Grothendieck Duality*

1 Gorenstein Rings

1.1 Injective envelopes and duals

Proposition 1 *Let A be any commutative ring, M an A-module. Then there is an injective A-module I_0 containing M such that for any non-zero submodule $N \subset I_0$, we have $N \cap M \neq 0$. This injective module I_0 is unique up to an isomorphism which is the identity on M.*

Proof: Let I be an injective A-module containing M. By Zorn's lemma, we can find an A-module M' with $M \subset M' \subset I$ such that

(i) for any A-submodule $N \subset I$, we have $N \cap M' \neq \phi \Rightarrow N \cap M \neq \phi$.

(ii) M' is maximal with respect to this property.

Further, again by Zorn, we can find a maximal $K \subset I$ such that $K \cap M' = 0$. If $\eta : I \to I/K$ is the natural map, then $\eta \mid_{M'}$ is a monomorphism, so that (by the injectivity of I) we can find $\psi : I/K \to I$ such that $\psi \circ \eta \mid_{M'}$ is the identity on M'. Thus

$$K = \ker \eta \subset \ker(\psi \circ \eta), \ M' \cap \ker(\psi \circ \eta) = 0,$$

and so by maximality of K, we have $K = \ker \eta = \ker(\psi \circ \eta)$. Hence ψ is an injection split by η. In particular, K and I/K are injective A-modules.

The inclusion $M' \hookrightarrow I/K$ has the property that any non-zero submodule $\overline{N}$ of I/K meets M'; else, $\psi^{-1}(\overline{N}) = N \subset I$ would be strictly larger than K

*Notes of C. P. Ramanujam, edited by V. Srinivas and will meet M' trivially, contradiciting the choice of K. Hence $\overline{N} \cap M \neq 0$. Thus any non-zero submodule of $\psi(I/K)$ meets M non-trivially; since $M' \subset \psi(I/K)$, the maximality of M' implies that $M' = \psi(I/K)$. Thus we have shown that $I = M' \oplus K$, and so $I_0 = M'$ is an injective A-module containing M with the desired property.

To prove the uniqueness, let $M \subset I_1$ be another inclusion into an injective A-module with the same property. Since the I_j are injective, there exist A-linear maps $\alpha : I_0 \to I_1$ and $\beta : I_1 \to I_0$ which are the identity on M. Since $\ker \alpha \cap M = 0$, we have $\ker \alpha = 0$. Regarding I_0 as a submodule of I_1 via the inclusion α, since I_0 is an injective module, α is a split inclusion, and we can write $I_1 = I_0 \oplus N$ for some submodule N. But then $N \cap M = 0$, so $N = 0$ *i.e.*, $\alpha : I_0 \to I_1$ is an isomorphism. □

Definition: A monomorphism $0 \to M \to I$ as in Proposition 1 is called an *injective hull* of M.

Let A be a commutative ring, and let M, N be A-modules; let I be an injective A-module. Let $X. \to M \to 0$ be a projective resolution of M over A. We have a natural isomorphism of complexes

$$\mathrm{Hom}_A(X_\bullet \otimes N, I) \cong \mathrm{Hom}_A(X_\bullet, \mathrm{Hom}_A(N, I)).$$

Since $\mathrm{Hom}_A(-, I)$ is exact, the cohomology groups on the left are the groups $\mathrm{Hom}_A(\mathrm{Tor}_i^A(M, N), I)$, and we get natural isomorphisms

$$\mathrm{Hom}_A(\mathrm{Tor}_i^A(M, N), I) \cong \mathrm{Ext}^i_A(M, \mathrm{Hom}_A(N, I)).$$

Now suppose that A is a Noetherian local ring, $\mathcal{M}$ its maximal ideal, $k = A/\mathcal{M}$ its residue field, and D an injective envelope of k as an A-module. For any A-module M, let $\mathcal{D}(M) = \mathrm{Hom}_A(M, D)$. Then from the above discussion, we get isomorphisms

$$\mathcal{D}(\mathrm{Tor}_i^A(M, N)) \cong \mathrm{Ext}^i(M, \mathcal{D}(N)).$$

Now $M \mapsto \mathcal{D}(M)$ is an exact, contravariant functor from the category of A-modules into itself. If $M \neq 0$, then there is a non-zero submodule $A/J \subset M$ for some ideal $J \subset \mathcal{M}$ (take a submodule Ax with $x \in M - \{0\}$). Then there is a surjection $\mathcal{D}(M) \twoheadrightarrow \mathcal{D}(A/J)$. Next, there is a surjection $A/J \twoheadrightarrow A/\mathcal{M} = k$, so that there is an injection $\mathcal{D}(k) = \mathrm{Hom}_A(k, D) \hookrightarrow \mathcal{D}(A/J)$. But $\mathcal{D}(k) \neq 0$, so we conclude that $\mathcal{D}(M) \neq 0$.

Define the *weak dimension* of N to be the smallest integer $d \geq 0$ such that $\mathrm{Tor}_i^A(M, N) = 0$ for any A-module M for $i > d$. If N is finitely generated, this equals the projective dimension of N. Then we have shown above that the weak dimension of N is also the smallest $d \geq 0$ such that $\mathrm{Ext}_A^i(M, \mathcal{D}(N)) = 0$ for any A-module M, for all $i > d$. Thus, we have:

Lemma 1 *The weak dimension of N equals the injective dimension of $\mathcal{D}(N)$.*

Claim: Any finitely generated submodule of D has finite length.

If M is a finitely generated submodule of D, it suffices to show that $\mathcal{M}$ is the only minimal prime of M. If not, M has a minimal prime $\mathcal{P} \neq \mathcal{M}$, and so a submodule $A/\mathcal{P} \subset M \subset D$. But then $k \cap A/\mathcal{P} = 0$, since there is no injection $k \to A/\mathcal{P}$ (of A-modules). This contradicts that D is the injective hull of k.

The claim implies that D may be considered as a module over the $\mathcal{M}$-adic completion $\widehat{A}$ of A. Hence $\mathcal{D}(M) = \mathrm{Hom}_A(M, D)$ is an $\widehat{A}$-module for any A-module M, and $M \mapsto \mathcal{D}(M)$ is an exact contravariant functor from the category $\mathcal{M}od(A)$ of A-modules to $\mathcal{M}od(\widehat{A})$. Since a strict chain of submodules of M is carried into a strict chain of quotients of $\mathcal{D}(M)$, we have the following lemma:

Lemma 2 *With the above notation,*

(i) $\mathcal{D}(M)$ is an Artinian $\widehat{A}$-module $\Rightarrow$ M is a Noetherian A-module, and

(ii) $\mathcal{D}(M)$ is a Noetherian $\widehat{A}$-module $\Rightarrow$ M is an Artinian A-module.

Now denote by $\widehat{\mathcal{D}} : \mathcal{M}od(\widehat{A}) \to \mathcal{M}od(\widehat{A})$ the functor $M \mapsto \mathrm{Hom}_{\widehat{A}}(M, D)$. We have a natural transformation (an A-homomorphism)

$$M \xrightarrow{\eta} \widehat{\mathcal{D}} \circ \mathcal{D}(M) = \mathrm{Hom}_{\widehat{A}}(\mathrm{Hom}_A(M, D), D),$$

$$\eta(m)(f) = f(m) \ \forall\ m \in M, f \in \mathrm{Hom}_A(M, D),$$

and since the target of η is an $\widehat{A}$-module, an induced natural $\widehat{A}$-homomorphism

$$\theta(M) : \widehat{A} \otimes_A M \to \widehat{\mathcal{D}}(\mathcal{D}(M)),$$

for any A-module M.

Theorem 2

(i) The mapping $\theta(M)$ is an isomorphism if M is Noetherian or Artinian. Further, if M is Noetherian (respectively Artinian) then $\mathcal{D}(M)$ is Artinian (respectively Noetherian).

(ii) $\widehat{A} = \mathrm{End}_A(D) = \mathrm{End}_{\widehat{A}}(D)$.

(iii) $\mathcal{D}$ gives an anti-equivalence between the category of Artinian A-modules and the category of Noetherian $\widehat{A}$-modules. If morever $A = \widehat{A}$ (i.e., A is complete*), then $\mathcal{D} = \widehat{\mathcal{D}}$ is its own quasi-inverse.*

Proof: Suppose α, β are two non-zero homomorphisms $k \to D$, where α is the inclusion given by the definition of D as an injective hull. Then $\beta(k) \cap k \neq 0$, and so $\beta(k) = k$, and $\beta = c\alpha$ for some $c \neq 0$ in k. Thus $\mathrm{Hom}_A(k, D) \cong k$, and hence also $\widehat{\mathcal{D}}(\mathcal{D}(k)) \cong k$. Clearly $\theta(k) : k \to \widehat{\mathcal{D}}(\mathcal{D}(k))$ is non-zero, and hence an isomorphism.

θ is a natural transformation between two exact functors

$$\mathcal{M}od(A) \to \mathcal{M}od(\widehat{A}).$$

Since $\theta(k)$ is an isomorphism, we deduce that $\theta(M)$ is an isomorphism if M has finite length.

Next, suppose that M is Noetherian. If $f \in \mathrm{Hom}_A(M, D)$, then $f(M)$ is a finitely generated submodule of D, and so has finite length; hence $f(\mathcal{M}^n M) = 0$ for some $n > 0$. Thus, we see that

$$\mathcal{D}(M) = \varinjlim_n \mathcal{D}(M/\mathcal{M}^n M),$$

and so

$$\widehat{\mathcal{D}}(\mathcal{D}(M)) = \varprojlim_n \widehat{\mathcal{D}}(\mathcal{D}(M/\mathcal{M}^n M))$$

(we have used the formula $\mathrm{Hom}\,(\varinjlim_n M_n, N) = \varprojlim_n \mathrm{Hom}\,(M_n, N)$, which is just a restatement of the universal property of the direct limit $\varinjlim_n M_n$). Since

$$\theta(M/\mathcal{M}^n M) : M/\mathcal{M}^n M \to \widehat{\mathcal{D}}(\mathcal{D}(M/\mathcal{M}^n M))$$

is an isomorphism, and we have a commutative diagram for each n (since θ is a natural transformation)

$$\begin{array}{ccc} \widehat{A} \otimes_A M & \stackrel{\theta(M)}{\longrightarrow} & \widehat{\mathcal{D}}(\mathcal{D}(M)) \\ \downarrow & & \downarrow \\ M/\mathcal{M}^n M & \stackrel{\theta(M/cM^n M)}{\longrightarrow} & \widehat{\mathcal{D}}(\mathcal{D}(M/\mathcal{M}^n M)) \end{array}$$

we see that $\theta(M)$ is an isomorphism.

This proves that $\theta(M)$ is an isomorphism for Noetherian M, and in particular that $\theta(A) : \widehat{A} \to \mathrm{Hom}_{\widehat{A}}(D, D)$ is an isomorphism. If $f \in \mathrm{Hom}_A(D, D)$, $d \in D$ and $\alpha \in \widehat{A}$, then for $a \in A$ which is a sufficiently good approximation to α,

$$f(\alpha d) = f(ad) = af(d) = \alpha f(d).$$

Thus $\mathrm{Hom}_A(D, D) = \mathrm{Hom}_{\widehat{A}}(D, D)$ *i.e.*, $\mathcal{D}(D) \cong \widehat{A}$, and so $\theta(D) : D \to \widehat{\mathcal{D}}(\mathcal{D}(D))$ is an isomorphism. Hence $\theta(D^{\oplus n})$ is an isomorphism for any $n > 0$.

Now if M is Noetherian, there is a surjection $A^{\oplus n} \twoheadrightarrow M$, and so an injection $D(M) \hookrightarrow \mathcal{D}(A^{\oplus n}) \cong D^{\oplus n}$ which is Artinian, since $\widehat{\mathcal{D}}(D^{\oplus n}) = \widehat{A}^{\oplus n}$ is Noetherian (see lemma 2). Hence $\mathcal{D}(M)$ is Artinian too. This proves (i) for Noetherian M.

Suppose that M is Artinian, and let $M \subset I$ be an injective hull of M. If $M \neq 0$, it has a non-zero finitely generated submodule, which has finite length; so we can find an element $x_1 \in M - \{0\}$ with annihilator $\mathcal{M}$ *i.e.*, a monomorphism $k \hookrightarrow I$. This extends to an A-linear map $i : D \to I$, whose kernel has trivial intersection with k, and so is zero. Since D is injective, i is a split inclusion, and we may write $I = D \oplus I_1$. If $I_1 \neq 0$, then $M_1 = M \cap I_1 \neq 0$, so that we may repeat the argument with an element $x_2 \in M_1 - \{0\}$ with annihilator $\mathcal{M}$, and obtain an isomorphism $I = D^{\oplus 2} \oplus I_2$, etc. This process must stop after a finite number of steps, since M is Artinian (else the chain of submodules $M \supset M_1 \supset M_2 \supset \cdots$ is a strictly decreasing infinite chain of submodules). Hence $I = D^{\oplus n}$ for some $n > 0$. Since D is Artinian, I/M is also Artinian, and we have an inclusion $I/M \hookrightarrow D^{\oplus m}$ for some $m > 0$ *i.e.*, an exact sequence

$$0 \to M \to D^{\oplus n} \to D^{\oplus m}.$$

As noted earlier, $\theta(D^{\oplus n})$ is an isomorphism for any $n > 0$; since θ is an exact functor, we see that $\theta(M)$ is an isomorphism, from the diagram with exact rows

$$\begin{array}{ccccc} 0 \to M & \longrightarrow & D^{\oplus n} & \to & D^{\oplus m} \\ \downarrow & & \cong\downarrow & & \cong\downarrow \\ 0 \to \widehat{\mathcal{D}}(\mathcal{D}(M)) & \to & \widehat{\mathcal{D}}(\mathcal{D}(D^{\oplus n})) & \to & \widehat{\mathcal{D}}(\mathcal{D}(D^{\oplus m})) \end{array}$$

Thus, when $A = \widehat{A}$, $\mathcal{D}$ defines an anti-equivalence of categories between the categories of Noetherian and Artinian A-modules, being its own quasi-inverse. When A is not complete, the categories of Artinian A-modules and Artinian $\widehat{A}$-modules are equivalent, so $\mathcal{D}$ gives an anti-equivalence between Artinian A-modules and Noetherian $\widehat{A}$-modules, since $\widehat{\mathcal{D}}$ is an anti-equivalence (from the case $A = \widehat{A}$ of (iii)). $\square$

Definition: If A is a Noetherian local ring, D an injective hull of k as an A-module, and M an A-module, the module $\mathcal{D}(M) = \mathrm{Hom}_A(M, D)$ is called the *dual* of M. Its isomorphism class is independent of the choice of D.

Remarks:

1. When A is not complete, the category of Noetherian A-modules is *not* anti-equivalent to the category of Artinian $\widehat{A}$- (or A-) modules.

2. Let A be a Noetherian local ring containing a coefficient field *i.e.*, a field k mapped isomorphically onto the residue field, which we again denote by k. For any module M, denote by $\overline{\mathrm{Hom}}_k(M, k)$ the set of k-linear maps $f : M \to k$ such that $f(\mathcal{M}^n M) = 0$ for some $n > 0$ *i.e.*,

$$\overline{\mathrm{Hom}}_k(M, k) = \varinjlim_n \mathrm{Hom}_k(M/\mathcal{M}^n M, k).$$

   Then $\overline{\mathrm{Hom}}_k(M, k)$ is an A-module in a natural way. Now set $D = \overline{\mathrm{Hom}}_k(A, k)$. It is easy to see that for any finitely generated module M, we have $\mathrm{Hom}_A(M, D) = \overline{\mathrm{Hom}}_k(M, k)$. Let I be an ideal of A, and $f : I \to k$ an element of $\overline{\mathrm{Hom}}_k(I, k)$, so that $f(\mathcal{M}^n I) = 0$ for some $n > 0$. By the Artin-Rees lemma, there exists $n' > 0$ such that $\mathcal{M}^{n'} \cap I \subset \mathcal{M}^n I$, and we get $f : I/(\mathcal{M}^{n'} \cap I) \to k$, which we may regard as a map $I + \mathcal{M}^{n'}/\mathcal{M}^{n'} \to k$. We may extend this to a k-linear map $g : A/\mathcal{M}^{n'} \to k$. Thus the element $f \in \overline{\mathrm{Hom}}_k(I, k)$ is the image of $g \in \overline{\mathrm{Hom}}_k(A, k)$ *i.e.*, D is an injective A-module. Further, $\mathrm{Hom}_A(k, D) = k$ and any element of D is killed by some $\mathcal{M}^n$. Hence D is the injective hull of k – if $x \in D - \{0\}$, then $\mathcal{M}^{r+1} x = 0$ while $\mathcal{M}^r x \neq 0$ for some $r \geq 0$, and then $\mathcal{M}^r x = k \subset D$ (since $\mathrm{Hom}_A(k, D) = k$); hence $Ax \cap k \neq 0$.

1.2 Gorenstein rings

Lemma 3 *Let A be a Noetherian local ring with residue field k, and M a finitely generated A-module. Suppose that for some integer $n > 0$, we have $\mathrm{Ext}^n_A(k, M) \neq 0$ and $\mathrm{Ext}^i_A(k, M) = 0$ for all $i > n$. Then M has injective dimension n, and we must have $n = \mathrm{depth}\, A$.*

Proof: To show that $n = \text{inj.dim.}\, M$, we have to show that for any module P of finite type, $\mathrm{Ext}^i(P, M) = 0$ for all $i > n$ (this follows by induction on n). Since we can find a composition series for P with quotients $A/\mathcal{P}$ where $\mathcal{P}$ is a prime ideal, it suffices to consider the case $P = A/\mathcal{P}$. If $\mathcal{P} = \mathcal{M}$, the maximal ideal, there is nothing to prove. If the assertion is false for some $\mathcal{P}$, then we may choose a $\mathcal{P}$ which is maximal with respect to this property. Choose $x \in (\mathcal{M} - \mathcal{P})$. Then x is a non zero divisor on $A/\mathcal{P}$, so that we have an exact sequence

$$0 \to A/\mathcal{P} \xrightarrow{x} A/\mathcal{P} \to A/(\mathcal{P} + Ax) \to 0.$$

Now $A/(\mathcal{P} + Ax)$ has a composition series with quotients $A/\mathcal{Q}$ where $\mathcal{Q}$ is prime and (strictly) contains $\mathcal{P}$ (since $(\mathcal{P}+Ax)A_{\mathcal{Q}} \neq A_{\mathcal{Q}}$). By the maximality of $\mathcal{P}$, we have $\mathrm{Ext}^i_A(A/\mathcal{Q}, M) = 0$ for $i > n$, and so $\mathrm{Ext}^i_A(A/(\mathcal{P}+Ax), M) = 0$ for $i > n$. Hence

$$\mathrm{Ext}^i_A(A/\mathcal{P}, M) \xrightarrow{x} \mathrm{Ext}^i(A/\mathcal{P}, M)$$

is surjective for $i > n$; hence

$$\mathcal{M}\mathrm{Ext}^i_A(A/\mathcal{P}, M) = \mathrm{Ext}^i_A(A/\mathcal{P}, M).$$

Since M, $A/\mathcal{P}$ are finite A-modules, $\mathrm{Ext}^i_A(A/\mathcal{P}, M)$ is finitely generated (we may compute it using a resolution of $A/\mathcal{P}$ by free A-modules of finite rank). By Nakayama's lemma, we conclude that $\mathrm{Ext}^i_A(A/\mathcal{P}, M) = 0$ for $i > n$, a contradiction.

Now if $x_1, x_2, \ldots, x_r$ is a maximal A-sequence in $\mathcal{M}$ (so that $\operatorname{depth} A = r$), we see by using the Koszul complex that

$$\operatorname{Ext}^r_A(A/(x_1, \ldots, x_r), M) \cong M/(x_1, \ldots, x_r)M \neq 0,$$

by Nakayama's lemma, since $M \neq 0$ is a finite A-module. Hence $r \leq n$. On the other hand, $\operatorname{depth} A/(x_1, \ldots, x_r) = \operatorname{depth} A - r = 0$, and so there is an exact sequence of A-modules

$$0 \to k \to A/(x_1, \ldots, x_r) \to P \to 0.$$

This gives an exact sequence of Ext groups

$$\operatorname{Ext}^n_A(A/(x_1, \ldots, x_r), M) \to \operatorname{Ext}^n_A(k, M) \to \operatorname{Ext}^{n+1}_A(P, M)$$

where the last term is zero, as seen above. Hence

$$\operatorname{Ext}^n_A(A/(x_1, \ldots, x_r), M) \twoheadrightarrow \operatorname{Ext}^n_A(k, M) \neq 0.$$

From the Koszul complex, this implies that $n \leq r$. Hence $n = r = \operatorname{depth} A$. $\square$

Recall that a submodule $N \subset M$ is called *irreducible* if we cannot write $N = P \cap Q$ for submodules $P, Q \subset M$ with $N \neq P$, $N \neq Q$.

Lemma 4 *Let A be a Noetherian local ring, M an Artinian A-module. Then we can find irreducible submodules $N_i \subset M$, $i = 1, \ldots, m$ such that $\cap_{i=1}^m N_i = (0)$, but (0) is not the intersection of any subfamily of the N_i. The integer m then equals $\dim_k \operatorname{Hom}_A(k, M)$.*

Proof: For an Artinian module M, we claim the following are equivalent:

(i) (0) is irreducible in M

(ii) $\dim_k \operatorname{Hom}_A(k, M) = 1$

(iii) $\mathcal{D}(M)$ is generated by one element.

Indeed, since M is Artinian, $\mathcal{D}(M)$ is a Noetherian $\widehat{A}$-module, by Theorem 2, and

$$\operatorname{Hom}_A(k, M) \cong \operatorname{Hom}_{\widehat{A}}(\mathcal{D}(M), \mathcal{D}(k)) = \operatorname{Hom}_{\widehat{A}}(\mathcal{D}(M), k) \cong \operatorname{Hom}_k(k \otimes_A \mathcal{D}(M), k).$$

Hence

$$\dim_k \operatorname{Hom}_A(k, M) = \dim_k k \otimes_A \mathcal{D}(M)$$

which is the minimal number of generators of $\mathcal{D}(M)$ as an $\widehat{A}$-module, and the second and third statements above are equivalent. Now suppose that (0) is irreducible in M, and $\alpha, \beta \in \operatorname{Hom}_A(k, M) - \{0\}$. If $\alpha(k) \neq \beta(k)$, then $\alpha(k) \cap \beta(k) = (0)$, contradicting irreducibility. Hence $\alpha(k) = \beta(k) \cong k$ and so $\alpha = c\beta$ for some $c \in k - \{0\}$. Thus $\dim_k \operatorname{Hom}_A(k, M) = 1$. Suppose that $M_1, M_2 \subset M$ are non-zero with $M_1 \cap M_2 = (0)$. We can find non-zero homomorphisms $\alpha : k \to M_1$, $\beta : k \to M_2$ since M_i are Artinian. Since the images of α and β have trivial intersection, α and β are k-linearly independent in $\operatorname{Hom}_A(k, M)$. Hence $\dim_k \operatorname{Hom}_A(k, M) > 1$, completing the proof of the claimed equivalence.

Now irreducible submodules $N \subset M$ correspond to $\widehat{A}$-submodules $\mathcal{D}(M/N) \subset \mathcal{D}(M)$ generated by 1 element, since N is irreducible in $M \Leftrightarrow$ 0 is irreducible in M/N. Further, $\cap_i N_i = (0) \Leftrightarrow M \to \oplus M/N_i$ is injective $\Leftrightarrow \oplus\mathcal{D}(M/N_i) \to \mathcal{D}(M)$ is surjective. Thus irredundant representations $(0) = \cap_i N_i$ with N_i irreducible correspond precisely to picking minimal sets of cyclic $\widehat{A}$-submodules of $\mathcal{D}(M)$ generating $\mathcal{D}(M)$ *i.e.*, to picking minimal sets of generators for $\mathcal{D}(M)$ as an $\widehat{A}$-module. □

Theorem 3 *Let A be a Noetherian local ring of dimension n with residue field k. The following are equivalent:*

(i) for any system of parameters $x_1, x_2, \ldots, x_n$ of A, the ideal $(x_1, \ldots, x_n)$ is irreducible in A

(ii) A is Cohen-Macaulay, and there is a system of parameters $x_1, \ldots, x_n$ such that $(x_1, \ldots, x_n)$ is irreducible in A

(iii) for $0 \leq i < n$, $\operatorname{Ext}^i_A(k, A) = 0$ and $\operatorname{Ext}^n_k(k, A) = k$

(iv) for large i, $\operatorname{Ext}^i_A(k, A) = 0$

(v) A has injective dimension n as an A-module

(vi) A has finite injective dimension as an A-module.

