^*(\mathbf{w}_i)$ *the* i**-th Stiefel-Whitney class** *of* E*.*

We now sketch a proof of parts (a) and (c) of this theorem; (b) will be proved later, since it involves considerations of oriented bundles. We begin with the following result; recall that $\mathbb{P}^n_k = \mathbb{G}_k(1, n+1)$ is the projective space of dimension n over k, parametrizing the set of lines in the vector space k^{n+1}.

Theorem 4.2. (1) *For* $n \geq 1$*, we have* $H^*(\mathbb{P}^n_{\mathbb{C}}, \mathbb{Z}) = \mathbb{Z}[x]/(x^{n+1})$*, where* x *is a homogeneous element of degree* 2*. Further,* $H^*(\mathbb{P}^\infty_{\mathbb{C}}, \mathbb{Z}) = H^*(\mathbb{G}_{\mathbb{C}}(1), \mathbb{Z}) = \mathbb{Z}[x]$ *is a graded polynomial algebra in 1 variable* x *of degree* 2*.*

(2) *For* $n \geq 1$*, we have* $H^*(\mathbb{P}^n_{\mathbb{R}}, \mathbb{Z}/2\mathbb{Z}) = (\mathbb{Z}/2\mathbb{Z})[y]/(y^{n+1})$*, where* y *is a homogeneous element of degree 1. Further,* $H^*(\mathbb{P}^\infty_{\mathbb{R}}, \mathbb{Z}/2\mathbb{Z}) = (\mathbb{Z}/2\mathbb{Z})[y]$ *is a graded polynomial algebra in 1 variable, with* $\deg y = 1$*.*

Proof. (Sketch) In both cases, the results for the finite dimensional projective spaces $\mathbb{P}^n$, and the compatibility with natural inclusions $\mathbb{P}^{n-1} \hookrightarrow \mathbb{P}^n$, will imply the result for the infinite projective spaces. So we will only discuss the finite dimensional cases.

Proof of (1): We may consider $\mathbb{P}^n_{\mathbb{C}}$ as the quotient space of $\mathbb{C}^{n+1} - \{0\}$ modulo the diagonal action of the multiplicative $\mathbb{C}^*$, or equivalently, as the quotient of the unit sphere $S^{2n+1} \subset \mathbb{C}^{n+1}$ modulo the action of the unit circle $S^1 \subset \mathbb{C}^*$.

The inclusion $\mathbb{C}^n \hookrightarrow \mathbb{C}^{n+1}$, as the subspace with vanishing last coordinate, induces an inclusion $\mathbb{P}^{n-1}_{\mathbb{C}} \hookrightarrow \mathbb{P}^n_{\mathbb{C}}$; the complement is the homeomorphic image of $\mathbb{C}^n \times \{1\} \subset \mathbb{C}^{n+1} - \{0\}$ under the quotient map $\mathbb{C}^{n+1} - \{0\} \to \mathbb{P}^n_{\mathbb{C}}$. This implies that the quotient space $\mathbb{P}^n_{\mathbb{C}}/\mathbb{P}^{n-1}_{\mathbb{C}}$ (obtained by collapsing $\mathbb{P}^{n-1}_{\mathbb{C}}$ to a point) is homeomorphic to the one-point compactification of $\mathbb{C}^n$, which is S^{2n}. By induction and the long exact sequence for the cohomology of the pair $(\mathbb{P}^n_{\mathbb{C}}, \mathbb{P}^{n-1}_{\mathbb{C}})$, we deduce that

$$H^i(\mathbb{P}^n_{\mathbb{C}}, \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } i = 2j \text{ with } 0 \leq j \leq n, \\ 0 & \text{otherwise.} \end{cases}$$

It remains to show that if x is a generator of $H^2(\mathbb{P}^n_{\mathbb{C}}, \mathbb{Z})$, then x^j is a generator of $H^{2j}(\mathbb{P}^n_{\mathbb{C}}, \mathbb{Z})$ for $2 \leq j \leq n$ (that x satisfies the relation $x^{n+1} = 0$ is clear, since $x^{n+1} \in H^{2n+2}(\mathbb{P}^n_{\mathbb{C}}, \mathbb{Z}) = 0$). This can be deduced by induction on n, and the Poincaré duality theorem, as follows. For $n = 1$ there is nothing to prove. If $i : \mathbb{P}^{n-1}_{\mathbb{C}} \to \mathbb{P}^n_{\mathbb{C}}$ is the inclusion, then the exact sequence for the pair $(\mathbb{P}^n_{\mathbb{C}}, \mathbb{P}^{n-1}_{\mathbb{C}})$ actually implies that $i^* : H^j(\mathbb{P}^n_{\mathbb{C}}, \mathbb{Z}) \to H^j(\mathbb{P}^{n-1}_{\mathbb{C}}, \mathbb{Z})$ is an isomorphism for $j < 2n$. Thus i^*x generates $H^2(\mathbb{P}^{n-1}_{\mathbb{C}}, \mathbb{Z})$, and if $(i^*x)^j = i^*(x^j)$ generates $H^{2j}(\mathbb{P}^{n-1}_{\mathbb{C}}, \mathbb{Z})$, then x^j generates $H^{2j}(\mathbb{P}^n_{\mathbb{C}}, \mathbb{Z})$ for $1 \leq j \leq n-1$. If we choose a generator $y \in H^{2n}(\mathbb{P}^n_{\mathbb{C}}, \mathbb{Z}) \cong \mathbb{Z}$ (corresponding to an *orientation* on the compact manifold $\mathbb{P}^n_{\mathbb{C}}$), then since x^{n-1} is a generator of $H^{2n-2}(\mathbb{P}^n_{\mathbb{C}}, \mathbb{Z}) \cong \mathbb{Z}$, Poincaré duality implies that there exists an element $z \in H^2(\mathbb{P}^n_{\mathbb{C}}, \mathbb{Z}) = \mathbb{Z}x$ such that $z \cup x^{n-1} = y$. Since $z = m \cdot x$ for some integer m, we have that $m \cdot x^n = y$. Since y is a generator of $H^{2n}(\mathbb{P}^n_{\mathbb{C}}, \mathbb{Z}) \cong \mathbb{Z}$, we must have $m = \pm 1$, and x^n is also a generator of $H^{2n}(\mathbb{P}^n_{\mathbb{C}}, \mathbb{Z})$. that $x^{n+1} = 0$ in $H^*(\mathbb{P}^n_{\mathbb{C}}, \mathbb{Z})$.

Proof of (2): This is along similar lines, using the description of $\mathbb{P}^n_{\mathbb{R}}$ as the quotient $\mathbb{R}^{n+1} - \{0\}/\mathbb{R}^* = S^n/(\mathbb{Z}/2\mathbb{Z})$, where the generator of $\mathbb{Z}/2\mathbb{Z}$ acts on S^n by the antipodal map $x \mapsto -x$. Again $\mathbb{P}^{n-1}_{\mathbb{R}} \hookrightarrow \mathbb{P}^n_{\mathbb{R}}$ with quotient space $\mathbb{P}^n_{\mathbb{R}}/\mathbb{P}^{n-1}_{\mathbb{R}}$ homeomorphic to the 1-point compactification of $\mathbb{R}^n$, namely S^n. This gives $H^i(\mathbb{P}^n_{\mathbb{R}}, \mathbb{Z}/2\mathbb{Z}) = 0$ for $i > n$. Further, S^n is simply connected, and the quotient map $S^n \to \mathbb{P}^n_{\mathbb{R}}$ is a covering space; hence $\mathbb{P}^n_{\mathbb{R}}$ has fundamental group $\mathbb{Z}/2\mathbb{Z}$. The long exact sequence of cohomology groups with $\mathbb{Z}/2\mathbb{Z}$-coefficients for the pair $(\mathbb{P}^n_{\mathbb{R}}, \mathbb{P}^{n-1}_{\mathbb{R}})$ implies that if $i : \mathbb{P}^{n-1}_{\mathbb{R}} \to \mathbb{P}^n_{\mathbb{R}}$ is the inclusion, then $i^* : H^j(\mathbb{P}^n_{\mathbb{R}}, \mathbb{Z}/2\mathbb{Z}) \to H^j(\mathbb{P}^{n-1}_{\mathbb{R}}, \mathbb{Z})$ is an isomorphism for $j < n-1$, and yields an exact sequence

$$0 \to H^{n-1}(\mathbb{P}^n_{\mathbb{R}}, \mathbb{Z}/2\mathbb{Z}) \xrightarrow{i^*} H^{n-1}(\mathbb{P}^{n-1}_{\mathbb{R}}, \mathbb{Z}/2\mathbb{Z}) \to H^n(S^n, \mathbb{Z}/2\mathbb{Z}) \to H^n(\mathbb{P}^n_{\mathbb{R}}, \mathbb{Z}/2\mathbb{Z}) \to 0.$$

Here $H^n(S^n, \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$ and $H^n(\mathbb{P}^n_{\mathbb{R}}, \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$ since S^n and $\mathbb{P}^n_{\mathbb{R}}$ are compact connected n-manifolds; this implies that

$$i^* : H^{n-1}(\mathbb{P}^n_{\mathbb{R}}, \mathbb{Z}/2\mathbb{Z}) \to H^{n-1}(\mathbb{P}^{n-1}_{\mathbb{R}}, \mathbb{Z}/2\mathbb{Z})$$

is an isomorphism. Now Poincaré duality and induction imply as before that if $x \in H^1(\mathbb{P}^n_{\mathbb{R}}, \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$ is a generator, then x^j is a generator of $H^j(\mathbb{P}^n_{\mathbb{R}}, \mathbb{Z}/2\mathbb{Z})$ for each $1 \leq j \leq n$. □

Remark 4.3. Alternate proofs of the above Theorem can be given, for example using multiplicative properties of the Serre spectral sequence for the Hopf fibration, which do not appeal to Poincaré duality.

Remark 4.4. As a consequence, we may define the following characteristic classes associated to *line bundles* (*i.e.*, vector bundles of rank 1), as follows. If $p : L \to X$ is a complex line bundle, and $[L] : X \to \mathbb{P}^\infty_\mathbb{C}$ is a classifying map, then define $[L]^*(x) = c_1(L)$, where $x \in H^2(\mathbb{P}^\infty_\mathbb{C}, \mathbb{Z})$ is the following generator: the inclusion $i : \mathbb{P}^1_\mathbb{C} \hookrightarrow \mathbb{P}^\infty_\mathbb{C}$ induces an isomorphism $i^* : H^2(\mathbb{P}^\infty_\mathbb{C}, \mathbb{Z}) \to H^2(\mathbb{P}^1_\mathbb{C}, \mathbb{Z})$; now $\mathbb{P}^1_\mathbb{C}$ is homeomorphic to the 2-sphere S^2, and $H^2(S^2, \mathbb{Z})$ has a standard generator y (corresponding to the standard orientation of S^2), and we take x to be the generator of $H^2(\mathbb{P}^\infty_\mathbb{C}, \mathbb{Z})$ to be the generator such that $i^*x = -y$ (the sign is to ensure compatibility with the definition of the *Euler class*, which we will define later). Similarly, $H^1(\mathbb{P}^\infty_\mathbb{R}, \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$ has a unique generator z; for any real line bundle $L \to X$, define $w_1(L) = [L]^*(z)$ for a classifying map $[L] : X \to \mathbb{P}^\infty_\mathbb{R}$.

