THE HODGE CHARACTERISTIC

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1. Introduction

The goal of this lecture is to discuss the proof of the following result, used in Kontsevich's proof of the theorem that the Hodge numbers of two birationally equivalent smooth Calabi-Yau varietes coincide.

Theorem 1.1. *There is a unique way of assigning to each complex algebraic variety X (not necessarily irreducible) a polynomial $f_X(u,v) \in \mathbb{Z}[u,v]$ such that we have the following properties.*

(i) *If X is smooth and proper over $\mathbb{C}$, then*

$$f_X(u,v) = \sum_{p,q \geq 0} h^{p,q}(X) u^p v^q$$

where

$$h^{p,q}(X) = \dim_{\mathbb{C}} H^q(X, \Omega^p_X) = \textit{dimension of the } (p,q)\textit{-th piece of the Hodge decomposition on } H^{p+q}(X,\mathbb{C}).$$

(ii) *If $Z \subset X$ is a closed subvariety, then*

$$f_X(u,v) = f_Z(u,v) + f_{X\setminus Z}(u,v).$$

(iii) $f_{X\times Y}(u,v) = f_X(u,v) f_Y(u,v)$.

The proof relies on Deligne's theory of Mixed Hodge Structures, as developed in his fundamental papers [1], [2]. We'll briefly review aspects of this theory, and sketch a proof that it implies the above theorem.

2. Pure Hodge Structures

Definition 2.1. A *pure Hodge Structure of weight n* consists of data $V = (V_{\mathbb{Z}}, \{V^{p,q}\}_{p+q=n})$, where

(i) $V_{\mathbb{Z}}$ is a finitely generated abelian group
(ii) each $V^{p,q} \subset V_{\mathbb{C}} := V_{\mathbb{Z}} \otimes \mathbb{C}$ is a $\mathbb{C}$-subspace, such that
  (a) $V_{\mathbb{C}} = \oplus_{p+q=n} V^{p,q}$ (this is called the *Hodge decomposition* of $V_{\mathbb{C}}$)
  (b) $\overline{V^{p,q}} = V^{q,p}$ for all p,q, where for any $\mathbb{C}$-subspace $W \subset V_{\mathbb{C}}$, $\overline{W}$ is the image of W under the $\mathbb{C}$-antilinear involution on $V_{\mathbb{C}} = V_{\mathbb{Z}} \otimes \mathbb{C}$ induced by complex conjugation on the $\mathbb{C}$-factor.

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An element $v \in V_{\mathbb{C}}$ can thus be uniquely written as a sum $v = \sum v^{p,q}$ with $v^{p,q} \in V^{p,q}$. If $v = v^{p,q}$ for some (p,q) we say v *is (purely) of type* (p,q).

Clearly pure Hodge structures of a given weight form an abelian category in a natural way, with morphisms being given by homomorphisms of the underlying abelian groups, whose complexifications respect the Hodge decompositions. Similarly we can form the graded abelian category of graded pure Hodge structures, where the n-th graded piece of an object is a pure Hodge structure of weight n.

Example 2.2. Let X be a proper smooth variety over $\mathbb{C}$. Then the Hodge Decomposition Theorem gives a decomposition of each singular cohomology group

$$H^n(X,\mathbb{C}) = \bigoplus_{\substack{\dim X \geq p,q \geq 0 \\ p+q=n}} H^{p,q}(X),$$

such that $H^n(X) = (H^n(X,\mathbb{Z}), \{H^{p,q}(X)\})$ is a pure Hodge structure of weight n. The assignment $X \mapsto H^*(X) = \oplus_n H^n(X)$ is a contravariant functor from smooth proper $\mathbb{C}$-varieties to graded pure Hodge structures.

We will be sloppy below, and often write "=" where we really mean "canonically isomorphic".

Lemma 2.3. *The category of graded pure Hodge structures admits tensor products, duals, and internal Hom's, with "expected" properties.*

Proof. We discuss the operations for pure Hodge structures; everything is then easily extended to graded pure Hodge structures. If V and V' are pure Hodge structures of weights m and n respectively, then $V \otimes V'$ is the pure Hodge structure of weight $m+n$ with underlying abelian group $V_{\mathbb{Z}} \otimes V'_{\mathbb{Z}}$, and Hodge decomposition induced by

$$(V \otimes V')^{a,b} = \bigoplus_{\substack{p+q=m, r+s=n, \\ p+r=a, q+s=b}} V^{p,q} \otimes_{\mathbb{C}} (V')^{r,s} \subset V_{\mathbb{C}} \otimes V'_{\mathbb{C}} = (V_{\mathbb{Z}} \otimes V'_{\mathbb{Z}})_{\mathbb{C}}.$$

Similarly, if V is pure of weight n, its dual V^* is the Hodge structure of weight $-n$ with

$$(V^*)_{\mathbb{Z}} = \mathrm{Hom}_{\mathbb{Z}}(V_{\mathbb{Z}}, \mathbb{Z}),$$

and Hodge decomposition

$$(V^*)^{-p,-q} = (V^{p,q})^* \subset \oplus_{r+s=n} (V^{r,s})^* = (\oplus_{r+s=n} V^{r,s})^*.$$

If V and V' are pure of weights m and n, then

$$\mathrm{Hom}_{\mathbb{Z}}(V_{\mathbb{Z}}, V'_{\mathbb{Z}})_{\mathbb{C}} = (V^* \otimes V')_{\mathbb{C}},$$

and we may take the Hodge decomposition already defined on the right side as the definition of the Hodge decomposition for $\mathrm{Hom}(V, V')$. □

Example 2.4. The functor $X \mapsto H_*(X,\mathbb{Z}) = \oplus_{n\geq 0} H_n(X,\mathbb{Z})$ is a covariant functor from smooth proper varieties to graded pure Hodge structures, where

$H_n(X,\mathbb{Z})$ carries the unique pure Hodge structure of weight $-n$ such that the natural (universal coefficient) homomorphism

$$H^n(X,\mathbb{Z}) \to \mathrm{Hom}(H_n(X,\mathbb{Z}),\mathbb{Z})$$

is a morhism of pure Hodge structures of weight n; here $\mathbb{Z} = H^0(\mathrm{point},\mathbb{Z})$ is the "trivial" Hodge structure, which is the unit with respect to tensor products of Hodge structures (i.e., it is purely of type $(0,0)$).

Definition 2.5. The *Tate* Hodge structure, denoted $\mathbb{Z}(1)$, is the Hodge structure on $H_2(\mathbb{P}^1_{\mathbb{C}},\mathbb{Z})$. Thus, it is a Hodge structure with underlying abelian group $\cong \mathbb{Z}$, whose complexification is defined to be purely of type $(-1,-1)$. We may identify $\mathbb{Z}(1)_{\mathbb{C}}$ with $\mathbb{C}$ so that $\mathbb{Z}(1)_{\mathbb{Z}} \subset \mathbb{C}$ is the cyclic additive subgroup generated by $2\pi\sqrt{-1}$.

We define

$$\mathbb{Z}(n) = \begin{cases} \mathbb{Z}(1)^{\otimes n} & \text{if } n \geq 0 \\ \mathbb{Z}(-n)^* & \text{if } n < 0 \end{cases}$$

Then there are canonical isomorphisms $\mathbb{Z}(m) \otimes \mathbb{Z}(n) \cong \mathbb{Z}(m+n)$.

For any pure Hodge structure V, let $V(n) := V \otimes \mathbb{Z}(n)$ for any $n \in \mathbb{Z}$. Then $V(n)$ is a pure Hodge structure of weight (weight V) $- 2n$. More generally, if $V = \oplus_{m\in\mathbb{Z}} V_m$ is a graded pure Hodge structure, where V_m is pure of weight m, we can define a new graded pure Hodge structure $V(n)[2n]$, whose m-th graded piece is $V_{m+2n}(n)$, which is again pure of weight m.