Proof: We proceed by induction on $n = \dim A$. Suppose first that $n = 0$ *i.e.*, A is Artinian. Then (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ the ideal (0) is irreducble in A $\Leftrightarrow$ $\dim_k \operatorname{Hom}_A(k, A) = 1$ *i.e.*, $\Leftrightarrow$ (iii). Further, by lemma 3, since $\dim A = \operatorname{depth} A = 0$, we have (iv) $\Leftrightarrow$ (v) $\Leftrightarrow$ (vi) $\Leftrightarrow$ A is injective as an A-module. Now, let D be the injective hull of k.

Suppose (0) is irreducible in A. We can find an injection $f : k \to A$. Since D is injective, f fits into a diagram

$$\begin{array}{ccccc} 0 \to & k & \xrightarrow{f} & A & \\ & i \searrow & & \swarrow \alpha & \\ & & D & & \end{array}$$

Since $\ker \alpha \cap k = (0)$ and (0) is irreducible, $\ker \alpha = (0)$ and α is injective; in particular, $\operatorname{length} A \leq \operatorname{length} D$. Applying the functor $\mathcal{D}$, we obtain a surjection

$$A = \widehat{A} = \mathcal{D}(D) \stackrel{\mathcal{D}(\alpha)}{\to} \mathcal{D}(A) = D.$$

Hence $\operatorname{length} A = \operatorname{length} D$, and so α is an isomorphism, and A is an injective A-module.

Conversely, if A is an injective A-module, then any monomorphism $f : k \to A$ extends to a monomorphism $\beta : D \to A$ (since it extends to a map β, with $\ker \beta \cap k = 0$). As above, applying $\mathcal{D}$, we conclude that in fact β is an isomorphism. Further, $\operatorname{Hom}_A(k, A) = \operatorname{Hom}_A(k, D) = k$, so (0) is irreducible. This proves the Theorem for $n = 0$.

Suppose that $n > 0$ and the Theorem holds for all rings of smaller dimension. We have in any case by lemma 3 the equivalences (iv) $\Leftrightarrow$ (vi) $\Leftrightarrow$ inj.dim. $A = \operatorname{depth} A$, and (v) $\Leftrightarrow$ (iv). We shall first show that each one of the hypothesis (i)-(vi) implies the existence of a non-zero divisor $x \in \mathcal{M}$, the maximal ideal of A; or equivalently, that $\operatorname{depth} A > 0$, or that $\mathcal{M} \notin \operatorname{Ass} A$. This is clear for (ii) (since $n > 0$) and (iii) (since $\operatorname{Hom}_A(k, A) = 0$). Thus it suffices to check that (i) and (vi) imply this.

Assume (i), and suppose that $\mathcal{M} \in \operatorname{Ass} A$, so that there exists $x \in A$, $x \neq 0$ with $\mathcal{M}x = (0)$. Let $y \in \mathcal{M}$ such that y does not lie in any minimal prime ideal of (0). Since $\cap_{k \geq 0} y^k A = (0)$, replacing y by y^k if necessary, we may assume that $x \notin Ay$. Now $B = A/Ay$ has $\dim B = n - 1$, and satisfies the hypothesis (i) with $n-1$ in place of n. Hence by induction, B satisfies (ii)-(vi); in particular, B is Cohen-Macaulay. But $\operatorname{depth} B = 0$, since the image of x in B is non-zero and is annihilated by the maximal ideal. Hence $\dim B = 0$, and $n = 1$. Suppose that $(0) = \mathcal{Q} \cap \mathcal{Q}'$ is a primary decomposition of (0) in A, where $\mathcal{Q}'$ is the intersection of the primary components for the minimal primes, and $\mathcal{Q}$ is the intersection of non-minimal primary components. Since $\operatorname{depth} A = 0$, $\mathcal{Q} \neq (0)$. Let $z \in \mathcal{Q}$ such that z does not lie in any minimal prime. Replacing z by z^n for large n, we may assume (since $\cap_n z^n A = (0)$) that $Az \neq \mathcal{Q}$, and $\mathcal{Q}' \not\subset Az$. Now $Az \subset (\mathcal{Q}+Az) \cap (\mathcal{Q}'+Az) = \mathcal{Q} \cap (\mathcal{Q}'+Az)$. If $t+\mu z \in \mathcal{Q} \cap (\mathcal{Q}'+Az)$, with $t \in \mathcal{Q}'$, then $\mu z \in \mathcal{Q} \Rightarrow t \in \mathcal{Q}$, so $t \in \mathcal{Q} \cap \mathcal{Q}' = (0)$. Hence $Az = \mathcal{Q} \cap (Az + \mathcal{Q}')$. But $Az \neq \mathcal{Q}$ and $Az \neq (Az + \mathcal{Q}')$, so Az is *not* irreducible. Since $\dim A = 1$ and $z \in A$ is a parameter (as it does not lie in any minimal prime), we see that (i) is contradicted.

Next, assume (vi), so that inj.dim. $A = \operatorname{depth} A$. We want to show $\operatorname{depth} A > 0$; if $\operatorname{depth} A = 0$, then A must itself be injective. Also, we have an injection $f : k \to A$ (since $\operatorname{depth} A = 0$), which factors (since A is injective) through $i : k \to D$, giving an injection $D \to A$. Since D is the injective hull of k, the map $D \to A$ must be a split inclusion. Applying $\mathcal{D}$ to the surjection $A \to D$, we obtain an injection $\widehat{A} = \mathcal{D}(D) \to \mathcal{D}(A) = D$. Hence $\widehat{A}$ is Artinian *i.e.*, A is Artinian, contradicting that $\dim A > 0$.

Thus, each of the hypothesis (i)-(vi) implies that there exists a non-zero divisor $x \in A$. Let us put $B = A/Ax$, and denote by (i)$'$,...,(vi)$'$ the hypotheses (i) to (vi) for B. Then, we have

$$\text{(i)} \Rightarrow \text{(i)}' \Leftrightarrow \text{(ii)}' \Leftrightarrow \text{(ii)},$$

where the middle equivalence is by the induction hypothesis.

Now, assume (ii) and let $x_1, \ldots, x_n$ be a system of parameters in A. Then $x_1, \ldots, x_n$ is a regular sequence, since we have assumed A is Cohen-Macaulay. From the exact sequences

$$0 \to A/(x_1, \ldots, x_i) \xrightarrow{x_{i+1}} A/(x_1, \ldots, x_i) \to A/(x_1, \ldots, x_{i+1}) \to 0,$$

we see by descending induction on i that

$$\operatorname{Ext}^j_A(k, A/(x_1, \ldots, x_i)) = 0 \ \forall j < n-i,$$

$$\operatorname{Ext}^{n-i}_A(k, A/(x_1, \ldots, x_i) \cong \operatorname{Hom}_A(k, A/(x_1, \ldots, x_n)).$$

In particular, for $i = 0$, we get

$$\operatorname{Ext}^i_A(k, A) = 0 \ \forall i < n, \ \ \operatorname{Ext}^n_A(k, A) \cong \operatorname{Hom}_A(k, A/(x_1, \ldots, x_n)).$$

By assumption, there is some system of parameters $y_1, \ldots, y_n$ such that $(y_1, \ldots, y_n)$ is irreducible in A *i.e.*, by lemma 4, we have $\dim_k \operatorname{Hom}_A(k, A/(y_1, \ldots, y_n) = 1$. Hence $\operatorname{Ext}_A^n(k, A) = k$. This implies that $\operatorname{Hom}_A(k, A/(x_1, \ldots, x_n)) = k$ *i.e.*, $(x_1, \ldots, x_n)$ is irreducible in A. Hence (ii) $\Rightarrow$ (i) and (ii) $\Rightarrow$ (iii).

Suppose (iii) holds, and let $y_1, \ldots, y_m$ be a maximal A-sequence. Then $m \leq n$. The long exact sequence of Ext groups associated to the exact sequences of A-modules

$$0 \to A/(x_1, \ldots, x_i) \stackrel{x_{i+1}}{\to} A/(x_1, \ldots, x_i) \to A/(x_1, \ldots, x_{i+1}) \to 0, \ 0 \leq i < m,$$

yields, by induction on i,

$$\operatorname{Ext}_A^j(k, A/(x_1, \ldots, x_i)) = 0 \text{ for } j < n - i,$$

$$\operatorname{Ext}_A^{n-i}(k, A/(x_1, \ldots, x_i)) \cong \operatorname{Ext}_A^n(k, A) = k.$$

On the other hand,

$$\operatorname{Ext}_A^0(k, A/(x_1, \ldots, x_m)) = \operatorname{Hom}_A(k, A/(x_1, \ldots, x_m)) \neq 0$$

since $\operatorname{depth} A = m$. We deduce that $m = n$, and $\operatorname{Hom}_A(k, A/(x_1, \ldots, x_n) = k$, which implies (ii). Hence we have shown:
(i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii).

Now (ii) implies that in the Artin ring $A/(x_1, \ldots, x_n) = C$, the ideal (0) is irreducible. Hence C is injective over itself, by the Theorem for rings of dimension 0. Further, we had seen in this case that C is an injective envelope of k as a C-module. On the other hand, if J is an injective A-module, then for any ideal I of A, and any A/I-module M, we have

$$\operatorname{Hom}_{A/I}(M, \operatorname{Hom}_A(A/I, J)) = \operatorname{Hom}_A(M, J);$$

hence $\operatorname{Hom}_A(A/I, J)$ is A/I-injective. In particular, if $J = D$, the injective hull of k as an A-module, we see that $\operatorname{Hom}_A(A/I, D)$ is an injective A/I module. Further, $\operatorname{Hom}_{A/I}(k, \operatorname{Hom}_A(A/I, D)) = \operatorname{Hom}_A(k, D) = k$, so $\operatorname{Hom}_A(A/I, D)$ is in fact the injective hull of k as an A/I-module. Applying this to C, we see that $\operatorname{Hom}_A(C, D) \cong C$. Thus, we see that

$$\mathcal{D}_A(C) \cong C = A/(x_1, \ldots, x_n).$$

Since C has projective dimension n over A, we see by lemma 1 that $\mathcal{D}(C) \cong C$ has injective dimension n over A. Hence, by descending induction on i, $\mathrm{Ext}^j_A(k, A/(x_1, \ldots, x_i) = 0$ for $j > n$. Thus (ii) $\Rightarrow$ (vi).

Finally, suppose that (vi) holds, and let $x \in A$ be a non-zero divisor. From the exact sequence

$$0 \to A \xrightarrow{x} A \to A/Ax \to 0,$$

we see that $\mathrm{Ext}^1_A(A/Ax, A) \cong A/Ax$, and $\mathrm{Ext}^i_A(A/Ax, A) = 0$ for $i \neq 1$. Let

$$0 \to A \to I^0 \to \cdots \to I^n \to 0$$

be a finite injective resolution of A. The complex $\mathrm{Hom}_A(A/Ax, I^\bullet)$ has $\mathrm{Ext}^i_A(A/Ax, A)$ as cohomology groups *i.e.*, has all cohomologies except the first equal to 0, and the first cohomology is $\mathrm{Ext}^1_A(A/Ax, A) \cong A/Ax$. Thus we have exact sequences (the bottom one defines Z^1)

$$0 \to \mathrm{Hom}_A(A/Ax, I^0) \to Z^1 \to \mathrm{Ext}^1_A(A/Ax, A) \to 0,$$

$$0 \to Z^1 \to \mathrm{Hom}_A(A/Ax, I^1) \to \mathrm{Hom}_A(A/Ax, I^2) \to \cdots \to \mathrm{Hom}_A(A/Ax, I^n) \to 0.$$

The bottom row gives an injective resolution for Z^1 as an A/Ax module, and so Z^1 has finite injective dimension over A/Ax. Since $\mathrm{Hom}_A(A/Ax, I^0)$ is an injective A/Ax-module, the top sequence splits; this shows that $\mathrm{Ext}^1_A(A/Ax, A) \cong A/Ax$ has finite injective dimension as an A/Ax-module, so (vi) is satisfied by A/Ax. This implies (ii) is satisfied by A/Ax, and hence by A. This completes the proof. □

Definition: A Noetherian local ring A satisfying any of the equivalent conditions (i)-(vi) of the Theorem is called a *Gorenstein* ring.

Remarks:

1. Any regular local ring is Gorenstein, since it has finite global dimension.

2. If A is Gorenstein and $x_1, \ldots, x_r$ is an A-sequence, then $A/(x_1, \ldots, x_r)$ is Gorenstein. In fact (ii) above is fulfilled.

3. More generally, let A be any Noetherian local ring, I an ideal in A, and $d = \dim A/I$. Then A/I is Gorenstein $\Leftrightarrow$ (a) $\mathrm{Ext}^d_A(k, A/I) = 0$ for $i < d$ and (b) $\mathrm{Ext}^d_A(k, A/I) = k$.

In fact, (a) is equivalent to the existence of an A/I-sequence $x_1, \dots, x_d$ of length d *i.e.*, to A/I being Cohen-Macaulay. Next, if A/I is Cohen-Macaulay and $x_1, \dots, x_d$ is a regular A/I-sequence, the Koszul complex yields

$$\mathrm{Ext}^d(k, A/I) \cong \mathrm{Hom}_A(k, A/(I + (x_1, \dots, x_d))),$$

so (b) $\Leftrightarrow I + (x_1, \dots, x_d)$ is irreducible in A $\Leftrightarrow I + (x_1, \dots, x_d)/I$ is irreducible in A/I. Thus (a) and (b) hold $\Leftrightarrow A/I$ satisfies the condition (ii).

4. Let A be a one dimensional non-normal Noetherian local domain with quotient field K such that its integral closure $\overline{A}$ (in K) is a finite A-module. Then A is Cohen-Macaulay, and $\mathrm{Ext}^1_A(k, A) \cong \mathcal{M}^{-1}/A$, as follows from the exact sequence

$$0 \to \mathcal{M} \to A \to k \to 0,$$

where $\mathcal{M}^{-1} = \{x \in K \mid x\mathcal{M} \subset A\}$ (thus $\mathrm{Hom}_A(k, K/A) = \mathcal{M}^{-1}/A$). If $\mathcal{M}^{-1}\mathcal{M} = A$, then there exist $x \in \mathcal{M}^{-1}$, $y \in \mathcal{M}$ such that $xy \notin \mathcal{M}$, so that xy is a unit; then for $z \in \mathcal{M}$, we have $z = zx(xy)^{-1}y \in Ay$, so that $\mathcal{M} = Ay$, and A is regular, a contradiction. Hence $\mathcal{M}^{-1}\mathcal{M} \subset \mathcal{M}$, and so $\mathcal{M}^{-1} \subset \overline{A}$. Thus $\mathrm{Hom}_A(k, \overline{A}/A) = \mathrm{Hom}_A(k, K/A) = \mathcal{M}^{-1}/A$.

Now if $I \subset A$ is the conductor, then by definition, $I = \mathrm{Ann}_A(\overline{A}/A)$, so that $\overline{A}/A$ is a faithful A/I-module. Hence so is its dual $\mathcal{D}(\overline{A}/A)$, since $\mathcal{D}(\mathcal{D}(\overline{A}/A) = \overline{A}/A$. Now A is Gorenstein $\Leftrightarrow$ $\mathrm{Ext}^1_A(k, A) = k$ $\Leftrightarrow$ $\mathrm{Hom}_A(k, \overline{A}/A) = k$ $\Leftrightarrow$ $\mathcal{D}(\overline{A}/A)$ is generated by one element $\Leftrightarrow$ $\mathcal{D}(\overline{A}/A) \cong A/I$. Now for any Artinian module M over an Artinian local ring B, length $(\mathcal{D}(M)) =$ length (M), since this is true for $M = k$, the residue field. Hence A is Gorenstein $\Rightarrow$ length $(\overline{A}/A) =$ length (A/I).

2 Local duality theory

Theorem 4 *(The local duality theorem) Let A be a Noetherian Cohen-Macaulay local ring of dimension n, with maximal ideal $\mathcal{M}$ and residue field k, and put*

$$\lim_{\overrightarrow{p}} \mathrm{Ext}^n_A(A/\mathcal{M}^p, A) = J.$$

Then we have a natural isomorphism

$$\varinjlim_{p} \operatorname{Ext}_A^{n-i}(A/\mathcal{M}^p, M) \cong \operatorname{Tor}_i^A(M, J) \qquad \cdots \qquad (\#)$$

for all A-modules M, for all $I \geq 0$.

Proof: Let $x_1, \ldots, x_n$ be a maximal A-sequence, and set $\mathcal{M}_p = (x_1^p, \ldots, x_n^p)$. Now $x_1^p, \ldots, x_n^p$ is also an A-sequence, and if $\mathcal{M}^r \subset \mathcal{M}_1$, then we have inclusions $\mathcal{M}^{npr} \subset \mathcal{M}_1^{np} \subset \mathcal{M}_p \subset \mathcal{M}^p$, so that

$$\varinjlim_{p} \operatorname{Ext}_A^j(A/\mathcal{M}^p, M) = \varinjlim_{p} \operatorname{Ext}_A^j(A/\mathcal{M}_p, M)$$

for any A-module M. Since $\mathcal{M}_p$ is generated by an A-sequence, the Koszul complex yields $\operatorname{Ext}_A^j(A/\mathcal{M}_p, M) = 0$ for $j > n$, and $\operatorname{Ext}_A^j(A/\mathcal{M}_p, A) = 0$ for $j < n$.

Define covariant functors T_i, $0 \leq i \leq n$, on the category of A-modules by

$$T_i(M) = \varinjlim_{p} \operatorname{Ext}_A^{n-i}(A/\mathcal{M}_p, M)$$

Then

(i) the T_i form a covariant ∂-functor in M *i.e.*, given a short exact sequence

$$0 \to M' \to M \to M'' \to 0,$$

we have a long exact sequence

$$0 \to T_n(M') \to T_n(M) \to T_n(M'') \xrightarrow{\partial} T_{n-1}(M'') \to \cdots$$

$$\cdots \to T_1(M'') \xrightarrow{\partial} T_0(M') \to T_0(M) \to T_0(M) \to 0;$$

given a commutative diagram of short exact sequences

$$\begin{array}{ccccccccc} 0 \to & M' & \to & M & \to & M'' & \to 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \to & N' & \to & N & \to & N'' & \to 0 \end{array}$$

the diagrams

$$\begin{array}{ccc} T_{i+1}(M'') & \xrightarrow{\partial} & T_i(M') \\ \downarrow & & \downarrow \\ T_{i+1}(N'') & \xrightarrow{\partial} & T_i(N') \end{array}$$

commute;

(ii) T_i commutes with direct sums, and $T_i(A) = 0$ for $i > 0$; hence T_i is *effaceable.*

The above shows that the T_i form a *universal ∂-functor* in the sense of Grothendieck.

Next, for any three A-modules M, N, P there is a *natural* homomorphism $\operatorname{Ext}^i_A(N,P) \otimes_A M \to \operatorname{Ext}^i(N, P \otimes M)$ for each i – if $m \in M$, there is an induced A-module map $\psi(m) : P \to P \otimes_A M$, $p \mapsto p \otimes m$, which induces a map

$$\psi(m)_* : \operatorname{Ext}^i_A(N,P) \to \operatorname{Ext}^i_A(N, P \otimes_A M);$$

one verifies that $\psi(\alpha \otimes m) = \psi(m)_*(\alpha)$ gives the desired map.

In particular, there is a natural map $M \otimes_A \operatorname{Ext}^n(A/\mathcal{M}^p, A) \to \operatorname{Ext}^n_A(A/\mathcal{M}^p, M)$ for each $p > 0$. Taking the direct limit over p, we obtain a natural transformation of functors of M

$$M \otimes_A J \to T_0(M).$$

This is an isomorphism for $M = A$, commutes with direct sums, and both functors are right exact. Hence, looking at a presentation of M as a cokernel of a mapping of free A-modules, we see that the above natural transformation is an isomorphism for all M. Since $\operatorname{Tor}^A_i(M,J)$ and $T_i(M)$ are both universal ∂-functors, this means that the above natural transformation extends uniquely to a natural isomorphism of ∂-functors *i.e.*, to natural isomorphisms

$$T_i(M) \cong \operatorname{Tor}^A_i(M,J)$$

which are compatible with the ∂-maps. $\square$

Since A is Cohen-Macaulay, $\operatorname{Ext}^i_A(k,A) = 0$, $0 \le i < n$, and hence $\operatorname{Ext}^i_A(N,A) = 0$ for $0 \le i < n$ for any A-module N of finite length. Thus, the natural homomorphisms

$$J_p = \operatorname{Ext}^n_A(A/\mathcal{M}_p, A) \to \operatorname{Ext}^n_A(A/\mathcal{M}_{p+1}, A) = J_{p+1}$$

are all injective, from the long exact sequence of Ext 's. Further,

$$\dim_k \operatorname{Hom}_A(k, J_p) = \dim_k \operatorname{Hom}_A(k, A/(x_1^p, \ldots, x_n^p)) = \dim_k \operatorname{Ext}^n_A(k,A)$$

is independent of p, and so $\operatorname{Hom}_A(k, J_p) \to \operatorname{Hom}_A(k, J_{p+1})$ is an isomorphism. Hence $k \otimes \mathcal{D}(J_{p+1}) \to k \otimes \mathcal{D}(J_p)$ is an isomorphism, and so $\mathcal{D}(J_{p+1}) \to$

$\mathcal{D}(J_p)$ is surjective (note that J_p, J_{p+1} have finite length, hence so do their duals). Let $m = \dim_k \operatorname{Ext}^n_A(k, A)$, and F a free $\widehat{A}$-module of rank m with basis $e_1, \ldots, e_m$. Inductively, we can find surjective $\widehat{A}$-homomorphisms $\varphi_p : F \to \mathcal{D}(J_p)$ such that the diagrams

$$\begin{array}{ccc} F & \xrightarrow{\varphi_p} & \mathcal{D}(J_p) \\ \varphi_{p+1} \searrow & & \swarrow \\ & \mathcal{D}(J_{p+1}) & \end{array}$$

commute. Hence we obtain a homomorphism $\varphi : F \to \mathcal{D}(J) = \varprojlim_p \mathcal{D}(J_p)$. Let $\operatorname{im}\varphi = G$, so that $G \subset \mathcal{D}(J)$ is a finitely generated $\widehat{A}$-submodule. If $G \neq \mathcal{D}(J)$, then we can find a finitely generated $\widehat{A}$-submodule $H \subset \mathcal{D}(J)$ which strictly contains G. Since $G \to \mathcal{D}(J_p)$ is surjective for each p, so is $H \to \mathcal{D}(J_p)$. Let $H_p = \ker H \to \mathcal{D}(J_p)$, so that $G + H_p = H$, and $H/H_p = \mathcal{D}(J_p)$ is Artinian; also, $\cap_p H_p = \ker(H \to \mathcal{D}(J)) = 0$. Hence for any $r > 0$ we can find $p(r)$ such that $H_{p(r)} \subset \mathcal{M}^r H$. Thus, $H \subset \cap_n (G + \mathcal{M}^n H) = G$, since G is closed in H for the $\mathcal{M}$-adic topology (as G, H are finite $\widehat{A}$-modules). Hence we must have $G = \mathcal{D}(J)$.

Thus, $\mathcal{D}(J) = \Omega_A$ is a finite $\widehat{A}$-module, and so J is Artinian. For any finite A-module M, $\operatorname{Tor}^A_i(M, J)$ is Artinian, since M has a resolution $F_\bullet \to M \to 0$ where F_i are free of finite rank, and $F_i \otimes_A J$ is Artinian for each i. Hence by Theorem 2 and the discussion preceeding lemma 1,

$$\operatorname{Tor}^A_i(M, J) \cong \widehat{A} \otimes_A \operatorname{Tor}^A_i(M, J) \cong \operatorname{Tor}^{\widehat{A}}_i(\widehat{A} \otimes_A M, J) \cong$$
$$\widehat{\mathcal{D}}(\operatorname{Ext}^i_{\widehat{A}}(\widehat{A} \otimes_A M, \mathcal{D}(J)) \cong \widehat{\mathcal{D}}(\operatorname{Ext}^i_{\widehat{A}}(\widehat{A} \otimes_A M, \Omega_A)).$$

Thus, we have:

Corollary 1 *The module $\Omega_A = \mathcal{D}(J)$ is finitely generated over $\widehat{A}$, and for any finitely generated A-module M, we have an isomorphism*

$$\varinjlim_p \operatorname{Ext}^j_A(A/\mathcal{M}^n, M) \cong \mathcal{D}(\operatorname{Ext}^{n-j}_{\widehat{A}}(\widehat{A} \otimes_A M, \Omega_A)) \quad \cdots \quad (\dagger).$$

The minimal number of generators of Ω as an $\widehat{A}$-module is $\dim_k \operatorname{Ext}^n(k, A)$. Further, $\operatorname{Ext}^i_{\widehat{A}}(k, \Omega_A) = 0$ for $0 \leq i < n$, $\operatorname{Ext}^n_{\widehat{A}}(k, \Omega_A) = k$, so that Ω_A is a Cohen-Macaulay module of dimension n over $\widehat{A}$, such that if $y_1, \ldots, y_n$ is any system of parameters of $\widehat{A}$, the submodule $(y_1, \ldots, y_n)\Omega_A$ is irreducible in Ω_A. Lastly, Ω_A is a faithful $\widehat{A}$-module.