We claim that for any two complex line bundles L_1, L_2, we have $c_1(L_1 \otimes L_2) = c_1(L_1) + c_1(L_2)$. Using suitable classifying maps for L_1 and L_2, we reduce to proving this in the following special case: $X = \mathbb{P}^m_\mathbb{C} \times \mathbb{P}^n_\mathbb{C}$, $L_1 = p_1^*\gamma_{1,m+1}$, $L_2 = \gamma_{1,n+1}$ where p_i, $i = 1, 2$ are the two projections, and $\gamma_{1,r+1} \to \mathbb{P}^r_\mathbb{C} = \mathbb{G}_\mathbb{C}(1, r+1)$ denotes the tautological line bundle, for any r (in fact we could further reduce to the case $n = m = 1$ if we please!). Now the natural map

$$H^2(\mathbb{P}^m_\mathbb{C}, \mathbb{Z}) \oplus H^2(\mathbb{P}^n_\mathbb{C}, \mathbb{Z}) \xrightarrow{p_1^*+p_2^*} H^2(\mathbb{P}^m_\mathbb{C} \times \mathbb{P}^n_\mathbb{C}, \mathbb{Z})$$

is an isomorphism, from the Künneth formula. For any point $(P, Q) \in \mathbb{P}^m_\mathbb{C} \times \mathbb{P}^n_\mathbb{C}$, if $i_Q : \mathbb{P}^m_\mathbb{C} \to \mathbb{P}^m_\mathbb{C} \times \mathbb{P}^n_\mathbb{C}$ is given by $t \mapsto (t, Q)$, and if $i_P : \mathbb{P}^n_\mathbb{C} \to \mathbb{P}^m_\mathbb{C} \times \mathbb{P}^n_\mathbb{C}$ is given by $s \mapsto (P, s)$, then i_P, i_Q are inclusions such that $p_1 \circ i_Q = 1_{\mathcal{P}^m}$, $p_2 \circ i_P = 1_{\mathcal{P}^n}$, while the other two composites $p_1 \circ i_P$ and $p_2 \circ i_Q$ are constant maps, and hence induce 0 on cohomology. Thus

$$i_P^* : H^2(\mathbb{P}^m_\mathbb{C} \times \mathbb{P}^n_\mathbb{C}, \mathbb{Z}) \to H^2(\mathbb{P}^n_\mathbb{C}, \mathbb{Z}),$$

$$i_Q^* : H^2(\mathbb{P}^m_\mathbb{C} \times \mathbb{P}^n_\mathbb{C}, \mathbb{Z}) \to H^2(\mathbb{P}^m_\mathbb{C}, \mathbb{Z})$$

are the two projections corresponding to the isomorphism $p_1^* + p_2^*$ considered above. Now $i_Q^*(L_1 \otimes L_2) \cong \gamma_{1,m+1}$ and $i_P^*(L_1 \otimes L_2) \cong \gamma_{1,n+1}$; hence $c_1(L_1 \otimes L_2)$ is the unique element of $H^2(\mathbb{P}^m_\mathbb{C} \times \mathbb{P}^n_\mathbb{C}, \mathbb{Z})$ which projects to $c_1(\gamma_{1,m+1})$ and $c_1(\gamma_{1,n+1})$; this element is clearly $p_1^*(c_1(\gamma_{1,m+1})) + p_2^*(c_1(\gamma_{1,n+1}))$, which is just $c_1(L_1) + c_1(L_2)$.

By an analogous argument, we also have $w_1(L_1 \otimes L_2) = w_1(L_1) + w_1(L_2)$ for any real line bundles L_1, L_2 on X.

Recall that if $p : E \to X$ is a k-vector bundle of rank r, then we may form the associated *projective bundle* $\pi : \mathbb{P}(E) \to X$, where $\mathbb{P}(E)$ is the quotient space $(E - 0_E(X))/k^*$ for the action of k^* by scalar multiplication on each fibre E_x. Thus the fibre over $x \in X$ of $\mathbb{P}(E) \to X$ is the projective space $\mathbb{P}(E_x) \cong \mathbb{P}^{r-1}_k$, which is the space of lines (1-dimensional k-vector subspaces) in E_x. There is an associated tautological line bundle on $\mathbb{P}(E)$, which restricts on each fibre $\mathbb{P}(E_x) \cong \mathbb{P}^{r-1}_k$ to the tautological line bundle $\gamma_{1,r}$; it is a subbundle of π^*E. Following the notation in algebraic geometry, we denote the tautological line bundle on $\mathbb{P}(E)$ by $\mathcal{O}_{\mathcal{P}(E)}(-1)$.

In the next theorem, the reader should keep in mind that (i) though the graded cohomology rings with $\mathbb{Z}$-coefficients may be non-commutative, homogeneous elements of even degree are central (ii) cohomology rings with $\mathbb{Z}/2\mathbb{Z}$-coefficients are always commutative. This follows from the general commutation formula $x \cup y = (-1)^{pq} y \cup x$, for homogeneous elements x, y of degrees p, q respectively.

Theorem 4.5. (Leray-Hirsch)

1) *Let* $p : E \to X$ *be a* complex *vector bundle of rank* n, *and* $\pi : \mathbb{P}(E) \to X$ *the associated* $\mathbb{P}^{n-1}_{\mathbb{C}}$*-bundle, with tautological line subbundle* $\mathcal{O}_{\mathbb{P}(E)}(-1) \hookrightarrow \pi^* E$. *Let* $\xi = c_1(\mathcal{O}_{\mathbb{P}(E)}(-1)) \in H^2(\mathbb{P}(E), \mathbb{Z})$. *Then the homomorphism on cohomology rings* $\pi^* : H^*(X, \mathbb{Z}) \to H^*(\mathbb{P}(E), \mathbb{Z})$ *makes* $H^*(\mathbb{P}(E), \mathbb{Z})$ *into a free module over* $H^*(X, \mathbb{Z})$ *with basis* $1, \xi, \dots, \xi^{n-1}$.
2) *Let* $p : E \to X$ *be a* real *vector bundle of rank* n, *and* $\pi : \mathbb{P}(E) \to X$ *the associated* $\mathbb{P}^{n-1}_{\mathbb{R}}$*-bundle, with tautological line subbundle* $\mathcal{O}_{\mathbb{P}(E)}(-1) \hookrightarrow \pi^* E$. *Let* $\xi = w_1(\mathcal{O}_{\mathbb{P}(E)}(-1)) \in H^1(\mathbb{P}(E), \mathbb{Z}/2\mathbb{Z})$. *Then the homomorphism on cohomology rings* $\pi^* : H^*(X, \mathbb{Z}/2\mathbb{Z}) \to H^*(\mathbb{P}(E), \mathbb{Z}/2\mathbb{Z})$ *makes* $H^*(\mathbb{P}(E), \mathbb{Z}/2\mathbb{Z})$ *into a free module over* $H^*(X, \mathbb{Z})$ *with basis* $1, \xi, \dots, \xi^{n-1}$.

Proof. (Sketch) We consider below the case of a complex vector bundle; the real case is similar. As usual we restrict to the case when X is compact Hausdorff, for simplicity; more generally the proof given will work for any finite vector bundle, in the sense of Remark 3.5. For the general case, one could use a spectral sequence argument, or (as in Spanier's book *Algebraic Topology*, Chapter 5, Sect. 7) use an argument very similar to ours to prove the corresponding result for homology, for arbitrary X, and then use formal arguments with free chain complexes to deduce the result for cohomology (Milnor's book also adopts a similar procedure).

If $E \to X$ is the trivial bundle, then the result is true by the Künneth formula (for the cohomology of a product space), and from the fomrula for the cohomology of a complex projective space (Theorem 4.2).

For any open subset $W \subset X$, we have maps

$$\oplus_{i=0}^{n-1} H^{j-2i}(W, \mathbb{Z}) \xrightarrow{\Phi^j_W} H^j(\mathbb{P}(E\mid_W), \mathbb{Z}),$$

$$(\alpha_0, \dots \alpha_{n-1}) \mapsto \sum_{i=0}^{n-1} \pi_W^*(\alpha_i) \cup (\xi_W)^i,$$

where $p_W : E\mid_W \to W$ is the restriction of the vector bundle E to the open set W, and $\pi_W : \mathbb{P}(W\mid_E) \to W$ is the corresponding projective bundle; $\xi_W \in H^2(\mathbb{P}(W\mid_E), \mathbb{Z})$ is c_1 of the corresponding tautological bundle, and is hence just the restriction to $\mathbb{P}(E\mid_W) \subset \mathbb{P}(E)$ of ξ.

From the Mayer-Vietoris exact sequence in cohomology and the 5-lemma, we see that if U, V are open subsets of X such that Φ^j_U, Φ^j_V and $\Phi^j_{U\cap V}$ are isomorphisms for all j, then $\Phi^j_{U\cup V}$ is also an isomorphism for all j. Now cover X by open subsets $V_1, \dots, V_r$ such that $E\mid_{V_i}$ is trivial for all i, and set $U_i = V_1 \cup V_2 \cup \cdots \cup V_i$. By induction on i, we then see that $\Phi^j_{U_i}$ is an isomorphism for all i, j; in particular, taking $i = r$, so that $U_i = X$, we have the theorem. $\square$

Corollary 4.6. 1) *For any complex vector bundle* E *of rank* n *on* X, *there is a unique relation*

$$\xi^n + \pi^*(\alpha_1)\xi^{n-1} + \pi^*(\alpha_2)\xi^{n-2} + \cdots + \pi^*(\alpha^n) = 0$$

in $H^*(\mathbb{P}(E), \mathbb{Z})$*, with* $\alpha_i \in H^{2i}(X, \mathbb{Z})$*.*

2) *For any real vector bundle* E *of rank* n *on* X*, there is a unique relation*

$$\xi^n + \pi^*(\alpha_1)\xi^{n-1} + \pi^*(\alpha_2)\xi^{n-2} + \cdots + \pi^*(\alpha^n) = 0$$

in $H^*(\mathbb{P}(E), \mathbb{Z}/2\mathbb{Z})$*, with* $\alpha_i \in H^i(X, \mathbb{Z}/2\mathbb{Z})$*.*

Definition 4.7. 1) For any complex vector bundle E of rank n on X, define its i-th *Chern class* to be $c_i(E) = \alpha_i$, where α_i is as in 1) of the above corollary 4.6.

2) For any real vector bundle E of rank n on X, define its i-th *Stiefel-Whitney class* to be $w_i(E) = \alpha_i$, where α_i is as in 2) of the above corollary.

Remark 4.8. (a) The new definition of c_1 of a complex line bundle agrees with the old one, since for a line bundle L, we have $\mathbb{P}(L) = X$, and the tautological line bundle on $\mathbb{P}(L)$ is L itself. Similarly there is no ambiguity in defining $w_1(L)$ for a real line bundle L.
(b) Since, as noted earlier, the Leray-Hirsch theorem is valid for bundles on arbitrary spaces, the definitions of the Chern classes and Stiefel-Whitney classes make sense for bundles on arbitrary base spaces X, for example, for the tautological bundles on infinite Grassmannians.

Another corollary of the Leray-Hirsch theroem is the following.

Corollary 4.9. (Splitting principle)

1) *Let* $p : E \to X$ *be a complex vector bundle which supports a positive definite Hermitian inner product (e.g., if* X *is compact Hausdorff). Then there exists a continuous map* $f : P \to X$ *such that (a)* f^*E *is a direct sum of complex line bundles on* P *(b)* $f^* : H^*(X, \mathbb{Z}) \to H^*(P, \mathbb{Z})$ *is injective.*
2) *Let* $p : E \to X$ *be a real vector bundle which supports a positive definite inner product (e.g., if* X *is compact Hausdorff). Then there exists a continuous map* $f : P \to X$ *such that (a)* f^*E *is a direct sum of real line bundles on* P *(b)* $f^* : H^*(X, \mathbb{Z}/2\mathbb{Z}) \to H^*(P, \mathbb{Z}/2\mathbb{Z})$ *is injective.*

Proof. The proof in the real and complex cases is similar, so we consider the latter. We work by induction on the rank of E, where we may take $P = X$ if E has rank 1. In general, if rank$E = n > 1$, note that $\pi : \mathbb{P}(E) \to X$ satisfies the condition that $\pi^* : H^*(X, \mathbb{Z}) \to H^(\mathbb{P}(E), \mathbb{Z})$ is injectie, and there is a line subbundle $\mathcal{O}_{\mathcal{P}(E)}(-1) \subset \pi^*E$. Choosing a Hermitian metric on E, we may write $\pi^*E = \mathcal{O}_{\mathcal{P}(E)}(-1) \oplus F$, where $q : F \to \mathbb{P}(E)$ has rank $n - 1$. Now by induction, there is a map $g : P \to \mathbb{P}(E)$ such that g^*F is a direct sum of complex line bundles, and g^* is injective on cohomology rings. Hence $f = g \circ \pi$ satisfies the desired conditions. ☐

Remark 4.10. We call a map $f : P \to X$ as in corollary 4.9 a *splitting map* for the vector bundle E. It is easy to see that if $E_1, \ldots, E_r$ are vector bundles, then there exists a continuous map $f : P \to X$ which is simultaneously a splitting map for each of the bundles E_i (for example, if $f_1 : P_1 \to X$ is a splitting map for E_1, and $f_2 : P_2 \to P_1$ is a splitting map for $f_1^*E_2$, then $f_2 \circ f_1 : P_2 \to$

X is simultaneously a splitting map for E_1 as well as E_2). Note that, by the above lemma, splitting maps exist for the tautological bundle γ_n on the infinite Grassmannian $\mathbb{G}_k(n)$.

Lemma 4.11.

1) *Let $L_1, \ldots, L_n$ be $\mathbb{C}$-line bundles, such that there exists a nowhere-vanishing section $s \in \Gamma(X, L_1 \oplus \cdots \oplus L_n)$ (i.e., $s(x) \neq 0_{E_x}$ for any $x \in X$, where $E = L_1 \oplus \cdots \oplus L_n$)). Then*
$$c_1(L_1) \cup \cdots \cup c_1(L_n) = 0$$
in $H^{2n}(X, \mathbb{Z})$.

2) *Let $L_1, \ldots, L_n$ be $\mathbb{R}$-line bundles, such that there exists a nowhere-vanishing section $s \in \Gamma(X, L_1 \oplus \cdots \oplus L_n)$ (i.e., $s(x) \neq 0_{E_x}$ for any $x \in X$, where $E = L_1 \oplus \cdots \oplus L_n$)). Then*
$$w_1(L_1) \cup \cdots \cup w_1(L_n) = 0$$
in $H^n(X, \mathbb{Z}/2\mathbb{Z})$.