Example 2.6. With the above definitions and conventions, we see that if X is smooth and proper over $\mathbb{C}$ of pure dimension d, then cap product with its fundamental class may be viewed as an isomorphism of pure Hodge structures of weight $-(2d-n)$

$$H^n(X,\mathbb{Z})(d) \to H_{2d-n}(X,\mathbb{Z}),$$

or equivalently, the Poincaré duality pairing may be viewed as a pairing of Hodge structures

$$H^n(X,\mathbb{Z}) \otimes H^{2d-n}(X,\mathbb{Z}) \to \mathbb{Z}(-d).$$

As a consequence, if X, Y are smooth, pure dimensional, proper $\mathbb{C}$-varieties, and $f : X \to Y$ is a morphism of relative dimension e (i.e., $\dim X - \dim Y = e$, which may be negative), then the Gysin homomorphism defines a morphism of graded pure Hodge structures

$$f_* : H^*(X,\mathbb{Z}) \to H^*(Y,\mathbb{Z})(e)[2e].$$

In particular, if X is smooth and proper over $\mathbb{C}$, $Z \subset X$ a subvariety purely of codimension p, and $\widetilde{Z} \to Z$ a resolution of singularities, giving a morphism $f : \widetilde{Z} \to X$ which is of relative dimension p, then there is an induced homomorphism of pure Hodge structures of weight 0

$$(\text{free abelian group on irreducible components of } Z) \;= H^0(\widetilde{Z},\mathbb{Z}) \to H^{2p}(X,\mathbb{Z}(p)).$$

This defines the *cycle class* of any algebraic cycle of codimension p in X which is supported on Z (i.e., which is a $\mathbb{Z}$-linear combination of irreducible components of Z).

In this language, the *Hodge conjecture* asserts, conversely, that for any element $\alpha \in H^{2p}(X, \mathbb{Z}(p))$ whose complexification is purely of type $(0,0)$ (so that α determines a *morphism of Hodge structures* $\mathbb{Z} \to H^{2p}(X, \mathbb{Z}(p))$), some non-zero integral multiple of α is the cycle class of some codimension p algebraic cycle on X, obtained by the procedure described in the previous paragraph.

3. The Hodge filtration

We'll use the following conventions regarding filtered objects (in an abelian category, say): we regard all filtrations as indexed by $\mathbb{Z}$; a *decreasing filtration* $\{F^pA\}$ is one where $F^{p+1}A \subset F^pA$ for all p, and we write $\mathrm{gr}\,^p_F A$ for $F^pA/F^{p+1}A$. Similarly an *increasing filtration* $\{W_mA\}$ is one where $W_{m-1}A \subset W_mA$ for all m, and we write $\mathrm{gr}\,^W_m A = W_mA/W_{m-1}A$. We may interchange between the two notions: if $\{W_mA\}$ is an increasing filtration, then $F^mA := W_{-m}A$ defines an increasing filtration such that $\mathrm{gr}\,^m_F A = \mathrm{gr}\,^W_{-m} A$.

A decreasing filtration $\{F^pA\}$ is called *finite* if $F^pA = 0$ and $F^{-p}A = A$ for all $p >> 0$; we can similarly definte finiteness for increasing filtrations.

Notice that if $\mathcal{A}$ is an abelian category, then we can form a new category $Fil^\bullet\mathcal{A}$ of pairs $(A, \{F^pA\})$, with $A \in \mathcal{A}$, and a finite decreasing filtration $\{F^pA\}$ on A, where morphisms $f : (A, \{F^pA\}) \to (B, \{F^pB\})$ are morphisms $f : A \to B$ in A which are compatible with the filtrations, i.e., satisfy $f(F^pA) \subset F^pB$. Then $Fil^\bullet\mathcal{A}$ is an additive category in a natural way, where every arrow has a kernel and cokernel, but $Fil^\bullet\mathcal{A}$ is *not* in general an abelian category: if $f : (A, \{F^pA\}) \to (B, \{F^pB\})$ is any morphism in $Fil^\bullet\mathcal{A}$, then $\mathrm{image}\,(f) \subset B$ inherits F-filtrations in two ways: as a quotient of A, and as a subobject of B. These need not coincide!

We may similarly consider the category $Fil_\bullet\mathcal{A}$ of objects in $\mathcal{A}$ with finite increasing filtrations; again it is naturally an additive category with kernels and cokernels, which need not be abelian.

Definition 3.1. A morphism $f : (A, \{F^pA\}) \to (B, \{F^pB\})$ in $Fil^\bullet\mathcal{A}$ is *strict* with respect to the F-filtrations if the two natural F-filtrations on $\mathrm{image}\,(f)$ coincide, i.e., $f(F^pA) = f(A) \cap F^pB$ for each $p \in \mathbb{Z}$.

Lemma 3.2. (a) *Let $V = (V_\mathbb{Z}, \{V^{p,q}\})$ be a pure Hodge structure of weight n. Define $F^pV_\mathbb{C} = \oplus_{p' \geq p} V^{p',n-p'} \subset V_\mathbb{C}$. Then $\{F^pV_\mathbb{C}\}$ is a finite, decreasing filtration by $\mathbb{C}$-subspaces, such that for each p,*

(i) *the natural map $F^pV_\mathbb{C} \oplus \overline{F^{n-p+1}V_\mathbb{C}} \to V_\mathbb{C}$, induced by inclusions on the two factors, is an isomorphism*

(ii) $F^pV_\mathbb{C} \cap \overline{F^{n-p}V_\mathbb{C}} = V^{p,n-p} \subset V_\mathbb{C}$.

(b) *Conversely, suppose $V_\mathbb{Z}$ is a finitely generated abelian group, and $\{F^pV_\mathbb{C}\}$ is a finite, decreasing filtration on $V_\mathbb{C} := V_\mathbb{Z} \otimes \mathbb{C}$ by $\mathbb{C}$-subspaces, such that we have the isomorphisms in (i) above for each p. Then, setting*

$$V^{p,n-p} = F^pV_\mathbb{C} \cap \overline{F^{n-p}V_\mathbb{C}},$$

the data

$$V = (V_\mathbb{Z}, \{V^{p,q}\}_{p+q=n})$$

define a pure Hodge structure of weight n.

(c) *The category of pure Hodge structures of weight n is equivalent to the category of finitely generated abelian groups whose complexifications have finite decreasing filtrations, such that the isomorphisms in (a) (i) above hold, with morphisms being homomorphisms f of abelian groups whose complexifications $f_{\mathbb{C}}$ preserve the F-filtrations (i.e., satisfy $f_{\mathbb{C}}(F^p) \subset F^p$).*

The filtration $\{F^pV_{\mathbb{C}}\}$ is called the *Hodge filtration* of the pure Hodge structure V. There is similarly a Hodge filtration on any graded pure Hodge structure. Morphisms of graded pure Hodge structures are homomorphisms f of graded abelian groups which preserve the Hodge filtrations, i.e., satisfy $f(F^p) \subset F^p$. From lemma 3.2 and the respective Hodge decompositions, it follows that

$$f(F^p) = (\mathrm{image}\, f) \cap F^p,$$

that is, "*any morphism of (graded) pure Hodge structures is strict with respect to the Hodge filtrations*".

We may similarly define the category of pure A-Hodge structures of weight n, where A is a Noetherian subring of the field $\mathbb{R}$ of reals, where instead of a finitely generated abelian group $V_{\mathbb{Z}}$, we're given a finitely generated A-module V_A, as well as a suitable filtration on $V_{\mathbb{C}} = V_A \otimes_A \mathbb{C}$ by $\mathbb{C}$-subspaces; the isomorphism property in lemma 3.2 (a)(i) above makes sense, since $A \subset \mathbb{R}$, so that $V_{\mathbb{C}}$ does admit a natural complex conjugation involution. Of particular use to us later is the notion of a *pure $\mathbb{Q}$-Hodge structure of weight n*.