Proof: The first assertion (*i.e.*, finite generation of Ω_A, and the isomorphism (†)) has already been proved. We have also shown above that (with the earlier notations) there is a surjection $F \twoheadrightarrow \Omega_A$, such that $F \otimes k \cong \Omega_A \otimes k \cong J_p \otimes k$ for all $p > 0$, where $F = \widehat{A}^{\oplus r}$, and $r = \dim_k \mathrm{Ext}^n(k, A)$. Hence Ω_A is minimally generated by r elements. Now, $\widehat{\mathcal{D}}(\Omega_A) \cong \widehat{A} \otimes_A J \cong J$, by Theorem 2, since J is Artinian. If $\alpha \in \widehat{A}$ kills Ω_A, then it kills each of the submodules $J_p = \widehat{A}/(x_1^p, \ldots, x_n^p)$ of $J = \widehat{\mathcal{D}}(\Omega_A)$, and so $\alpha \in \cap(x_1^p, \ldots, x_n^p) = 0$. Hence Ω_A is a faithful $\widehat{A}$-module.

Now to calculate $\mathrm{Ext}^i_{\widehat{A}}(k, \Omega_A)$. We apply the duality isomorphism (†) with $M = k$. For each $p > 0$, set as before $\mathcal{M}_p = (x_1^p, \ldots, x_n^p)$ and $K_\bullet(x_1^p, \ldots, x_n^p)$ the Koszul complex over $\widehat{A}$ with respect to $x_1^p, \ldots, x_n^p$. Then $K_\bullet(x_1^p, \ldots, x_n^p)$ resolves $\widehat{A}/\mathcal{M}_p$, and we have a map of complexes $\psi_p : K_\bullet(x_1^{p+1}, \ldots, x_n^{p+1}) \to K_\bullet(x_1^p, \ldots, x_n^p)$ lifting $\widehat{A}/\mathcal{M}_{p+1} \to \widehat{A}/\mathcal{M}_p$, given by

$$e_{i_1} e_{i_2} \cdots e_{i_k} \mapsto x_{i_1} x_{i_2} \cdots x_{i_k} e_{i_1} e_{i_2} \cdots e_{i_k}.$$

Hence

$$\mathrm{Hom}\,(\psi_p, k) : \mathrm{Hom}\,(K_r(x_1^p, \ldots, x_n^p), k) \to \mathrm{Hom}\,(K_r(x_1^{p+1}, \ldots, x_n^{p+1}), k)$$

is 0 for $r \geq 1$, so that $\mathrm{Ext}^r_A(A/\mathcal{M}_p, k) \to \mathrm{Ext}^r_A(A/\mathcal{M}_{p+1}, k)$ is 0 for $0 \leq r < n$, and $\mathrm{Hom}\,(A/\mathcal{M}_p, k) \otimes k \to \mathrm{Hom}\,(A/\mathcal{M}_{p+1}, k) \otimes k$ is an isomorphism. Thus

$$\mathrm{Ext}^i_{\widehat{A}}(k, \Omega_A) = 0, \; 0 \leq i < n,$$
$$\mathrm{Ext}^n_{\widehat{A}}(k, \Omega_A) \cong k.$$

This evidently implies the remaining assertions. □

Corollary 2 *The following assertions are equivalent on a Cohen-Macaulay ring A:*

(i) *A is a Gorenstein ring*

(ii) $J \cong D$

(iii) $\Omega_A \cong \widehat{A}$

(iv) *Ω_A is generated by 1 element as an $\widehat{A}$-module*

(v) *J is injective*

(vi) Ω_A *is* $\widehat{A}$*-free.*

Proof: A is Gorenstein $\Leftrightarrow$ $\mathrm{Ext}^n(k, A) = k$ $\Leftrightarrow$ Ω_A is generated by 1 element as an $\widehat{A}$-module $\Leftrightarrow$ $\Omega_A \cong \widehat{A}$ (since Ω_A is a faithful $\widehat{A}$-module) $\Leftrightarrow$ $J \cong D$. Thus, (i), (ii), (iii), (iv) are equivalent, and they clearly imply (v) and (vi), and (v) $\Leftrightarrow$ (vi). On the other hand, since $\mathrm{Ext}^n_{\widehat{A}}(k, \Omega_A) = k$, (vi) $\Rightarrow$ (iii). $\square$

Remarks:

1. Ω_A has finite injective dimension (as an $\widehat{A}$-module). Indeed, we may assume A is complete; by lemma 1, it suffices to show that J has weak dimension $\leq n$. But the duality theorem (#) implies that for any A-module M, $\mathrm{Tor}^A_i(M, J) = 0$ for $i > n$, since $\mathrm{Ext}^{n-i}_A(A/\mathcal{M}^p, M) = 0$ for all $p > 0$.

2. The associated prime ideals to $\widehat{A}$ and Ω_A (or what is the same, the minimal prime ideals of (0) in $\widehat{A}$ and Ω_A, since $\dim \Omega_A = \dim A$ and both are Cohen-Macaulay) are the same.

   To see this, note that since Ω_A is Cohen-Macaulay of dimension n, $\mathrm{Ass}\,(\Omega_A) \subset \mathrm{Ass}\,(\widehat{A})$. If $\mathcal{P} \in \mathrm{Ass}\,(\widehat{A})$, $\mathcal{P} \notin \mathrm{Ass}\,(\Omega_A)$, choose an $x \notin \mathcal{P}$ such that x lies in all the associated primes of Ω_A. Then $x^m \neq 0$ for any $m > 0$, but x^m kills Ω_A for large m. This is impossible since Ω_A is a faithful $\widehat{A}$-module.

3. We want to compute $\mathrm{End}_{\widehat{A}}(\Omega_A)$. We may assume that A is complete without loss of generality. Then from Corollary 1 (with $j = 0$, $M = \Omega_A$), we have

$$\mathcal{D}(\mathrm{End}_A(\Omega_A)) \cong \varinjlim_p \mathrm{Ext}^n_A(A/\mathcal{M}^p, \Omega_A),$$

   and so

$$\mathrm{End}_A(\Omega_A) \cong \mathcal{D}(\varinjlim_p \mathrm{Ext}^n(A/\mathcal{M}^p, \Omega_A)) \cong \varprojlim_p \mathcal{D}(\mathrm{Ext}^n_A(A/\mathcal{M}^p, \Omega_A))$$

$$\cong \varprojlim_p (\varinjlim_q \mathrm{Hom}_A(A/\mathcal{M}^q, A/\mathcal{M}^p)) \cong \varprojlim_p \mathrm{Hom}_A(A/\mathcal{M}^p, A/\mathcal{M}^p) \cong A.$$

   Thus, $\mathrm{End}_{\widehat{A}}(\Omega_A) \cong \widehat{A}$.

4. Since Ω_A has finite injective dimension over $\widehat{A}$, for any prime ideal $\mathcal{P}$ of $\widehat{A}$, the localised module $(\Omega_A)_{\mathcal{P}}$ has finite injective dimension over $\widehat{A}_{\mathcal{P}}$. Indeed, if N is an $\widehat{A}_{\mathcal{P}}$-module of finite type, then there exists an $\widehat{A}$-module N_1 of finite type such that $N \cong (N_1)_{\mathcal{P}}$, and $\mathrm{Ext}\,^i_{\widehat{A}_{\mathcal{P}}}(N, (\Omega_A)_{\mathcal{P}}) \cong \mathrm{Ext}\,^i_{\widehat{A}}(N_1, \Omega_A) \otimes_{\widehat{A}} \widehat{A}_{\mathcal{P}} = 0$ for i sufficiently large.

   Hence, if $\mathcal{P}$ is any minimal prime of (0) in $\widehat{A}$, then $(\Omega_A)_{\mathcal{P}}$ has finite injective dimension over the Artin ring $\widehat{A}_{\mathcal{P}}$ so that (since it is of finite type) it is injective over $\widehat{A}_{\mathcal{P}}$ (by lemma 3). Hence, $(\Omega_A)_{\mathcal{P}}$ is a direct sum of finitely many copies of the injective hull over $\widehat{A}_{\mathcal{P}}$ of its residue field. But since $\mathrm{End}\,_{\widehat{A}}(\Omega_A) = \widehat{A}$, we have

$$(\Omega_A)_{\mathcal{P}} = \text{ injective hull of residue field of } \widehat{A}_{\mathcal{P}} \text{ over } \widehat{A}_{\mathcal{P}}.$$

   In particular,

$$\text{length}\,(\widehat{A}_{\mathcal{P}}) = \text{length}\,((\Omega_A)_{\mathcal{P}}).$$

   Thus, if A is a domain, Ω_A is of rank 1 over $\widehat{A}$.

   Here we made use of the fact that if A is an Artin ring and M an A-module of finite type, then length (M) = length $(\mathcal{D}(M))$, since this holds for $M = k$ and both sides are additive on short exact sequences.

Definition: A module Ω_0 of finite type over a Cohen-Macaulay local ring A is said to be a *dualising module* if $\widehat{A} \otimes_A \Omega_0 \cong \Omega_A$.

Note that if Ω_0 is dualising for A, we have the duality ismorphism

$$\varinjlim_{p} \mathrm{Ext}\,^j_A(A/\mathcal{M}^p, M) \cong \mathcal{D}(\mathrm{Ext}\,^{n-j}_A(M, \Omega_0))$$

for any finitely generated A-module M. This follows immediately from Corollary 1.

Remark: A dualising module need not always exist for a local Cohen-Macaulay ring A, if A is not complete. However, we shall see that if it exists for A, then it exists for any localisation of A and any Cohen-Macaulay quotient A/I. Since it exists for Gorenstein rings ($\Omega_0 = A$), it exists for the localisations of Cohen-Macaulay quotients of A. Note that Ω_0 is unique up to isomorphism.

The next Proposition will be used to give a characterisation of the dualising module.

Proposition 5 *Let A be a Noetherian Cohen-Macaulay local ring and M an A-module of finite type which is Cohen-Macaulay of dimension equal to* $\dim A$ *and of finite injective dimension. Then there is an integer r such that* $\widehat{A} \otimes_A M \cong \Omega_A^{\oplus r}$.

Proof: Let $r = \dim \mathrm{Ext}\,^n_A(k, M)$ where $n = \dim A$.

First note that since $\mathrm{Ext}\,^i_A(k, M) = 0$ for $i < n$ (since M is Cohen-Macaulay), $\mathrm{Ext}\,^i_A(N, M) = 0$ for $i < n$ and N of finite length. Thus,

$$\mathrm{Ext}\,^n_A(A/\mathcal{M}^{p-1}, M) \to \mathrm{Ext}\,^n_A(A/\mathcal{M}^p, M)$$

is injective for every p, and

$$\mathcal{D}(\mathrm{Ext}\,^n_A(A/\mathcal{M}^{p+1}, M)) \to \mathcal{D}(\mathrm{Ext}\,^n_A(A/\mathcal{M}^p, M))$$

is surjective for every p. Choose a basis $y_1, \ldots, y_r$ of $\mathcal{D}(\mathrm{Ext}\,^n_A(k, M)) \cong \mathrm{Ext}\,^n_A(k, M)$. It follows that we can find $z_1, \ldots, z_r \in \varprojlim_p \mathcal{D}(\mathrm{Ext}\,^n_A(A/\mathcal{M}^p, M))$ whose images in $\mathcal{D}(\mathrm{Ext}\,^n_A(k, M))$ are $y_1, \ldots, y_r$ respectively. Now, we have a natural isomorphism (from (†))

$$\varprojlim_p \mathcal{D}(\mathrm{Ext}\,^n_A(A/\mathcal{M}^p, M)) \cong \mathrm{Hom}\,_{\widehat{A}}(\widehat{A} \otimes_A M, \Omega_A) \cong \mathrm{Hom}\,_A(M, \Omega_A),$$

and we get elements $\alpha_1, \ldots, \alpha_r \in \mathrm{Hom}\,_A(M, \Omega_A)$, and hence a homomorphism $\alpha = (\alpha_1, \ldots, \alpha_r) : M \to \Omega_A^{\oplus r}$. Now, we know that $\mathrm{Hom}\,_{\widehat{A}}(\Omega_A, \Omega_A) \cong \widehat{A}$, and under the composite

$$\mathrm{Hom}\,_{\widehat{A}}(\Omega_A, \Omega_A) \cong \varprojlim_p \mathcal{D}(\mathrm{Ext}\,^n_A(A/\mathcal{M}^p, \Omega_A)) \to \mathcal{D}(\mathrm{Ext}\,^n_A(k, \Omega_A)) \cong \mathcal{D}(k) \cong k,$$

the image of $1 \in \mathrm{End}\,(\Omega_A)$ is a non-zero element (this is because the composite map $\mathrm{End}\,_{\widehat{A}}(\Omega_A) \to k$ is surjective. Also, the diagram

$$\begin{array}{rcl} \mathrm{Hom}\,_A(M, \Omega_A) & \to & \mathcal{D}(\mathrm{Ext}\,^n_A(k, M)) \\ \mathrm{Hom}\,(\alpha_i, \Omega_A) \downarrow & & \downarrow \mathcal{D}(\mathrm{Ext}\,^n(k, \alpha_i)) \\ \mathrm{Hom}\,_{\widehat{A}}(\Omega_A, \Omega_A) & \to & \mathcal{D}(\mathrm{Ext}\,^n_A(k, \Omega_A)) \end{array}$$

is commutative, by the naturality of the duality isomorphism. Thus, we deduce that

$$\operatorname{Ext}^n_A(k,\alpha) : \operatorname{Ext}^n_A(k,M) \to \operatorname{Ext}^n_A(k,\Omega_A^{\oplus r})$$

is an isomorphism. Since $\operatorname{Ext}^i_A(k,M) = \operatorname{Ext}^i_A(k,\Omega_A) = 0$ for $i \neq n$, we deduce (by the five-lemma and induction on $\operatorname{length}(N)$) that for any A-module N of finite length,

$$\operatorname{Ext}^n_A(N,M) \stackrel{\operatorname{Ext}^n_A(N,\alpha)}{\longrightarrow} \operatorname{Ext}^n_A(N,\Omega_A^{\oplus r})$$

is an isomorphism. If $x_1,\ldots,x_n$ is a system of parameters and $p>0$ an integer, then we have an isomorphism

$$\operatorname{Ext}^n_A(A/(x_1^p,\ldots,x_n^p),M) \cong M/(x_1^p,\ldots,x_n^p)M,$$

which is natural in M, since M is Cohen-Macaulay. Thus, as p vaaries, we have a compatible family of isomorphisms

$$\frac{M}{(x_1^p,\ldots,x_n^p)M} \cong \frac{\Omega_A^{\oplus r}}{(x_1^p,\ldots,x_n^p)\Omega_A^{\oplus r}}.$$

Passing to the inverse limit over p, $\widehat{M} \cong \Omega_A^{\oplus r}$, as desired. □

Corollary 3 *Let A be as in the Proposition, and M an A-module such that*

(i) M *is Cohen-Macaulay of dimension* $n = \dim A$

(ii) M *is of finite injective dimension over* A

(iii) $\operatorname{End}_A(M) = A$.

Then M is a dualising module. The same conclusion holds if (iii) is replaced by
(iii)′ $\operatorname{Ext}^n_A(k,M) = k$.

Proof: We know that $\widehat{M} \cong \Omega_A^{\oplus r}$, where $r \geq 1$. If $r > 1$, then $\operatorname{End}_A(M) \otimes_A \widehat{A} \cong \operatorname{End}_{\widehat{A}}(\Omega_A^{\oplus r}) \cong M_r(\widehat{A})$, where $M_r(\widehat{A})$ is the ring of $r \times r$ matrices over $\widehat{A}$. In particular, $\operatorname{End}_A(M) \neq A$. Thus $r = 1$, proving that M is dualising for A. Since $r = \dim_k \operatorname{Ext}^n_A(k,M)$, we may replace (iii) by (iii)′. □

Corollary 4 *Let A be as in the Proposition, and suppose a dualising module Ω_0 exists for A. Then, we have*

(i) for any prime ideal $\mathcal{P}$ of A, $\Omega_0 \otimes_A A_{\mathcal{P}}$ is a dualising module for $A_{\mathcal{P}}$

(ii) if I is any ideal of A such that A/I is Cohen-Macaulay and $\operatorname{ht} I = h$, the $\operatorname{Ext}^h_A(A/I, \Omega_0)$ is a dualising module for A/I.

Proof: (i) In fact, $\Omega_0 \otimes_A A_{\mathcal{P}}$ is Cohen-Macaulay of dimension equal to $\dim A_{\mathcal{P}}$, oof finite injective dimension, and $\operatorname{End}_{\mathcal{P}}(\Omega_0 \otimes_A A_{\mathcal{P}}) \cong A_{\mathcal{P}}$.
(ii) First, we check that $\operatorname{Ext}^i_A(A/I, \Omega_0) = 0$ if $i \neq h$. This follows from:

Sublemma 1 *Let A be as in the Proposition. Suppose M is a finite A-module of finite injective dimension which is Cohen-Macaulay of dimension equal to $\dim A$, and N is Cohen-Macaulay of dimension r. Then $\operatorname{Ext}^i_A(N, M) = 0$ for $i \neq n - r$.*

Proof: We proceed by induction on r. If $r = 0$, we are through since N is Artinian and the result holds for $N = k$. Suppose $r > 0$, and the result holds for smaller values of r. Let x be a non-zero divisor on N. Then we have the exact sequence

$$\operatorname{Ext}^i_A(N, M) \xrightarrow{x} \operatorname{Ext}^i_A(N, M) \to \operatorname{Ext}^{i+1}(N/xN, M)$$

and by the induction hypothesis, the last group is 0 if $i+1 \neq n-(r-1)$ *i.e.*, $i \neq n - r$. Since $\operatorname{Ext}^i_A(N, M)$ is then a finite A-module on which multiplication by x is surjective, we are done by Nakayama's lemma. □

Now to the proof of (ii). Let $0 \to \Omega_0 \to I^\bullet$ be a finite injective resolution of Ω_0. Then the sequences

$$0 \to \operatorname{Hom}_A(A/I, I^0) \to \cdots \to \operatorname{Hom}_A(A/I, I^{h-1}) \to B^h \to 0$$

and

$$0 \to Z^h \to \operatorname{Hom}_A(A/I, I^h) \to \operatorname{Hom}_A A(A/I, I^{h+1}) \to \cdots$$

are exact (this defines B^h, Z^h) and $Z^h/B^h \cong \operatorname{Ext}^h_A(A/I, \Omega_0)$. Further, $\operatorname{Hom}_A(A/I, I^j)$ is an injective A/I-module for each j. Thus Z^h, B^h and hence $\operatorname{Ext}^h_A(A/I, \Omega_0)$ have finite injective dimension over A/I; further, B^h is in fact injective, so that $Z^h \cong B^h \oplus \operatorname{Ext}^h_A(A/I, \Omega_0)$. Now $\operatorname{Ext}^i_A(k, \Omega_0) = 0$ for $i \neq n$, and $\operatorname{Ext}^n_A(k, \Omega_0) = k$, where $\operatorname{Ext}^i_A(k, \Omega_0)$ is the i^{th}-cohomology of the complex $\mathrm{Hom}_A(k, I^\bullet) \cong \mathrm{Hom}_{A/I}(k, \mathrm{Hom}_A(A/I, I^\bullet))$. Hence we deduce that if $M = \mathrm{Ext}^h_A(A/I, \Omega_0)$, then

$$\mathrm{Ext}^i_{A/I}(k, M) = 0 \quad 0 \leq i < n-h,$$

$$\mathrm{Ext}^{n-h}_{A/I}(k, M) = k.$$

In view of Corollary 3 we are through. □

3 Local cohomology

3.1 Sheaf theoretic preliminaries

We start with some preliminary definitions. A map $f : X \to Y$ of topological spaces is said to be an *immersion* if f factors as $X \xrightarrow{g} Z \xrightarrow{i} Y$, where g is a homeomorphism, Z a locally closed subspace of Y, and i the inclusion. The immersion f is said to be *closed* or *open* if $Z \subset Y$ is closed or open, respectively.

Let $\mathcal{F}$ be a sheaf of abelian groups on a topological space X, $U \subset X$ an open set, and $\sigma \in \mathcal{F}(U)$ a section over U. Then the *support* of σ, denoted $\mid \sigma \mid$, is the set

$$\mid \sigma \mid = \{x \in U \mid \sigma_x \neq 0\},$$

where σ_x is the image of σ in the stalk $\mathcal{F}_x$ of $\mathcal{F}$ at x. Clearly $\mid \sigma \mid$ is closed in U. Similarly we define the support of the sheaf $\mathcal{F}$ (which we denote $\mathrm{supp}\,\mathcal{F}$) as

$$\mathrm{supp}\,\mathcal{F} = \{x \in X \mid \mathcal{F}_x \neq 0\}.$$

Recall the standard sheaf operations: if $f : X \to Y$ is a map of topological spaces, and $\mathcal{F}$ is a sheaf on X, its *direct image* $f_*\mathcal{F}$ is the sheaf

$$f_*(\mathcal{F})(U) = \mathcal{F}(f^{-1}(U)),$$

and if $\mathcal{G}$ is a sheaf on Y, its *inverse image* $f^{-1}\mathcal{G}$ is the sheaf on X associated to the presheaf

$$U \mapsto \varinjlim_{\substack{V \supset f(U) \\ V \text{ open in } Y}} \mathcal{G}(V).$$

Then $f^{-1}\mathcal{G}$ has stalks $(f^{-1}\mathcal{G})_x = \mathcal{G}_{f(x)}$. In the case when f is the inclusion of an open set, $(f^{-1}\mathcal{G})(U) = \mathcal{G}(U)$ for all open sets $U \subset X \subset Y$. The functors f^{-1} and f_* are adjoint *i.e.*, there are natural isomorphisms

$$\mathrm{Hom}_{\mathcal{O}_X}(f^{-1}\mathcal{G}, \mathcal{F}) \cong \mathrm{Hom}_{\mathcal{O}_Y}(\mathcal{G}, f_*\mathcal{F}),$$

for any sheaves of abelian groups $\mathcal{F}$, $\mathcal{G}$ on X, Y respectively.

Let $(X, \mathcal{O}_X)$ be a ringed space. We denote the category of $\mathcal{O}_X$-modules by $\mathcal{M}_X$, and for a locally closed subspace $Y \subset X$, we denote by $\mathcal{M}_{X,Y}$ the full subcategory of $\mathcal{M}_X$ consisting of sheaves with support contained in Y.

Proposition 6 *Let $(X, \mathcal{O}_X)$ be a ringed space, $i : Y \to X$ the inclusion of a locally closed subset, and $\mathcal{M}_Y = i^{-1}\mathcal{M}_X$. Then the restriction of $i^{-1} : \mathcal{M}_X \to \mathcal{M}_Y$ to $\mathcal{M}_{X,Y}$ gives an equivalence of categories $i^{-1} : \mathcal{M}_{X,Y} \to \mathcal{M}_Y$.*

Proof: We have to construct a quasi-inverse functor $\widetilde{i} : \mathcal{M}_Y \to \mathcal{M}_{X,Y}$ (*i.e.*, a functor $\widetilde{i}$ such that $\widetilde{i} \circ i^{-1}$ and $i^{-1} \circ \widetilde{i}$ are naturally isomorphic to the respective identity functors). If $\mathcal{F} \in \mathcal{M}_X$ and $x \in Y$, then $(i^{-1}\mathcal{F})_x = \mathcal{F}_x$, so $\mathrm{supp}\, i^{-1}\mathcal{F} = \mathrm{supp}\, \mathcal{F} \cap Y$. If $\mathcal{F} \in \mathcal{M}_{X,Y}$ then (if $\widetilde{i}$ exists) we have $\widetilde{i} \circ i^{-1}\mathcal{F} \cong \mathcal{F}$, and if $\mathcal{G} \in \mathcal{M}_Y$, then $i^{-1} \circ \widetilde{i}\mathcal{G} \cong \mathcal{G}$. Hence $\widetilde{i} : \mathcal{M}_Y \to \mathcal{M}_{X,Y}$ must preserve supports. Hence, it suffices to construct $\widetilde{i}$ when i is either a closed immersion or an open immersion, since an arbitrary immersion i is the composite $i_1 \circ i_2$ where i_2 is an open immersion and i_1 is a closed immersion, and we may then define $\widetilde{i} = \widetilde{i_1} \circ \widetilde{i_2}$.