Proof. We consider the complex case, since the real case is similar. Let $s_i \in \Gamma(X, L_i)$ be the component of s in L_i, and $U_i = \{x \in X \mid s_i(x) \neq 0_{(L_i)_x}\}$ be the locus where s_i does not vanish. Then we are given that $\{U_i\}_{i=1}^n$ is an open cover of X. Now for each i, we have that $c_1(L_i) \mapsto 0$ under $H^2(X, \mathbb{Z}) \to H^2(U_i, \mathbb{Z})$, since we have a trivialization $L_i \mid_{U_i} \cong U_i \times \mathbb{C}$ (using the section s_i), and c_1 of the trivial line bundle vanishes. Hence we can find relative cohomology classes $\widetilde{c_1}(L_i) \in H^2(X, U_i; \mathbb{Z})$ such that $\widetilde{c_1}(L_i) \mapsto c_1(L_i)$. Then the cup product
$$\widetilde{c_1}(L_1) \cup \cdots \cup \widetilde{c_1}(L_n) \in H^{2n}(X, U_1 \ldots, U_n; \mathbb{Z})$$
maps to $c_1(L) \cup \cdots \cup c_1(L_n)$ under the natural map
$$H^{2n}(X, U_1 \ldots, U_n; \mathbb{Z}) \to H^{2n}(X, \mathbb{Z}).$$
But $U_1 \cup \cdots \cup U_n = X$, so that $H^{2n}(X, U_1 \cup \ldots, \cup U_n; \mathbb{Z}) = 0$. □

Corollary 4.12. 1) *If $p : E \to X$ is a complex vector bundle of rank n, and $f : P \to X$ a splitting map for E, with $f^*E = L_1 \oplus \cdots \oplus L_n$, then*
$$f^*(c_i(E)) = \text{ i-th elementary symmetric function in } c_1(L_1), \ldots, c_1(L_n).$$

2) *If $p : E \to X$ is a real vector bundle of rank n, and $f : P \to X$ a splitting map for E, with $f^*E = L_1 \oplus \cdots \oplus L_n$, then*
$$f^*(w_i(E)) = \text{ i-th elementary symmetric function in } w_1(L_1), \ldots, w_1(L_n).$$

Proof. As usual, we consider the case of complex vector bundles, and leave the (very similar) case of real bundles to the reader.

Let Q be the fibre product
$$\begin{array}{ccc} Q = P \times_X \mathbb{P}(E) & \xrightarrow{g} & \mathbb{P}(E) \\ \eta \downarrow & & \downarrow \pi \\ P & \xrightarrow{f} & X \end{array}$$

Then the inclusion of the tautological line subbundle $L = \mathcal{O}_{\mathcal{P}(E)}(-1) \subset \pi^* E$ induces an inclusion of a line subbundle

$$g^* L \hookrightarrow g^* \pi^* E = \eta^* f^* E = \eta^* (L_1 \oplus \cdots \oplus L_n).$$

Thus we have an inclusion of a trivial line subbundle

$$Q \times \mathbb{C} \hookrightarrow \left((\eta^*(L_1) \otimes g^*(L^\vee)) \oplus \cdots \oplus (\eta^*(L_n) \otimes g^*(L^\vee)) \right),$$

where $L^\vee$ denotes the dual line bundle. The inclusion of a trivial line bundle is equivalent to giving a section which does not vanish anywhere, and so by lemma 4.11, we have an identity in $H^*(Q, \mathbb{Z})$

$$\prod_{i=1}^{n} (\eta^*(c_1(L_i) + g^*\xi) = 0.$$

Thus we have a relation

$$g^*(\xi^n) + \eta^* s_1(c_1(L_1), \ldots, c_1(L_n)) g^*(\xi^{n-1}) + \eta^* s_2(c_1(L_1), \ldots, c_1(L_n)) g^*(\xi^{n-2}) + \cdots + \eta^* s_n(c_1(L_1), \ldots, c_1(L_n)) = 0,$$

where s_i denotes the i-th elementary symmetric polynomial (note that the classes $c_1(L_i)$ are in the centre of the cohomology ring, and so it makes sense to evaluate a polynomial on the $c_1(L_i)$). We also have a relation

$$g^*(\xi^n) + g^*\pi^*(c_1(E)) g^*(\xi^{n-1}) + g^*\pi^*(c_2(E)) g^*(\xi^{n-2}) + \cdots + g^*\pi^*(c_n(E)) = 0,$$

which we may rewrite as

$$g^*(\xi^n) + \eta^* f^*(c_1(E)) g^*(\xi^{n-1}) + \eta^* f^*(c_2(E)) g^*(\xi^{n-2}) + \cdots + \eta^* f^*(c_n(E)) = 0.$$

Since $Q = \mathbb{P}(f^*E)$ is a projective bundle over P, the elements $g^*(\xi^j) = g^*(\xi)^j$, $0 \leq j \leq n-1$ are linearly independent over $H^*(P, \mathbb{Z})$, by Theorem 4.5, and so the above two monic relations satisfied by $g^*(\xi)$ must coincide. Thus, comparing coefficients, and using the injectivity on cohomology of η^*, we get that

$$f^* c_i(E) = s_i(c_1(L_1), \ldots, c_1(L_n)).$$

□

Corollary 4.13. (Whitney sum formula)

1) *Let $E \to X$, $F \to X$ be two complex vector bundles. Then we have a formula*

$$\sum_{i \geq 0} c_i(E \oplus F) = \left(\sum_{i \geq 0} c_i(E))(\sum_{i \geq 0} c_i(F) \right)$$

in $H^*(X, \mathbb{Z})$.

2) *Let $E \to X$, $F \to X$ be two real vector bundles. Then we have a formula*

$$\sum_{i \geq 0} w_i(E \oplus F) = \left(\sum_{i \geq 0} w_i(E))(\sum_{i \geq 0} w_i(F) \right)$$

in $H^*(X, \mathbb{Z}/2\mathbb{Z})$.

Proof. Notice that if E is a complex vector bundle which is a direct sum of line bundles, $E \cong L_1 \oplus \cdots \oplus L_n$, then from corollary 4.12, we have an expression

$$\sum_{i\geq 0} c_i(E) = \prod_{i=1}^{n}(1 + c_1(L_i)).$$

We now prove 1): by the splitting principle, we reduce to considering the case when E and F are both direct sums of complex line bundles, say $E \cong L_1 \oplus \cdots \oplus L_r$, $F \cong M_1 \oplus \cdots \oplus M_s$; then we have that

$$E \oplus F \cong L_1 \oplus \oplus L_r \oplus M_1 \oplus \cdots M_s$$

is also a direct sum of line bundles, and so we have formulas

$$\sum_{i\geq 0} c_i(E) = \prod_{i=1}^{r}(1 + c_1(L_i)),$$

$$\sum_{i\geq 0} c_i(F) = \prod_{j=1}^{s}(1 + c_1(M_j)),$$

$$\sum_{i\geq 0} c_i(E \oplus F) = \left(\prod_{i=1}^{r}(1 + c_1(L_i))\right)\left(\prod_{j=1}^{s}(1 + c_1(M_j))\right);$$

from these formulas, the desired formula in 1) is obvious. The proof of 2) is very similar. □

Remark 4.14. If E is a complex vector bundle of rank n on X, and $f : Y \to X$ is a splitting map for E, with $f^*E = L_1 \oplus \cdots \oplus L_n$, then $x_i = c_1(L_i) \in H^2(Y, \mathbb{Z})$ are called *Chern roots* for E. This terminology is because we have a factorization in the polynomial algebra $H^*(Y, \mathbb{Z})[t]$

$$t^n - f^*c_1t^{n-1} + \cdots + (-1)^n f^*c_n = \prod_{i=1}^{n}(t - x_i);$$

thus if we let $C(t)$ denote the polynomial on the left, the roots of $C(t) = 0$ are the x_i, $1 \leq i \leq n$. An equivalent way of expressing the above factorization is

$$t^n + f^*c_1t^{n-1} + \cdots + f^*c_n = \prod_{i=1}^{n}(t + x_i).$$

Example 4.15. Show that the Chern classes of a tensor product $E \otimes F$ of two complex vector bundles are given by 'universal' polynomials with integer coefficients in the Chern classes of E and F.

We now give a proof of Theorem 4.1(a), assuming that the classification of vector bundles via homotopy classes of maps to an infinite Grassmannian, the formula for the cohomology ring of a projective bundle, the resulting formalism of Chern classes, and the splitting principle, are all valid even when the base space X is a "sufficiently good" non-compact Hausdorff space; this can be rigorously justified, but we do not do this here. We will need that the above results hold even when X is an infinite dimensional CW-complex, with finitely many cells of any given dimension. The idea is that if X is such a space, and X_n is its n-skeleton, then X_n is a compact Hausdorff space, $\cup_{n\geq 0} X_n = X$, and any compact subset of X lies in some X_n; further, a map $X \to Y$ is continuous if and only if its restriction to each X_n is continuous. Thus, using the theory developed above, applied to each of the "finite dimensional approximations" X_n, one can extend its validity to such spaces X as well.

Another way to make our arguments rigorous is to use the fact that for any $i \geq 0$, the natural maps

$$H^i(X_{n+1}, A) \to H^i(X_n, A),$$

and hence also

$$H^i(X, A) \to H^i(X_n, A),$$

are isomorphisms for $n > i$. Thus, any conclusions regarding cohomology of any infinite dimensional CW complex X as above can be obtained by considering the cohomology groups of the finite dimensional approximations X_n. This approach avoids the need for constructing classifying maps for vector bundles on such an infinite dimensional space X.

Consider the space $X = (\mathbb{P}^\infty_{\mathbb{C}})^n = \mathbb{P}^\infty_{\mathbb{C}} \times \mathbb{P}^\infty_{\mathbb{C}} \times \cdots \times \mathbb{P}^\infty_{\mathbb{C}}$. If $p_i : X \to \mathbb{P}^\infty_{\mathbb{C}}$ is the i-th projection, then there is a vector bundle $E = p_1^*\gamma_{1,\infty} \oplus p_2^*\gamma_{1,\infty} \oplus \cdots \oplus p_n^*\gamma_{1,\infty}$ on X of rank n. Let $f : X \to \mathbb{G}_{\mathbb{C}}(n)$ be a classifying map for this bundle. We claim that if γ_n is the tautological bundle on $\mathbb{G}_{\mathbb{C}}(n)$, and $g : P \to \mathbb{G}_{\mathbb{C}}(n)$ is a splitting map for γ_n, then there is a continuous map $h : P \to X$ giving a diagram, commutative up to homotopy,

$$\begin{array}{ccc} P & \xrightarrow{h} & X \\ & \searrow g & \downarrow f \\ & & \mathbb{G}_{\mathbb{C}}(n) \end{array}$$

Hence the natural map on cohomology $f : H^*(\mathbb{G}_{\mathbb{C}}(n), \mathbb{Z}) \to H^*(X, \mathbb{Z})$ is *injective*, and f is itself a splitting map for γ_n. If $\sigma : X \to X$ is any permutation of the factors, then there is a natural isomorphism $\sigma^* E \cong E$; hence $f \circ \sigma$ must be homotopic to f, and so $f^* = \sigma^* \circ f^*$ on $H^*(\mathbb{G}_{\mathbb{C}}(n), \mathbb{Z})$. This means that the subring

$$f^*(H^*(\mathbb{G}_{\mathbb{C}}(n), \mathbb{Z})) \subset H^*(X, \mathbb{Z}) \cong \otimes_{i=1}^n H^*(\mathbb{P}^\infty_{\mathbb{C}}, \mathbb{Z})) = \mathbb{Z}[t_1, \ldots, t_n]$$

(where $t_i = p_i^*(x)$, for the generator $x \in H^2(\mathbb{P}^\infty_{\mathbb{C}}, \mathbb{Z})$) is contained in the ring of invariants for the permutation group S_n on n symbols, acting by permuting the variables t_i. Hence if $s_i(t_1, \ldots, t_n)$ denotes the i-th elementary symmetric polynomial, then

$$f^*(H^*(\mathbb{G}_{\mathbb{C}}(n), \mathbb{Z})) \subset \mathbb{Z}[s_1(t_1, \ldots, t_n), \ldots, s_n(t_1, \ldots, t_n)].$$

But by corollary 4.12, $s_i(t_1, \ldots, t_n) = c_i(E) = f^*(c_i(\gamma_n))$. Since f^* is injective, we deduce that $H^*(\mathbb{G}_{\mathbb{C}}(n), \mathbb{Z})$ is the polynomial algebra in the n (algebraically independent) elements $c_1(\gamma_n), \ldots, c_n(\gamma_n)$.