4. Mixed Hodge Structures

We begin with some further remarks on filtered objects in an abelian category. If $(A, \{F^pA\}) \in Fil^{\bullet}\mathcal{A}$, and $C \subset B \subset A$ are subobjects in $\mathcal{A}$, there is a unique induced F-filtration on the subquotient B/C. Indeed, at first sight there are two ways of defining $F^p(B/C)$, namely as "a subobject of a quotient object" or as a "quotient object of a subobject", that is,

$$(\mathrm{image}\, F^pA \to \frac{A}{C}) \cap \frac{B}{C} = \frac{(F^pA + C) \cap B}{C},$$

or as

$$\frac{F^pA \cap B}{F^pA \cap C} \cong \frac{(F^pA \cap B) + C}{C} \subset \frac{B}{C}.$$

However,

$$(F^pA + C) \cap B = (F^pA \cap B) + C$$

since $C \subset B$. Thus, both recipes give the same F-filtrations on B/C.

As a consequence, if $\{F^pA\}$ and $\{G^qA\}$ are two decreasing filtrations on A, then

(i) there is a well-defined F-filtration on $\mathrm{gr}_G^q A$ for each $q \in \mathbb{Z}$, so we can form the subquotient $\mathrm{gr}_F^p \mathrm{gr}_G^q A$ for each $p, q \in \mathbb{Z}$
(ii) there is a well-defined G-filtration on $\mathrm{gr}_F^p A$ for each $p \in \mathbb{Z}$, so we can form the subquotient $\mathrm{gr}_G^q \mathrm{gr}_F^p A$

(iii) there is in fact a *canonical isomorphism* between these subquotients:

$$\mathrm{gr}_F^p \mathrm{gr}_G^q A \cong \mathrm{gr}_G^q \mathrm{gr}_F^p A,$$

since $\mathrm{gr}_F^p \mathrm{gr}_G^q A$ may be identified with

$$\frac{\frac{F^pA\cap G^qA}{F^pA\cap G^{q+1}A}}{\frac{F^{p+1}A\cap G^qA}{F^{p+1}A\cap G^{q+1}A}} \cong \frac{F^pA\cap G^qA}{(F^pA\cap G^{q+1}A)+(F^{p+1}A\cap G^qA)}$$

where the roles of the F and G filtrations in the new expression may clearly be interchanged.

Since we may re-index an increasing filtration to convert it into a decreasing one, similar remarks apply when an object has two filtrations, one or both of which may be increasing.

We are now ready to give the main definition of this lecture, that of a mixed Hodge structure.

Definition 4.1. A *mixed Hodge structure* $V = (V_{\mathbb{Z}}, \{W_nV_{\mathbb{Q}}\}, \{F^pV_{\mathbb{C}}\})$ consists of data

(i) a finitely generated abelian group $V_{\mathbb{Z}}$
(ii) a finite, *increasing* filtration $\{W_nV_{\mathbb{Q}}\}$ on $V_{\mathbb{Q}} = V_{\mathbb{Z}}\otimes\mathbb{Q}$ by $\mathbb{Q}$-subspaces, called the *weight filtration*,
(iii) a finite *decreasing* filtration $\{F^pV_{\mathbb{C}}\}$ on $V_{\mathbb{C}}$ by $\mathbb{C}$-subspaces, called the *Hodge filtration*,

which satisfy the condition that

$$(\mathrm{gr}_{\bullet}^W V_{\mathbb{Q}}, \{F^p\mathrm{gr}_{\bullet}^W V_{\mathbb{Q}}\})$$

is a graded pure $\mathbb{Q}$-Hodge structure, whose graded piece of degree n (which is a pure $\mathbb{Q}$-Hodge structure of weight n) is $(\mathrm{gr}_n^W V_{\mathbb{Q}}, \{F^p\mathrm{gr}_n^{W\otimes\mathbb{C}} V_{\mathbb{C}}\})$.

A *morphism of mixed Hodge structures* is defined to be a homomorphism of abelian groups which, when tensored with $\mathbb{Q}$, is compatible with the weight filtrations $W_\bullet$, and which when tensored with $\mathbb{C}$, is compatible with the Hodge filtrations $F^\bullet$.

We will abuse notation, and write $W_\bullet V_{\mathbb{C}} = \{W_nV_{\mathbb{C}}\}$ for the complexification of the weight filtration as well. Thus

$$\mathrm{gr}_F^p\mathrm{gr}_n^W V_{\mathbb{C}}$$

is the $(p, n-p)$-th piece of the Hodge decomposition of the pure $\mathbb{Q}$-Hodge structure on $\mathrm{gr}_n^W V_{\mathbb{Q}}$.

The category MHS has natural tensor products, duals and internal Hom's, obtained from the corresponding operations on the underlying abelian groups, and invoking the similar properties for pure Hodge structures. For example, one finds that

$$F^p\mathrm{Hom}(V_{\mathbb{C}}, V'_{\mathbb{C}}) = \{f\in\mathrm{Hom}(V_{\mathbb{C}}, V'_{\mathbb{C}}) \mid f(F^iV_{\mathbb{C}})\subset F^{i+p}V'_{\mathbb{C}}\ \ \forall i\in\mathbb{Z}\}.$$

For any subring $A\subset\mathbb{R}$, we may also define the notion of an A-mixed Hodge structure, abbreviated A-MHS, in the obvious way, where $W_\bullet$ is now a filtration by $A\otimes\mathbb{Q}$-submodules of $V_{A\otimes\mathbb{Q}}$; of interest to us is the case $A=\mathbb{Q}$.

A remarkable property of mixed Hodge structures is the following, which is crucial in the proof of Theorem 1.1, and in most applications of mixed Hodge structures.

Proposition 4.2. *Let* $V = (V_{\mathbb{Z}}, W_\bullet V_{\mathbb{Q}}, F^\bullet V_{\mathbb{C}})$ *and* $V' = (V'_{\mathbb{Z}}, W_\bullet V'_{\mathbb{Q}}, F^\bullet V'_{\mathbb{C}})$ *be mixed Hodge structures,* $f : V \to V'$ *a morphism of mixed Hodge structures. Then*

(i) $f_{\mathbb{Q}} : V_{\mathbb{Q}} \to V'_{\mathbb{Q}}$ *is strict with respect to the weight filtrations*
(ii) $f_{\mathbb{C}} : V_{\mathbb{C}} \to V'_{\mathbb{C}}$ *is strict with respect to the Hodge filtrations.*

Proof. See [1], Théorème (2.3.5). □

Corollary 4.3. *The category MHS of mixed Hodge structures is an abelian category, on which the functors* W_n, $\mathrm{gr}\,^W_n$, F^p, $\mathrm{gr}\,^p F$, $\mathrm{gr}\,^p_F \mathrm{gr}\,^W_n = \mathrm{gr}\,^W_n \mathrm{gr}\,^p_F$ *are exact functors (the first two to* $\mathbb{Q}$*-vector spaces, and the last three to* $\mathbb{C}$*-vector spaces).*

Corollary 4.4. *A morphism between MHS is an isomorphism* $\iff$ *the underlying homomorphism of abelian groups is an isomorphism. Thus, the forgetful functor from MHS to abelian groups is a faithful functor reflecting isomorphisms.*

We may view a pure Hodge structure of weight n as an object of MHS for which the weight filtration has at most 1 non-trivial jump: $W_{n-1} = 0$, $W_n =$ (everything). Similarly, if $\oplus_{n\in\mathbb{Z}} V_n$ is a graded pure Hodge structure (with a finite number of non-zero weighted pieces), it may be viewed as a MHS with $W_n = \oplus_{m\leq n}(V_m)_{\mathbb{Q}} \subset V_{\mathbb{Q}}$.

The following example illustrates why there are "naturally occuring" interesting mixed Hodge structures in algebraic geometry, which are not just graded pure Hodge structures.