When i is a closed immersion, we can take $\widetilde{i} = i_*$, since we evidently have $\mathrm{supp}\, i_*\mathcal{G} \subset Y$ and $i^{-1} \circ i_*\mathcal{G} \cong \mathcal{G}$ for $\mathcal{G} \in \mathcal{M}_Y$, and for $\mathcal{F} \in \mathcal{M}_{X,Y}$, the natural map $\mathcal{F} \to i_* \circ i^{-1}\mathcal{F}$ is an isomorphism. Note that for $\mathcal{G} \in \mathcal{M}_Y$, we consider $i_*\mathcal{G}$ as an $\mathcal{O}_X$-module via the homomorphism $\mathcal{O}_X \to i_*\mathcal{O}_Y = i_* \circ i^{-1}\mathcal{O}_X$.

Suppose that i is an open immersion, and $\mathcal{G} \in \mathcal{M}_Y$. Then define $\widetilde{i}\mathcal{G}$ by

$$\widetilde{i}(\mathcal{G})(U) = \{\sigma \in \mathcal{G}(Y \cap U) \mid \mid \sigma \mid \text{is closed in } U\}.$$

One verifies easily that the sheaf conditions are satisfied. This is an $\mathcal{O}_X$-module in a natural way, and we have evidently $i^{-1} \circ \widetilde{i} = 1_{\mathcal{M}_Y}$, the identity functor. Suppose on the other hand that $\mathcal{F} \in \mathcal{M}_{X,Y}$. We have an evident injection of $\mathcal{O}_X$-modules $\widetilde{i} \circ i^{-1}\mathcal{F} \hookrightarrow \mathcal{F}$ and this is an isomorphism, since both sides have support in Y and we get the identity on applying i^{-1} to both sides. □

Definition: For a locally closed subset Y of X and an $i^{-1}\mathcal{O}_X$-module $\mathcal{G}$ on Y, if $\widetilde{i} : \mathcal{M}_Y \to \mathcal{M}_{X,Y}$ is the quasi-inverse to $i^{-1} : \mathcal{M}_{X,Y} \to \mathcal{M}_Y$, we put $\mathcal{G}_Y = \widetilde{i}\mathcal{G}$. Further, for any $\mathcal{F} \in \mathcal{M}_X$, we shall put $\mathcal{F}_Y = \widetilde{i}i^{-1}\mathcal{F}$.

Note that $(\widetilde{i}\mathcal{G})_y = \mathcal{G}_y$ for any $y \in Y$, while $(\widetilde{i}\mathcal{G})_x = 0$ for $x \notin Y$.

Now, if $Y \xrightarrow{i} X$, $Z \xrightarrow{j} Y$ are immersions, we have $(i \circ j)^{-1} = j^{-1} \circ i^{-1}$; hence $(\widetilde{i \circ j} = \widetilde{i} \circ \widetilde{j}$. Further, i^{-1} and $\widetilde{i}$ are both exact functors, hence so is $\widetilde{i} \circ i^{-1} : \mathcal{M}_X \to \mathcal{M}_{X,Y}$, $\mathcal{F} \mapsto \mathcal{F}_Y$.

If $i : Y \to X$ is a closed immersion, then $\widetilde{i} = i_*$, so that for any $\mathcal{F} \in \mathcal{M}_X$, $\mathcal{G} \in \mathcal{M}_Y$ we have natural isomorphisms

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \widetilde{i}\mathcal{G}) \cong \mathrm{Hom}_{\mathcal{O}_Y}(i^{-1}\mathcal{F}, \mathcal{G}) \qquad \cdots \qquad (1)$$

where Y is closed in X, $\mathcal{F} \in \mathcal{M}_X$, $\mathcal{G} \in \mathcal{M}_Y$, and in paticular, for any $\mathcal{F}$, a natural transformation of functors $\mathcal{F} \to \widetilde{i} \circ i^{-1}\mathcal{F} = \mathcal{F}_Y$, which is evidently surjective.

On the other hand, suppose $Y \subset X$ is open. It follows from the definition of $\widetilde{i}$ in this case that there is a natural transformation $\widetilde{i} \circ i^{-1}\mathcal{F} = \mathcal{F}_Y \to \mathcal{F}$ which is injective, and an isomorphism when restricted to Y. Hence $\mathcal{F}/\mathcal{F}_Y$ has support in the closed set $X - Y$, so that if $\mathcal{F} \in \mathcal{M}_{X,Y}$, $\mathcal{F}' \in \mathcal{M}_X$, then

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F}'/\mathcal{F}'_Y) = 0$$

and so

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F}') = \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F}'_Y).$$

Hence, if $\mathcal{G} \in \mathcal{M}_Y$, $\mathcal{F} \in \mathcal{M}_X$, we have an isomorphism of functors

$$\mathrm{Hom}_{\mathcal{O}_Y}(\mathcal{G}, i^{-1}\mathcal{F}) \cong \mathrm{Hom}_{\mathcal{O}_X}(\widetilde{i}\mathcal{G}, \widetilde{i} \circ i^{-1}\mathcal{F}) \cong \mathrm{Hom}_{\mathcal{O}_X}(\widetilde{i}\mathcal{G}, \mathcal{F}_Y)$$
$$\cong \mathrm{Hom}_{\mathcal{O}_X}(\widetilde{i}\mathcal{G}, \mathcal{F}),$$

that is,

$$\mathrm{Hom}_{\mathcal{O}_Y}(\mathcal{G}, i^{-1}\mathcal{F}) \cong \mathrm{Hom}_{\mathcal{O}_X}(\widetilde{i}\mathcal{G}, \mathcal{F}) \qquad \cdots \qquad (2)$$

if $i : Y \to X$ is the inclusion of an open set.

Finally, if $Z \xrightarrow{j} Y \xrightarrow{i} X$ where $Z \subset Y$ is locally closed, and $Y \subset X$ is locally closed, then $Z \subset X$ is locally closed, and the relations $(i \circ j)^{-1} = j^{-1} \circ i^{-1}$ and $(\widetilde{i \circ j}) = \widetilde{i} \circ \widetilde{j}$ imply

$$(\widetilde{i \circ j} \circ (i \circ j)^{-1} \circ \widetilde{i} \circ i^{-1} = (\widetilde{i \circ j} \circ j^{-1} \circ i^{-1} \circ \widetilde{i} \circ i^{-1}$$

$$= (\widetilde{i \circ j} \circ j^{-1} \circ i^{-1} = (\widetilde{i \circ j} \circ (i \circ j)^{-1}$$

i.e., there is a natural isomorphism

$$(\mathcal{F}_Y)_Z \cong \mathcal{F}_Z$$

for any $\mathcal{F} \in \mathcal{M}_X$. More generally, for any two locally closed subsets $Y, Z \subset X$, and $\mathcal{F} \in \mathcal{M}_X$, we have

$$(\mathcal{F}_Y)_Z \cong ((\mathcal{F}_Y)_Z)_{Y \cap Z} \cong (\mathcal{F}_Y)_{Y \cap Z} \cong \mathcal{F}_{Y \cap Z} \qquad \cdots \quad (3).$$

Suppose now that Y is locally closed in X, and Z is closed in Y. Then we have natural transformations

$$\mathcal{F}_{Y-Z} = (\mathcal{F}_Y)_{Y-Z} \hookrightarrow \mathcal{F}_Y \quad \text{(since } Y - Z \subset Y \text{ is open)}$$

and

$$\mathcal{F}_Y \twoheadrightarrow (\mathcal{F}_Y)_Z \cong \mathcal{F}_Z \quad \text{(since } Z \text{ is closed in } Y).$$

The sequence of sheaves

$$0 \to \mathcal{F}_{Y-Z} \to \mathcal{F}_Y \to \mathcal{F}_Z \to 0 \qquad \cdots \quad (4)$$

is exact.

Proposition 7 *For any locally closed subspace Y of X, the functor*

$$\widetilde{i} \circ i^{-1} : \mathcal{M}_X \to \mathcal{M}_X, \quad \mathcal{F} \mapsto \mathcal{F}_Y,$$

has a right adjoint $\mathcal{H}^0_Y(-) : \mathcal{M}_X \to \mathcal{M}_X$, so that there are natural isomorphisms

$$\operatorname{Hom}_{\mathcal{O}_X}(\mathcal{F}_Y, \mathcal{F}') \cong \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{H}^0_Y(\mathcal{F}')) \qquad \cdots \quad (5)$$

Proof: In view of (3), it suffices to prove the statement when Y is open or Y is closed in X. When Y is open, we have by (2),

$$\begin{aligned} \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{F}_Y, \mathcal{F}') &\cong \operatorname{Hom} - \mathcal{O}_X(\widetilde{i} \circ i^{-1}\mathcal{F}, \mathcal{F}') \cong \operatorname{Hom}_{\mathcal{O}_Y}(i^{-1}\mathcal{F}, i^{-1}\mathcal{F}') \\ &\cong \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{F}, i_* \circ i^{-1}\mathcal{F}'), \end{aligned}$$

so we can take $\mathcal{H}^0_Y(\mathcal{F}') = i_* \circ i^{-1}\mathcal{F}'$.

When Y is closed in X, define

$$\mathcal{H}_Y^0(\mathcal{F})(U) = \{\sigma \in \mathcal{F}(U) \mid \mid \sigma \mid \subset Y\}.$$

Clearly, $\mathcal{H}_Y^0(\mathcal{F})$ is the maximal subsheaf of $\mathcal{F}$ whose support is contained in Y. Hence for $\mathcal{F} \in \mathcal{M}_{X,Y}$, $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F}') = \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{H}_Y^0(\mathcal{F}'))$ and in particular,

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}_Y, \mathcal{F}') = \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}_Y, \mathcal{H}_Y^0(\mathcal{F}')) \cong \mathrm{Hom}_{\mathcal{O}_Y}(i^{-1}\mathcal{F}, i^{-1}\mathcal{H}_Y^0(\mathcal{F}'))$$

$$\cong \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, i_* i^{-1}\mathcal{H}_Y^0(\mathcal{F}')) \cong \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{H}_Y^0(\mathcal{F}')).$$

This verifies that $\mathcal{H}_Y^0$ is right adjoint to $\widetilde{i} \circ i^{-1}$. □

Corollary 5 *If $Z \subset Y \subset X$ are immersions, then*

$$\mathcal{H}_Z^0(\mathcal{H}_Y^0(\mathcal{F})) \cong \mathcal{H}_Z^0(\mathcal{F}). \qquad \cdots \quad (6)$$

Proof: This follows from (3) and (5). □

Corollary 6 *For Y closed in X we have a natural isomorphism of functors*

$$\mathrm{Hom}_{\mathcal{O}_X}(\widetilde{i}\mathcal{G}, \mathcal{F}) \cong \mathrm{Hom}_{\mathcal{O}_Y}(\mathcal{G}, i^{-1}\mathcal{H}_Y^0(\mathcal{F})) \qquad \cdots \quad (7)$$

for all $\mathcal{G} \in \mathcal{M}_Y$, $\mathcal{F} \in \mathcal{M}_X$.

Proof: This follows on substituting $\mathcal{F} = \widetilde{i}\mathcal{G}$ in (5), and noting that $\mathcal{H}_Y^0(\mathcal{F})$ has support in Y. □

Remarks:

1. Since for Y open in X, $\mathcal{H}_Y^0(\mathcal{F}) = i_* \circ i^{-1}\mathcal{F}$, and for Y closed in X, $\mathcal{H}_Y^0(\mathcal{F})(U) = \{\sigma \in \mathcal{F}(U) \mid \mid \sigma \mid \subset Y\}$, if $Y \subset U \subset X$ with Y closed in U and U open in X, then we have the explicit description

   $$\mathcal{H}_Y^0(\mathcal{F})(V) = \{\sigma \in \mathcal{F}(U \cap V) \mid \mid \sigma \mid \subset Y\} \qquad \cdots \quad (8)$$

   since $\mathcal{H}_Y^0(\mathcal{F}) = \mathcal{H}_Y^0(\mathcal{H}_U^0(\mathcal{F}))$. For Y, Z locally closed in X, we have

   $$\mathcal{H}_Y^0 \circ \mathcal{H}_Z^0 = \mathcal{H}_{Y \cap Z}^0 \qquad \cdots \quad (9)$$

   as follows immediately from $(\mathcal{F}_Y)_Z \cong \mathcal{F}_{Y \cap Z}$ (see (3)).

2. The functors $\widetilde{i}$, $\mathcal{F} \mapsto \mathcal{F}_Y$, $\mathcal{H}^0_Y$ are all 'independent' of the structure sheaf $\mathcal{O}_X$, in the sense that they commute with the 'restriction of scalars' functors from the category of $\mathcal{O}_X$-modules to the category of $\widetilde{\mathcal{O}_X}$-modules, for a homomorphism of sheaves of rings $\widetilde{\mathcal{O}_X} \to \mathcal{O}_X$, and the corresponding restrictions to Y, etc. Thus, we can form $\widetilde{i}(\mathcal{F})$, $\mathcal{F}_Y$, $\mathcal{H}^0_Y(\mathcal{F})$ as sheaves of abelian groups and get the same resulting sheaves (take $\widetilde{\mathcal{O}_X} = \mathbf{Z}_X$, the constant sheaf associated to the ring $\mathbf{Z}$ of integers).

Proposition 8 *(i) For Y locally closed in X, the functor $\mathcal{H}^0_Y : \mathcal{M}_X \to \mathcal{M}_X$ is left exact and takes injectives to injectives.*

(ii) If $Y \subset X$ is locally closed and $Z \subset Y$ is closed, we have natural transformations

$$\mathcal{H}^0_Z \to \mathcal{H}^0_Y, \quad \mathcal{H}^0_Y \to \mathcal{H}^0_{Y-Z} \qquad \cdots \quad (10)$$

and for any $\mathcal{F} \in \mathcal{M}_X$ the sequence

$$0 \to \mathcal{H}^0_Z(\mathcal{F}) \to \mathcal{H}^0_Y(\mathcal{F}) \to \mathcal{H}^0_{Y-Z}(\mathcal{F}) \qquad \cdots \quad (11)$$

is exact.

Proof: (i) is clear from (5) and the fact that $\mathcal{F} \mapsto \mathcal{F}_Y$ is exact.

The natural transformations (10), and the exactness of (11), follow from (5) and the existence of the natural tranformations $\mathcal{F}_{Y-Z} \to \mathcal{F}_Y$ and $\mathcal{F}_Y \to \mathcal{F}_Z$ (see the discussion preceeding Proposition 7), and the exact sequence associated with these natural transformations (note that if we have a short exact sequence of functors $\mathcal{A} \to \mathcal{B}$ which all have right adjoints, the corresponding 'dual' sequence of adjoint functors $\mathcal{B} \to \mathcal{A}$ need not be exact on the right, in general). □

Lemma 5 *Suppose $\mathcal{F}$ is a flasque sheaf on X. Then*

(i) $\mathcal{H}^0_Y(\mathcal{F})$ is flasque for any locally closed subspace Y of X

(ii) for $Y \subset X$ locally closed and $Z \subset Y$ closed, the sequence

$$0 \to \mathcal{H}^0_Z(\mathcal{F}) \to \mathcal{H}^0_Y(\mathcal{F}) \to \mathcal{H}^0_{Y-Z}(\mathcal{F}) \to 0 \qquad \cdots \quad (12)$$

is exact.

Proof: (i) If $Y = U \cap F$ where $U \subset X$ is open, $F \subset X$ is closed, we have $\mathcal{H}^0_Y(\mathcal{F}) = \mathcal{H}^0_U(\mathcal{H}^0_F(\mathcal{F}))$. Hence it suffices to prove (i) if Y is open or closed. If Y is open, $\mathcal{H}^0_Y(\mathcal{F}) = i_* \circ i^{-1}(\mathcal{F})$, and $i^{-1}(\mathcal{F})$, $i_* \circ i^{-1}(\mathcal{F})$ are flasque, so we are through. Suppose that $Y \subset X$ is closed, and let $\sigma \in \mathcal{H}^0_Y(\mathcal{F})(U)$ *i.e.*, $\sigma \in \mathcal{F}(U)$, and $| \sigma | \subset Y \cap U$. Let $Z = Y - (Y \cap U)$. We can define $\tau \in \mathcal{F}(X - Z)$ by taking $\tau \mid_U = \sigma$, and $\tau \mid_{X-Y} = 0$. Extend τ to a section $\eta \in \mathcal{F}(X)$ ($\mathcal{F}$ is flasque). Then $| \eta | \subset Y$, since $\eta \mid_{X-Y} = \tau \mid_{X-Y} = 0$, so $\eta \in \mathcal{H}^0_Y(\mathcal{F})(X)$, and clearly $\eta \mid_U = \sigma$.

(ii) Since $Z = Y \cap F$ where F is closed in X (take $F =$ closure of Z in X), the sequence (12) can be rewritten as

$$0 \to \mathcal{H}^0_F(\mathcal{H}^0_Y(\mathcal{F})) \to \mathcal{H}^0_Y(\mathcal{F}) \to \mathcal{H}^0_{X-F}(\mathcal{H}^0_Y(\mathcal{F})) \to 0.$$

Since $\mathcal{H}^0_Y(\mathcal{F})$ is flasque by (i) we are reduced to considering the case $Y = X$, where $Z \subset X$ is closed. By (11) we are reduced to showing that for any U, $\mathcal{F}(U) \to \mathcal{H}^0_{X-Z}(\mathcal{F})(U) = \mathcal{F}(U \cap (X - Z))$ is surjective, which is clear since $\mathcal{F}$ is flasque. □

Definition: For an $\mathcal{O}_X$-module $\mathcal{F}$ and $Y \subset X$ a locally closed subset, define

$$H^0_Y(\mathcal{F}) = \mathcal{H}^0_Y(\mathcal{F})(X).$$

It follows from the above that $H^0_Y(\mathcal{F})$ is left exact in $\mathcal{F}$, and if

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$

is exact with $\mathcal{F}'$ flasque (in particular, if $\mathcal{F}'$ is $\mathcal{O}_X$-injective), then

$$0 \to H^0_Y(\mathcal{F}') \to H^0_Y(\mathcal{F}) \to H^0_Y(\mathcal{F}'') \to 0$$

is exact. Further, if $\mathcal{F}$ is flasque, and $Z \subset Y$ is closed, then

$$0 \to H^0_Z(\mathcal{F}) \to H^0_Y(\mathcal{F}) \to H^0_{Y-Z}(\mathcal{F}) \to 0$$

is exact.

Definition: For $p \geq 0$, $H^p_Y(\mathcal{F})$ and $\mathcal{H}^p_Y(\mathcal{F})$ are the right derived functors of $H^0_Y(\mathcal{F})$, $\mathcal{H}^0_Y(\mathcal{F})$.

Now, if

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$

is exact, and $\mathcal{F}'$, $\mathcal{F}''$ are flasque, then $\mathcal{F}$ is flasque; if $\mathcal{F}'$ is flasque and $\mathcal{F}$ is injective, then $\mathcal{F}''$ is flasque. It now follows by standard arguments that by induction on p, $H^p_Y(\mathcal{F}) = 0$ and $\mathcal{H}^p_Y(\mathcal{F}) = 0$ (as a sheaf) for $p > 0$, if $\mathcal{F}$ is flasque. This shows in particular that the objects $H^p_Y(\mathcal{F})$, $\mathcal{H}^p_Y(\mathcal{F})$ are 'independent of the structure sheaf $\mathcal{O}_X$' since they may be computed as $\mathbf{Z}_X$-modules, where $\mathbf{Z}_X$ is the constant sheaf $\mathbf{Z}$ on X. This is because an injective $\mathcal{O}_X$-resolution is a flasque resolution by $\mathbf{Z}_X$-modules (*i.e.*, by flasque sheaves of abelian groups).

If $U \subset X$ is open, then

$$\mathcal{H}^0_{Y\cap U}(\mathcal{F}_U) = \mathcal{H}^0_{Y\cap U}(\mathcal{F})_U = \mathcal{H}^0_U(\mathcal{H}^0_Y(\mathcal{F}))_U = \mathcal{H}^0_Y(\mathcal{F})_U,$$

so that we have a natural map

$$H^0_Y(\mathcal{F}) \stackrel{\rho^X_U}{\to} H^0_{Y\cap U}(\mathcal{F}_U) = H^0_{Y\cap U}(\mathcal{F}\mid_U).$$

Since $\mathcal{F}$ flasque $\Rightarrow \mathcal{F}\mid_U$ flasque, we have for all $p \geq 0$ a map

$$H^p_Y(\mathcal{F}) \stackrel{\rho^X_U}{\to} H^p_{Y\cap U}(\mathcal{F}\mid_U),$$

and clearly $\rho^V_W \circ \rho^U_V = \rho^U_W$ for $U \supset V \supset W$. Hence we obtain a presheaf

$$U \mapsto H^p_{Y\cap U}(\mathcal{F}\mid_U)$$

on X. Let $(\mathcal{F})^p_Y$ be the associated sheaf. Then for any exact sequence

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$

of $\mathcal{O}_X$-modules, there is a long exact sequence

$$\cdots \to (\mathcal{F}')^p_Y \to (\mathcal{F})^p_Y \to (\mathcal{F}'')^p_Y \to (\mathcal{F}')^{p+1}_Y \to \cdots$$

which is deduced from the long exact sequence of presheaves. Further, $\mathcal{F}$ injective $\Rightarrow \mathcal{F}\mid_U$ flasque for any open $U \subset X \Rightarrow (\mathcal{F})^p_Y = 0$ for $p > 0$, since the presheaf is itself 0. Hence $\mathcal{F} \mapsto (\mathcal{F})^p_Y$ is a universal ∂-functor in the sense of Grothendieck. Lastly, $(\mathcal{F})^0_Y = \mathcal{H}^0_Y(\mathcal{F})$, from the definition of $H^0_{Y\cap U}(\mathcal{F}\mid_U)$. We deduce that $(\mathcal{F})^p_Y \cong \mathcal{H}^p_Y(\mathcal{F})$. Hence we have proved:

Lemma 6 *The sheaves $\mathcal{H}^p_Y(\mathcal{F})$ are associated to the presheaves*

$$U \mapsto H^p_{Y\cap U}(\mathcal{F}\mid_U).$$

Suppose that $Y \subset X$ is locally closed, and $Z \subset Y$ is closed. Let $0 \to \mathcal{F} \to \mathcal{I}^\bullet$ be an injective resolution of $\mathcal{F}$. In view of our earlier remarks (see (12)) we have short exact sequences of complexes

$$0 \to \mathcal{H}^0_Z(\mathcal{I}^\bullet) \to \mathcal{H}^0_Y(\mathcal{I}^\bullet) \to \mathcal{H}^0_{Y-Z}(\mathcal{I}^\bullet) \to 0,$$

$$0 \to H^0_Z(\mathcal{I}^\bullet) \to H^0_Y(\mathcal{I}^\bullet) \to H^0_{Y-Z}(\mathcal{I}^\bullet) \to 0,$$

and so we get long exact sequences

$$\begin{aligned} 0 \to \mathcal{H}^0_Z(\mathcal{F}) \to \mathcal{H}^0_Y(\mathcal{F}) \to \mathcal{H}^0_{Y-Z}(\mathcal{F}) \to \cdots \to \mathcal{H}^p_Z(\mathcal{F}) \\ \to \mathcal{H}^p_Y(\mathcal{F}) \to \mathcal{H}^p_{Y-Z}(\mathcal{F}) \to \mathcal{H}^{p+1}_Z(\mathcal{F}) \to \cdots \end{aligned} \qquad \cdots \quad (13)$$

$$\begin{aligned} 0 \to H^0_Z(\mathcal{F}) \to H^0_Y(\mathcal{F}) \to H^0_{Y-Z}(\mathcal{F}) \to \cdots \to H^p_Z(\mathcal{F}) \\ \to H^p_Y(\mathcal{F}) \to H^p_{Y-Z}(\mathcal{F}) \to H^{p+1}_Z(\mathcal{F}) \to \cdots \end{aligned} \qquad \cdots \quad (14)$$

Since $\mathcal{H}^0_Y$ takes injectives to injectives (Proposition 8), and $\mathcal{H}^0_Y \circ \mathcal{H}^0_Z = \mathcal{H}^0_{Y\cap Z}$, and $\Gamma \circ \mathcal{H}^0_Y = H^0_Y$, we get convergent spectral sequences (of composite functors)

$$E_2^{p,q} = \mathcal{H}^p_Y(\mathcal{H}^q_Z(\mathcal{F})) \Rightarrow \mathcal{H}^{p+q}_{Y\cap Z}(\mathcal{F}) \qquad \cdots \quad (15)$$

$$E_2^{p,q} = H^p(X, \mathcal{H}^q_Y(\mathcal{F})) \Rightarrow H^{p+q}_Y(\mathcal{F}) \qquad \cdots \quad (16)$$

We recall some well known facts on the functors Ext and $\mathcal{E}xt$. Let $\mathcal{F}$, $\mathcal{G}$ be $\mathcal{O}_X$-modules, and define a sheaf $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ by

$$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})(U) = \mathrm{Hom}_{\mathcal{O}_X|_U}(\mathcal{F}|_U, \mathcal{G}|_U).$$

Then $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ and $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ are left exact covariant functors in $\mathcal{G}$ for fixed $\mathcal{F}$. For $p \geq 0$ define $\mathrm{Ext}^p_{\mathcal{O}_X}(\mathcal{F}, -)$, $\mathcal{E}xt^p_{\mathcal{O}_X}(\mathcal{F}, -)$ to be the p^{th} right derived functors of $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, -)$, $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, -)$ respectively.