In a similar way, we may formally deduce the structure of $H^*(\mathbb{G}_{\mathbb{R}}(n), \mathbb{Z}/2\mathbb{Z})$ (*i.e.*, theorem 4.1(c)) from the theory of Steifel-Whitney classes applied to an analogous bundle on $X = (\mathbb{P}^\infty_{\mathbb{R}})^n$, and the splitting principle applied to the universal bundle on the infinite Grassmanian $\mathbb{G}_{\mathbb{R}}(n)$.

5. Oriented Bundles, the Thom Class and the Euler Class

Recall that an *orientation* for a real vector space V of dimension n is an equivalence class of bases $\{v_1, \dots, v_n\}$, where this basis is equivalent to $\{v'_1, \dots, v'_n\}$ if the transition matrix $[a_{ij}]$, determined by the relations $v'_i = \sum_j a_{ij}v_j$, has a positive determinant. An equivalent way to do this is to choose a connected component of $\overset{n}{\wedge} V - \{0\}$; another is to choose a generator for the homology group $H_n(V, V - \{0\}; \mathbb{Z}) \cong \mathbb{Z}$, or equivalently for $H^n(V, V - \{0\}; \mathbb{Z})$. A basis of an oriented vector space V is called an *oriented basis* of V if it is a member of the equivalence class of bases giving the orientation of V.

If V is oriented, then so is its dual $V^\vee$, in a natural way, via dual bases. If V_1, V_2 are oriented real vector spaces, then their direct sum $V_1 \oplus V_2$, tensor product $V_1 \otimes V_2$ and internal Hom $\mathrm{Hom}(V_1, V_2)$ are also oriented in standard ways, the latter in such a way that the canonical isomorphism $V_1^\vee \otimes V_2 \to \mathrm{Hom}(V_1, V_2)$ is orientation preserving. The orientations chosen for the direct sum and tensor product (and hence Hom) is a matter of convention, but are functorial. If v_{1i} $1 \le i \le m$, and v_{2j}, $1 \le j \le n$ are oriented bases for V_1, V_2 respectively, then $v_{11}, v_{12}, \dots, v_{1m}, v_{21}, \dots, v_{2n}$ is taken to be an oriented basis for $V_1 \oplus V_2$. For the tensor product, one standard choice for an orientation is the class of the "reverse lexicographically ordered" basis $v_{1i} \otimes v_{2j}$, where v_{1i} is an oriented basis for V_1, and v_{2j} is an oriented basis for V_2: thus $v_{1i} \otimes v_{2j}$ precedes $v_{1i'} \otimes v_{2j'}$ if $j < j'$, or $j = j'$ and $i < i'$.

Example 5.1. Let V be a complex vector space of dimension n. Then the underlying real vector space has a standard orientation: if $v_1, \dots, v_n$ is any $\mathbb{C}$-basis for V, then an oriented $\mathbb{R}$-basis is $v_1, \sqrt{-1}v_1, v_2, \sqrt{-1}v_2, \dots, v_n, \sqrt{-1}v_n$. This is also a matter of convention, but has the advantages that (i) $\mathbb{C}$ acquires its standard orientation (ii) the orientation of a complex vector space is independent of the choice of $\mathbb{C}$-basis, and is compatible with direct sums of vector spaces (iii) any complex linear automorphism of a $\mathbb{C}$-vector space V preserves orientation.

An *orientation* of a real vector bundle $p : E \to X$ of rank n is a choice of orientation of each fiber vector space $E_x = p^{-1}(x)$, such that under any local trivialization $\phi : p^{-1}(U) \to U \times \mathbb{R}^n$ (compatible with the structure of E as a vector bundle) such that U is connected, the induced orientation on $\mathbb{R}^n$ from the linear isomorphism $\phi_x : E_x \to \mathbb{R}^n$ is independent of $x \in U$ (*i.e.*, the orientation on E_x "varies continuously with x"). A vector bundle is called *orientable* if it has at least 1 orientation; if the base space X is connected, then an orientable bundle has precisely two orientations.

Example 5.2. A C^∞ manifold is M called *orientable* if its tangent bundle TM is orientable. This is seen to be consistent with the definition used in algebraic topology, namely that there exists a locally consistent choice of a generator of $H_n(M, M - \{x\}; \mathbb{Z}) \cong \mathbb{Z}$ for each $x \in M$. Indeed, we may identify a neighbourhood of 0_x in the tangent space T_xM with a neighbourhood of x in M, by choosing a Riemannian metric on M, and using the corresponding exponential mapping $\mathrm{Exp}_x : (T_xM(\varepsilon), 0_x) \to (M, x)$, where $T_xM(\varepsilon)$ is the ball of radius ε in T_xM; for small ε, this is a diffeomorphism onto its image, and yields an isomorphism of $H_n(M, M - \{x\}; \mathbb{Z})$ with $H_n(T_xM, T_xM - 0_x; \mathbb{Z})$, which is in fact independent of the metric. One shows that for points x, y in a small closed coordinate neighbourhood U in M, the composite identification

$$H_n(M, M - \{x\}; \mathbb{Z}) \xrightarrow{\cong} H_n(M, M - U; \mathbb{Z}) \xleftarrow{\cong} H_n(M, M - \{y\}; \mathbb{Z})$$

is compatible with the identification

$$H_n(T_xM, T_xM - 0_x; \mathbb{Z}) \xrightarrow{\cong} H_n(TM\mid_U, TM\mid_U - 0_M; \mathbb{Z}) \xleftarrow{\cong} H_n(T_yM, T_yM - 0_y; \mathbb{Z})$$

induced by the local trivialization of $TM\mid_U$ (since this is a local property, we can reduce to showing this for $\mathbb{R}^n$ with the Euclidean metric, where it is easy). Thus the choice of a locally compatible set of generators of the groups

$$\{H_n(M, M - \{x\}; \mathbb{Z})\}_{x \in M}$$

is equivalent to the choice of an orientation for TM. Part of the Poincaré duality theorem then asserts the equivalence of these choices with the choice of a fundamental homology class in $H_n(M, \mathbb{Z})$.

From the equivalent characterizations stated earlier of orientation of a real vector space, we obtain the following equivalent characterizations of orientation of a real vector bundle $p : E \to X$ of rank n:

(i) an equivalence class of trivializations of the real line bundle $\overset{n}{\wedge} E$, where two trivializations determined by nowhere vanishing sections $s, t \in \Gamma(X, \overset{n}{\wedge} E)$ are equivalent if $t = us$ for a continuous, everywhere positive function $u \in C_{\mathbb{R}}(X)$.

(ii) a choice of a generator of $H_n(E_x, E_x - \{0_x\}; \mathbb{Z}) \cong \mathbb{Z}$ which "varies continuously with x", *i.e.*, a trivialization of the local coefficient system determined by $\{H_n(E_x, E_x - \{0_x\}; \mathbb{Z})\}_{x \in X}$. We may instead rephrase this in terms of the dual local coefficient system $\{H^n(E_x, E_x - \{0_x\}; \mathbb{Z})\}_{x \in X}$.

There is an evident notion of pull-back of oriented bundles, giving rise to a category of oriented bundles, together with a forgetful functor to the category of real vector bundles; the category of oriented bundles has duals, direct sums, tensor products and internal Hom's (*i.e.*, if $E_i \to X$ are oriented bundles, then the bundle $\mathrm{Hom}(E_1, E_2)$ is oriented in a functorial way).

Recall that if $p : E \to X$ is a vector bundle, then $0_X \subset E$ denotes the image of the 0-section.

Theorem 5.3. (Thom isomorphism) *Let $p : E \to X$ be an oriented vector bundle of rank n, with orientation $\{\xi(x) \in H^n(E_x, E_x - \{0_x\}; \mathbb{Z})\}_{x \in X}$. Then*

$$H^i(E, E - 0_X; \mathbb{Z}) = \begin{cases} 0 & \text{if } i < n \\ \mathbb{Z} & \text{if } i = n, \end{cases}$$

and $H^n(E, E - 0_X; \mathbb{Z})$ has a unique generator $\xi = \xi(E)$ which maps to the orientation element $\xi(x) \in H^n(E_x, E_x - \{0_x\}; \mathbb{Z})$ for each $x \in X$. The ring (without *identity)* $H^*(E, E - 0_X; \mathbb{Z})$ *is a free* $H^*(X, \mathbb{Z})$*-module of rank 1 with* ξ *as a generator; equivalently, the maps*

$$H^j(X, \mathbb{Z}) \to H^{j+n}(E, E - 0_X; \mathbb{Z}),$$

$$a \mapsto \xi \cup p^* a,$$

are isomorphisms for all $j \geq 0$.

Proof. (Sketch) The proof for compact X (or more generally, for finite vector bundles) is similar to the proof of Theorem 4.5, via a Mayer-Vietoris argument. First it is easy to prove for a trivial bundle, using the Künneth formula. Next, one shows that if it is true for the restrictions of $p : E \to X$ (with its chosen orientation) to open subsets U, V and $U \cap V$, then it is true for the restriction to $U \cup V$ as well. This allows one to obtain the result when X is compact, *e.g.*, a finite CW complex.

The proof for arbitrary X uses a limit argument, which is however a little delicate (see Milnor's book, Chapter 10, for details). $\square$

Definition 5.4. The distinguished cohomology class $\xi(E) \in H^n(E, E - 0_X; \mathbb{Z})$ determined in Theorem 5.3 is called the **Thom class** of the oriented bundle $p : E \to X$. Since $p^* : H^i(X, \mathbb{Z}) \to H^i(E, \mathbb{Z})$ is an isomorphism for all i (with inverse given by pull-back along the 0-section), the image of ξ in $H^n(E; \mathbb{Z})$ is of the form $p^*(e(E))$ for a unique element $e(E) \in H^n(X, \mathbb{Z})$, called the **Euler class** of the oriented bundle E.

One easy, but important, corollary of the Thom isomorphism theorem is the Gysin exact sequence, stated below.

Corollary 5.5. (Gysin exact sequence) *Let* $p : E \to X$ *be an oriented* $\mathbb{R}$*-vector bundle of rank* n*, and* $e(E) \in H^n(X; \mathbb{Z})$ *its Euler class. Then there is a long exact sequence*

$$\cdots \to H^i(X; \mathbb{Z}) \xrightarrow{-\cup e(E)} H^{i+n}(X; \mathbb{Z}) \xrightarrow{p^*} H^{i+n}(E - 0_X; \mathbb{Z}) \to H^{i+1}(X; \mathbb{Z}) \to \cdots$$

Proof. This is just the long exact sequence for the integral cohomology of the pair $(E, E - 0_X)$, where we have replaced $H^j(E; \mathbb{Z})$ with the group isomorphic to it (via p^*) $H^j(X; \mathbb{Z})$, and similarly used the Thom isomorphism to replace $H^j(E, E - 0_X; \mathbb{Z})$ with $H^{j-n}(X; \mathbb{Z})$; the induced maps then have the stated description, by an obvious functoriality of the cup product. $\square$

Some further obvious functoriality properties of the Thom and Euler classes are listed below. The proofs are immediate, and left as an exercise to the reader.

Lemma 5.6. (i) *The Thom and Euler classes of an oriented real vector bundle* $p : E \to X$ *are characteristic classes, i.e., for any continuous map* $f : Y \to X$*, the pull-back bundle* $f^* E \to Y$ *has a canonical induced orientation, and the Thom class and Euler class of* $f^* E$ *are respectively the pull-backs of the Thom class and Euler class of* E.

(ii) *If* ξ *and* e *are the Thom class and Euler class, respectively, of an oriented bundle* $p : E \to X$*, then the Thom class and Euler class of the same bundle with reversed orientation are* $-\xi$ *and* $-e$.

(iii) *If* $p_i : E_i \to X$, $i = 1, 2$ *are oriented real vector bundles of ranks* n_1, n_2 *respectively, then their direct sum* $E_1 \oplus E_2$ *is oriented in a natural way, such that the orientation of each fiber* $(E_1 \oplus E_2)_x = (E_1)_x \oplus (E_2)_x$ *has the direct sum orientation. If* $f_i : E_1 \oplus E_2 \to E_i$ *are the projections, then the Thom class of* $E_1 \oplus E_2$ *satisfies* $\xi(E_1 \oplus E_2) = f_1^*\xi(E_1) \cup f_2^*\xi(E_2)$*, and the Euler class satisfies* $e(E_1 \oplus E_2) = e(E_1) \cup e(E_2)$.

Example 5.7. Let $p : E \to X$ be an oriented vector bundle of odd rank n. Then multiplication by -1 is an isomorphism of the oriented bundle E with E', defined to be the same vector bundle but with the opposite orientation. Hence if e is the Euler class of the oriented bundle E, then $e = -e$ in $H^n(X, \mathbb{Z})$, so that e is a 2-torsion class. Thus, it is usually more interesting to consider Euler classes only for bundles of even rank.

Next, we sketch the proof of a result which motivates the choice of the term "Euler class". Let $< \, , \, >$ denote the Kronecker index (*i.e.*, the evaluation of a cohomology class on a homology class of the same degree).