Example 4.5. Let X be a smooth projective curve over $\mathbb{C}$, and x, y distinct points of X. Consider the long exact sequence of the pair $(X, \{x, y\})$ in singular cohomology, part of which is

$$H^0(X, \mathbb{Z}) \xrightarrow{\rho} H^0(\{x, y\}, \mathbb{Z}) \to H^1(X, \{x, y\}, \mathbb{Z}) \to H^1(X, \mathbb{Z}) \to 0.$$

This gives rise to an extension

$$0 \to \mathbb{Z} \to H^1(X, \{x, y\}, \mathbb{Z}) \to H^1(X, \mathbb{Z}) \to 0.$$

Here, we may regard

$$\mathrm{coker}\,(H^0(X, \mathbb{Z}) \to H^0(\{x, y\}, \mathbb{Z})) \cong \mathbb{Z}$$

as the trivial Hodge structure $\mathbb{Z} = \mathbb{Z}(0)$, and give $H^1(X, \mathbb{Z})$ its standard pure Hodge structure of weight 1. Suppose we were to define a MHS on the relative cohomology $H^1(X, \{x, y\}, \mathbb{Z})$, such that the above extension is in the category MHS; then we would obtain an extension class in $\mathrm{Ext}\,^1_{MHS}(H^1(X, \mathbb{Z}), \mathbb{Z})$, which has a natural abelian group structure.

It is an exercise in "linear algebra" (left to the reader!) to show that if $V = (V_{\mathbb{Z}}, F^\bullet V_{\mathbb{C}})$ is a pure Hodge structure of weight 1, where $V_{\mathbb{Z}}$ is torsion-free, then there is a natural isomorphism of abelian groups

$$\mathrm{Ext}\,^1_{MHS}(V, \mathbb{Z}) \cong \frac{V_{\mathbb{C}}}{F^1 V_{\mathbb{C}} + V_{\mathbb{Z}}} \cong \frac{\mathrm{gr}\,^0_F V_{\mathbb{C}}}{\mathrm{image}\, V_{\mathbb{Z}}} \cong \frac{V^{0,1}}{\mathrm{image}\, V_{\mathbb{Z}}}.$$

In particular, we have that

$$\operatorname{Ext}^1_{MHS}(H^1(X,\mathbb{Z}),\mathbb{Z}) \cong \frac{H^{0,1}(X)}{\operatorname{image} H^1(X,\mathbb{Z})} = \frac{H^1(X,\mathcal{O}_X)}{\operatorname{image} H^1(X,\mathbb{Z})}.$$

Here, we observe that the Hodge decomposition on $H^1(X,\mathbb{Z})$ is such that the map $H^1(X,\mathbb{C}) \twoheadrightarrow H^{0,1}(X) = H^1(X,\mathcal{O}_X)$ is just the natural map on cohomology induced by the sheaf map $\mathbb{C}_X \to \mathcal{O}_X$ from the constant sheaf $\mathbb{C}_X$ on X determined by $\mathbb{C}$, to the (analytic) structure sheaf of holomorphic functions on X. Thus the induced map $H^1(X,\mathbb{Z}) \to H^1(X,\mathcal{O}_X)$ is similarly induced by the sheaf map $\mathbb{Z}_X \to \mathcal{O}_X$. From the cohomology sequence associated to the *exponential sheaf sequence*

$$0 \to \mathbb{Z}_X \to \mathcal{O}_X \overset{f\mapsto e^{2\pi\sqrt{-1}f}}{\longrightarrow} \mathcal{O}_X^* \to 0$$

we thus identify the quotient

$$\frac{H^1(X,\mathcal{O}_X)}{\operatorname{image} H^1(X,\mathbb{Z})}$$

with

$$\ker H^1(X,\mathcal{O}_X^*) \to H^2(X,\mathbb{Z}),$$

i.e., with the group of isomorphism class of line bundles on X of degree 0, which is the *Jacobian variety* of the curve X. Thus, associated to an ordered pair[1] (x,y) of distinct points of X, we would obtain an isomorphism class of a line bundle of degree 0 on X. Clearly, the "natural" choice of MHS on the relative cohomology $H^1(X,\{x,y\},\mathbb{Z})$ is the one where this line bundle is just $\mathcal{O}_X([x]-[y])$! With this choice, the *Abel-Jacobi mapping* on the curve X acquires a description in terms of extension classes in MHS. In particular, the MHS on $H^1(X,\{x,y\},\mathbb{Z})$ is *not* a graded pure Hodge structure, if X has genus > 0.

A result giving the existence of many such mixed Hodge structures is the following theorem, which summarizes several results in [1], [2]. This will suffice to obtain Theorem 1.1.

Theorem 4.6. (Main Theorem on existence of MHS)

There is a unique way to assign a MHS to the singular cohomology groups $H^n(X,Y,\mathbb{Z})$, where X is an arbitrary $\mathbb{C}$-variety (possibly reducible), $Y \subset X$ a closed subvariety (possibly empty), and $n \geq 0$ an integer, such that the following properties hold.

(i) *If X is smooth and proper over $\mathbb{C}$, the MHS on $H^n(X,\emptyset,\mathbb{Z}) = H^n(X,\mathbb{Z})$ is the standard pure Hodge structure of weight n.*

(ii) *If $f : (X,Y) \to (X',Y')$ is a morphism of pairs (i.e., $f : X \to X'$ is a morphism of varieties such that $f(Y) \subset Y'$), then the natural map $f^* : H^n(X',Y',\mathbb{Z}) \to H^n(X,Y,\mathbb{Z})$ is a morphism of MHS.*

(iii) *If (X,Y) and (X',Y') are pairs, then the external product map*

$$H^n(X,Y,\mathbb{Z}) \otimes H^m(X',Y',\mathbb{Z}) \to H^{m+n}(X\times X',(Y\times X' \cup X\times Y'),\mathbb{Z})$$

is a morphism of MHS.

[1] When we identify $\operatorname{coker}\rho$ with $\mathbb{Z}$, we are ordering the points.

(iv) *If (X,Y) is a pair, then the boundary maps in the long exact sequence for singular cohomology of the pair*

$$\partial : H^n(Y,\mathbb{Z}) \to H^{n+1}(X,Y,\mathbb{Z})$$

are morphisms of MHS.

(v) *For any pair (X,Y) with $\dim X = d$, we have*

$$W_{-1}H^n(X,Y,\mathbb{Q}) = 0, \quad W^{2d}H^n(X,Y,\mathbb{Q}) = H^n(X,Y,\mathbb{Q}),$$
$$F^0H^n(X,Y,\mathbb{C}) = H^n(X,Y,\mathbb{C}), \quad F^{d+1}H^n(X,Y,\mathbb{C}) = 0.$$

Remark 4.7. The statement in (v) above can be sharpened, see [2](8.2.4) and (8.3.10) (particularly the top of page 45).

Corollary 4.8. *If $Z \subset Y \subset X$ are closed subvarieties, then the long exact sequence in singular cohomology for the triple (X,Y,Z) is an exact sequence in MHS, for the MHS's on relative cohomology given in Theorem 4.6.*

Proof. The long exact sequence in question has the form

$$\cdots \to H^n(X,Y,\mathbb{Z}) \to H^n(X,Z,\mathbb{Z}) \to H^n(Y,Z,\mathbb{Z}) \xrightarrow{\partial} H^{n+1}(X,Y,\mathbb{Z}) \to \cdots$$

where the functoriality assertion (ii) in Theorem 4.6 implies that the first two homomorphisms are morphisms of MHS, induced by morphisms of pairs $(X,Z) \subset (X,Y)$ and $(Y,Z) \subset (X,Z)$. The boundary map ∂ in the exact sequence, in the topological context, is defined to be the composition

$$H^n(Y,Z,\mathbb{Z}) \to H^n(Y,\mathbb{Z}) \xrightarrow{\partial} H^{n+1}(X,Y).$$

This factorization also shows it is a morphism of MHS. □

If X is an arbitrary variety, let $\overline{X}$ be a proper variety over $\mathbb{C}$ which contains X as a Zariski open subset, and define $\partial\overline{X} = \overline{X} \setminus X$. We'll call $(\overline{X}, \partial\overline{X})$ a *compactification* of X, with boundary $\partial\overline{X}$. The compactifications of X form a category, where morphisms are morphisms of the proper varieties which "are the identity on the open subset X".