Suppose now that $\mathcal{G}$ is injective. We assert that $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ is flasque. In fact, if $\sigma \in \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})(U)$, then σ is an $\mathcal{O}_X|_U$-linear sheaf map $\mathcal{F}|_U \to \mathcal{G}|_U$, which we may regard as an $\mathcal{O}_X$-map $\mathcal{F}_U \to \mathcal{G}$. Since $\mathcal{F}_U$ is a subsheaf of $\mathcal{F}$, this extends to an $\mathcal{O}_X$-map $\tau : \mathcal{F} \to \mathcal{G}$. Then $\tau \in \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})(X)$ and $\tau|_U = \sigma$. Since $\Gamma(X, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})) = \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$, we get a spectral sequence

$$E_2^{p,q} = H^p(X, \mathcal{E}xt^q_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})) \Rightarrow \mathrm{Ext}^{p+q}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \qquad \cdots \quad (17)$$

Note that since $i^{-1} : \mathcal{M}_X \to \mathcal{M}_U$ (U open) admits a left adjoint $\widetilde{i} : \mathcal{M}_U \to \mathcal{M}_X$ (see (2)) which is exact, we see that for any injective $\mathcal{O}_X$-module $\mathcal{I} \in \mathcal{M}_X$, $i^{-1}\mathcal{I} \in \mathcal{M}_U$ is an injective $\mathcal{O}_U$-module. It follows that

$$\mathcal{E}xt^{\,p}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \mid_U = \mathcal{E}xt^{\,p}_{\mathcal{O}_U}(\mathcal{F} \mid_U, \mathcal{G} \mid_U) \qquad \cdots \quad (18)$$

where $\mathcal{O}_U = \mathcal{O}_X \mid_U$. If

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$

is an exact sequence of $\mathcal{O}_X$-modules, and

$$0 \to \mathcal{G} \to \mathcal{I}^{\bullet}$$

is an injective resolution of $\mathcal{G}$, then there are exact sequences of complexes

$$0 \to \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{F}', \mathcal{I}^{\bullet}) \to \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{I}^{\bullet}) \to \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{F}'', \mathcal{I}^{\bullet}) \to 0$$

and

$$0 \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}', \mathcal{I}^{\bullet}) \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{I}^{\bullet}) \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}'', \mathcal{I}^{\bullet}) \to 0,$$

which yield long exact sequences

$$\cdots \to \operatorname{Ext}^{\,p}_{\mathcal{O}_X}(\mathcal{F}', \mathcal{G}) \to \operatorname{Ext}^{\,p}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \to \operatorname{Ext}^{\,p}_{\mathcal{O}_X}(\mathcal{F}'', \mathcal{G}) \to \operatorname{Ext}^{\,p+1}_{\mathcal{O}_X}(\mathcal{F}', \mathcal{G}) \to \cdots \qquad \cdots \quad (19)$$

and

$$\cdots \to \mathcal{E}xt^{\,p}_{\mathcal{O}_X}(\mathcal{F}', \mathcal{G}) \to \mathcal{E}xt^{\,p}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \to \mathcal{E}xt^{\,p}_{\mathcal{O}_X}(\mathcal{F}'', \mathcal{G}) \to \mathcal{E}xt^{\,p+1}_{\mathcal{O}_X}(\mathcal{F}', \mathcal{G}) \to \cdots \qquad \cdots \quad (20)$$

Again, for any open $U \subset X$, we have restrictions

$$\rho^X_U : \operatorname{Ext}^{\,p}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \to \operatorname{Ext}^{\,p}_{\mathcal{O}_{X|_U}}(\mathcal{F} \mid_U, \mathcal{G} \mid_U)$$

induced by the restrictions on Hom, and we get a presheaf

$$U \mapsto \operatorname{Ext}^{\,p}_{\mathcal{O}_{X|_U}}(\mathcal{F} \mid_U, \mathcal{G} \mid_U)$$

for every $p > 0$. If $\widetilde{\mathrm{Ext}}^p(\mathcal{F}, \mathcal{G})$ is the associated sheaf, then $\widetilde{\mathrm{Ext}}^p(\mathcal{F}, \mathcal{I})$ vanishes for $\mathcal{I}$ injective (since $\mathcal{I}\mid_U$ is injective for every U), and for a given $\mathcal{F}$, there is a functorial long exact sequence associated to any exact sequence of $\mathcal{G}$'s. Hence $\widetilde{\mathrm{Ext}}^p(\mathcal{F}, -)$ is a universal ∂-functor in Grothendieck's sense; since $\widetilde{\mathrm{Ext}}^0(\mathcal{F}, \mathcal{G}) = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$, we deduce that there is a natural isomorphism $\widetilde{\mathrm{Ext}}^p(\mathcal{F}, \mathcal{G}) \cong \mathcal{E}xt^p_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$. Thus we have proved:

Lemma 7 *The sheaves $\mathcal{E}xt^p$ are associated to the presheaves*

$$U \mapsto \mathrm{Ext}^p_{\mathcal{O}_{X\mid_U}}(\mathcal{F}\mid_U, \mathcal{G}\mid_U).$$

Lemma 8 *(i) The natural map*

$$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})_x \to \mathrm{Hom}_{\mathcal{O}_{X,x}}(\mathcal{F}_x, \mathcal{G}_x)$$

induces a natural map for each $p > 0$

$$\mathcal{E}xt^p_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})_x \to \mathrm{Ext}^p_{\mathcal{O}_{X,x}}(\mathcal{F}_x, \mathcal{G}_x) \qquad \cdots \qquad (21)$$

which is a morphism of ∂-functors in $\mathcal{G}$ (i.e., is compatible with the respective long exact sequences associated to a short exact sequences of $\mathcal{G}$'s).

(ii) Suppose that $\mathcal{O}_X$ is $\mathcal{O}_X$-coherent and $\mathcal{F}$ is $\mathcal{O}_X$-coherent. Then the maps in (21) are isomorphisms for all $p \geq 0$ for all $\mathcal{G} \in \mathcal{M}_X$.

Proof: (i) Since $\mathcal{E}xt^p_{\mathcal{O}_X}(\mathcal{F}, -)_x$ and $\mathrm{Ext}^p_{\mathcal{O}_{X,x}}(\mathcal{F}_x, -)$ are both δ-functors, and the first is universal (since it is effaceable), the natural transformation for $p = 0$ induces a unique natural transformation of ∂-functors.

(ii) For $p = 0$, the result is clear for $\mathcal{F} = \mathcal{O}_X$. Next, for any coherent $\mathcal{F}$, since the problem is local, by (18), we may assume that there exists an exact sequence

$${\mathcal{O}_X}^{\oplus m} \to {\mathcal{O}_X}^{\oplus n} \to \mathcal{F} \to 0,$$

and the result follows from the 5-lemma.

We assume that the result holds for a given value of p for all coherent $\mathcal{F}$, and prove it for $p+1$. Given a coherent sheaf $\mathcal{F}$, then again after replacing X by an open neighbourhood of x, we can find an exact sequence

$$0 \to \mathcal{F}' \to {\mathcal{O}_X}^{\oplus n} \to \mathcal{F} \to 0$$

for some n, with $\mathcal{F}'$ coherent.

We claim that this yields a commutative diagram

$$\begin{array}{ccccccc}
\mathcal{E}xt^{\,p}_{\mathcal{O}_X}({\mathcal{O}_X}^{\oplus n},\mathcal{G})_x & \to & \mathcal{E}xt^{\,p}_{\mathcal{O}_X}(\mathcal{F}',\mathcal{G})_x & \to & \mathcal{E}xt^{\,p+1}_{\mathcal{O}_X}(\mathcal{F},\mathcal{G})_x & \to & \mathcal{E}xt^{\,p+1}_{\mathcal{O}_X}({\mathcal{O}_X}^{\oplus n},\mathcal{G})_x \\
\cong\downarrow & & \cong\downarrow & & \downarrow & & \downarrow \\
\mathrm{Ext}^{\,p}_{\mathcal{O}_{X,x}}({\mathcal{O}_{X,x}}^{\oplus n},\mathcal{G}_x) & \to & \mathrm{Ext}^{\,p}_{\mathcal{O}_{X,x}}(\mathcal{F}'_x,\mathcal{G}_x) & \to & \mathrm{Ext}^{\,p+1}_{\mathcal{O}_{X,x}}(\mathcal{F}_x,\mathcal{G}_x) & \to & \mathrm{Ext}^{\,p+1}_{\mathcal{O}_{X,x}}({\mathcal{O}_{X,x}}^{\oplus n},\mathcal{G}_x)
\end{array}$$

Now $\mathcal{H}om_{\,\mathcal{O}_X}(\mathcal{O}_X,\mathcal{G})\cong\mathcal{G}$, so that $\mathcal{H}om_{\,\mathcal{O}_X}(\mathcal{O}_X,-)$ is the identity functor on $\mathcal{M}_X$. Hence $\mathcal{E}xt^{\,i}_{\mathcal{O}_X}(\mathcal{O}_X,-)=0$ for $i>0$. Thus $\mathcal{E}xt^{\,p+1}_{\mathcal{O}_X}({\mathcal{O}_X}^{\oplus n},\mathcal{G})=0$. Similarly, $\mathrm{Ext}^{\,p+1}_{\mathcal{O}_{X,x}}({\mathcal{O}_{X,x}}^{\oplus n},\mathcal{G}_x)=0$; thus, granting the claim, the result follows.

To prove the claim, let $0\to\mathcal{G}\to\mathcal{I}^\bullet$ be an injective resolution of $\mathcal{G}$. Let $0\to\mathcal{G}_x\to I^\bullet$ be an injective resolution of $\mathcal{G}_x$ over $\mathcal{O}_{X,x}$. We can find a map of complexes $\mathcal{I}^\bullet_x\to I$ lifting the identity map on $\mathcal{G}_x$, since $I^\bullet$ is a complex of injectives. Using the natural transformation (in $\mathcal{G}$)

$$\mathcal{H}om_{\,\mathcal{O}_X}(\mathcal{F},\mathcal{G})_x\to\mathrm{Hom}_{\,\mathcal{O}_{X,x}}(\mathcal{F}_x,\mathcal{G}_x)$$

we then have a commutative diagram, whose rows are exact sequences of complexes,

$$\begin{array}{ccccc}
0\to\mathcal{H}om_{\,\mathcal{O}_X}(\mathcal{F},\mathcal{I}^\bullet)_x & \to & \mathcal{H}om_{\,\mathcal{O}_X}({\mathcal{O}_X}^{\oplus n},\mathcal{I}^\bullet)_x & \to & \mathcal{H}om_{\,\mathcal{O}_X}(\mathcal{F}',\mathcal{I}^\bullet)_x\to 0 \\
\downarrow & & \downarrow & & \downarrow \\
0\to\mathrm{Hom}_{\,\mathcal{O}_{X,x}}(\mathcal{F}_x,I^\bullet) & \to & \mathrm{Hom}_{\,\mathcal{O}_{X,x}}({\mathcal{O}_{X,x}}^{\oplus n},I^\bullet) & \to & \mathrm{Hom}_{\,\mathcal{O}_{X,x}}(\mathcal{F}'_x,I^\bullet)\to 0
\end{array}$$

where the rows yield the long exact sequences used in the earlier diagram, and the vertical maps induce the vertical maps of that diagram. This proves the claim. □

Note that (ii) does not follow from the general result about ∂-functors, since $\mathcal{G}\mapsto\mathrm{Ext}^{\,p}_{\mathcal{O}_{X,x}}(\mathcal{F},\mathcal{G})$ is not known to be effaceable.

3.2 Inductive limits and dimension

Lemma 9 *Let X be a Noetherian topological space, I a directed set, and let*

$$\mathcal{F}_i,\ \ i\in I,$$

$$\psi_{ij}:\mathcal{F}_j\to\mathcal{F}_i\ \ \forall\, i,j\in I,\ i\ge j$$

be an inductive (=direct) system of sheaves of abelian groups on X. Let

$$\mathcal{F} = \varinjlim_{i \in I} \mathcal{F}_i,$$

and let Y be a locally closed subset of X. Then we have

(i) for any open set $U \subset X$,

$$\mathcal{F}(U) = \varinjlim_{i \in I} \mathcal{F}_i(U).$$

(ii) $\varinjlim_{i \in I} H^0_Y(\mathcal{F}_i) = H^0_Y(\mathcal{F})$.

(iii) if $\mathcal{F}_i$ is flasque for each i, the $\mathcal{F}$ is flasque.

Proof: (i) Let

$$\mathcal{G}(U) = \varinjlim_{i \in I} \mathcal{F}_i(U).$$

Then $U \mapsto \mathcal{G}(U)$ is a presheaf, such that the direct limit $\mathcal{F}$ is the associated sheaf. We claim that in fact $\mathcal{G}$ is a sheaf, so that $\mathcal{G} = \mathcal{F}$ (see Hartshorne, *Algebraic Geometry*, Ch. II, Exercise 1.11). Suppose $U = \cup U_\alpha$ where $\{U_\alpha\}_{\alpha \in \mathcal{A}}$ is a family of open sets, and let $\sigma \in \mathcal{G}(U)$ with $\sigma \mapsto 0 \in \mathcal{G}(U_\alpha)$ for all α. Since U is Noetherian, it is quasi-compact, and we may replace the cover $\{U_\alpha\}$ by a finite subcover, say $\{U_1, \cdots, U_n\}$. Now σ is the image of some $\sigma_i \in \mathcal{F}_i(U)$. For each $1 \leq t \leq n$, we can find an index $i_t \geq i$ such that $\sigma_i \mid_{U_t} \mapsto 0 \in \mathcal{F}_{i_t}(U_t)$. We can find $j \geq i_t$ for all $1 \leq t \leq n$, since any two elements of I have a common upper bound. If $\sigma_i \mapsto \sigma_j \in \mathcal{F}_j(U)$, then $\sigma_j \mid_{U_t} = 0$ for each t, so $\sigma_j = 0$. Hence $\sigma = 0$.

On the other hand, suppose given $\sigma_\alpha \in \mathcal{G}(U_\alpha)$ with

$$\sigma_\alpha \mid_{U_\alpha \cap U_\beta} = \sigma_\beta \mid_{U_\alpha \cap U_\beta} \quad \forall\, \alpha, \beta \in \mathcal{A};$$

we wish to find $\sigma \in \mathcal{G}(U)$ with $\sigma \mid_{U_\alpha} = \sigma_\alpha$. Let $A \subset \mathcal{A}$ be a finite subset such that $\cup_{\alpha \in A} U_\alpha = U$. As before, we can find an index $i \in I$ which is sufficiently large so that for each $\alpha \in A$, there exist $\sigma_{i,\alpha} \in \mathcal{F}_i(U_\alpha)$ such that $\sigma_{i,\alpha} \mapsto \sigma_\alpha$. Further, for $\alpha, \beta \in A$,

$$\sigma_{i,\alpha} \mid_{U_\alpha \cap U_\beta} - \sigma_{i,\beta} \mid_{U_\alpha \cap U_\beta}$$

restricts to zero on $U_\alpha \cap U_\beta$. Hence, by replacing i by a still larger index, we may assume that the sections $\sigma_{i,\alpha}$ patch to yield a section $\sigma_i \in \mathcal{F}_i(U)$. Let $\sigma \in \mathcal{G}(U)$ be the image of σ_i; it suffices to show that $\sigma \mid_{U_\alpha} = \sigma_\alpha$, where we know this for $\alpha \in A$. But for any $\alpha \in \mathcal{A}$, U_α is covered by $U_\alpha \cap U_\beta$ with $\beta \in A$, and

$$\sigma \mid_{U_\alpha \cap U_\beta} = \sigma_\beta \mid_{U_\alpha \cap U_\beta} = \sigma_\alpha \mid_{U_\alpha \cap U_\beta},$$

so that $\sigma \mid_{U_\alpha} - \sigma_\alpha$ is a 'locally zero' element of $\mathcal{G}(U_\alpha)$. By the argument given above, this shows it is zero, as desired.

(ii) For any sheaf $\mathcal{G}$, $H^0_Y(\mathcal{G})$ depends only on the restriction of $\mathcal{G}$ to an open neighbourhood of Y in X; also, $\varinjlim_i (\mathcal{F}_i \mid_U) = \mathcal{F} \mid_U$ for any open set U; hence we may replace X by an open subset containing Y, and so we may assume that Y is closed in X. If $U = X - Y$, then (see (14)) for any sheaf $\mathcal{G}$, $H^0_Y(\mathcal{G}) = \ker(\rho^X_U : \mathcal{G}(X) \to \mathcal{G}(U))$. Since $\varinjlim$ is exact, we have a commutative diagram with exact rows

$$\begin{array}{ccccc} 0 \to \varinjlim_i H^0_Y(\mathcal{F}_i) \to & & \varinjlim_i \mathcal{F}_i(X) & & \to \varinjlim_i \mathcal{F}_i(U) \\ \downarrow & & \downarrow\cong & & \downarrow\cong \\ 0 \to H^0_Y(\mathcal{F}) \to & & \mathcal{F}(X) & & \to \mathcal{F}(U) \end{array}$$

where the second and third vertical arrows are isomorphisms. This proves the first arrow is one too.

(iii) If $\mathcal{F}_i$ is flasque for each $i \in I$, then for any open $U \subset X$, we have an exact sequence $\mathcal{F}_i(X) \to \mathcal{F}_i(U) \to 0$; since $\varinjlim$ is exact,
$\varinjlim_i \mathcal{F}_i(X) \to \varinjlim_i \mathcal{F}_i(U) \to 0$ is exact *i.e.*, $\mathcal{F}(X) \to \mathcal{F}(U) \to 0$ is exact. □

We recall the *Godement resolution* of any sheaf of abelian groups by flasque sheaves. For any abelian sheaf $\mathcal{F}$ define

$$\mathcal{G}od^0(\mathcal{F})(U) = \prod_{x \in U} \mathcal{F}_x,$$

and let $i = i(\mathcal{F}) : \mathcal{F} \to \mathcal{G}od^0(\mathcal{F})$ be the map given by

$$i_U : \mathcal{F}(U) \to \mathcal{G}od^0(\mathcal{F})(U) = \prod_{x \in U} \mathcal{F}_x, \ \ \sigma \mapsto (\sigma_x)_{x \in U},$$

where $\sigma \mapsto \sigma_x \in \mathcal{F}_x$. Then i is injective. Having defined

$$0 \to \mathcal{F} \xrightarrow{i} \mathcal{G}od^0(\mathcal{F}) \xrightarrow{i_0} \mathcal{G}od^1(\mathcal{F}) \xrightarrow{i_1} \cdots \xrightarrow{i_{n-1}} \mathcal{G}od^n(\mathcal{F}),$$

define $\mathcal{G}od^{n+1}(\mathcal{F}) = \mathcal{G}od^0(\operatorname{coker} i_{n-1})$, and let $i_n : \mathcal{G}od^n(\mathcal{F}) \to \mathcal{G}od^{n+1}(\mathcal{F})$ be the composite

$$\mathcal{G}od^n(\mathcal{F}) \twoheadrightarrow \operatorname{coker}(i_{n-1}) \stackrel{i(\operatorname{coker}(i_{n-1}))}{\longrightarrow} \mathcal{G}od^{n+1}(\mathcal{F}).$$

Clearly $\mathcal{G}od^0(\mathcal{F})$ is flasque for any $\mathcal{F}$, and hence so is $\mathcal{G}od^n(\mathcal{F})$ for each n. Further, by induction on n, $\mathcal{F} \to \mathcal{G}od^n(\mathcal{F})$ is an exact functor on the category of abelian sheaves, since $\mathcal{G}od^0$ is one. Hence $\mathcal{F} \mapsto \mathcal{G}od^\bullet(\mathcal{F})$ is an exact functor from the category of abelian sheaves on X to the category of flasque complexes; also $i(\mathcal{F}) : \mathcal{F} \to \mathcal{G}od^0(\mathcal{F})$ is a natural transformation.

Proposition 9 *Let X be a Noetherian space, $Y \subset X$ a locally closed set, and*

$$\mathcal{F}_i, \ \ i \in I,$$

$$\psi_{ij} : \mathcal{F}_j \to \mathcal{F}_i \ \ \forall\, i, j \in I, \ i \geq j$$

be an inductive (=direct) system of abelian sheaves. Then for each $p > 0$, we have natural isomorphisms

$$\varinjlim_i H^p_Y(\mathcal{F}_i) \cong H^p_Y(\varinjlim_i \mathcal{F}_i).$$

Proof: Let $\mathcal{F} = \varinjlim_i \mathcal{F}_i$. Since $\varinjlim$ is exact, the above remarks on the Godement resolution imply that we have a resolution

$$0 \to \mathcal{F} \to \varinjlim_i \mathcal{G}od^\bullet(\mathcal{F}_i),$$

which by lemma 9(ii) is in fact a flasque resolution. By lemma 9(i), there is a natural isomorphism

$$H^0_Y(\varinjlim_i \mathcal{G}od^n(\mathcal{F}_i)) \cong \varinjlim_i H^0_Y(\mathcal{G}od^n(\mathcal{F}_i))$$

for each n, and hence an isomorphism of complexes

$$H^0_Y(\varinjlim_i \mathcal{G}od^\bullet(\mathcal{F}_i)) \cong \varinjlim_i H^0_Y(\mathcal{G}od^\bullet(\mathcal{F}_i)).$$

Since $\varinjlim_i$ is exact, it commutes with taking cohomology, and so we obtain isomorphisms

$$H^n_Y(\mathcal{F}) = H^n(H^0_Y(\varinjlim_i \mathcal{G}od^\bullet(\mathcal{F}_i))) \cong \varinjlim_i H^n(H^0_Y(\mathcal{G}od^\bullet(\mathcal{F}_i))) = \varinjlim_i H^n_Y(\mathcal{F}_i).$$

□

Theorem 10 *(i) Let X be a Noetherian space of (combinatorial)*[1] *dimension n, Y a locally closed subspace and $\mathcal{F}$ any abelian sheaf on X. Then*

$$H^p_Y(X, \mathcal{F}) = 0 \;\; \forall\, p > n.$$

(ii) If X and $\mathcal{F}$ are as above and

$$0 \to \mathcal{F} \to \mathcal{F}^0 \to \mathcal{F}^1 \to \cdots \to \mathcal{F}^{n-1} \to \mathcal{F}^n \to 0$$

is exact with $\mathcal{F}^i$ flasque for $0 \leq i < n$, then $\mathcal{F}^n$ is also flasque.

Proof: Assume (i) for Y closed. We shall deduce (ii). Since $\mathcal{F}^i$ is flasque for $0 \leq i < n$, by splitting the given exact sequence into short exact sequences, we obtain isomorphisms $H^p_Y(\mathcal{F}^n) \cong H^{n+p}_Y(\mathcal{F}) = 0$ for all $p > 0$. From the exact sequence (see (14))

$$H^0(X, \mathcal{F}^n) \to H^0(X - Y, \mathcal{F}^n) \to H^1_Y(\mathcal{F}^n)$$

we see that $\mathcal{F}^n(X) \to \mathcal{F}^n(X - Y)$ is surjective for every closed set Y *i.e.*, $\mathcal{F}^n$ is flasque. On the other hand, (ii) implies (i), since we may apply $H^0_Y(-)$ to such a finite flasque resolution to compute the $H^p_Y(\mathcal{F})$.