Theorem 5.8. *Let* M *be a compact, oriented* C^∞ *manifold of dimension* n *(so that the tangent bundle* TM *is oriented), and let* $[M] \in H_n(M, \mathbb{Z})$ *be the corresponding fundamental class. Then*

$$< e(TM), [M] >= \sum_{i \geq 0} (-1)^i \mathrm{rank}\, H_i(M, \mathbb{Z}) = \text{ Euler characteristic of } M.$$

Proof. (Sketch) We reduce immediately to the case when M is connected. Let $\Delta_M : M \to M \times M$ be the diagonal map, and let $p_i : M \times M \to M$, $i = 1, 2$, be the two projections.

Since M is oriented, we are given a generator of $H^n(M, M - \{x\}; \mathbb{Z}) \cong \mathbb{Z}$ for each $x \in M$. We claim there is a unique class

$$\widetilde{[M]} \in H^n(M \times M, M \times M - \Delta_M(M); \mathbb{Z})$$

whose restriction to the subset $\{x\} \times M \cong M$ yields the chosen generator of $H^n(M, M - \{x\}; \mathbb{Z})$, for each $x \in M$ (by analogy, $\widetilde{[M]}$ is sometimes called the *Thom class* of the oriented manifold; it can also be defined for oriented topological manifolds).

Indeed, fix a Riemannian metric on M, and let $\pi : TM \to M$ be the bundle projection. The exponential map $\mathrm{Exp} : TM \to M$ (defined everywhere since M is compact) yields a map $\psi = (\pi, \mathrm{Exp}) : TM \to M \times M$, such that for each $x \in M$, $\psi(T_xM, 0_x) = \{x\} \times (M, x)$, and in particular, ψ maps the 0-section $0_M \subset TM$ isomorphically onto the diagonal $\Delta(M) \subset M \times M$. Hence for some $\varepsilon > 0$, ψ yields a diffeomorphism of TM_ε with $N_\varepsilon \subset M \times M$, where TM_ε is the open ε-ball around the 0-section in the tangent bundle, and N_ε is a neighbourhood of the diagonal in $M \times M$. We may take $\widetilde{[M]}$ to be the image of the Thom class $\xi(TM)$ under the induced composite isomorphism

$$\begin{aligned} H^n(TM, TM - 0_M; \mathbb{Z}) &\overset{\cong}{\leftarrow} H^n(TM_\varepsilon, TM_\varepsilon - 0_M; \mathbb{Z})) \overset{(\psi^{-1})^*}{\longrightarrow} \\ &H^n(N_\varepsilon, N_\varepsilon - \Delta_M(M); \mathbb{Z}) \overset{\cong}{\leftarrow} H^n(M \times M, M \times M - \Delta_M(M); \mathbb{Z}), \end{aligned}$$

where the first and last isomorphism are by excision. The restriction of $\widetilde{[M]}$ to $\{x\}\times M$ is the chosen generator, since $\psi(T_xM, 0_x) = \{x\}\times(M, x)$, and the Thom class has a similar restriction property with respect to the fibre T_xM; since this restriction property uniquely characterizes the Thom class, the class $\widetilde{[M]}$ is also uniquely characterized.

We also see that the image of $\widetilde{[M]}$ under the composition

$$H^n(M\times M, M\times M - \Delta_M(M);\mathbb{Z}) \to H^n(M\times M;\mathbb{Z}) \xrightarrow{\Delta_M^*} H^n(M,\mathbb{Z})$$

is just the Euler class $e(TM)$ of the tangent bundle.

Let $u_M \in H^n(M\times M;\mathbb{Z})$ denote the image of $\widetilde{[M]}$. We claim next that $p_1^*a\cup u_M = p_2^*a\cup u_M$ for any $a\in H^*(M,\mathbb{Z})$, where $p_i : M\times M\to M$ are the projections. Indeed, it suffices to prove a similar formula with $\widetilde{[M]}$ in place of u_M, and by excision, it further suffices to show equality of the two expressions in $H^{*+n}(N_\varepsilon, N_\varepsilon - \Delta_M(M);\mathbb{Z})$. Now $\Delta_M(M)$ is a strong deformation retract of N_ε (because the 0-section of the normal bundle is a strong deformation retract of the total space of the bundle). Hence $p_1\mid_{N_\varepsilon}$ and $p_2\mid_{N_\varepsilon}$ are homotopic, since $p_1 = p_2$ on $\Delta_M(M)$. Thus $p_1^*a = p_2^*a$ in $H^*(N_\varepsilon,\mathbb{Z})$.

We now work with cohomology with rational coefficients. Then we have a Künneth decomposition

$$H^n(M\times M;\mathbb{Q}) = \oplus_{i=0}^n H^i(M,\mathbb{Q})\otimes H^{n-i}(M,\mathbb{Q}).$$

Let $b_1,\ldots,b_r$ be a basis for the graded vector space $H^*(M,\mathbb{Q})$ such that each b_j is homogeneous (*i.e.*, lies in some $H^i(M,\mathbb{Q})$). Then there are unique homogeneous elements $c_1,\ldots,c_r\in H^*(M,\mathbb{Q})$ such that $\deg b_j + \deg c_j = n$ for all j, and

$$u_M = \sum_{j=1}^r p_1^*b_j\cup p_2^*c_j$$

(here we also let u_M denote the image of u_M in rational cohomology). For any $x\in M$, if $j_x : M\hookrightarrow M\times M$ is given by $j_x(y) = (x,y)$, then we know that $j_x^*\widetilde{[M]}\in H^n(M, M-\{x\};\mathbb{Z})$ is the distinguished generator, given by the orientation of M. This implies that $j_x^*u_M\in H^n(M;\mathbb{Q}) = \mathbb{Q}$ is the dual of the fundamental class, *i.e.*, the Kronecker index $< j_x^*u_M, [M] >$ is 1, and $j_x^*u_M$ is the unique generator of $H^n(M;\mathbb{Q})$ with this property. Note that for any $x\in M$, the composition

$$\oplus_{i=0}^n H^i(M;\mathbb{Q})\otimes H^{n-i}(M;\mathbb{Q}) \xrightarrow{\cong} H^n(M\times M;\mathbb{Q}) \xrightarrow{j_x^*} H^n(M;\mathbb{Q})$$

is just the projection onto the summand $H^0(M;\mathbb{Q})\otimes H^n(M;\mathbb{Q}) = H^n(M;\mathbb{Q})$ (the identification is by $1\otimes a\mapsto a$). In particular, if we assume (as we may, without loss of generality) that $b_1 = 1\in H^0(M;\mathbb{Q}) = \mathbb{Q}$ is the unit element of the cohomology ring, then $c_1 = j_x^*u_M$ for any $x\in M$ is this distinguished generator of $H^n(M;\mathbb{Q})$, since each b_j, $j>1$, is necessarily homogeneous of degree >0, and so $b_1\otimes c_1 = 1\otimes c_1$ is precisely the component of u_M in $H^0(M;\mathbb{Q})\otimes H^n(M;\mathbb{Q})$.

Now for any homogeneous element $a \in H^*(M,\mathbb{Q})$, the equation $p_1^*a \cup u_M = p_2^*a \cup u_M$ becomes

$$\sum_{j=1}^{r} p_1^*(a \cup b_j) \cup p_2^*c_j = \sum_{j=1}^{r} (-1)^{\deg a \deg b_j} p_1^*b_j \cup p_2^*(a \cup c_j).$$

In other words, in $H^*(M;\mathbb{Q}) \otimes H^*(M;\mathbb{Q})$, we have an identity

$$\sum_{j=1}^{r} (a \cup b_j) \otimes c_j = \sum_{j=1}^{r} (-1)^{\deg a \deg b_j} b_j \otimes (a \cup c_j)$$

for all homogeneous elements $a \in H^*(M;\mathbb{Q})$.

Suppose $a \in H^i(M;\mathbb{Q})$, and consider the components of both sides in the summand $H^i(M;\mathbb{Q}) \otimes H^n(M;\mathbb{Q})$. On the left side, this component is clearly $a \otimes c_1$. On the right, it is

$$\sum_{\{j \mid \deg b_j = i\}} (-1)^i b_j \otimes (a \cup c_j).$$

Now $a \cup c_j =< a \cup c_j, [M] > c_1$. Hence we have an identity

$$a = \sum_{\{j \mid \deg b_j = i\}} (-1)^i < a \cup c_j, [M] > b_j$$

for all $a \in H^i(M;\mathbb{Q})$. But the set $\{b_j \mid \deg b_j = i\}$ is a basis for the $\mathbb{Q}$-vector space $H^i(M;\mathbb{Q})$. Hence we deduce that, with respect to the pairing

$$H^i(M;\mathbb{Q}) \otimes H^{n-i}(M;\mathbb{Q}) \to \mathbb{Q},$$

$$a \otimes c \mapsto < a \cup c, [M] >,$$

the set $\{(-1)^i c_j \mid \deg b_j = i\}$, taken in the same order, is a dual set of vectors, *i.e.*,

$$(-1)^i < b_{j'} \cup c_j, [M] >= \delta_{j'j},$$

the Kronecker delta. In particular, the above pairing is non-singular on the left. But i is any index between 0 and n, and the corresponding pairing for $n-i$ in place of i is the equivalent to the above one, upto interchanging the factors and a sign $(-1)^{i(n-i)}$. Hence the pairing is non-degenerate, for each i. Incidentally, this gives a proof of the Poincaré duality theorem for $H^*(M;\mathbb{Q})$.

We now conclude that

$$\Delta_M^* u_M = \sum_{j=1}^{r} b_j \cup c_j,$$

where $< b_j \cup c_j, [M] >= (-1)^{\deg b_j}$. Hence

$$< \Delta_M^* u_M, [M] >= \sum_{j=1}^{r} (-1)^{\deg b_j} = \sum_{i=0}^{n} (-1)^i \dim_{\mathbb{Q}} H^i(M;\mathbb{Q}) = \chi(M).$$

□

Remark 5.9. ("Localization" of the Euler class) We know that if $p : E \to X$ is an oriented vector bundle of rank n, then its Euler class $e(E) \in H^n(X, \mathbb{Z})$ vanishes if E has a nowehere vanishing section, since we would then have a decomposition $E = E' \oplus (X \times \mathbb{R})$. More generally, if s is any section of E with zero set $Z(s) \subset X$, we can define an associated "localized Euler class" $e(E, s) \in H^n(X, X - Z(s); \mathbb{Z})$ by the formula $e(E, s) = s^*\xi(E)$, where $\xi(E) \in H^n(E, E - 0_X; \mathbb{Z})$ is the Thom class. The image of $e(E, s)$ in $H^n(X, \mathbb{Z}$ is just the pull-back under $s : X \to E$ of the image of $\xi(E)$ in $H^n(E, \mathbb{Z}$. But the map $s : X \to E$ is clearly homotopic to the 0-section $0_E : X \to E$, and so $e(E, s)$ maps to the Euler class $e(E)$ in $H^n(X, \mathbb{Z})$.

In particular, suppose M is an n-manifold, and $p : E \to M$ is an oriented real vector bundle of rank n. Let $s \in \Gamma(M, E)$ be a section with an isolated zero set $Z(s) = \{x_1, \ldots, x_m, \ldots\}$. Then $H^n(M, M - Z(s); \mathbb{Z}) = \prod_{i\geq 1} H^n(M, M - \{x_i\}; \mathbb{Z}) \cong \prod_{i\geq 1} \mathbb{Z}$. Thus $e(E, s) = (e(E, s, x_i))_{i\geq 1}$, where $e(E, s, x_i) \in H^n(M, M - \{x_i\}; \mathbb{Z})$ is a further localization of the Euler class; the "global" Euler class $e(E) \in H^n(M, \mathbb{Z})$ is then

$$\sum_{i\geq 1} \text{image}\,(e(E, s, x_i)).$$

This expression may appear strange, since we have a sum which seems to be infinite; however, $H^n(M, \mathbb{Z}) = \prod_j H^n(M_j, \mathbb{Z})$ where M_j are the connected components of M, where if M_j is connected and non-compact, $H^n(M_j, \mathbb{Z}) = 0$, while for compact M_j, only finitely many x_i lie in M_j. Hence the above infinite sum is meaningful!

In fact, if x is any isolated zero of a section s we may define $e(E, s, x)$ as follows, consistent with the earlier usage: choose a local trivialization for E over a disk coordinate neighbourhood U of x in M, compatible with the chosen orientation of E, so that s corresponds a vector valued function $f : U \to \mathbb{R}^n$ with $f(x) = 0$, $f(U - \{x\}) \subset \mathbb{R}^n - \{0\}$; then $e(E, s, x)$ is the image of the standard generator of $H^n(\mathbb{R}^n, \mathbb{R}^n - \{0\}; \mathbb{Z})$ under the composition

$$H^n(\mathbb{R}^n, \mathbb{R}^n - \{0\}; \mathbb{Z}) \xrightarrow{f^*} H^n(U, U - \{x\}; \mathbb{Z}) \cong H^n(M, M - \{x\}; \mathbb{Z}).$$

This definition is easily seen to be independent of the choice of oriented local trivialization of E on U. If M is oriented, so that we have a chosen generator $[M]_x \in H_n(M, M - \{x\}; \mathbb{Z})$, the class $e(E, s, x)$ is determined by the local Kronecker index $< e(E, s, x), [M]_x >$, an integer, which is just the degree of the map germ $(U, x) \to (\mathbb{R}^n, 0)$. In this case, we call this Kronecker index $< e(E, s, x), [M]_x > \in \mathbb{Z}$ the *index* of s at x. In particular, if $E = TM$ is the tangent bundle of an oriented manifold, we have the notion of the *index of a vector field* v on M at any isolated zero of v.