We claim that the relative cohomology group $H^n(\overline{X}, \partial\overline{X}, A)$, for any coefficient abelian group A, depends only on X (upto canonical isomorphism). Indeed, if $(\widetilde{X}, \partial\widetilde{X})$ is another compactification, the closure of the image of the "diagonal" inclusion of X yields another compactification which maps to both $\overline{X}$ and $\widetilde{X}$, in the sense of morphisms of compactifications. So it suffices to remark that any morphism of compactifications yields a relative homeomorphism of pairs of topological spaces (which are "reasonable", i.e., are homeomorphic to finite CW pairs of complexes, for example), and hence induces an isomorphism on relative singular cohomology, by the excision theorem in topology.

We define the *compactly supported cohomology groups $H^n_c(X,A)$* to be the relative cohomology groups $H^n(\overline{X}, \partial\overline{X}, A)$ for any compactification $(\overline{X}, \partial\overline{X})$ and any abelian group A. The argument above shows that this is well-defined, yielding a finitely generated abelian group when $A = \mathbb{Z}$, which vanishes unless $0 \leq n \leq 2\dim X$; further, combining Theorem 4.6 with Corollary 4.4, it follows that $V = H^n_c(X, \mathbb{Z})$ also carries a natural mixed Hodge structure, such that the indices (p, q) with $V^{p,q} \neq 0$ are a finite set of pairs of *non-negative* numbers.

Lemma 4.9. *For any two varieties X, Y there is a Kunneth isomorphism*

$$\oplus_{r+s=n} H^r_c(X, \mathbb{Q}) \otimes H^s_c(Y, \mathbb{Q}) \to H^n_c(X \times Y, \mathbb{Q})$$

which is an isomorphism of $\mathbb{Q}$-MHS.

Proof. If X, Y have compactifications $(\overline{X}, \partial\overline{X})$ and $(\overline{Y}, \partial\overline{Y})$ respectively, then the product pair

$$(\overline{X \times Y}, \partial(\overline{X \times Y})) := (\overline{X} \times \overline{Y}, \partial\overline{X} \times \overline{Y} \cup \overline{X} \times \partial\overline{Y})$$

is a compactification of $X \times Y$. Theorem 4.6(iii) implies that the direct sum of the external product maps

$$\oplus_{r+s=n} H^r(\overline{X}, \partial\overline{X}, \mathbb{Z}) \otimes H^s(\overline{Y}, \partial\overline{Y}, \mathbb{Z}) \to H^n(\overline{X \times Y}, \partial(\overline{X \times Y}), \mathbb{Z})$$

is a morphism of MHS, which becomes an isomorphism of rational vector spaces when tensored with $\mathbb{Q}$, from the topological Kunneth theorem; from Proposition 4.2 it is then an isomorphism of $\mathbb{Q}$-MHS. $\square$

Lemma 4.10. *Let X be a variety, $Y \subset X$ a closed subvariety, and $U = X \setminus Y$ the complementary open set. Then there is a long exact sequence in MHS*

$$0 \to H^0_c(U, \mathbb{Z}) \to H^0_c(X, \mathbb{Z}) \to H^0_c(Y, \mathbb{Z}) \to \cdots$$
$$\to H^n_c(U, \mathbb{Z}) \to H^n_c(X, \mathbb{Z}) \to H^n_c(Y, \mathbb{Z}) \to H^{n+1}_c(U, \mathbb{Z}) \to \cdots$$

Proof. Let $(\overline{X}, \partial\overline{X})$ be a compactification of X, and let $\overline{Y}$ be the (Zariski, or equivalently Euclidean) closure of Y in $\overline{X}$. Then $(\overline{Y} \cup \partial X, \partial\overline{X})$ is a compactification of Y, and $(\overline{X}, \partial\overline{X} \cup \overline{Y})$ is a compactification of U. Hence we have a well-defined triple $(\overline{X}, \partial\overline{X} \cup \overline{Y}, \partial\overline{X})$ consisting of a variety and a chain of closed subvarietes. The associated long exact sequence, which by Corollary 4.8 is a sequence of MHS, is the sequence whose existence is asserted in the lemma. $\square$

Remark 4.11. This lemma only used the existence and properties of MHS for pairs (X, Y) where X is proper over $\mathbb{C}$.

Let $K_{\mathbb{Q}H}$ denote the Grothendieck ring of mixed $\mathbb{Q}$-Hodge structures. Since any $\mathbb{Q}$-MHS has a canonical finite filtration (the weight filtration itself!) by sub-MHS whose associated graded object is a graded pure $\mathbb{Q}$-MHS, such that passage to any gven filtration level is an exact functor, it follows easily that $K_{\mathbb{Q}H}$ coincides with the Grothendieck ring of graded pure $\mathbb{Q}$-Hodge structures.

Proposition 4.12. *The assignment $X \mapsto F_X \in K_{\mathbb{Q}H}$ given by*

$$F_X = \sum_{n \geq 0} (-1)^n [H^n_c(X, \mathbb{Q})] = \sum_{n=0}^{2\dim X} (-1)^n \sum_{i=0}^{2\dim X} [\mathrm{gr}^W_i H^n_c(X, \mathbb{Q})]$$

satisfies the following properties.

(i)
$$F_X = \sum_{n\geq 0}(-1)^n[H^n(X,\mathbb{Q})]$$
if X is proper (not necessarily smooth).

(ii)
$$F_{X\times Y} = F_X \cdot F_Y$$
where on the right, $\cdot$ denotes the multiplication on $K_{\mathbb{Q}H}$ induced by tensor products of MHS.

(iii) *If X is any $\mathbb{C}$-variety, Y a closed subvariety, $U = X \setminus Y$, then $F_X = F_U + F_Y$ in $K_{\mathbb{Q}H}$.*

Proof. From Theorem 4.6(i), if X is proper, then $(X, \emptyset)$ is a compactification of X, and $H^n_c(X,\mathbb{Z}) = H^n(X,\mathbb{Z})$ as MHS. Next, if X is a variety, $Y \subset X$ a closed subvariety, and $U = X \setminus Y$, then the exact sequence in lemma 4.10 immediately yields the relation $F_X = F_Y + F_U$ in $K_{\mathbb{Q}H}$. Finally the compatibility with products follows from lemma 4.9. □

Remark 4.13. Again we only need the existence and properties of MHS for pairs (X, Y) where X is proper over $\mathbb{C}$.

Now notice that there is a natural ring homomorphism
$$\chi : K_{\mathbb{Q}H} \to \mathbb{Z}[u, u^{-1}, v, v^{-1}]$$
obtained by mapping a pure $\mathbb{Q}$-Hodge structure V of weight n to
$$\chi([V]) = (-1)^n \sum_{p+q=n} (\dim_{\mathbb{C}} V^{p,q}) u^p v^q.$$
If V is a $\mathbb{Q}$-MHS, then we clearly must have
$$\chi([V]) = \sum_n \chi([\mathrm{gr}^W_n V]) = \sum_{p,q} (-1)^{p+q} (\dim_{\mathbb{C}} \mathrm{gr}^p_F \mathrm{gr}^{p+q}_W V_{\mathbb{C}}) u^p v^q.$$
This last formula is compatible with exact sequences of MHS, and so χ is well-defined as an additive homomorphism. Since $\chi(V_1 \otimes V_2) = \chi(V_1)\chi(V_2)$ for pure $\mathbb{Q}$-Hodge structures V_i, we see that χ is in fact a ring homomorphism.

Corollary 4.14. *The assignment*
$$X \mapsto f_X(u,v) = \sum_{p,q\geq 0} (-1)^{p+q} \sum_{n\geq 0} (-1)^n \dim_{\mathbb{C}} \mathrm{gr}^p_F \mathrm{gr}^W_{p+q} H^n_c(X,\mathbb{C}) u^p v^q$$
satisfies the conditions of Theorem 1.1.