Thus it suffices to prove (i) when Y is closed. Since this is clear for $n = 0$, we may assume $n > 0$ and that the theorem holds for all X of smaller dimension. Further, by Noetherian induction, we may assume the theorem is valid if X is replaced by any proper closed subset. Now, if $\mathcal{S}$ is the class of all sheaves for which the theorem holds, then $\mathcal{S}$ is closed under extensions *i.e.*, if

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$

[1] Any open cover has a refinement such that all $n+2$-fold intersections of distinct open sets vanish, and n is the smallest such integer. If X is irreducible, this means any open cover has a refinement consisting of at most $n+1$ open sets.

is an exact sequence with $\mathcal{F}', \mathcal{F}'' \in \mathcal{S}$, then $\mathcal{F} \in \mathcal{S}$. Further, by Proposition 9, $\mathcal{S}$ is closed under inductive limits. Now any sheaf $\mathcal{F}$ on X is the inductive limit of quotients of finite direct sums $\oplus_{i=1}^{r} \mathbf{Z}_{U_i}$, where the U_i are open in X, and $\mathbf{Z}_{U_i}$ is the constant sheaf $\mathbf{Z}$ on U_i, extended by zero to all of X. Arguing by induction on r, it suffices to prove the theorem when $\mathcal{F}$ is a quotient of $\mathbf{Z}_U$ for some open $U \subset X$. Suppose then that

$$0 \to \mathcal{G} \to \mathbf{Z}_U \to \mathcal{F} \to 0$$

is exact. Identify $(\mathbf{Z}_U)_x$ with $\mathbf{Z}$ for all $x \in U$. For any $r > 0$, the sets $E_r = \{x \in U \mid r\mathcal{F}_x = 0\}$ are open in U. If all the E_r are empty, then $\mathcal{G} = 0$ and $\mathcal{F} = \mathbf{Z}_U$. If not, choose the smallest r such that $V = E_r$ is non-empty. Then for $x \in V$, $\mathcal{G}_x = r\mathbf{Z}$, so that $\mathcal{F} \mid_V = (\mathbf{Z}/r\mathbf{Z}) \mid_V$, and we have an exact sequence

$$0 \to (\mathbf{Z}/r\mathbf{Z})_V \to \mathcal{F} \to \mathcal{F}' \to 0$$

with $\mathcal{F}'$ supported on a proper closed subset $F \subset U$. Since $H^p_Y(\mathcal{F}') = H^p_{Y\cap F}(\mathcal{F}' \mid_F)$, by the induction hypothesis it suffices to consider the sheaf $(\mathbf{Z}/r\mathbf{Z})_V$. Thus, in any case, we may assume $\mathcal{F} = A_V$, where A is an abelian group, and $V \subset X$ is open. If X is reducible, we can express $\mathcal{F}$ as an extension of two sheaves having supports contained in proper closed subsets of X, and we are through. Hence we may assume X is irreducible. We then have an exact sequence

$$0 \to A_V \to A_X \to A_{X-V} \to 0.$$

Since X is irreducible, A_X is flasque, and if $F = X - V$, then $\dim F < \dim X$. Since $H^p_Y(A_F) = H^p_{Y\cap F}(A_F \mid_F) = 0$ for $p > n-1$, the exact sequence

$$H^p_Y(A_F) \to H^{p+1}_Y(A_V) \to H^{p+1}_Y(A_X)$$

finishes the proof. □

Corollary 7 *If X is Noetherian of dimension n, Y locally closed on X, then the sheaves $\mathcal{H}^p_Y(\mathcal{F})$ are 0 for $p > n$.*

Proof: Immediate from the Theorem and lemma 6. □

3.3 Application to schemes

Lemma 10 *Let X be a Noetherian scheme, $\mathcal{QC}(X)$ the category of quasi-coherent $\mathcal{O}_X$-modules on X, and $\mathcal{I}$ an injective object of $\mathcal{QC}(X)$. Then for any open $U \subset X$, $\mathcal{I}\mid_U$ is an injective object of $\mathcal{QC}(U)$.*

Proof: We first make a remark: on a Noetherian scheme Y, in order that $\mathcal{F} \in \mathcal{QC}(Y)$ be an injective object, it is sufficient to assume that if $0 \to \mathcal{G} \to \mathcal{G}'$ is an exact sequence of *coherent* sheaves on Y, any homomorphism $\mathcal{G} \to \mathcal{F}$ extends to $\mathcal{G}'$. Indeed, suppose this condition holds, and let $0 \to \mathcal{G}_1 \to \mathcal{G}_2$ be an exact sequence in $\mathcal{QC}(Y)$ and $\alpha_1 : \mathcal{G}_1 \to \mathcal{F}$ a homomorphism. By Zorn's lemma we can find a maximal quasi-coherent subsheaf $\mathcal{G}_3$ of $\mathcal{G}_2$ to which α_1 extends; replacing $(\mathcal{G}_1, \alpha_1)$ by $\mathcal{G}_3$ and the extension, we can assume that $\mathcal{G}_1$ is itself maximal. If $\mathcal{G}_1 \neq \mathcal{G}_2$, we can find a coherent subsheaf $\mathcal{G}_4$ of $\mathcal{G}_2$ such that $\mathcal{G}_4$ is not a subsheaf of $\mathcal{G}_1$. Let $\mathcal{G}_5 = \mathcal{G}_4 \cap \mathcal{G}_1$; now $\alpha \mid_{\mathcal{G}_5}$ extends to $\beta : \mathcal{G}_4 \to \mathcal{F}$; hence α extends to the subsheaf of $\mathcal{G}_2$ generated by $\mathcal{G}_1$ and $\mathcal{G}_4$, contradicting the maximality of $\mathcal{G}_1$.

Now to the proof of the lemma. Let $0 \to \mathcal{F}_1 \to \mathcal{G}_1$ be an exact sequence of coherent sheaves on U. Now $\mathcal{G}_1$ can be extended to a coherent sheaf $\mathcal{G}$ on X, and $\mathcal{F}_1$ can be extended to a coherent subsheaf of $\mathcal{G}$. Suppose $\alpha : \mathcal{F}_1 \to \mathcal{I}\mid_U$ is a homomorphism. If $\mathcal{J}$ is any ideal sheaf of definition for $X - U$ in X, then since $\mathcal{F}$ is coherent and $\mathcal{F}\mid_U = \mathcal{F}_1$, we can extend α to a homomorphism $\alpha_1 : \mathcal{J}^n\mathcal{F} \to \mathcal{I}$ for some sufficiently large n (where $\mathcal{J}^n\mathcal{F} = \mathrm{im}\,(\mathcal{J}^n \otimes \mathcal{F} \to \mathcal{F})$). Since $\mathcal{J}^n\mathcal{F}$ is a subsheaf of $\mathcal{G}$, α_1 extends to a homomorphism $\beta : \mathcal{G} \to \mathcal{I}$, whose restriction to U is the desired extension. □

Lemma 11 *Let X be a Noetherian scheme and $\mathcal{I}$ an injective object of $\mathcal{QC}(X)$. Then $\mathcal{I}$ is an injective object in the category $\mathcal{M}_X$ of all $\mathcal{O}_X$-modules.*

Proof: To check that an $\mathcal{O}_X$-module $\mathcal{I}$ is an injective object of $\mathcal{M}_X$, it suffices to check that given a sheaf of ideals $\mathcal{J} \subset \mathcal{O}_X$ and a homomorphism $\mathcal{J} \to \mathcal{I}$, it extends to $\mathcal{O}_X$. Indeed, suppose this holds, and $\mathcal{F} \subset \mathcal{G}$ are any $\mathcal{O}_X$-modules, and $\alpha : \mathcal{F} \to \mathcal{I}$ a homomorphism. By Zorn's lemma there is a maximal subsheaf of $\mathcal{G}$ to which α extends, so we may assume that $\mathcal{F}$ is itself maximal. If $\mathcal{F} \neq \mathcal{G}$, then we can find a homomorphism $\beta : (\mathcal{O}_X)_U \to \mathcal{G}$ with $\mathrm{im}\,\beta \not\subset \mathcal{F}$. Let $\mathcal{J} = \beta^{-1}(\mathcal{F})$, so that $\mathcal{J} \subset (\mathcal{O}_X)_U \subset \mathcal{O}_X$ is a sheaf of ideals (perhaps not coherent); if $\gamma : \mathcal{J} \to \mathcal{I}$ is the induced map, it extends to $(\mathcal{O}_X)_U$ (as it does to all of $\mathcal{O}_X$), giving an extension of α to $\mathcal{F} + \operatorname{im}\beta$, contradicting maximality.

Now, let $\mathcal{I}$ be an injective object in $\mathcal{QC}(X)$, $\mathcal{J}$ a sheaf of ideals and $\alpha : \mathcal{J} \to \mathcal{I}$ a homomorphism. By Zorn's lemma, we may assume that α does not extend to any strictly larger ideal sheaf. Suppose $\mathcal{J} \neq \mathcal{O}_X$, and let $\mathcal{F} = \mathcal{O}_X/\mathcal{J}$. Then $F = \operatorname{supp}(\mathcal{F})$ is closed, since it equals the support of the image of the section $1 \in \Gamma(X, \mathcal{O}_X)$. Let x be the generic point of some component of F, and suppose $f_{1x}, \ldots, f_{nx}$ generate $\mathcal{J}_x$ over $\mathcal{O}_{X,x}$. We can choose an affine open neighbourhood U of x such that

(i) $F \cap U = F_1$ is irreducible

(ii) there exist $f_1, \ldots, f_n \in \mathcal{J}(U)$ whose images in $\mathcal{J}_x$ are the f_{ix}

(iii) if $A = \Gamma(U, \mathcal{O}_X)$, $J = \sum_i Af_i$, and $\eta : A \to A_x = \mathcal{O}_{X,x}$ the canonical map, then $\eta^{-1}(\mathcal{J}_x) = J$.

Let $V = U - F$. Then $\mathcal{J}\mid_V = \mathcal{O}_V$, so $(\mathcal{O}_U)_V \subset \mathcal{J}\mid_U$.

Claim: $\mathcal{J}\mid_U = \widetilde{J} + (\mathcal{O}_U)_V$, where $\widetilde{J}$ is the coherent sheaf of ideals on U associated to $J \subset A$.

Granting the claim, let $\alpha_U = \alpha\mid_U$; since $\mathcal{I}\mid_U$ is (by lemma 10) an injective object of $\mathcal{QC}(U)$, we see that $(\alpha_U)\mid_{\widetilde{J}}: \widetilde{J} \to \mathcal{I}\mid_U$ extends to a map $\beta : \mathcal{O}_U \to \mathcal{I}\mid_U$. Next, since $\mathcal{I}\mid_U \in \mathcal{QC}(U)$, we see that $(\alpha_U)\mid_{(\mathcal{O}_U)_V}: (\mathcal{O}_U)_V \to \mathcal{I}\mid_U$ extends to a map $\gamma : \mathcal{K} \to \mathcal{I}\mid_U$, where $\mathcal{K}$ is a defining ideal for F_1 in U. Then β and γ both yield maps $\widetilde{J} \cap \mathcal{K} \to \mathcal{I}\mid_U$ which have the same restriction to V. Hence they have the same restriction to $\mathcal{K}^n(\widetilde{J} \cap \mathcal{K})$, for some $n \geq 0$. By Artin-Rees, $\widetilde{J} \cap \mathcal{K}^N = \mathcal{K}^{N-r}(\widetilde{J} \cap \mathcal{K}^r) \subset \mathcal{K}^n(\widetilde{J} \cap \mathcal{K})$ for sufficiently large N. Thus, β and γ yield a well defined map $\delta : \mathcal{K}^N + \widetilde{J} \to \mathcal{I}\mid_U$, which restricts to α_U on $\mathcal{J}\mid_U = (\mathcal{O}_U)_V + \widetilde{J}$. Since $\mathcal{I}\mid_U$ is an injective object in $\mathcal{QC}(U)$, δ extends to a map $\mathcal{O}_U \to \mathcal{I}\mid_U$; this means α extends to a map $\mathcal{J} + (\mathcal{O}_X)_U \to \mathcal{I}$. But this is a contradiction, since $\mathcal{J} + (\mathcal{O}_X)_U$ is a strictly larger ideal sheaf than $\mathcal{J}$.

To prove the claim, note that since $J \subset \Gamma(U, \mathcal{J})$, we have $\widetilde{J} \subset \mathcal{J}$, and so for any point $y \in U$, $J_y = (\widetilde{J})_y \subset \mathcal{J}_y \subset A_y = \mathcal{O}_{X,y}$. Next, $\widetilde{J}_y + ((\mathcal{O}_U)_V)_y =$

$\mathcal{J}_y = A_y$ for $y \notin F$. For $y \in F_1$, note that there is a commutative diagram

$$\begin{array}{ccc} A_y & \stackrel{\psi}{\to} & A_x \\ \chi \nwarrow & & \nearrow \eta \\ & A & \end{array}$$

since A_x is a localisation of A_y (as $y \in F_1$, and $x \in F_1$ is the generic point). Further, $\psi(\mathcal{J}_y) \subset \mathcal{J}_x$; hence $\chi^{-1}(\mathcal{J}_y) \subset \eta^{-1}(\mathcal{J}_x) = J$. Since $\mathcal{J}_y \subset A_y$ satisfies $\mathcal{J}_y = A_y\chi(\chi^{-1}(\mathcal{J}_y))$ (this is true of any ideal in A_y), we have $\mathcal{J}_y \subset J_y$ *i.e.*, $\mathcal{J}_y = J_y$. Since $\mathcal{J} \mid_U, \widetilde{J} + (\mathcal{O}_U)_V$ are ideal sheaves in $\mathcal{O}_U$ with the same stalks, they are equal. □

Lemma 12 *Let X be a Noetherian scheme, $\mathcal{F} \in \mathcal{QC}(X)$ a quasi-coherent sheaf on X. Then there is a monomorphism $0 \to \mathcal{F} \to \mathcal{I}$ where $\mathcal{I}$ is an injective object in $\mathcal{M}_X$ which is quasi-coherent.*

Proof: When X is affine this is clear - if $X = \operatorname{Spec} A$, $\mathcal{F} = \widetilde{M}$, choose an injection $M \hookrightarrow I$ where I is an injective A-module; then $\mathcal{F} \hookrightarrow \widetilde{I}$ is the desired monomorphism. In the general case, since X is Noetherian, we may cover it by a finite number of open affines, say $X = \cup_i U_i$; let $\mu_i : U_i \to X$ be the inclusions. Choose monomorphisms $\mathcal{F} \mid_{U_i} \hookrightarrow \mathcal{I}_i$, leading to an injection $\mathcal{F} \hookrightarrow \oplus_i \mu_{i*}\mathcal{I}_i$. But $\mu_{i*}\mathcal{I}_i$ is an injective object in $\mathcal{QC}(X)$ for each i (since μ_{i*} has a left adjoint μ_i^{-1}), and by lemma 11, is then in fact an injective object of $\mathcal{M}_X$. □

Corollary 8 *Let X be a Noetherian scheme, Y a locally closed subscheme, $\mathcal{F}$ a coherent sheaf and $\mathcal{G}$ a quasi-coherent sheaf. Then for any $p \geq 0$,*

(i) $\mathcal{E}xt^p_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ is quasi-coherent

(ii) $\mathcal{H}^p_Y(\mathcal{G})$ is quasi-coherent.

Proof: Choose an injective resolution $0 \to \mathcal{G} \to \mathcal{I}^\bullet$ with $\mathcal{I}^n$ quasi-coherent injective. Then we claim that $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{I}^\bullet)$ and $\mathcal{H}^0_Y(\mathcal{I}^\bullet)$ are complexes of quasi-coherent sheaves, so the cohomology sheaves are quasi-coherent. This is clear for $\mathcal{H}om$ because $\mathcal{F}$ is coherent.

So it suffices to show that if $\mathcal{G}$ is quasi-coherent, then $\mathcal{H}^0_Y(\mathcal{G})$ is too. It suffices to check this separately for Y open and Y closed, for if $Y =$

$U \cap F$ with U open, F closed, then by (9), $\mathcal{H}_Y^0 = (\mathcal{H}_U^0)_F$. When Y is open, $\mathcal{H}_Y^0(\mathcal{G}) = i_* \circ i^{-1}(\mathcal{G})$ is quasi-coherent, since i^{-1} and i_* preserve quasi-coherence. Suppose that Y is closed. Since $\mathcal{G}$ is the union of its coherent subsheaves $\mathcal{G}_i$ and $\cup_i \mathcal{H}_Y^0(\mathcal{G}_i) = \mathcal{H}_Y^0(\mathcal{G})$ (by lemma 9), and a union of coherent subsheaves of a quasi-coherent sheaf is quasi-coherent, we may assume that $\mathcal{G}$ is coherent. But if $\mathcal{J}$ is the defining ideal of Y in X and n is sufficiently large, then

$$\mathcal{H}_Y^0(\mathcal{G}) = \varinjlim_n \mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X/\mathcal{J}^n, \mathcal{G}) = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X/\mathcal{J}^N, \mathcal{G})$$

if N is sufficiently large, and the last sheaf is coherent. □

Corollary 9 *If $X = \operatorname{Spec} A$ is affine, M and N are A-modules with M finitely generated, then*

$$\mathcal{E}xt^p_{\mathcal{O}_X}(M, N) = \widetilde{\operatorname{Ext}}^p_A(M, N) \qquad \cdots \qquad (22)$$

(where $\widetilde{\operatorname{Ext}}^p_A(M, N)$ denotes the quasi-coherent $\mathcal{O}_X$-module associated to the A-module $\operatorname{Ext}^p_A(M, N)$).

Proof: Choose an injective resolution $0 \to N \to I^\bullet$ of N as an A-module. Then

$$\mathcal{H}om_{\mathcal{O}_X}(\widetilde{M}, \widetilde{I^n}) = \widetilde{\operatorname{Hom}}_A(M, I^n)$$

for each n, and $0 \to \widetilde{N} \to \widetilde{I^\bullet}$ is an injective resolution of $\widetilde{N}$. □

Let X be a Noetherian scheme, Y a closed subset, $\mathcal{J}$ a defining ideal of Y, and $n > 0$ an integer. Let $\mathcal{G} \in \mathcal{QC}(X)$ be a quasi-coherent sheaf on X. We have natural homomorphisms

$$\operatorname{Hom}_{\mathcal{O}_X}(\mathcal{O}_X/\mathcal{J}^n, \mathcal{G}) \to H_Y^0(\mathcal{G}),$$

$$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X/\mathcal{J}^n, \mathcal{G}) \to \mathcal{H}_Y^0(\mathcal{G})$$

and hence homomorphisms

$$\varinjlim_n \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{O}_X/\mathcal{J}^n, \mathcal{G}) \to H_Y^0(\mathcal{G}),$$

$$\varinjlim_n \mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X/\mathcal{J}^n, \mathcal{G}) \to \mathcal{H}_Y^0(\mathcal{G}).$$

We assert that these are isomorphisms. It suffices to check this locally and this is clear when X is affine, by taking $\mathcal{G} = \widetilde{M}$. Now, substituting for $\mathcal{G}$ any injective quasi-coherent resolution of $\mathcal{G}$ and using the fact that cohomology commutes with inductive limits, we get isomorphisms

$$\varinjlim_n \operatorname{Ext}^p_{\mathcal{O}_X}(\mathcal{O}_X/\mathcal{J}^n, \mathcal{G}) \overset{\cong}{\to} H^p_Y(\mathcal{G}), \qquad \cdots \quad (23)$$

$$\varinjlim_n \mathcal{E}xt^p_{\mathcal{O}_X}(\mathcal{O}_X/\mathcal{J}^n, \mathcal{G}) \overset{\cong}{\to} \mathcal{H}^p_Y(\mathcal{G}). \qquad \cdots \quad (24)$$

In view of the fact that the maps (21) are isomorphisms for $\mathcal{F}$ coherent, we get the following: if X is a Noetherian scheme, Y closed in X, $x \in X$, A_x the local ring at x, $I \subset A_x$ any defining ideal of Y at x, then

$$\varinjlim_n \operatorname{Ext}^p_{A_x}(A_x/I^n, \mathcal{G}_x) \cong \mathcal{H}^p_Y(\mathcal{F})_x \qquad \cdots \quad (25)$$

Note that by Corollary 7, $\mathcal{H}^p_Y(\mathcal{F})_x = 0$ for $p > \dim X$.

Proposition 11 *Let X be a Noetherian scheme, $Y \subset X$ a closed subset, $\mathcal{F}$ coherent on X, and $x \in X$. Let $A_x = \mathcal{O}_{X,x}$. Then the following are equivalent:*

(i) for every finitely generated A_x-module N such that $\operatorname{supp} N \subset Y_x$,

$$\operatorname{Ext}^i_{A_x}(N, \mathcal{F}_x) = 0 \ \text{ for } i < p$$

(ii) for one finitely generated A_x-module N with $\operatorname{supp} N = Y_x$,

$$\operatorname{Ext}^i_{A_x}(N, \mathcal{F}_x) = 0 \ \text{ for } i < p$$

(iii) if I is some (or any) defining ideal of Y at x, there are elements $f_1, \ldots, f_p$ in I such that f_i is a non-zero divisor in $\mathcal{F}_x/(f_1, \ldots, f_{i-1})\mathcal{F}_x$ for $i = 1, \ldots, p$

(iv) for any prime ideal $\mathcal{P} \supset I$, we have

$$\operatorname{depth}_{(A_x)_{\mathcal{P}}}(\mathcal{F}_x)_{\mathcal{P}} \geq p$$

(v) $\mathcal{H}^i_Y(\mathcal{F})_x = 0$ for $i < p$.

Proof: We proceed by induction on p. The proposition has content only for $p > 0$; suppose first that $p = 1$. Clearly (i)$\Rightarrow$(ii). Suppose that (ii) holds, and let $J = \mathrm{Ann}_{A_x} N$, so that J defines Y at x, and so $\sqrt{J} = \sqrt{I}$. If (iii) is false for I, there is a $\mathcal{P} \in \mathrm{Ass}(\mathcal{F}_x)$ with $J \subset \mathcal{P}$. Now, $\mathrm{Hom}_{A_x}(N, A_x/\mathcal{P}) = \mathrm{Hom}_{A_x}(N/\mathcal{P}N, A_x/\mathcal{P}) \neq 0$ since $N/\mathcal{P}N$ is a finitely generated faithful $A_x/\mathcal{P}$-module. Since $\mathcal{P} \in \mathrm{Ass}(\mathcal{F}_x)$ there is a monomorphism $A_x/\mathcal{P} \hookrightarrow \mathcal{F}_x$; hence $\mathrm{Hom}_{A_x}(N, \mathcal{F}_x) \neq 0$, contradicting (ii). Hence (ii) $\Rightarrow$(iii). Clearly (iii) $\Rightarrow$(iv). We shall show that (iv) $\Rightarrow$(i). Since N admits a composition series with quotients $A_x/\mathcal{P}$ with $\mathcal{P} \in Y_x$, we may assume $N = A_x/\mathcal{P}$ with $\mathcal{P} \in Y_x$. If $\mathrm{Hom}_{A_x}(A_x/\mathcal{P}, \mathcal{F}_x) \neq 0$, then there is a $\mathcal{Q} \in \mathrm{Ass}(\mathcal{F}_x)$ with $\mathcal{P} \subset \mathcal{Q}$, and hence $I \subset \mathcal{Q}$, so that $\mathrm{depth}_{(A_x)_\mathcal{Q}}(\mathcal{F}_x)_\mathcal{Q} = 0$, contradicting (iv).

Hence (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv). Now

$$\mathrm{Hom}_{A_x}(A_x/I^n, \mathcal{F}_x) \hookrightarrow \mathrm{Hom}_{A_x}(A_x/I^{n+1}, \mathcal{F}_x)$$

and the union of this increasing sequence is $\mathcal{H}^0_Y(\mathcal{F})_x$. Thus, (v) is false $\Leftrightarrow$ there exists $n > 0$ with $\mathrm{Hom}_{A_x}(A_x/I^n, \mathcal{F}_x) \neq 0$. Hence (i) $\Rightarrow$(v) $\Rightarrow$(ii).

Thus we are through for $p = 1$.

Suppose now that $p > 1$, and the assertion of the proposition holds for all smaller values of p. If (a) denotes any one of (i)-(v), let (a)$'$ denote the same condition for $p - 1$ instead of p. Now, (i) $\Rightarrow$(ii) is trivial. Assume (ii). By what we have already shown (the case $p = 1$), there is an $f \in \mathrm{Ann}_{A_x}(N)$ such that f is a non-zero divisor on $\mathcal{F}_x$. The exact sequence

$$0 \to \mathcal{F}_x \xrightarrow{f} \mathcal{F}_x \to \mathcal{F}/f\mathcal{F}_x \to 0$$

gives that $\mathrm{Ext}^i_{A_x}(N, \mathcal{F}_x/f\mathcal{F}_x) = 0$ for $i < p-1$, so that (ii)$'$ holds for $\mathcal{F}_x/f\mathcal{F}_x$. Replacing f by a power if necessary, we may assume $f \in I$. Now (ii)$'$ $\Leftrightarrow$ (iii)$'$, so there exist $f_2, \ldots, f_p$ in I such that f_i is a non-zero divisor on $\mathcal{F}_x/(f, f_2 \ldots, f_{i-1})\mathcal{F}_x$. Thus (ii) $\Rightarrow$(iii). Again trivially (iii) $\Rightarrow$(iv).