Now suppose M is a compact, oriented C^∞ manifold of dimension n, with fundamental class $[M]$, and let v be a vector field with (isolated) zero set $\{x_1, \ldots, x_m\}$. Let $[M]_x \in H_n(M, M - \{x\}; \mathbb{Z})$ denote the orientation class, for any $x \in M$, so that $[M] \mapsto [M]_x$ under $H_n(M,\mathbb{Z}) \to H_n(M, M-\{x\};\mathbb{Z})$. Then we compute that

$$\chi(M) =< e(TM), [M] >= \sum_{i=1}^{n} < e(TM, v, x_i), [M]_{x_i} >= \sum_{i=1}^{m} (\text{index of } v \text{ at } x_i).$$

This is the *Poincaré-Hopf theorem.*

6. Pontryagin Classes

In this section, we exploit the connections between oriented real vector bundles and complex vector bundles, in order to study the Pontryagin classes of real vector bundles. In particular, we will prove Theorem 4.1(b).

We begin by noting that, using the convention of Example 5.1, one obvious way of obtaining an oriented real vector bundle of even rank $2n$ is to consider the underlying real vector bundle of a complex vector bundle $p : E \to X$ of $\mathbb{C}$-rank n. In particular, the Euler class $e(E) \in H^{2n}(X,\mathbb{Z})$ gives a characteristic class for complex vector bundles E of $\mathbb{C}$-rank n. We have already seen that such a characteristic class must be a polynomial in the Chern classes (at least on the categories of bundles on compact Hausdorff spaces, or manifolds). In fact, we have the following result.

Lemma 6.1. *If $p : E \to X$ is a complex vector bundle of $\mathbb{C}$-rank n, then the Euler class of the underlying oriented real bundle $e(E) \in H^{2n}(X,\mathbb{Z})$ coincides with the n-th Chern class $c_n(E)$.*

Proof. We will give the proof when X is a compact Hausdorff space. From lemma 5.6 and the splitting principle, we are reduced to considering the case when $p : E \to X$ is a complex line bundle. In this case, we are further reduced to the case when $X = \mathbb{P}^n_{\mathbb{C}}$ and E is the tautological line bundle $\gamma_{1,n} = \mathcal{O}_{\mathbb{P}^n}(-1)$. Now $H^2(\mathbb{P}^n,\mathbb{Z}) = \mathbb{Z}$ is generated by $c_1(\gamma_{1,n})$ (we took this as the definition of c_1, basically). Hence $e(\gamma_{1,n})$ is an integer multiple $a\, c_1(\gamma_{1,n})$, where (since $\gamma_{1,n}$ restricts to $\gamma_{1,1}$ on any linear $\mathbb{P}^1_{\mathbb{C}} \subset \mathbb{P}^n_{\mathbb{C}}$), the coefficient a is a universal constant, which we can determine by computing it for $\gamma_{1,1}$ on $\mathbb{P}^1_{\mathbb{C}} \cong S^2$.

Equivalently, we can compute it for an arbitrary complex line bundle on $\mathbb{P}^1_{\mathbb{C}}$; we choose to do it for the tangent bundle, which has a natural structure of a complex line bundle. In this case, we already know by Theorem 5.6 that the Euler class equals $2y$, where $y \in H^2(\mathbb{P}^1_{\mathbb{C}};\mathbb{Z})$ is the distinguished generator determined by the orientation on $\mathbb{P}^1_{\mathbb{C}}$, since $\chi(\mathbb{P}^1_{\mathbb{C}}) = \chi(S^2) = 2$.

We claim that, as a complex line bundle, $T\mathbb{P}^1_{\mathbb{C}} \cong (\gamma_{1,1}^{\otimes 2})^{\vee}$, the dual of the tensor square of $\gamma_{1,1}$. Granting this, $c_1(T\mathbb{P}^1_{\mathbb{C}})$ is (by our definition of c_1) equal to $(-2)(-y) = 2y$, since $c_1(\gamma_{1,1}) = -y$; this agrees with the Euler class, which will complete the proof of this lemma.

One way to understand $T\mathbb{P}^1_{\mathbb{C}}$ as a complex line bundle is directly via a transition function. In fact $\mathbb{P}^1_{\mathbb{C}}$ has a covering by 2 holomorphic coordinate charts (U, z) and (V, w), each of which identifies the corresponding open subset of $\mathbb{P}^1_{\mathbb{C}}$ with the complex plane $\mathbb{C}$, such that $U \cap V$ is identified with $\mathbb{C}^* = \mathbb{C} - \{0\}$, and the transition function is $w = w(z) = z^{-1}$. Considered as a diffeomorphism of

$\mathbb{R}^2 - \{0\}$, with coordinates x, y, this transition function is given by the formula

$$(x, y) \mapsto (u(x, y), v(x, y)) = (x/(x^2 + y^2), -y/(x^2 + y^2)),$$

and so the Jacobian matrix is

$$J(x, y) = \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix} = \begin{bmatrix} \frac{y^2-x^2}{(x^2+y^2)^2} & \frac{-2xy}{(x^2+y^2)^2} \\ \frac{2xy}{(x^2+y^2)^2} & \frac{y^2-x^2}{(x^2+y^2)^2} \end{bmatrix}.$$

In other words, on the fiber over $z = x + \sqrt{-1}y$, this is just the complex linear transformation $\mathbb{C} \to \mathbb{C}$ given by scalar multiplication by $-1/z^2$. Ignoring the factor -1 does not change the complex line bundle, and the bundle $\gamma_{1,1}$ has a transition function (with respect to the same open cover, and an obvious choice of local trivializations) given by multiplication by z. Hence $T\mathbb{P}^1_{\mathbb{C}} \cong (\gamma_{1,1}^{\otimes 2})^\vee$, as claimed (for line bundles, the dual and tensor power operations have the obvious description on transition functions). □

Example 6.2. (Tangent bundle of $\mathbb{P}^n_{\mathbb{C}}$) In the last part of the preceeding proof, we computed using the definitions that the tangent bundle of $\mathbb{P}^n_{\mathbb{C}}$ is $(\gamma_{1,1}^{\otimes 2})^\vee$. More generally, we show how to compute the tangent bundle of $\mathbb{P}^n_{\mathbb{C}}$, and hence determine its Chern classes. This will also have applications to Pontryagin classes.

Let ω_n be the orthogonal complement of $\gamma_{1,n} \subset \mathbb{P}^n_{\mathbb{C}} \times \mathbb{C}^{n+1}$, where we fix the standard Hermitian metric on $\mathbb{C}^{n+1}$ (and thus on the trivial bundle, and on the sub-bundle $\gamma_{1,n}$). Thus $\gamma_{1,n} \oplus \omega_n = \mathbb{P}^n_{\mathbb{C}} \times \mathbb{C}^{n+1}$. We claim the tangent bundle of $\mathbb{P}^n_{\mathbb{C}}$ is identified, as a complex vector bundle, with $\mathrm{Hom}(\gamma_{1,n}, \omega_n)$. In fact, if L is any complex line in $\mathbb{C}^{n+1}$, with orthogonal complement $L^\perp$, then we have a direct sum decomposition $\mathbb{C}^{n+1} = L \oplus L^\perp$, and hence a map $\phi_L : \mathrm{Hom}(L, L^\perp) \to \mathbb{P}^n_{\mathbb{C}}$, $\phi_L(f) = \{z + f(z) \mid z \in L\}$. One checks that this is a homeomorphism onto its image, and gives a holomorphic coordinate chart on $\mathbb{P}^n_{\mathbb{C}}$. Thus the tangent space to $\mathbb{P}^n_{\mathbb{C}}$ at L is identified with the complex vector space $\mathrm{Hom}(L, L^\perp)$. One verifies that this defines an isomorphism of complex vector bundles $T\mathbb{P}^n_{\mathbb{C}} \to \mathrm{Hom}(\gamma_{1,n}, \omega_n)$.

Now we see that

$$T\mathbb{P}^n_{\mathbb{C}} \oplus \mathbb{P}^n_{\mathbb{C}} \times \mathbb{C} \cong \mathrm{Hom}(\gamma_{1,n}, \omega_n) \oplus \mathrm{Hom}(\gamma_{1,n}, \gamma_{1,n})$$

$$\cong \mathrm{Hom}(\gamma_{1,n}, \mathbb{P}^n_{\mathbb{C}} \times \mathbb{C}^{n+1}) \cong (\gamma_{1,n}^\vee)^{\oplus n+1}.$$

Hence we have an equality between total Chern classes

$$c(T\mathbb{P}^n_{\mathbb{C}}) = c(\gamma_{1,n}^\vee)^{n+1} = (1 + y)^{n+1},$$

where $y \in H^2(\mathbb{P}^n_{\mathbb{C}}; \mathbb{Z})$ is the generator which restricts to the orientation class in $H^2(\mathbb{P}^1_{\mathbb{C}}; \mathbb{Z}) = H^2(S^2; \mathbb{Z})$.

Now in another direction, if $p : E \to X$ is a real vector bundle of rank n, then $E \oplus E$ has a complex structure given by $J(v, w) = (w, -v)$ on any fiber. In other words, we can form $E_{\mathbb{C}} = E \otimes_{\mathbb{R}} \mathbb{C} \to X$, which is a $\mathbb{C}$-vector bundle of rank n. The underlying real bundle, as already noted, is $E \oplus E$. If E itself is the real bundle of rank $2n$ underlying a $\mathbb{C}$-vector bundle of rank n, then $E_{\mathbb{C}} = E \oplus \overline{E}$ (here if E has complex structure J_E, recall that $\overline{E}$ is the same $\mathbb{R}$-vector bundle of rank $2n$ with the new complex structure $-J_E$). Note that $\overline{E_{\mathbb{C}}} \cong E_{\mathbb{C}}$.

Lemma 6.3. *For any complex vector bundle E, we have $c_i(\overline{E}) = (-1)c_i(E)$ for all i.*

Proof. This follows from the splitting principle, and the case of the universal line bundle on $\mathbb{P}^\infty_\mathbb{C}$, which in turn follows from the case of $\gamma_{1,1}$ on $\mathbb{P}^1_\mathbb{C}$; now $\overline{\gamma_{1,1}}$ has the same underlying real bundle with the opposite orientation, and hence the negative of its Euler class. □

Corollary 6.4. *For any real vector bundle E, $2c_i(E_\mathbb{C}) = 0$ for odd i.*

Definition 6.5. For any real vector bundle $p : E \to X$ of rank n, define its i-th *Pontryagin class* to be $p_i(E) = (-1)^i c_{2i}(E_\mathbb{C}) \in H^{4i}(X;\mathbb{Z})$. Define the total Pontryagin class of E to be the unit $p(E) = 1 + p_1(E) + \cdots \in H^{4*}(X;\mathbb{Z})$.

Remark 6.6. The Pontryagin classes are functorial for pull-backs. From the formula for the total Chern class of a direct sum, we see that

$$p(E \oplus F) - p(E)p(F)$$

is a 2-torsion cohomology class; in particular, if we consider the total Pontryagin class with values in $H^{4*}(X; Z[1/2])$, then we do get the formula $p(E \oplus F) = p(E)p(F)$, just as for Chern classes.