Proof. We first define $f_X = \chi(F_X) \in \mathbb{Z}[u, u^{-1}, v, v^{-1}]$. Then f_X satisfies properties (ii) and (iii) in Theorem 1.1, from Proposition 4.12 above. If X is smooth and proper over $\mathbb{C}$, then $H^n_c(X,\mathbb{Z}) = H^n(X,\mathbb{Z})$ carries the standard pure Hodge structure of weight n, and so $\mathrm{gr}^{p+q}_W H^n(X,\mathbb{C}) = 0$ unless $n = p + q$, and further $\mathrm{gr}^p_F \mathrm{gr}^W_{p+q} H^{p+q}(X,\mathbb{C})$ is naturally identifed with the Hodge piece $H^{p,q}(X)$ in this case. Hence
$$f_X(u,v) = \sum_{p,q\geq 0} \dim_{\mathbb{C}} H^{p,q}(X) u^p v^q$$

as desired. Finally, since

$$[H^n_c(X,\mathbb{Q})] = \sum_{i\geq 0}[\mathrm{gr}^W_i H^n_c(X,\mathbb{Q})]$$

in $K_{\mathbb{Q}H}$, we obtain the explicit formula for f_X stated in the corollary. □

5. Some ideas in the proof of Theorem 4.6

The construction of the MHS on the relative cohomology groups $H^n(X,Y,\mathbb{Z})$ of a pair (X,Y) use the notions of *hypercohomology*, and *simplicial schemes*. We will try to brifely review these notions as needed below, to give an idea of the desired constructions of MHS.

5.1. **Hypercohomology.** If $\mathcal{C}^\bullet$ is a (cochain) complex of sheaves of abelian groups on a topological space X, bounded below (i.e., with $\mathcal{C}^n = 0$ for $n << 0$), then we can find a complex of injective sheaves $\mathcal{I}^\bullet$, also bounded below, and a map of complexes $\mathcal{C}^\bullet \to \mathcal{I}^\bullet$, which induces an isomorphism on cohomology sheaves (i.e., is a *quasi-isomorphism* of complexes of sheaves). If $\mathcal{C}^\bullet$ consists of a single non-zero sheaf $\mathcal{C} = \mathcal{C}^0$ occuring in degree 0 (say), this complex $\mathcal{I}^\bullet$ may be taken to be an injective resolution of $\mathcal{C}$. The general assertion can be proved using this case, by an induction argument, and using the universal property of injectives (proof left to the reader). Further, given any two such injective complexes $\mathcal{I}^\bullet$, there is a unique chain homotopy class of maps between them (relative to the maps from $\mathcal{C}^\bullet$), which is itself a chain homotopy equivalence. Thus, the cohomology group $H^n(\Gamma(X,\mathcal{I}^\bullet))$ depends only on $\mathcal{C}^\bullet$, and is defined to be the *n-th hypercohomology group* of $\mathcal{C}^\bullet$ on X, denoted $\mathbb{H}^n(X,\mathcal{C}^\bullet)$; further, if $\mathcal{C}^\bullet \to \mathcal{D}^\bullet$ is a quasi-isomorphism between (bounded below) complexes, it induces an isomorphism on hypercohomology (if $\mathcal{D}^\bullet \to \mathcal{I}^\bullet$ is a quasi-isomorphism to a (bounded below) complex of injectives, then the composition $\mathcal{C}^\bullet \to \mathcal{D}^\bullet \to \mathcal{I}^\bullet$ is also a quasi-isomorphism).

One next shows: if $\mathcal{C}^\bullet \to \mathcal{D}^\bullet$ is any morphism of complexes, one can "lift" it to a compatible morphism between quasi-isomorphic complexes of injectives, and the "lifted" map is unique upto chain homotopy. This makes the hypercohomology groups functors on the category of complexes. Further, if

$$0 \to \mathcal{C}^\bullet_1 \to \mathcal{C}^\bullet_2 \to \mathcal{C}^\bullet_3 \to 0$$

is an exact sequence of complexes, we can find compatible quasi-isomorphic injective complexes $\mathcal{C}^\bullet_j \to \mathcal{I}^\bullet_j$ fitting into an exact sequence of complexes

$$0 \to \mathcal{I}^\bullet_1 \to \mathcal{I}^\bullet_2 \to \mathcal{I}^\bullet_3 \to 0$$

where the sequences of terms

$$0 \to \mathcal{I}^n_1 \to \mathcal{I}^n_2 \to \mathcal{I}^n_3 \to 0$$

are in fact split exact (as with the familiar special case where one lifts a short exact sequence of sheaves to a short exact sequence, termwise split, of injective resolutions, one first constructs $\mathcal{C}_j \to \mathcal{I}^\bullet_j$ for $j = 1, 3$, then one defines the terms of the middle resolution $\mathcal{I}_2^n$ by $\mathcal{I}_2^n = \mathcal{I}_1^n \oplus \mathcal{I}_3^n$, then one inductively constructs the differentials and maps....). This yields: if

$$0 \to \mathcal{C}_1^\bullet \to \mathcal{C}_2^\bullet \to \mathcal{C}_3^\bullet \to 0$$

is an exact sequence of (bounded below) complexes, there is an associated long exact sequence

$$\cdots \mathbb{H}^n(X, \mathcal{C}_1^\bullet) \to \mathbb{H}^n(X, \mathcal{C}_2^\bullet) \to \mathbb{H}^n(X, \mathcal{C}_3^\bullet) \to \mathbb{H}^{n+1}(X, \mathcal{C}_1^\bullet) \to \cdots$$

"Concretely" one may compute hypercohomology of a complex $\mathcal{C}^\bullet$ using a quasi-isomorphic complex of $\Gamma(X, -)$-acyclic sheaves, just as for sheaf cohomology, one may use acyclic resolutions instead of injective resolutions. One functorial way to do it is to use the "canonical" Godement flasque (=flabby) resolution, which we recall. If $\mathcal{C}$ is any sheaf of abelian groups on a topological space X, and $i : X_{disc} \to X$ is the identity map on the set X, where X_{disc} has the discrete topology, then define $\mathcal{G}^0(\mathcal{C}) = i_* i^{-1}\mathcal{C}$. One sees that $\Gamma(U, \mathcal{G}^0(\mathcal{C})) = \prod_{x \in U} \mathcal{C}_x$ where $\mathcal{C}_x$ is the stalk at x, and restriction to a smaller open set is given by projection; thus $\mathcal{G}^0(\mathcal{C})$ is flasque. There is a canonical inclusion $\mathcal{C} \hookrightarrow \mathcal{G}^0(\mathcal{C})$. We now inductively define maps $\mathcal{G}^i(\mathcal{C}) \to \mathcal{G}^{i+1}(\mathcal{C})$ by

$$\mathcal{G}^1(\mathcal{C}) = \mathcal{G}\left(\operatorname{coker} \mathcal{C} \to \mathcal{G}^0(\mathcal{C})\right), \quad \mathcal{G}^{i+1}(\mathcal{C}) = \mathcal{G}\left(\operatorname{coker} \mathcal{G}^{i-1}(\mathcal{C}) \to \mathcal{G}^i(\mathcal{C})\right) \text{ for } i \geq 1.$$

Then $\mathcal{C} \to \mathcal{G}^\bullet(\mathcal{C})$ is a flasque resolution of $\mathcal{C}$.

Applying this to the terms of a (bounded below) complex $\mathcal{C}^\bullet$, we obtain a double complex $\mathcal{G}^\bullet(\mathcal{C}^\bullet)$, with associated total complex $Tot^\bullet(\mathcal{G}^\bullet(\mathcal{C}^\bullet))$, whose terms are

$$Tot^n(\mathcal{G}^\bullet(\mathcal{C}^\bullet)) = \oplus_{i+j=n} \mathcal{G}^i(\mathcal{C}^j)$$

(these are *finite* direct sums, by our boundedness hypothesis), such that there is a quasi-isomorphism of (bounded below) complexes $\mathcal{C}^\bullet \to Tot^\bullet(\mathcal{G}^\bullet(\mathcal{C}^\bullet))$. So we may compute hypercohomology as

$$\mathbb{H}^n(X, \mathcal{C}^\bullet) = H^n(\Gamma(X, Tot^\bullet(\mathcal{G}^\bullet(\mathcal{C}^\bullet)))).$$

This may not be a very practical way of actually computing hypercohomology, but is good for theoretical purposes, since we won't have to worry about any "higher homotopies" in our constructions below involving simplicial schemes.