Now (iv) $\Rightarrow$(since $p > 1$) that I is not contained in any $\mathcal{P} \in \mathrm{Ass}(\mathcal{F}_x)$, for any defining ideal I of Y at x; in particular, if N is any finite A_x-module with $\mathrm{supp}(N) \subset Y_x$, then $\mathrm{Ann}(N) \not\subset \mathcal{P}$ for any $\mathcal{P} \in \mathrm{Ass}(\mathcal{F}_x)$. Thus we can find $f \in \mathrm{Ann}(N)$ such that f is a non-zero divisor on $\mathcal{F}_x$. For any $\mathcal{P} \supset I$,

$$\mathrm{depth}_{(A_x)_\mathcal{P}}(\mathcal{F}_x/f\mathcal{F}_x)_\mathcal{P} = \mathrm{depth}_{(A_x)_\mathcal{P}}(\mathcal{F}_x)_\mathcal{P} - 1 \geq p - 1,$$

so that by induction hypothesis, $\mathrm{Ext}^i_{A_x}(N, \mathcal{F}_x/f\mathcal{F}_x) = 0$ for $i < p-1$. Hence the sequence

$$0 \to \mathrm{Ext}^i_{A_x}(N, \mathcal{F}_x) \xrightarrow{f} \mathrm{Ext}^i_{A_x}(N, \mathcal{F}_x)$$

is exact for $i < p$; since $f \in \mathrm{Ann}\,(N)$, this implies (i). Finally, as before, (i) $\Rightarrow$(v) since

$$\lim_{\overrightarrow{n}} \mathrm{Ext}\,^i_{A_x}(A_x/I^n, \mathcal{F}_x) = \mathcal{H}^i_Y(\mathcal{F})_x.$$

On the other hand, suppose (v) holds, so that (v) $\Rightarrow$(v)$'$ $\Rightarrow$(i)$'$. We have exact sequences

$$\mathrm{Ext}\,^{i-1}_{A_x}(I^n/I^{n+1}, \mathcal{F}_x) \to \mathrm{Ext}\,^i_{A_x}(A_x/I^n, \mathcal{F}_x) \to \mathrm{Ext}\,^i_{A_x}(A_x/I^{n+1}, \mathcal{F}_x)$$

where (i)$'$ $\Rightarrow \mathrm{Ext}\,^{i-1}_{A_x}(I^n/I^{n+1}, \mathcal{F}_x) = 0$ for $i-1 < p-1$. Hence (v) $\Rightarrow$(i)$'$ $\Rightarrow \mathrm{Ext}\,^i_{A_x}(A_x/I, \mathcal{F}_x) \hookrightarrow \mathcal{H}^i_Y(\mathcal{F})_x = 0$ for $i < p$ $\Rightarrow$(ii) is valid with $N = A_x/I$. $\square$

4 Global duality theory

Theorem 12 *Let X be a scheme of dimension n, proper over a field k. Then there is a complex*

$$0 \to \mathcal{I}_n \to \mathcal{I}_{n-1} \to \cdots \to \mathcal{I}_0 \to 0$$

of injective quasi-coherent sheaves on X such that for any quasi-coherent sheaf $\mathcal{F}$ on X, we have a natural isomorphism

$$H^p(X, \mathcal{F})^* \xrightarrow{\cong} H_p(\mathrm{Hom}\,_{\mathcal{O}_X}(\mathcal{F}, \mathcal{I}_\bullet)) \qquad \cdots \quad (**)$$

(where M^* denotes the k-linear dual of M).

Any complex $\mathcal{I}_\bullet$ of quasi-coherent injectives on X satisfying (**) has the following properties.

(a) The homology sheaves $\mathcal{H}_p(\mathcal{I}_\bullet)$ are independent of the particular complex $\mathcal{I}_\bullet$, and are coherent.

(b) If X' is again proper over k, $\mathcal{I}'_\bullet$ a similar complex on X', $U \subset X$, $U' \subset X'$ open subsets and $f : U \to U'$ an isomorphism, then there is an isomorphism

$$\mathcal{H}_p(\mathcal{I}_\bullet)\mid_U \xrightarrow{\cong} \mathcal{H}_p(\mathcal{I}'_\bullet)\mid_{U'}$$

over f.

(c) If X is Cohen-Macaulay at a point where $\dim_x X = n$, then $\mathcal{H}_p(\mathcal{I}_\bullet)_x = 0$ for $p \neq n$ and $\mathcal{H}_n(\mathcal{I}_\bullet)_x = \Omega_{\mathcal{O}_{X,x}}$ is the dualising module[2] of $\mathcal{O}_{X,x}$.

[2]Since $\mathcal{O}_{X,x}$ is a quotient of a regular local ring, it has a dualising module.

Proof: First, we prove the existence of such a finite complex $\mathcal{I}_\bullet$ (concentrated in degrees ≥ 0). Let $\mathcal{F}$ be quasi-coherent on X, $U = \operatorname{Spec} A$ an affine open subset of X, and $M = \Gamma(U, \mathcal{F})$. Let $i_U : U \to X$ be the inclusion. Then we have a sequence of natural isomorphisms

$$\Gamma(U,\mathcal{F})^* = M^* = \operatorname{Hom}_A(M, \operatorname{Hom}_k(A,k)) \cong \operatorname{Hom}_{\mathcal{O}_U}(\mathcal{F}\mid_U, \widetilde{\operatorname{Hom}}_k(A,k))$$

$$\cong \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{F}, i_{U*}\widetilde{\operatorname{Hom}}_k(A,k)) = \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{I}_U)$$

where $\mathcal{I}_U = i_{U*}\widetilde{\operatorname{Hom}}_k(A,k)$. The above composite isomorphism shows that $\mathcal{I}_U$ is an *injective* $\mathcal{O}_X$-module (since it is an injective object of $\mathcal{QC}(X)$ - see lemma 11). Hence, if $\mathcal{U} = \{U_i\}_{i\in I}$ is a finite affine open cover of X (since X is proper over k, it is Noetherian), then there is an injective quasi-coherent sheaf

$$\mathcal{I}_p = \bigoplus_{i_0,\ldots,i_p\in I} \mathcal{I}_{U_{i_0,\ldots,i_p}},$$

where $U_{i_0,\ldots,i_p} = U_{i_0}\cap U_{i_1}\cap\cdots\cap U_{i_p}$ (which is affine). By its definition, $\mathcal{I}_p$ has the property that if $\check{C}^p(\mathcal{U},\mathcal{F})$ is the p^{th} term of the Čech complex of alternating cochains with values in $\mathcal{F}$, we have a natural isomorphism

$$\check{C}^p(\mathcal{U},\mathcal{F})^* \xrightarrow{\cong} \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{F},\mathcal{I}_p).$$

Further, the natural transformations

$$\delta^p : \check{C}^p(\mathcal{U},\mathcal{F}) \to \check{C}^{p+1}(\mathcal{U},\mathcal{F})$$

induce natural transformations $\check{C}^{p+1}(\mathcal{U},\mathcal{F})^* \to \check{C}^p(\mathcal{U},\mathcal{F})^*$, hence homomorphisms $\psi_p : \mathcal{I}_{p+1} \to \mathcal{I}_p$. We clearly have $\psi_p \circ \psi_{p+1} = 0$, so that $\mathcal{I}_\bullet$ is a finite (because $\mathcal{U}$ is finite) complex of quasi-coherent injective sheaves, such that there are natural isomorphisms

$$H^p(X,\mathcal{F})^* \cong H_p(\check{C}^p(\mathcal{U},\mathcal{F})^*) \cong H_p(\operatorname{Hom}_{\mathcal{O}_X}(\mathcal{F},\mathcal{I}_\bullet)).$$

Now, for $p > n$,

$$(0) = H^p(X,\mathcal{F})^* \cong H_p(\operatorname{Hom}_{\mathcal{O}_X}(\mathcal{F},\mathcal{I}_\bullet)).$$

If $\mathcal{Z}_p = \ker(\mathcal{I}_p \to \mathcal{I}_{p-1})$, then $\mathcal{Z}_p$ is quasi-coherent, so applying the above vanishing statement with $\mathcal{F} = \mathcal{Z}_p$, we see that the inclusion $\mathcal{Z}_p \to \mathcal{I}_p$ represents 0 in $H_p(\operatorname{Hom}_{\mathcal{O}_X}(\mathcal{Z}_p,\mathcal{I}_\bullet))$, so that $\mathcal{I}_{p+1} \to \mathcal{Z}_p$ is a split surjection;

hence $\mathcal{H}_p(\mathcal{I}_\bullet) = 0$ for $p > n$. Let m be the largest integer p such that $\mathcal{I}_p \neq 0$. If $m > n$, then $\mathcal{H}_m(\mathcal{I}_\bullet) = 0 \Rightarrow \mathcal{I}_m \hookrightarrow \mathcal{I}_{m-1}$, and since $\mathcal{I}_m$ is an injective $\mathcal{O}_X$-module, this inclusion is split. Writing $\mathcal{I}_{m-1} = \mathcal{I}_m \oplus \mathcal{I}'_{m-1}$, we thus obtain a shorter complex

$$0 \to \mathcal{I}'_{m-1} \to \mathcal{I}_{m-2} \to \cdots \mathcal{I}_0 \to 0$$

with the same property as $\mathcal{I}_\bullet$. Repeating this procedure, we end up with a complex of length n

$$0 \to \mathcal{I}_n \to \mathcal{I}_{n-1} \to \cdots \to \mathcal{I}_0 \to 0$$

with the requisite property (**).

Now, let $\mathcal{I}_\bullet$ be any complex of quasi-coherent injectives having the desired property (**), and $U = \operatorname{Spec} A$ an affine open subset of X such that the closed subset $Y = X - U$ has defining ideal $\mathcal{J}$. Then we have

$$\begin{aligned} \mathcal{H}_p(\mathcal{I}_\bullet)(U) = \;& H_p(\mathcal{I}_\bullet(U)) \;\; \text{(since } U \text{ is affine)} \\ \cong \;& H_p(\varinjlim_n \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{J}^n, \mathcal{I}_\bullet)) \cong \varinjlim_n H_p(\operatorname{Hom}_{\mathcal{O}_X}(\mathcal{J}^n, \mathcal{I}_\bullet)) \\ \cong \;& \varinjlim_n H^p(X, \mathcal{J}^n)^* \qquad\qquad \cdots \quad (*) \end{aligned}$$

This can be considered as an A-module in the following manner. Any $f \in A$ defines a homomorphism $\mathcal{J}^m \to \mathcal{O}_X$ for a suitably large m, hence a homomorphism $\mathcal{J}^{m+r} \to \mathcal{J}^r$, hence $H^p(X, \mathcal{J}^{m+r}) \to H^p(X, \mathcal{J}^p)$, and finally $H^p(X, \mathcal{J}^r)^* \to H^p(X, \mathcal{J}^{m+r})^*$. Hence, in the inductive limit, we get that $\varinjlim_n H^p(X, \mathcal{J}^n)^*$ is an A-module. One checks easily that this A-module structure is independent of the choice of the ideal of definition $\mathcal{J}$ and the homomorphism $\mathcal{J}^m \to \mathcal{O}_X$ representing f.

Now suppose that $U' \subset U$ is a smaller open set, and let $Y' = X - U'$; let $\mathcal{J}'$ be a defining ideal for Y' with $\mathcal{J}' \subset \mathcal{J}$. Then we have homomorphisms $(\mathcal{J}')^n \to \mathcal{J}^n$ for $n > 0$, hence homomorphisms $H^p(X, \mathcal{J}^n)^* \to H^p(X, (\mathcal{J}')^n)^*$, and hence a homomorphism of inductive limits. Now, it is easy to check that $(*)$ is an isomorphism of A-modules, and is compatible with restrictions. This shows, in particular, that the sheaves $\mathcal{H}_p$ are independent of the choice of $\mathcal{I}_\bullet$.

We prove (b). We can clearly find an X'' proper over both X and X' and an open subset U'' of X'' mapping isomorphically onto U and U' (take X'' to be the closure in $X \times_k X'$ of the graph of the isomorphism $f : U \to U'$, and U'' to be this graph). Thus we may assume without loss of generality that we have a morphism $f : X \to X'$ such that $f^{-1}(U') = U$ and $f : U \to U'$ is an isomorphism. To prove that there is an isomorphism of $\mathcal{H}_p(\mathcal{I}_\bullet)(U)$ and $\mathcal{H}_p(\mathcal{I}'_\bullet)(U')$ over f, it suffices to exhibit, for any V' affine open in U', an isomorphism

$$\mathcal{H}_p(\mathcal{I}'_\bullet)(V') \xrightarrow{\cong} \mathcal{H}^p(\mathcal{I}_\bullet)(f^{-1}(V'))$$

which is compatible with restrictions for inclusions $V'' \subset V'$, $f^{-1}(V'') \subset f^{-1}(V')$ of affine open sets. Then, replacing U' by V' and U by $V = f^{-1}(V')$, it suffices to consider the case when U' and U are affine. Let $\mathcal{J}'$ be a defining ideal of $Y' = X' - U'$, so that $\mathcal{J} = \mathrm{im}\,(f^*\mathcal{J}' \to \mathcal{O}_X)$ is a defining ideal of $Y = X - U = f^{-1}(Y')$. We then have natural homomorphisms $(\mathcal{J}')^n \to f_*\mathcal{J}^n$, hence homomorphisms

$$H^p(X', (\mathcal{J}')^n) \to H^p(X', f_*\mathcal{J}^n) \to H^p(X, \mathcal{J}^n),$$

and on dualising and passing to the direct limit, a homomorphism

$$\varinjlim_n H^p(X, \mathcal{J}^n)^* \to \varinjlim_n H^p(X', (\mathcal{J}')^n)^*. \qquad \cdots \quad (\dagger)$$

This is a homomorphism of $\Gamma(U, \mathcal{O}_X)$-modules, compatible with restriction to affine open subsets $V' \subset U'$, $V = f^{-1}(V') \subset U$. It suffices to show that $(\dagger)$ is an isomorphism.

Let us recall the following theorem from E.G.A. III:

Theorem 0 *Let $f : X \to Y$ be a proper morphism of Noetherian schemes, $\mathcal{F}$ a coherent sheaf on X and $\mathcal{I}$ a sheaf of ideals on Y. Then for any $q \geq 0$, $\oplus_{n\geq 0} R^q f_*(\mathcal{I}^n\mathcal{F})$ can be considered as a graded sheaf of modules over the sheaf of rings $\oplus_{n\geq 0}\mathcal{I}^n$, and as such it is finitely generated. In particular, there exists $m_0 \geq 0$ such that $\mathcal{I}^k R^q f_*(\mathcal{I}^m\mathcal{F}) = R^q(\mathcal{I}^{m+k}\mathcal{F})$ for all $k \geq 0$,$m \geq m_0$.*

In the theorem, the action of $\oplus_{n\geq 0}\mathcal{I}^n$ on $R^q f_*(\mathcal{I}^n\mathcal{F})$ is defined as follows: if $x \in \mathcal{I}^m(U)$ and $y \in H^q(f^{-1}(U), \mathcal{I}^n\mathcal{F})$, x defines a homomorphism $(\mathcal{I}^n\mathcal{F}) \mid_U \xrightarrow{\widetilde{x}} (\mathcal{I}^{m+n}\mathcal{F}) \mid_U$, and $x \cdot y$ is the image in $H^q(f^{-1}(U), \mathcal{I}^{m+n}\mathcal{F})$ of y with respect to $\widetilde{x}$.

Now, the homomorphism $(\dagger)$ factorises as

$$\varinjlim_n H^p(X, \mathcal{J}^n)^* \xrightarrow{g} \varinjlim_n H^p(X', f_*(\mathcal{J}^n))^* \xrightarrow{h} \varinjlim_n H^p(X', (\mathcal{J}')^n)^*.$$

Let us first show that h is an isomorphism.

Choose m_0 as in the above Theorem where $Y = X'$, $\mathcal{F} = \mathcal{O}_X$, $q = 0$ and $\mathcal{I} = \mathcal{J}'$. Thus $(\mathcal{J}')^k f_*(\mathcal{J}^m) = f_*(\mathcal{J}^{m+k})$ for $m \geq m_0$.

Let $\mathcal{K}$, $\mathcal{L}$, $\mathcal{C}$ be the kernel, image and cokernel of $(\mathcal{J}')^{m_0} \to f_*\mathcal{J}^{m_0}$ respectively. Since $f^{-1}(U') \to U'$ is an isomorphism, and $\mathcal{K}$, $\mathcal{C}$ are coherent sheaves with support in $X' - U'$, $\mathcal{K}$ and $\mathcal{C}$ are both annihilated by $(\mathcal{J}')^{m_1}$ for some $m_1 > 0$. We have the following exact sequences

$$0 \to \mathcal{K} \cap (\mathcal{J}')^{m_0+k} \to (\mathcal{J}')^{m_0+k} \to (\mathcal{J}')^k\mathcal{L} \to 0,$$

$$0 \to (\mathcal{J}')^k f_*(\mathcal{J}^{m_0}) \cap \mathcal{L} \to (\mathcal{J}')^k f_*(\mathcal{J}^{m_0}) = f_*(\mathcal{J}^{m_0+k}) \to (\mathcal{J}')^k\mathcal{C} \to 0,$$

(where the middle equality is by the choice of m_0 made above) and by Artin-Rees, this reduces for $k \geq k_0$ to the pair of isomorphisms

$$(\mathcal{J}')^{m_0+k} \cong (\mathcal{J}')^k\mathcal{L},$$

$$(\mathcal{J}')^k f_*(\mathcal{J}^{m_0}) \cap \mathcal{L} = (\mathcal{J}')^{k-k_0}((\mathcal{J}')^{k_0} f_*(\mathcal{J}^{m_0}) \cap \mathcal{L}) \xrightarrow{\cong} (\mathcal{J}')^k f_*(\mathcal{J}^{m_0}) = f_*(\mathcal{J}^{m_0+k}).$$

Thus, it suffices to show that

$$\varinjlim_k H^p(X', (\mathcal{J}')^{k-k_0}((\mathcal{J}')^{k_0} f_*(\mathcal{J}^{m_0}) \cap \mathcal{L}))^* \to \varinjlim_k H^p(X', (\mathcal{J}')^k\mathcal{L})^*$$

is an isomorphism. This follows from the inclusions

$$(\mathcal{J}')^k\mathcal{L} \subset (\mathcal{J}')^{k-k_0}((\mathcal{J}')^{k_0} f_*(\mathcal{J}^{m_0}) \cap \mathcal{L}) \subset (\mathcal{J}')^{k-k_0}\mathcal{L} \subset (\mathcal{J}')^{k-2k_0}((\mathcal{J}')^{k_0} f_*(\mathcal{J}^{m_0}) \cap \mathcal{L}).$$

Thus we are left with proving that g is an isomorphism. For every $m > 0$, consider the Leray spectral sequence $E_r^{p,q}(m)$,

$$E_2^{p,q}(m) = H^p(X', R^q f_*\mathcal{J}^m) \Rightarrow H^{p+q}(X, \mathcal{J}^m).$$

For $m' > m$, we have a morphism of spectral sequences

$$E_r^{p,q}(m') \to E_r^{p,q}(m).$$

Further, $E_r^{p,q}(m) = E_\infty^{p,q}$ for $r > p + q$ and every m, and the homomorphism

$$H^p(X', f_*\mathcal{J}^m) = E_2^{p,0} \twoheadrightarrow E_\infty^{p,0} \hookrightarrow H^p(X, \mathcal{J}^m)$$

is the natural homomorphism. Thus, to prove g is an isomorphism, it suffices to show that $\lim\limits_{\overrightarrow{m}} E_2^{p,q}(m) = 0$ for $q > 0$. This would follow if we can show that for any $m > 0$ and $q > 0$, the map

$$H^p(X', R^q f_* \mathcal{J}^m)^* \to H^p(X', R^q f_* \mathcal{J}^{m'})^*$$

is 0 for some $m' > m$. Now, $R^q f_* \mathcal{J}^m$ has support in $X' - U'$ for $q > 0$, since $f^{-1}(U') \to U'$ is an isomorphism. Further, for m sufficiently large and $k \geq 0$, $R^q f_* \mathcal{J}^{m+k} = (\mathcal{J}')^k R^q f_* \mathcal{J}^m$. Hence, for k large, the map $R^q f_* \mathcal{J}^{m+k} \to R^q f_* \mathcal{J}^m$ is 0 (take k so large that $(\mathcal{J}')^k$ annihilates $R^q f_* \mathcal{J}^m$). This proves (b).

In view of (b), to prove the coherence statement in (a), which is an assertion local on X, we may assume that X is a closed subset of $(\mathbf{P}^1)^N = \mathbf{P}^1 \times \cdots \times \mathbf{P}^1$, since any point of x has an open neighbourhood which is a closed subvariety of $\mathbf{A}^N$ for some N. Further, if

$$0 \to \mathcal{I}_N \to \cdots \to \mathcal{I}_1 \to \mathcal{I}_0 \to 0$$

is a complex of quasi-coherent injectives on $(\mathbf{P}^1)^N$ with the requisite property $(**)$ on $(\mathbf{P}^1)^N$, and X is a closed subscheme of $(\mathbf{P}^1)^N$, $i : X \hookrightarrow (\mathbf{P}^1)^N$ the inclusion, then clearly the complex of quasi-coherent injective sheaves

$$0 \to i^{-1}\mathcal{H}om_{(\mathbf{P}^1)^N}(i_*\mathcal{O}_X, \mathcal{I}_N) \to \cdots \to i^{-1}\mathcal{H}om_{(\mathbf{P}^1)^N}(i_*\mathcal{O}_X, \mathcal{I}_1) \to i^{-1}\mathcal{H}om_{(\mathbf{P}^1)^N}(i_*\mathcal{O}_X, \mathcal{I}_0) \to 0$$

is a complex of quasi-coherent injective $\mathcal{O}_X$-modules on X with the requisite property $(**)$ on X. So it suffices to prove that

(i) $\mathcal{H}_i(\mathcal{I}_\bullet) = 0$ for $i < N$

(ii) $\mathcal{H}_N(\mathcal{I}_\bullet)$ is a coherent sheaf on $(\mathbf{P}^1)^N$.

Indeed, granting (i) and (ii), $\mathcal{I}_\bullet$ is an injective resolution of the coherent sheaf $\mathcal{H}_N(\mathcal{I}_\bullet)$, so that

$$\mathcal{H}_i(\mathcal{H}om\,(i_*\mathcal{O}_X, \mathcal{I}_\bullet)) \xrightarrow{\cong} \mathcal{E}xt^{N-i}_{(\mathbf{P}^1)^N}(i_*\mathcal{O}_X, \mathcal{H}_N(\mathcal{I}_\bullet))$$

which is coherent.

Let $\mathcal{I}$ denote the ideal sheaf of the point $\{\infty\} \in \mathbf{P}^1$, and let $\mathcal{J}$ be the ideal sheaf of the divisor $\cup_i p_i^*(\infty)$ where $p_i : (\mathbf{P}^1)^N \to \mathbf{P}^1$ is the projection onto the i^{th} factor. Then

$$\mathcal{J} = \otimes_i p_i^* \mathcal{I} = \mathcal{I} \boxtimes \mathcal{I} \boxtimes \cdots \boxtimes \mathcal{I}.$$

It is sufficient to show that

(i) $\varinjlim_n H^p((\mathbf{P}^1)^N, \mathcal{J}^n)^* = 0$ for $0 \leq p < N$,

(ii) $\varinjlim_n H^N((\mathbf{P}^1)^N, \mathcal{J}^n)^*$ is a finitely generated $k[x_1, \ldots, x_N]$-module.

This is because $(\mathbf{P}^1)^N$ is covered by affine open subsets isomorphic to $(\mathbf{P}^1)^N - \cup_i p_i^*(\infty)$.

Now, one knows that $H^0(\mathbf{P}^1, \mathcal{I}^n) = 0$ for $n > 0$. Hence (i) follows from the Kunneth formula. Also,

$$H^N((\mathbf{P}^1)^N, \mathcal{J}^n) \cong H^1(\mathbf{P}^1, \mathcal{I}^n) \otimes_k \cdots \otimes_k H^1(\mathbf{P}^1, \mathcal{I}^n).$$

Thus, it suffices to show that $\varinjlim_n H^1(\mathbf{P}^1, \mathcal{I}^n)$ is a finitely generated $k[x]$-module, where x is the coordinate on $\mathbf{P}^1 - \{\infty\}$. Using the standard covering $U_0 = \mathbf{P}^1 - \{\infty\}, U_\infty = \mathbf{P}^1 - \{0\}$, since $\mathcal{I}\mid_{U_0} = \mathcal{O}_{U_0}$, and $\mathcal{I}\mid_{U_\infty} = x^{-1}\mathcal{O}_{U_\infty}$, we see that any element

$$\begin{aligned} \xi \in\ & H^1(\mathbf{P}^1, \mathcal{I}^n) \cong \Gamma(U_0 \cap U_\infty \mathcal{I}^n)/(\Gamma(U_0, \mathcal{I}^n) + \Gamma(U_\infty, \mathcal{I}^n)) \\ \cong\ & k[x, x^{-1}]/(k[x] + x^{-n}k[x^{-1}]) \end{aligned}$$

has a unique representing cocycle $\xi \in \Gamma(U_0 \cap U_\infty, \mathcal{I}^n)$ of the form

$$\xi' = a_1 x^{-1} + a_2 x^{-2} + \cdots + a_{n-1} x^{-n+1}$$

Further, multiplication by x,

$$(x) : H^1(\mathbf{P}^1, \mathcal{I}^n) \to H^1(\mathbf{P}^1, \mathcal{I}^{n-1}),$$

is represented by

$$(x) : a_1 x^{-1} + \cdots a_{n-1} x^{1-n} \mapsto a_2 x^{-1} + a_3 x^{-2} + \cdots + a_{n-1} x^{-n+2}.$$

Now $a_1, \ldots, a_{n-1}$, considered as linear forms on the vector space $H^1(\mathbf{P}^1, \mathcal{I}^n)$, form a basis of this space, and the inclusion of $H^1(\mathbf{P}^1, \mathcal{I}^n)^* \hookrightarrow H^1(\mathbf{P}^1, \mathcal{I}^{n+1})^*$ induced by $\mathcal{I}^{n+1} \subset \mathcal{I}^n$ takes a_i to a_i for all i. Hence $\varinjlim_n H^1(\mathbf{P}^1, \mathcal{I}^n)$ has a basis given by a_i, $i > 0$; the action of x induced by dualising (x) takes a_i to a_{i+1}. Hence this direct limit is a free module over $k[x]$ generated by a_1. This proves (a) of the theorem.