If E is an oriented real vector bundle of rank $2n$, then the real bundle underlying $E_\mathbb{C}$ is $E \oplus E$, with an orientation change of $(-1)^n$ (*i.e.*, with the same orientation, if n is even, and opposite orientation, if n is odd); this is because if $v_1, \ldots, v_n$ is an oriented basis in a fiber E_x, then an oriented bais of $E_x \oplus E_x$ coming from $E_\mathbb{C}$ is $(v_1, 0), (0, v_1), (v_2, 0), (0, v_2), \ldots, (0, v_{2n})$, while one for $E \oplus E$ is $(v_1, 0), (v_2, 0), \ldots, (v_{2n}, 0), (0, v_1), \ldots, (0, v_{2n})$. Hence by lemma 6.1, we compute that

$$p_n(E) = (-1)^n c_{2n}(E_\mathbb{C}) = e(E \oplus E) = e(E) \cup e(E) = e(E)^2 \in H^{4n}(X;\mathbb{Z}).$$

If E is a complex vector bundle of rank n, then the complexification of the underlying real bundle is (as noted earlier) $E \oplus \overline{E}$, which has total Chern class $(1 + c_1(E) + c_2(E) + \cdots + c_n(E))(1 - c_1(E) + c_2(E) - \cdots + (-1)^n c_n(E))$. Hence we have a formula

$$\begin{aligned}(1 - p_1(E) + p_2(E) - \cdots + (-1)^n p_n(E)) &= \\ (1 + c_1(E) + c_2(E) + \cdots + c_n(E))&(1 - c_1(E) + c_2(E) - \cdots + (-1)^n c_n(E)).\end{aligned}$$

This implies formulas

$$p_k(E) = c_k(E)^2 - 2c_{k-1}(E)c_{k+1}(E) + \cdots \pm 2c_1(E)c_{2k-1}(E) \mp 2c_{2k}(E).$$

For example, this allows us to compute the Pontryagin classes of the tangent bundle of $\mathbb{P}^n_\mathbb{C}$ to be

$$p_k(T\mathbb{P}^n_\mathbb{C}) = \binom{n+1}{k} a^{2k},\ 1 \le k \le \left[\frac{n}{2}\right],$$

where $a \in H^2(\mathbb{P}^n_\mathbb{C};\mathbb{Z})$ is the standard generator (which restricts to the orientation class in $H^2(\mathbb{P}^1_\mathbb{C};\mathbb{Z})$).

Let $\mathbb{G}^+_{\mathbb{R}}(n,m)$ be the space of oriented n-dimensional subspaces of $\mathbb{R}^m$, and $\mathbb{G}^+_{\mathbb{R}}(n) = \varinjlim_m \mathbb{G}^+_{\mathbb{R}}(n,m)$. Note that there are obvious maps $\mathbb{G}^+_{\mathbb{R}}(n,m) \to \mathbb{G}_{\mathbb{R}}(n,m)$ and $\mathbb{G}^+_{\mathbb{R}}(n) \to \mathbb{G}_{\mathbb{R}}(n)$, which are in fact covering spaces of degree 2. Let γ_n^+ be the induced oriented vector bundle of rank n on $\mathbb{G}^+_{\mathbb{R}}(n)$, and $\gamma^+_{n,m}$ its restriction to $\mathbb{G}^+_{\mathbb{R}}(n,m)$ (which equals the pullback of $\gamma_{n,m}$).

With respect to the standard correspondence between covering spaces of degree 2 of a space X and elements of $H^1(X;\mathbb{Z}/2\mathbb{Z})$, the covering $\mathbb{G}^+_{\mathbb{R}}(n) \to \mathbb{G}_{\mathbb{R}}(n)$ is the one associated to $w_1(\gamma_n)$, the universal first Stiefel-Whitney class; $\mathbb{G}^+_{\mathbb{R}}(n,m) \to \mathbb{G}_{\mathbb{R}}(n,m)$ has a similar interpretation, and is the pull-back for the inclusion $\mathbb{G}_{\mathbb{R}}(n,m) \hookrightarrow \mathbb{G}_{\mathbb{R}}(n)$.

Let τ be the involution of $\mathbb{G}^+_{\mathbb{R}}(n)$ obtained by "reversing the orientation of the n-plane"; then τ yields a free $\mathbb{Z}/2\mathbb{Z}$ action on $\mathbb{G}^+_{\mathbb{R}}(n)$ such that the covering space $\mathbb{G}^+_{\mathbb{R}}(n) \to \mathbb{G}_{\mathbb{R}}(n)$ is the quotient map. Hence $H^*(\mathbb{G}_{\mathbb{R}}(n);\mathbb{Z}[1/2]) \subset H^*(\mathbb{G}^+_{\mathbb{R}}(n);\mathbb{Z}[1/2])$ as the subring of elements invariant under τ. Hence Theorem 4.1(b) follows from the following result, which determines the possible characteristic classes for oriented bundles in the cohomology ring with $\mathbb{Z}[1/2]$-coefficients.

Theorem 6.7. *The cohomology ring $H^*(\mathbb{G}^+_{\mathbb{R}}(n);\mathbb{Z}[1/2])$ is generated, as a $\mathbb{Z}[1/2]$-algebra, by the Pontryagin classes $p_i(\gamma_n^+)$, $1 \le i \le \left[\frac{n}{2}\right]$, and the Euler class $e(\gamma_n^+)$; these generators are subject to a single relation: $e(\gamma_n^+) = 0$ if n is odd, and $e(\gamma_n^+)^2 = p_{n/2}(\gamma_n^+)$ if n is even. The action of τ on the cohomology ring fixes the Pontryagin classes, and maps $e(\gamma_n^+)$ to $-e(\gamma_n^+)$.*

Proof. The proof is in several steps. Note that $\tau(e(\gamma_n^+)) = -e(\gamma_n^+)$, since τ reverses the orientation on γ_n^+. Also τ fixes the Pontryagin classes since these are pulled back from $H^*(\mathbb{G}_{\mathbb{R}}(n),\mathbb{Z}[1/2])$.

For $n = 1$, there is nothing to prove, since $\mathbb{G}^+_{\mathbb{R}}(1) = S^\infty$, the universal covering of $\mathbb{G}_{\mathbb{R}}(1) = \mathbb{P}^\infty_{\mathbb{R}}$, and S^∞ is contractible.

Next, consider the case $n = 2$. Now any complex line bundle determines an oriented line bundle; conversely, we claim that an oriented $\mathbb{R}$-vector bundle $p : E \to X$ of rank 2 equipped with a Euclidean metric is, in a natural way, also a complex line bundle, with the following complex structure: if v, w is any oriented orthonormal basis for a fiber E_x, then $Jv = w$, $Jw = -v$. One verifies that this depends only on the orientation and the metric; the isomorphism class of this complex line bundle is independent of the metric, since we can deform any Euclidean metric m into another one m' through the 1-parameter family of metrics $tm + (1-t)m'$, with $t \in [0,1]$, so that the classifying maps $X \to \mathbb{P}^\infty_{\mathbb{C}}$ associated to m, m' are homotopic.

Thus, for compact Hausdorff spaces or manifolds, the classification of oriented $\mathbb{R}$-vector bundles of rank 2 is equivalent to that of complex line bundles; in particular, the natural inclusion on classifying spaces $i : \mathbb{P}^\infty_{\mathbb{C}} \hookrightarrow \mathbb{G}^+_{\mathbb{R}}(2)$ (given by associating the underlying oriented real vector bundle to $\gamma_1 \to \mathbb{P}^\infty_{\mathbb{C}}$) is a homotopy equivalence. Thus the cohomology algebra of $\mathbb{G}^+_{\mathbb{R}}(2)$ is a polynomial algebra in $e(\gamma_2^+)$, and $i^*e(\gamma_2^+) = c_1(\gamma_1)$. The subalgebra of τ-invariant elements is the polynomial algebra in $e(\gamma_2^+)^2 = p_1(\gamma_2^+)$.

Now we consider the case when $n > 2$. We make use of the Gysin exact sequence associated to the universal oriented bundle γ_n^+ of rank n (we suppress the coefficient ring $\mathbb{Z}[1/2]$)

$$\cdots \to H^i(\mathbb{G}_{\mathbb{R}}^+(n)) \xrightarrow{e(\gamma_n^+)} H^{i+n}(\mathbb{G}_{\mathbb{R}}^+(n)) \to H^{i+n}(\gamma_n^+ - 0_{\mathbb{G}_{\mathbb{R}}^+(n)}) \to \cdots$$

Note that there is a continuous map

$$\alpha : \gamma_n^+ - 0_{\mathbb{G}_{\mathbb{R}}^+(n)} \to \mathbb{G}_{\mathbb{R}}^+(n-1)$$

given by $(W, x) \mapsto W \cap (\mathbb{R}x)^{\perp}$ (where we fix the positive definite inner product on $\mathbb{R}^{\infty} = \varinjlim_{m} \mathbb{R}^m$). We claim this map α induces an isomorphism on integral cohomology. One way to see this is to observe that it is locally trivial, and its fiber over W' is homeomorphic to $(0,\infty) \times \mathbb{G}_{\mathbb{R}}^+(1)$, which is contractible. Another way, avoiding the question of local triviality, is to compare the map for a given H^i with the analogous map obtained from the Grassmannian $\mathbb{G}_{\mathbb{R}}^+(n,m)$ for large enough m (depending on i), and identify this map with a map from another Gysin sequence, for $(\gamma_{n-1,m}^+)^{\perp} \to \mathbb{G}_{\mathbb{R}}^+(n-1,m)$ (see Milnor's book, pages 162 and 180).

Thus one has an exact sequence

$$\cdots \to H^i(\mathbb{G}_{\mathbb{R}}^+(n)) \xrightarrow{e(\gamma_n^+)} H^{i+n}(\mathbb{G}_{\mathbb{R}}^+(n)) \xrightarrow{\widetilde{\alpha}} H^{i+n}(\mathbb{G}_{\mathbb{R}}^+(n-1)) \to \cdots,$$

where $\widetilde{\alpha}$ is induced by the isomorphism α^* on cohomology. Further, if $\pi : \gamma_n^+ - 0_{\mathbb{G}_{\mathbb{R}}^+(n)} \to \mathbb{G}_{\mathbb{R}}^+(n)$ is the projection, then by construction, we have an isomorphism

$$\pi^*\gamma_n^+ \cong \alpha^*\gamma_{n-1}^+ \oplus \Big((\gamma_n^+ - 0_{\mathbb{G}_{\mathbb{R}}^+(n)}) \times \mathbb{R}\Big)$$

as oriented bundles. Thus $\widetilde{\alpha}(p_j(\gamma_n^+)) = p_j(\gamma_{n-1}^+)$ (and of course $\widetilde{\alpha}(e(\gamma_n^+)) = 0$).

In particular, we see by induction on n that $H^*(\mathbb{G}_{\mathbb{R}}^+(n); \mathbb{Z}[1/2])$ is generated over $\mathbb{Z}[1/2]$ by the Pontryagin classes and the Euler class of γ_n^+. Also, the relations between these classes, stated in the theorem, have been proved. It remains to show that the Pontryagin classes are algebraically independent elements of the cohomology algebra, and that when n is even, the only relation satisfied by $e(\gamma_n^+)$ and the Pontryagin classes is the stated one, that $e(\gamma_n^+)^2 = p_{n/2}(\gamma_n^+)$.

Let $(\mathbb{P}_{\mathbb{C}}^{\infty})^m$ denote the product of m copies of $\mathbb{P}_{\mathbb{C}}^{\infty}$. Consider the following diagram, commutative up to homotopy, where $r = \left[\frac{n}{2}\right]$.

$$\begin{array}{ccc} (\mathbb{P}_{\mathbb{C}}^{\infty})^r & \xrightarrow{f} & (\mathbb{P}_{\mathbb{C}}^{\infty})^n \\ \pi \downarrow & & \downarrow \eta \\ \mathbb{G}_{\mathbb{R}}^+(n) & \xrightarrow{i} & \mathbb{G}_{\mathbb{C}}(n) \end{array}$$

Here the maps are defined as follows. $(\mathbb{P}_{\mathbb{C}}^{\infty})^r$ supports the oriented vector bundle E of rank $2r$ underlying the direct sum of the r pull-back complex line bundles $\pi_j^*(\gamma_1)$, where $\pi_j : (\mathbb{P}_{\mathbb{C}}^{\infty})^r \to \mathbb{P}_{\mathbb{C}}^{\infty}$, $1 \leq j \leq r$, are the projections. The maps π is the classifying map for E, if $n = 2r$, and the classifying map for $E \oplus ((\mathbb{P}_{\mathbb{C}}^{\infty})^r \times \mathbb{R})$ (with the standard orientation on the trivial line bundle), if $n = 2r+1$. The map i is the classifying map for the complex vector bundle $(\gamma_n^+)_{\mathbb{C}}$. The map η is the splitting map for the universal bundle $\gamma_n \to \mathbb{G}_{\mathbb{C}}(n)$, given by the classifying map for the complex vector bundle $\oplus_{j=1}^n q_j^*(\gamma_1)$, where $q_j : (\mathbb{P}_{\mathbb{C}}^{\infty})^n \to \mathbb{P}_{\mathbb{C}}^{\infty}$, $1 \leq j \leq n$, are the projections. Finally f is the classifying map for $E_{\mathbb{C}}$, if $n = 2r$, and for $(E \oplus ((\mathbb{P}^{\infty}_{\mathbb{C}})^r \times \mathbb{R}))_{\mathbb{C}}$, if $n = 2r + 1$. Note that in fact $E_{\mathbb{C}} \cong \oplus_{j=1}^r (p_j^*(\gamma_1) \oplus p_j^*(\overline{\gamma_1}))$.