Finally, we discuss the spectral sequences in hypercohomology. One way to view these is through filtered complexes (we discuss decreasing filtrations, the case of increasing filtrations is obtained by re-indexing). Let $\mathcal{C}^\bullet$ be a bounded below complex, and $\{F^p\mathcal{C}^\bullet\}$ a decreasing filtration by subcomplexes, which induces a finite filtration on each term $\mathcal{C}^n$ (such a filtration will be called *finite*). Then we have the following.

(i) There is an induced finite filtration on hypercohomology

$$F^p\mathbb{H}^n(X, \mathcal{C}^\bullet) = \operatorname{image} \mathbb{H}^n(X, F^p\mathcal{C}^\bullet) \to \mathbb{H}^n(X, \mathcal{C}^\bullet)$$

(ii) There are short exact sequences of complexes

$$0 \to F^{p+1}\mathcal{C}^\bullet \to F^p\mathcal{C}^\bullet \to \operatorname{gr}_F^p \mathcal{C}^\bullet \to 0$$

and associated families of long exact sequences

$$\cdots \mathbb{H}^n(X, F^{p+1}\mathcal{C}^\bullet) \xrightarrow{i} \mathbb{H}^n(X, F^p\mathcal{C}^\bullet) \xrightarrow{j} \mathbb{H}^n(X, \operatorname{gr}_F^p\mathcal{C}^\bullet) \xrightarrow{\partial} \mathbb{H}^{n+1}(X, F^{p+1}\mathcal{C}^\bullet) \to \cdots$$

These may be assembled together into an *exact couple*

$$\begin{array}{ccc} D & \xrightarrow{i} & D \\ {}^{\partial}\nwarrow & & \swarrow_{j} \\ & E & \end{array}$$

with $D = \oplus_{n,p} H^n(X, F^p\mathcal{C}^\bullet)$, $E = \oplus_{n,p} H^n(X, \operatorname{gr}_F^p\mathcal{C}^\bullet)$ and maps induced by those with the same labels in the exact sequences.

In a standard way, we obtain an induced spectral sequence

$$E_1^{p,q} = \mathbb{H}^{p+q}(X, \operatorname{gr}_F^p\mathcal{C}^\bullet) \implies \mathbb{H}^{p+q}(X, \mathcal{C}^\bullet).$$

Because our complexes are bounded below, and the filtration by subcomplexes is finite, the spectral sequence is *convergent*, i.e., for any p, q, for all $n >> 0$, we have

$$E_n^{p,q} \cong E_\infty^{p,q} = \operatorname{gr}_F^p\mathbb{H}^{p+q}(X, \mathcal{C}^\bullet),$$

where $F^p\mathbb{H}^n(X, \mathcal{C}^\bullet)$ is the filtration in (i) above. The E_1 differentials

$$\mathbb{H}^{p+q}(X, \operatorname{gr}_F^p\mathcal{C}^\bullet) = E_1^{p,q} \to E_1^{p+1,q} = \mathbb{H}^{p+q+1}(X, \operatorname{gr}_F^{p+1}\mathcal{C}^\bullet)$$

also have a "concrete description" as the composition

$$\mathbb{H}^{p+q}(X, \operatorname{gr}_F^p\mathcal{C}^\bullet) \xrightarrow{\partial} \mathbb{H}^{p+q+1}(X, F^{p+1}\mathcal{C}^\bullet) \xrightarrow{j} \mathbb{H}^{p+q+1}(X, \operatorname{gr}_F^{p+1}\mathcal{C}^\bullet)$$

of maps from two "adjacent" exact sequences.

Two examples of such filtrations on a (bounded below) complex $\mathcal{C}^\bullet$ are as follows; these are needed later.

(i) Let $F^p\mathcal{C}^\bullet \subset \mathcal{C}^\bullet$ be defined by

$$F^p\mathcal{C}^n = \begin{cases} \mathcal{C}^n & \text{if } n \geq p \\ 0 & \text{otherwise.} \end{cases}$$

The induced spectral sequence looks like

$$E_1^{p,q} = H^q(X, \mathcal{C}^p) \implies \mathbb{H}^{p+q}(X, \mathcal{C}^\bullet).$$

This is sometimes called the *first spectral sequence* for hypercohomology. It depends on the complex itself, in general, and not just on the quasi-isomorphism class of the complex (or its class "in the derived category"). We will often write $\mathcal{C}^{\bullet \geq p}$ to denote the subcomplex $F^p\mathcal{C}^\bullet$ in this filtration.

(ii) Let $W^{-p}\mathcal{C}^\bullet \subset \mathcal{C}^\bullet$ be defined by

$$W^{-p}\mathcal{C}^n = \begin{cases} \mathcal{C}^n & \text{if } n < p \\ 0 & \text{if } n > p \\ \ker \mathcal{C}^p \to \mathcal{C}^{p+1} & \text{if } n = p. \end{cases}$$

(The funny indexing is because we want to get a decreasing filtration.) Now the spectral sequence looks like

$$E_1^{p',q'} = H^{2p'+q'}(X, \mathcal{H}^{-p'}(\mathcal{C}^\bullet)) \implies \mathbb{H}^{p'+q'}(X, \mathcal{C}^\bullet).$$

Here $\mathcal{H}^i(\mathcal{C}^\bullet)$ is the i-th cohomology sheaf of $\mathcal{C}^\bullet$. We may make a substitution $p = 2p' + q'$, $q = -p'$. Then $p' + q' = p + q$, and we may view the above spectral sequence as being of the form

$$E_2^{p,q} = H^p(X, \mathcal{H}^q(\mathcal{C}^\bullet)) \implies \mathbb{H}^{p+q}(X, \mathcal{C}^\bullet).$$

Note that, with the original indexing, the differential $d_r : E_r^{p',q} \to E_r^{p'+r,q'-r+1}$ becomes, with the new numering, a map $E_{r+1}^{p,q} \to E_{r+1}^{p+r+1,q-r}$, which is what it should be for an E_{r+1} differential! This (renumbered) spectral sequence is sometimes called the *second spectral sequence for hypercohomology*. It depends only on the quasi-isomorphism class of the complex $\mathcal{C}^\bullet$.

5.2. **MHS on the cohomology of a smooth variety.** We now return to Hodge theory. First, we interpret the Hodge decomposition on a smooth proper $\mathbb{C}$-variety in terms of hypercohomology. If X is, more generally, a complex manifold, and $\Omega_X^\bullet$ is the de Rham complex of sheaves of holomorphic differentials (with exterior derivative maps), then it is a resolution of the constant sheaf $\mathbb{C}_X$ associated to $\mathbb{C}$ on X, i.e., $\mathbb{C}_X \to \Omega_X^\bullet$ is a quasi-isomorphism (this is usually called the *holomorphic Poincaré lemma*). Hence there is a canonical isomorphism

$$H^n(X, \mathbb{C}) \cong \mathbb{H}^n(X, \Omega_X^\bullet),$$

where we have also identified singular cohomology with cohomology of the corresponding constant sheaf. Thus we have a spectral sequence

$$E_1^{p,q} = H^q(X, \Omega_X^p) \implies H^{p+q}(X, \mathbb{C}),$$

sometimes called the "Hodge to de Rham spectral sequence". The induced filtration

$$F^pH^n(X, \mathbb{C}) = \operatorname{image} \mathbb{H}^n(X, F^p\Omega_X^\bullet) \to \mathbb{H}^n(X, \Omega_X^\bullet) = H^n(X, \mathbb{C})$$

is called the Hodge filtration. If X is a compact, "bimeromorphically Kahler" manifold, e.g., a proper smooth variety over $\mathbb{C}$, then Hodge theory implies that $(H^n(X, \mathbb{Z}), \{F^pH^n(X, \mathbb{C})\})$ is a pure Hodge structure of weight n, and this is the "standard" Hodge structure on $H^n(X, \mathbb{C})$. As a consequence of the Hodge decomposition, it follows that the above spectral sequence degenerates at E_1, giving isomorphisms

$$H^q(X, \Omega_X^p) = E_1^{p,q} \cong E_\infty^{p,q} = \operatorname{gr}_F^p H^{p+q}(X, \mathbb{C}).$$

This degeneration assertion can now be proved "algebraically", by first invoking Serre's GAGA to reinterpret it in terms of hypercohomology (in the Zariski topology) of the complex of algebraic Kahler differentials, and then by reduction to prime characteristic, for example following Deligne and Illusie [3]. The Hodge decomposition is obtained by also using complex conjugation on $H^n(X, \mathbb{C}) = H^n(X, \mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}$, which (presumably) is "not algebraically defined".