Now to the proof of (c). The set of points $x \in X$ such that $\dim_x X = n$ and $\mathcal{O}_{X,x}$ is Cohen-Macaulay is an open set (E.G.A. IV 6.11.2). Let us call this set U. Suppose we show that for every closed point $x \in U$, $\mathcal{H}_p(\mathcal{I}_\bullet)_x = 0$ for $p < n$, and $\mathcal{H}_n(\mathcal{I}_\bullet)_x$ is the dualising module of $\mathcal{O}_{X,x}$, then it follows that (i) $\mathcal{H}_p(\mathcal{I}_\bullet) \mid_U = 0$ for $p < n$ (since $\mathcal{H}_p(\mathcal{I}_\bullet) \mid_U$ is a coherent sheaf with vanishing stalks at all closed points), and (ii) for any $x \in U$, $\mathcal{H}_n(\mathcal{I}_\bullet)_x$ is of finite injective dimension (because a localisation of an injective module is injective), $\operatorname{End}_{\mathcal{O}_{X,x}}(\mathcal{H}_n(\mathcal{I}_\bullet)_x) = \mathcal{O}_{X,x}$, and $\mathcal{H}_n(\mathcal{I}_\bullet)_x$ is Cohen-Macaulay of dimension equal to $\dim \mathcal{O}_{X,x}$. By our characterisation of the dualising module (Corollary 3), it would follow that $\mathcal{H}_n(\mathcal{I}_\bullet)_x$ is the dualising module, and (c) would be valid for all $x \in U$. So it suffices to prove (c) for closed points $x \in U$.

Let $f_1, \ldots, f_n \in \mathcal{O}_{X,x}$ be a maximal $\mathcal{O}_{X,x}$-sequence, and $\mathcal{F} = \mathcal{O}_{X,x}/(f_1, \ldots, f_n)$ considered as a coherent sheaf on X with support at x (*i.e.*, as a skyscraper sheaf at x). Then $H^i(X, \mathcal{F}) = 0$ for $i > 0$. On the other hand, if we put $\mathcal{J}^p = \mathcal{I}_{n-p}$, the complex $\mathcal{J}^\bullet$ is concentrated in degrees ≥ 0, and we claim there is a spectral sequence

$$E_2^{p,q} = \operatorname{Ext}^p_{\mathcal{O}_X}(\mathcal{F}, \mathcal{H}^q(\mathcal{J}^\bullet)) \Rightarrow H^{p+q}(\operatorname{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{J}^\bullet)).$$

To see this, let $\mathcal{J}^\bullet \to \mathcal{J}^{\bullet,\bullet}$ be a Cartan-Eilenberg resolution *i.e.*, a double complex of injectives, such that

(i) $\mathcal{J}^m \to \mathcal{J}^{m,\bullet}$ is a resolution for each m, and

(ii) if $\mathcal{I}^{m,n} = \mathcal{H}_I^m(\mathcal{J}^{\bullet m})$, then $\mathcal{I}^{m,\bullet}$ is an inejctive resolution of $\mathcal{H}^m(\mathcal{J}^\bullet)$.

The desired spectral sequence is obtained by considering spectral sequences of the double complex

$$A^{m,n} = \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{J}^{m,n}).$$

Since $\mathcal{F}$, $\mathcal{H}^q(\mathcal{J}^m)$ are coherent, and $\mathcal{F}$ is concentrated at the closed point x, the spectral sequence of Sec. 2, (17) and lemmas 7 and 8 imply that in the above spectral sequence, we have

$$E_2^{p,q} = \operatorname{Ext}^p_{\mathcal{O}_{X,x}}(\mathcal{F}_x, \mathcal{H}^q(\mathcal{J}^\bullet)_x).$$

Let q_0 be the largest integer such that $\mathcal{H}^q(\mathcal{J}^\bullet)_x \neq 0$. From the Koszul resolution for $\mathcal{F}_x$, we have

$$\operatorname{Ext}^n_{\mathcal{O}_{X,x}}(\mathcal{F}_x, \mathcal{H}^{q_0}(\mathcal{J}^\bullet)_x) = \mathcal{H}^{q_0}(\mathcal{J}^\bullet)_x/(f_1, \ldots, f_n)\mathcal{H}^{q_0}(\mathcal{J}^\bullet)_x \neq 0$$

by Nakayama's lemma. It follows from the spectral sequence that $H^{n+q_0}(\operatorname{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{J}^\bullet)) \neq 0$; but $H^{n+q_0}(\operatorname{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{J}^\bullet)) \cong H^{n-(n+q_0)}(X, \mathcal{F})^*$, so that $q_0 = 0$. Hence $\mathcal{H}_p(\mathcal{I}_\bullet)_x = \mathcal{H}^{n-p}(\mathcal{J}^\bullet) = 0$ for $p < n$. Thus, on U, we have an exact sequence

$$0 \to \mathcal{H}_n(\mathcal{I}_\bullet) \to \mathcal{I}_n \to \mathcal{I}_{n-1} \to \cdots \to \mathcal{I}_0 \to 0$$

i.e., an injective resolution of $\mathcal{H}_n(\mathcal{I}_\bullet)$ on U. Hence, for any coherent $\mathcal{F}$ with support in U, we have

$$\operatorname{Ext}^p_{\mathcal{O}_X}(\mathcal{F}, \mathcal{H}_n(\mathcal{I}_\bullet)) = H_p(\operatorname{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{I}_\bullet)) \cong H^{n-p}(X, \mathcal{F})^*$$

(since for such $\mathcal{F}$, $\operatorname{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) = \operatorname{Hom}_{\mathcal{O}_U}(\mathcal{F}\mid_U, \mathcal{G}\mid_U)$ for any $\mathcal{G}$). Taking $\mathcal{F} = k(x)$, the residue field of x, we deduce that (with $\Omega = \mathcal{H}_n(\mathcal{I}_\bullet)_x$) $\operatorname{Ext}^p_{\mathcal{O}_{X,x}}(k(x), \Omega) = 0$ for $p < n$, and $\dim_k \operatorname{Ext}^n_{\mathcal{O}_{X,x}}(k(x), \Omega)$ is a $k(x)$-vector space of dimension $[k(x) : k] = \dim H^0(X, k(x))$ over k. Since Ω is of finite injective dimension, it is the dualising module of $\mathcal{O}_{X,x}$. □

Definition: A complex $0 \to I_N \to I_{N-1} \to \cdots \to \mathcal{I}_0 \to 0$ of quasi-coherent injective $\mathcal{O}_X$-modules is called a *dualising complex* on X if there are natural isomorphisms

$$H^p(X, \mathcal{F})^* \xrightarrow{\cong} H_p(\operatorname{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{I}_\bullet))$$

for any quasi-coherent sheaf $\mathcal{F}$ on X.

Then the formula $(*)$ in the proof of the Theorem shows that $\mathcal{H}_i(\mathcal{I}_\bullet) = 0$ for $i > n = \dim X$, so we may split off an exact direct summand of $\mathcal{I}_\bullet$ so that the remaining summand is concentrated in degrees between 0 and n, and is also a dualising complex.

Definition: If X is Cohen-Macaulay of dimension n everywhere, $\mathcal{I}_\bullet$ a dualising complex on X, then $\mathcal{H}_n(\mathcal{I}_\bullet) = \Omega_X$ is called a *dualising sheaf* on X.

Note that in this case, a dualising complex yields an injective resolution of Ω_X, so that (**) yields a natural isomorphism

$$H^i(X, \mathcal{F})^* \xrightarrow{\cong} \operatorname{Ext}^{n-i}_{\mathcal{O}_X}(\mathcal{F}, \Omega_X)$$

for all $i \geq 0$, for any quasi-coherent sheaf $\mathcal{F}$ on X.

Corollary 10 *Let*

$$0 \to \mathcal{I}_n \to \cdots \to \mathcal{I}_0 \to 0$$

be a dualising complex on X, $Y \subset X$ a closed subscheme with defining ideal sheaf $\mathcal{J}$. Then

$$0 \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{J}, \mathcal{I}_n) \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{J}, \mathcal{I}_{n-1}) \to \cdots \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{J}, \mathcal{I}_0) \to 0$$

is a dualising complex on Y. In particular, if X and Y are equidimensional and Cohen-Macaulay of dimensions n and m respectively, and $h = n - m = \operatorname{codim}_X Y$, then

$$\Omega_Y = \mathcal{E}xt^h_{\mathcal{O}_X}(\mathcal{O}_Y, \Omega_X).$$

Proof: If $\mathcal{F}$ is a quasi-coherent $\mathcal{O}_Y$-module, $i : Y \to X$ the inclusion, then we have natural isomorphisms for each p

$$H^p(Y, \mathcal{F}) \cong H^p(X, i_*\mathcal{F}),$$

$$\operatorname{Hom}_{\mathcal{O}_Y}(\mathcal{F}, \mathcal{H}om_{\mathcal{O}_X}(i_*\mathcal{O}_Y, \mathcal{I}_p)) \cong \operatorname{Hom}_{\mathcal{O}_X}(i_*\mathcal{F}, \mathcal{I}_p).$$

Hence $i^{-1}\mathcal{H}om_{\mathcal{O}_X}(i_*\mathcal{O}_Y, \mathcal{I}_\bullet)$ is a dualising complex for Y, and its m^{th} homology sheaf Ω_Y. But $\mathcal{I}^\bullet = \mathcal{I}_{n-\bullet}$ is an injective resolution for Ω_X, so the m^{th} homology sheaf of $i^{-1}\mathcal{H}om_{\mathcal{O}_X}(i_*\mathcal{O}_Y, \mathcal{I}_\bullet)$ is just $\mathcal{E}xt^{n-m}_{\mathcal{O}_X}(i_*\mathcal{O}_Y, \Omega_X)$. □

Corollary 11 *Suppose X is equidimensional Cohen-Macaulay of dimension n. Then any stalk $\Omega_{X,x} = (\Omega_X)_x$ is Cohen-Macaulay. Further, if U is any open subset of X with* $\dim(X_U) \leq n-2$, *then* $\Omega_X \to i_*(\Omega_X \mid_U) = i_* \circ i^{-1}(\Omega_X)$ *is an isomorphism, where $I : U \hookrightarrow X$ is the inclusion.*

Proof: The first assertion follows from the fact that $\Omega_{X,x}$ is the dualising module of $\mathcal{O}_{X,x}$, and the second because $\Omega_{X,x}$ is Cohen-Macaulay (proof similar to that of $A = \cap_{\mathrm{ht}\,\mathcal{P}=1} A_{\mathcal{P}}$). □

Remark: This corollary is useful for the following reason. We shall show below that if U consists of smooth points of X over k, then $\Omega_X \mid_U$ is the sheaf of Kähler n-forms $\Omega^n_{U/k}$. By the above corollary, if X is equidimensional and Cohen-Macaulay, and is non-singular in the complement of a closed subset of codimension ≥ 2, then Ω_X is the sheaf of (meromorphic) n-forms on X which are regular at all smooth points of X over k.

Corollary 12 *Let $f : X \to Y$ be a birational finite morphism of Cohen-Macaulay varieties (i.e., k-irreducible reduced schemes) which are proper over k, and let Ω_X, Ω_Y be the respective dualising sheaves. Then $f_*(\Omega_X)$ can be identified with the maximal $f_*(\mathcal{O}_X)$-submodule of Ω_Y (this makes sense, since Ω_Y is a torsion free $\mathcal{O}_Y$-module, and $\mathcal{O}_Y \hookrightarrow f_*(\mathcal{O}_X)$ is an isomorphism over an open set).*

Proof: On the category of coherent $\mathcal{O}_X$-modules, we have natural isomorphisms of functors of the coherent sheaf $\mathcal{F}$

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \Omega_X) \cong H^n(X, \mathcal{F})^* \cong H^n(Y, f_*\mathcal{F}) \cong \mathrm{Hom}_{\mathcal{O}_Y}(f_*\mathcal{F}, \Omega_Y). \qquad \cdots \qquad (*)$$

In this take $\mathcal{F} = \Omega_X$; the image of the identity map on Ω_X is an $\mathcal{O}_Y$-linear map $\eta : f_*\Omega_X \to \Omega_Y$. Since $1 \neq 0$ on Ω_X, $\eta \neq 0$. Since Ω_X, Ω_Y are Cohen-Macaulay, they are torsion free, and they are also of rank 1. Hence η is injective, and its image is an $f_*(\mathcal{O}_X)$-submodule of Ω_Y. Let $\mathcal{F}$ be a maximal (= maximum) $\mathcal{O}_X$-submodule of Ω_Y, *i.e.*,

$$\mathcal{F}_y = \{m_y \in \Omega_{Y,y} \mid \ f_*(\mathcal{O}_X)_y m_y \subset \Omega_{Y,y}\}$$

(since $f_*(\mathcal{O}_X)$ is a sheaf of rings, this is an $f_*(\mathcal{O}_X)$-submodule). Then we must have a factorisation

$$\eta = i \circ \lambda, \ \ f_*(\Omega_X) \xrightarrow{\lambda} \mathcal{F} \xrightarrow{i} \Omega_Y.$$

Further, since f is finite, there is a coherent $\mathcal{O}_X$-module $\mathcal{G}$ with $\mathcal{F} = f_*(\mathcal{G})$, and an $\mathcal{O}_X$-linear map $\mu : \Omega_X \to \mathcal{G}$ such that $\lambda = f_*(\mu)$. Also, $\mathcal{G}$ is torsion free of rank one since $f_*(\mathcal{G}) = \mathcal{F}$ is. By $(*)$ above, the inclusion

$i : f_*(\mathcal{G}) = \mathcal{F} \hookrightarrow \Omega_Y$ corresponds to an $\mathcal{O}_X$-linear mapping $j : \mathcal{G} \to \Omega_X$. By the naturality of $(*)$ above, we have a commutative diagram, whose horizontal arrows are isomorphisms $(*)$,

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{G}, \Omega_X) & \xrightarrow{\cong} & \mathrm{Hom}_{\mathcal{O}_Y}(\mathcal{F}, \Omega_Y) \\ (-)\circ\mu \downarrow & & \downarrow (-)\circ\lambda \\ \mathrm{Hom}_{\mathcal{O}_X}(\Omega_X, \Omega_X) & \xrightarrow{\cong} & \mathrm{Hom}_{\mathcal{O}_Y}(f_*(\Omega_X), \Omega_Y) \end{array}$$

Since $\eta = i \circ \lambda$, $j \circ \mu$ is the identity on Ω_X, hence (since Ω_X, $\mathcal{G}$ are torsion free of rank one) j, μ are isomorphisms, and $\lambda = f_*(\mu)$ is one too. □

Remark: Suppose further in Corollary 12, Y is Gorenstein, so that Ω_Y is locally free of rank one. Let $\mathcal{C} = \mathcal{A}nn_{\mathcal{O}_Y}(f_*(\mathcal{O}_X)/\mathcal{O}_Y)$ be the *conductor*. Then we have clearly

$$f_*(\Omega_X) = \mathcal{C} \cdot \Omega_Y.$$

Lemma 13 *Let X', X'' be proper k-schemes with dualising complexes $\mathcal{I}'_\bullet$, $\mathcal{I}''_\bullet$, and let $X = X' \times_k X''$. Let $\mathcal{I}_\bullet$ be a dualising complex on X. Then we have isomorphisms*

$$\mathcal{H}_n(\mathcal{I}_\bullet) \cong \bigoplus_{p+q=n} \mathcal{H}_p(\mathcal{I}'_\bullet) \boxtimes_k \mathcal{H}_q(\mathcal{I}''_\bullet).$$

Proof: Let U', U'' be affine open subsets of X', X'' respectively with $F' = X' - U'$, $F'' = X'' - U''$, and let $\mathcal{J}'$, $\mathcal{J}''$ be defining ideal sheaves for F', F'' respectively. Then $\mathcal{J}' \boxtimes_k \mathcal{J}''$ is a defining ideal of $X - (U' \times_k U'')$. We have therefore isomorphisms compatible with restrictions

$$\begin{aligned} \mathcal{H}_n(\mathcal{I}_\bullet)(U) &= \varinjlim_m H^n(X' \times X'', (\mathcal{J}')^m \boxtimes_k (\mathcal{J}'')^m)^* \\ &= \varinjlim_m \oplus_{p+q=n} H^p(X', (\mathcal{J}')^m)^* \otimes_k H^q(X'', (\mathcal{J}'')^m)^* \\ &= \oplus_{p+q=n} (\varinjlim_m H^p(X', (\mathcal{J}')^m)^*) \otimes_k (\varinjlim_m H^q(X'', (\mathcal{J}'')^m)^*) \\ &= \oplus_{p+q=n} \mathcal{H}_p(\mathcal{I}'_\bullet)(U') \otimes_k \mathcal{H}_q(\mathcal{I}''_\bullet)(U'') \\ &= \oplus_{p+q=n} (\mathcal{H}_p(\mathcal{I}'_\bullet) \boxtimes_k \mathcal{H}_q(\mathcal{I}''_\bullet))(U). \end{aligned}$$

□

Corollary 13 *Let X be a proper k-scheme and U an open subset of X consisting of Gorenstein points of dimension n. Set $\Omega = \mathcal{H}_n(\mathcal{I}_\bullet)$ where $\mathcal{I}_\bullet$ is a dualising complex on X. Let $\Delta : X \hookrightarrow X \times_k X$ be the diagonal embedding. Then $\Delta^* \mathcal{E}xt^n_{\mathcal{O}_{X\times X}}(\Delta_* \mathcal{O}_X, \mathcal{O}_{X\times X}) \mid_U$ is a locally free $\mathcal{O}_X$-module of rank one, and there is an isomorphism of invertible $\mathcal{O}_U$-modules*

$$\Omega \mid_U \cong (\Delta^* \mathcal{E}xt^n_{\mathcal{O}_{X\times X}}(\Delta_* \mathcal{O}_X, \mathcal{O}_{X\times X}) \mid_U)^*.$$

Proof: Let $\mathcal{I}_\bullet$ be a dualising complex on $X \times X$. By lemma 13, $\mathcal{H}_p(\mathcal{I}_\bullet) \mid_{U\times U} = 0$ for $p < 2n = \dim X \times X$, and

$$\mathcal{H}_{2n}(\mathcal{I}_\bullet) \mid_{U\times U} \cong (\Omega \mid_U) \boxtimes_k (\Omega \mid_U).$$

Now apply Corollary 10 to the diagonal embedding of X in $X \times X$, to obtain

$$\Omega \mid_U \cong \Delta^* \mathcal{E}xt^n_{\mathcal{O}_{U\times U}}((\Delta \mid_U)_* \mathcal{O}_U, \Omega \mid_U \boxtimes_k \Omega \mid_U).$$

Since U is Gorenstein, $\Omega \mid_U$ is an invertible sheaf, and we get

$$\begin{aligned} \Omega \mid_U \cong\ & \Delta^*(\mathcal{E}xt^n_{\mathcal{O}_{X\times X}}(\Delta_*(\mathcal{O}_X), \mathcal{O}_{X\times X}) \otimes_{\mathcal{O}_{X\times X}} (\Omega \boxtimes_k \Omega) \mid_{U\times U}) \\ \cong\ & \Delta^*(\mathcal{E}xt^n_{\mathcal{O}_{X\times X}}(\Delta_*(\mathcal{O}_X), \mathcal{O}_{X\times X})) \otimes_{\mathcal{O}_X} \Delta^*((\Omega \boxtimes_k \Omega) \mid_{U\times U}) \\ \cong\ & \Delta^*(\mathcal{E}xt^n_{\mathcal{O}_{X\times X}}(\Delta_*(\mathcal{O}_X), \mathcal{O}_{X\times X})) \otimes_{\mathcal{O}_X} (\Omega \otimes_{\mathcal{O}_X} \Omega) \mid_U \end{aligned}$$

Since $\Omega \mid_U$ is an invertible $\mathcal{O}_U$-module, we may 'cancel' one factor of $\Omega \mid_U$ from both sides, to obtain the desired result. □

Corollary 14 *With assumptions as in Corollary 13, if in addition U is smooth over k, then*

$$\Omega \mid_U \cong \Omega^n_{X/k} \mid_U .$$

Proof: One has to exhibit, for a scheme X which is pure of dimension n and smooth over k, an isomorphism

$$\Delta^*(\mathcal{E}xt^n_{\mathcal{O}_{X\times X}}(\Delta_*(\mathcal{O}_X, \mathcal{O}_{X\times X}))^* \cong \Omega^n_{X/k},$$

where $\Delta : X \to X \times X$ is the diagonal embedding. Now $\Delta(X) \subset X \times X$ is a local complete intersection subvariety. If $\mathcal{I}$ is its sheaf of ideals, $\Delta^*(\mathcal{I}/\mathcal{I}^2) \cong \Omega^1_{X/k}$. Thus, the lemma follows from the next one. □

Lemma 14 *Let A be a commutative ring with 1 and I an ideal in A generated by an A-sequence $f_1, \ldots, f_n$. Then there is a natural isomorphism, compatible with localisations,*

$$\operatorname{Ext}^n_A(A/I, A) \cong \operatorname{Hom}_{A/I}(\overset{n}{\wedge} I/I^2, A/I).$$

Proof: The Koszul complex $K_\bullet(f_1, \ldots, f_n)$ over A gives a free resolution of A/I which we may use to compute $\operatorname{Ext}^n_A(A/I, A)$. Let $f : F = A^{\oplus n} \to A$ be the mapping with $f(e_i) = f_i$, where e_i is the i^{th} basis vector. Let $g : A \to F^*$ be the induced mapping on duals. Then the Koszul complex is

$$0 \to \overset{n}{\wedge} F \overset{\delta_n}{\to} \overset{n-1}{\wedge} F \overset{\delta_{n-1}}{\to} \cdots \overset{\delta_3}{\to} \overset{2}{\wedge} F \overset{\delta_2}{\to} F \overset{\delta_1 = f}{\to} A \to 0$$

where the last differential $\delta_n : \overset{n}{\wedge} F \to \overset{n-1}{\wedge} F$ is

$$v_1 \wedge \cdots \wedge v_n \mapsto \sum_{i=1}^{n} f(v_i)(-1)^i (v_1 \wedge \cdots \wedge \widehat{v_i} \wedge \cdots \wedge v_n)$$

Then $\operatorname{Ext}^n_A(A/I, A)$ is the cokernel of the dual mapping to δ_n, which is defined (in terms of the generators for $\overset{n-1}{\wedge} F^*$ obtained from the dual basis $\{e_i^*\}$ of the basis $\{e_i\}$) by

$$e_1^* \wedge \cdots \wedge \widehat{e_i^*} \wedge \cdots \wedge e_n^* \mapsto (-1)^i f_i \, e_1^* \wedge \cdots \wedge e_n^*.$$

Thus there is an isomorphism $\phi : \operatorname{Ext}^n_A(A/I, A) \overset{\cong}{\to} \operatorname{Hom}_A(\overset{n}{\wedge} F, A/I)$. generated by the image of $e_1^* \wedge \cdots \wedge e_n^*$. However $F/IF \cong I/I^2$ which is a free A/I-module of rank n, and so

$$\operatorname{Ext}^n_{A/I}(A/I, A) \cong \operatorname{Hom}_A(\overset{n}{\wedge} F, A/I) \cong \operatorname{Hom}_{A/I}(\overset{n}{\wedge} F/IF, A/I) \cong \operatorname{Hom}_{A/I}(\overset{n}{\wedge} I/I^2, A/I).$$

This composite isomorphism is natural, and is clearly compatible with localisation. □