The above homotopy commutative diagram induces a commutative diagram on cohomology algebras. Here the cohomology algebras 3 of the terms have the following descriptions, as polynomial algebras:

$$H^*((\mathbb{P}^{\infty}_{\mathbb{C}})^r; \mathbb{Z}[1/2]) = \mathbb{Z}[1/2][\pi_1^* c_1(\gamma_1), \dots, \pi_r^* c_1(\gamma_1)],$$

$$H^*((\mathbb{P}^{\infty}_{\mathbb{C}})^n; \mathbb{Z}[1/2]) = \mathbb{Z}[1/2][q_1^* c_1(\gamma_1), \dots, q_n^* c_1(\gamma_1)],$$

$$H^*(\mathbb{G}_{\mathbb{C}}(n); \mathbb{Z}[1/2]) = \mathbb{Z}[1/2][c_1(\gamma_n), \dots, c_n(\gamma_n)].$$

With these descriptions, the maps on cohomology have the following descriptions, from our results proved so far:

(i) $f^*(q_1^*(c_1(\gamma_1))) = -f^*(q_2^*(c_1(\gamma_1)) = \pi_1^* c_1(\gamma_1), \dots,$ $f^*(q_{2r-1}^* c_1(\gamma_1)) = -f^*(q_{2r}^*(c_1(\gamma_1))) = \pi_r^* c_1(\gamma_1)$, and if $n = 2r+1$, $f^*(q_n^*(c_1(\gamma_1)) = 0$;

(ii) $\eta^* c_m(\gamma_n) =$ m-th elementary symmetric function in $q_j^*(c_1(\gamma_1))$, $1 \leq j \leq n$;

(iii) $i^* c_{2m}(\gamma_n) = (-1)^m p_m(\gamma_n^+), i^* c_{2m+1}(\gamma_n) = 0$;

(iv) $\pi^* p_m(\gamma_m^+) = m$-th elementray symmetric function in $p_1(\pi_i^* \gamma_1)$, $1 \leq i \leq r$, where $p_1(\pi^* \gamma_1) = \pi_i^*(c_1(\gamma_1))^2$;

(v) $\pi^*(e(\gamma_n^+)) = \prod_{j=1}^r \pi_j^*(c_1(\gamma_1))$, if $n = 2r$. If $n = 2r + 1$, of course $e(\gamma_n^+) = 0$ in $H^n(\mathbb{G}_{\mathbb{R}}^+; \mathbb{Z}[1/2])$.

Combining all of these formulas, we see easily that $p_1(\gamma_n^+), \dots, p_r(\gamma_n^+)$ are algebraically independent elements of $H^*(\mathbb{G}_{\mathbb{R}}^+; \mathbb{Z}[1/2])$, since these map to the elementary symmetric polynomials $s_1, \dots, s_r$ in the squares of the variables in a polynomial ring in r variables; further, if $n = 2m$, the image of the universal Euler class e is the product of these variables, and the only relation satisfied by (the image of) e over the above subring of symmetric polynomials $\mathbb{Z}[s_1, \dots, s_r]$ is the obvious one, that $e^2 = s_r$. □

7. The Signature Theorem

If M is a compact, oriented manifold of dimension n, define its *signature* $\sigma(M)$ as follows: if n is not a multiple of 4, define $\sigma(M) = 0$; if $n = 4m$, the pairing

$$H^{2m}(M; \mathbb{Q}) \otimes H^{2m}(M; \mathbb{Q}) \to \mathbb{Q}, \quad x \otimes y \mapsto < x \cup y, [M] >,$$

where $[M] \in H_n(M; \mathbb{Z})$ is the fundamental class, is in fact a symmetric bilinear form, which is non-degenerate, by Poincaré duality. The signature of the corresponding real bilinear form, *i.e.*, the integer

(number of positive eigenvalues) − (number of negative eigenvalues),

is defined to be $\sigma(M)$.

This has the following obvious properties; the third property follows from Poincaré duality for manifolds with boundary.

Lemma 7.1. (i) *$\sigma(M \coprod M') = \sigma(M) + \sigma(M')$.*
(ii) *$\sigma(M \times M') = \sigma(M)\sigma(M')$.*

(iii) *if M is the oriented boundary of another oriented manifold B, then $\sigma(M) = 0$.*

Hirzebruch's signature theorem gives a formula for the signature of a $4m$-dimensional oriented C^∞ manifold M in terms of the *Pontryagin numbers* of the manifold M, which are defined to be the Kronecker indices

$$p_{i_1,\dots,i_r}(M) =< p_{i_1}(TM) \cup p_{i_2}(TM) \cup \cdots \cup p_{i_r}(TM), [M] >\in \mathbb{Z},$$

where TM is the tangent bundle, $[M]$ is the fundamental class in $H_{4m}(M;\mathbb{Z})$, and $i_1,\dots,i_r$ are non-negative integers such that $i_1+\cdots+i_r = m$ (here we define the 0-th Pontryagin class to be the unit element $1 \in H^0(M;\mathbb{Z})$). If $I = \{i_1,\dots,i_r\}$ we also use $p_I(M)$ to denote $p_{i_1,\dots,i_r}(M)$. The Pontryagin numbers are clearly invariants of the oriented C^∞ manifold M, which change sign on reversing the orientation (since the Pontryagin classes are invariant under change of orientation, while the fundamental class changes sign). In contrast, note that by similar reasoning, the Euler number $< e(TM), [M] >$ is invariant under reversing orientation, which is of course also clear since it is just the Euler characteristic of M.

Before stating the signature theorem in detail, we first note that, as with the signature, the Pontryagin numbers of a compact oriented C^∞ manifold M vanish if it is the oriented boundary of another manifold B. Indeed, by considering the inward normal direction to M within B, we have an isomorphism of vector bundles $TB\mid_M = TM \oplus (M \times \mathbb{R})$. Hence the Pontryagin classes of TB restrict to those of TM. If $[B] \in H_{n+1}(B, M;\mathbb{Z})$ is the fundamental class of the oriented pair (B, M), then under the connecting homomorphism $\partial : H_{n+1}(B, M;\mathbb{Z}) \to H_n(M;\mathbb{Z})$, we have $\partial[B] = [M]$. Hence if $\delta : H^n(M;\mathbb{Z}) \to H^{n+1}(B, M;\mathbb{Z})$ is the connecting homomorphism in cohomology, then for any $a \in H^n(M;\mathbb{Z})$, we have an equality between Kronecker indices $< a, [M] >=< \delta(a), [B] >$. Since the Pontrygin classes of TM lie in the image of $H^*(B;\mathbb{Z}) \to H^*(M;\mathbb{Z})$, and hence in the kernel of the connecting homomorphism, we see that for any appropriate set of non-negative integers $I = \{i_1,\dots,i_r\}$, we have that $\delta(p_{i_1}(TM) \cup \cdots \cup p_{i_r}(TM)) = 0$, and so $p_I(M) = 0$.

Note that we have a similar definition for *Steifel-Whitney numbers* of a C^∞ manifold M, as Kronecker indices with values in $\mathbb{Z}/2\mathbb{Z}$, defined similarly using the $\mathbb{Z}/2\mathbb{Z}$-orientation and the Stiefel-Whitney classes of the tangent bundle; we have a similar vanishing argument for Stiefel-Whitney numbers of M, provided M is a boundary of a C^∞ manifold B.

Recall that two compact, oriented C^∞ n-manifolds M, M' are *oriented cobordant* if $M \coprod M'$ is the oriented boundary of a C^∞ manifold B. For $n \geq 0$, let Ω_n^+ denote the set of oriented cobordism classes of compact, oriented C^∞ n-manifolds, which is an abelian group with respect to oriented disjoint union (as addition), with 0 element being the class of any oriented boundary. An orientation of a 0-manifold M is defined to be an function with values in $\{1, -1\}$, with the obvious notions of oriented cobordism, etc. For any compact oriented manifold M, the oriented boundary of $M \times [0, 1]$ is just $M \coprod(-M)$, where $-M$ denotes the manifold underlying M with its opposite orientation; thus the additive inverse of the cobordism class of M is that of $-M$.

Then $\Omega_0^+ = \mathbb{Z}$, and $\oplus_{n\geq 0}\Omega_n^+$ has the structure of a graded ring, when endowed with the oriented product as multiplication; this is graded commutative, in the sense that the class of $M \times M'$ equals $(-1)^{nn'}$ times that of $M' \times M$, where $\dim M = n$, $\dim M' = n'$.

From lemma 7.1, the signature determines a ring homomorphism $\Omega^+ \to \mathbb{Z}$, and hence a homomorphism $\Omega^+ \otimes \mathbb{Q} \to \mathbb{Q}$. A fundamental theorem of Thom, which we do not prove here, states the following (see Milnor's book, Chapter 18, for more details).

Theorem 7.2. *For each n, Ω_n^+ is a finitely generated abelian group, which is finite if n is not a multiple of 4. The $\mathbb{Q}$-algebra $\Omega^+ \otimes \mathbb{Q}$ is the polynomial algebra $\mathbb{Q}[[\mathbb{P}^{2n}_{\mathbb{C}}],\ n \geq 0]$ in a countable set of variables, namely the oriented (rational) cobordism classes of the even dimensional complex projective spaces. Further, any $\mathbb{Q}$-linear functional on Ω_{4m}^+ is a linear combination of the functionals determined by Pontryagin numbers $p_I(M)$ for all partitions $I = \{_1, \ldots, i_r\}$ of m.*

In particular, the signature functional on each $\Omega_{4m}^+ \otimes \mathbb{Q}$ must be so expressible, for each m, and since signature is a ring homomorphism, the corresponding Pontryagin number functionals must have appropriate multiplicativity properties.

Consider the Taylor series expansion

$$f(t) = \frac{\sqrt{t}}{\tanh\sqrt{t}} = 1 + \sum_{k\geq 1}\lambda_k t^k = 1 + \frac{1}{3}t - \frac{1}{45}t^2 + \cdots + (-1)^{k-1}\frac{2^{2k}B_k}{(2k)!}t^k + \cdots,$$

where B_k are the Bernoulli numbers, which may be defined by the identity

$$\frac{x}{e^x - 1} = 1 - \frac{x}{2} + \frac{B_1}{2!}x^2 - \frac{B_2}{4!}x^4 + \frac{B_3}{6!}x^6 - \cdots$$

(the Taylor series expansion above is then easily proved using the definition of tanh). Associated to this power series $f(t)$, we can define a sequence of polynomials $L_{k,n} = L_{k,n}(x_1, \ldots, x_n) \in \mathbb{Q}[x_1, \ldots, x_n]$ as follows: let $y_1, \ldots, y_n$ be indeterminates such that x_i is the i-th elementary symmetric polynomial in $y_1, \ldots, y_n$; now $L_{k,n}$ is defined by the condition that we have an equality between formal power series in t with coefficients in $\mathbb{Q}[y_1, \ldots, y_n]$

$$L(x_1, x_2, \ldots, x_n, t) = 1 + \sum_{k\geq 1} L_{k,n}(x_1, \ldots, x_n)t^k = \prod_{i=1}^{n} f(y_i t).$$

Note that $\prod_{i=1}^n f(y_i t)$ is indeed a formal power series in t, such that the coefficient of t^k is a homogeneous symmetric polynomial over $\mathbb{Q}$ in $y_1, \ldots, y_n$ of degree k. Note that, as a result, for $n > k$, the polynomial $L_{k,n}$ depends only on $x_1, \ldots, x_k$, and coincides with $L_{k,k}(x_1, \ldots, x_k)$, which we denote by just L_k. Further, for $n < k$, $L_{k,n}(x_1, \ldots, x_n) = L_k(x_1, \ldots, x_n, 0, \ldots, 0)$.

For a compact oriented C^∞ manifold M of dimension $4k$, the *L-genus* of M is defined to be the rational number

$$L[M] =< L_k(p_1(TM), \ldots, p_k(TM)), [M] > .$$

From the definitions, one can show that, if we define $L[M] = 0$ when $\dim M$ is not a multiple of 4, then the L-genus does determine a homomorphism of $\mathbb{Q}$-algebras $\Omega^+ \otimes \mathbb{Q} \to \mathbb{Q}$.

Theorem 7.3. (Hirzebruch Signature Theorem) *The signature $\sigma(M)$ of a compact, oriented C^∞ manifold M of dimension $4k$ coincides with its L-genus.*

As a corollary, the L-genus is in fact an integer; this implies certain divisibility properties for Pontryagin numbers. We do not prove this theorem here, but one proof is to deduce it from Thom's theorem stated above, describing the oriented rational cobordism ring $\Omega^+ \otimes \mathbb{Q}$. This reduces one to checking the theorem for the generators of this $\mathbb{Q}$-algebra, *i.e.*, for the even dimensional complex projective spaces $\mathbb{P}^{2k}_{\mathbb{C}}$, where on the one hand, we obviously have $\sigma(\mathbb{P}^{2k}_{\mathbb{C}}) = 1$, and on the other hand, from the definitions and the formulas for $p_i(T\mathbb{P}^{2k}_{\mathbb{C}})$ for all i, one can verify, after some calculation, that indeed $L[\mathbb{P}^{2k}_{\mathbb{C}}] = 1$.

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