If now X is a smooth variety over $\mathbb{C}$ (which we may assume to be connected), then one can find a smooth proper compactification $(\overline{X}, Y)$ with boundary Y

which is a divisor in $\overline{X}$ with simple normal crossings. Then one has the *logarithmic de Rham complex* $\Omega^\bullet_{\overline{X}}(\log Y)$ of meromorphic differentials on $\overline{X}$ which are holomorphic on X, and have at worst logarithmic poles along Y. We now recall its definition.

Let $Y = \cup_{i=1}^t Y_i$ where Y_i are the (non-singular) irreducible components of the divisor Y. If $j : X \to \overline{X}$ is the inclusion (of complex manifolds), then we may regard $j_*\Omega^\bullet_X$ as a differential graded $\mathcal{O}_{\overline{X}}$-algebra[2]; now define $\Omega^\bullet_{\overline{X}}(\log Y)$ to be the graded $\mathcal{O}_{\overline{X}}$-sub-algebra of $j_*\Omega^\bullet_X$ generated by the subsheaf $\Omega^1_{\overline{X}}(\log Y) \subset j_*\Omega^1_X$, where

$$\Omega^1_{\overline{X}}(\log Y) = \ker(\Omega^1_{\overline{X}} \to \oplus \Omega^1_{Y_i}) \otimes_{\mathcal{O}_{\overline{X}}} \mathcal{O}_{\overline{X}}(Y).$$

One verifies that this subalgebra in fact defines a sub-DGA, and in particular, a complex of sheaves on $\overline{X}$. Further, $\Omega^1_{\overline{X}}(\log Y)$ is easily seen to be a locally free $\mathcal{O}_{\overline{X}}$-module of rank $N = \dim \overline{X} = \dim X$, so that the terms of the log de Rham complex are all locally free as well, just as in the de Rham complex of $\overline{X}$.

Lemma 5.1. *There is a natural identification of $H^n(X, \mathbb{C})$ with the hypercohomology $\mathbb{H}^n(\overline{X}, \Omega^\bullet_{\overline{X}}(\log Y))$.*

Proof. Locally near a point $x \in \overline{X}$, we may choose holomorphic coordinates $z_1, \ldots, z_N$ so that the ideal of Y is generated (near x) by the product function $z_1 \cdots z_r$, say, for some $0 \le r \le N$. Then $\Omega^1_{\overline{X}}(\log Y)$ is, near x, the free $\mathcal{O}_{\overline{X}}$-module generated by

$$\frac{dz_1}{z_1}, \ldots, \frac{dz_r}{z_r}, dz_{r+1}, \ldots, dz_N.$$

Further, restricting to a small enough polydisc neighbourhood

$$U(\varepsilon) = \{y \mid |z_i(y)| < \varepsilon\},$$

we may (using the z_i, and the holomorphic inverse function theorem, for example) identify $U(\varepsilon)$ with Δ^N, where $\Delta = \{z \in \mathbb{C} \mid |z| < 1\}$ is the unit disk in the plane. Then we may identify $j : X \cap U(\varepsilon) \hookrightarrow U(\varepsilon)$ with

$$(\Delta^*)^r \times \Delta^{N-r} \hookrightarrow \Delta^N,$$

where $\Delta^* = \Delta \setminus \{0\}$ is the punctured unit disc.

Notice that, since $(\Delta^*)^r \times \Delta^{N-r}$ is a Stein manifold, the "Hodge" cohomology groups $H^p(\Omega^i)$ vanish for all i and all $p > 0$. This means that $H^p(U(\varepsilon) \cap X, \Omega^i_X) = 0$ for all i, and all $p > 0$, for any small enough $\varepsilon > 0$, and hence, by considering stalks, we have that

$$R^p j_* \Omega^i_X = 0 \quad \forall\, p > 0.$$

This implies that $H^q(X, \Omega^p_X) \cong H^q(\overline{X}, j_*\Omega^p_X)$ for all p, q. Hence, by comparing the two first hypercohomology spectral sequences, we get isomorphisms

$$H^n(X, \mathbb{C}) = \mathbb{H}^n(X, \Omega^\bullet_X) = \mathbb{H}^n(\overline{X}, j_*\Omega^\bullet_X).$$

The lemma will now follow if we prove a local assertion, that

$$\Omega^\bullet_{\overline{X}}(\log Y) \hookrightarrow j_*\Omega^\bullet_X$$

[2] I mean here that $j_*\Omega^\bullet_X$ is a graded $\mathcal{O}_{\overline{X}}$-algebra, as well as a sheaf of DGA's over $\mathbb{C}$.

is a quasi-isomorphism.

Since $\Omega^\bullet_X$ is a resolution of the constant sheaf $\mathbb{C}_X$ by sheaves which are j_*-acyclic, the cohomology sheaves of $j_*\Omega^\bullet_X$ may be interpreted as the higher direct image sheaves $R^pj_*\mathbb{C}_X$. The "second spectral sequence" for the hypercohomology of $\Omega^\bullet_{\overline{X}}(\log Y)$ thus takes the form

$$E_2^{p,q} = H^p(\overline{X}, R^qj_*\mathbb{C}_X) \implies \mathbb{H}^{p+q}(\overline{X}, j_*\Omega^\bullet_X) = H^{p+q}(X, \mathbb{C}). \tag{5.1}$$

This is just the *Leray spectral sequence*, for the continuous map $j : X \to \overline{X}$, and the sheaf $\mathbb{C}_X$ (in fact the Leray spectral sequence for any continuous map $f : X \to Y$, and any sheaf $\mathcal{F}$ on X, which takes the form

$$E_2^{p,q} = H^p(Y, R^qf_*\mathcal{F}) \implies H^{p+q}(X, \mathcal{F}),$$

may similarly be viewed as the second spectral sequence in hypercohomology on Y of $f_*\mathcal{C}^\bullet$, where $\mathcal{F} \to \mathcal{C}^\bullet$ is any resolution of $\mathcal{F}$ by sheaves $\mathcal{C}^i$ satisfying $R^pf_*\mathcal{C}^i = 0$ for $p > 0$, for all i). In particular, (5.1) is obtained from a similar spectral sequence

$$E_2^{p,q} = H^p(\overline{X}, R^qj_*\mathbb{Z}_X) \implies H^{p+q}(X, \mathbb{Z})$$

by tensoring with $\mathbb{C}$. Hence the induced filtration on $H^n(X, \mathbb{C}) = H^n(X, \mathbb{Q}) \otimes \mathbb{C}$ is obtained from a filtration on $H^n(X, \mathbb{Q})$ by $\mathbb{Q}$-subspaces, on tensoring with $\mathbb{C}$.

From our local topological description of the map j as an inclusion

$$(\Delta^*)^r \times \Delta^{N-r} \hookrightarrow \Delta^N,$$

we see easily that $R^pj_*\mathbb{C}$ is described by a simple formula (recall $Y = Y_1 \cup \cdots \cup Y_t$):

$$R^pj_*\mathbb{C}_X = \bigoplus_{1 \le i_1 < i_2 \cdots < i_p \le t} \mathbb{C}_{Y_{i_1} \cap \cdots \cap Y_{i_p}}, \quad \forall\, p \ge 1.$$

This may be related to the log-de Rham complex as follows. If $I \subset \{1, \ldots, t\}$ is non-empty, let

$$Y_I = \sum_{i \in I} Y_i, \quad Y(I) = \cap_{i \in I} Y_i.$$

Then $Y(I)$ is either empty, or a smooth complete intersection in X of pure codimension $\#I$, where $\#I$ is the cardinality of I, while Y_I is a divisor in $\overline{X}$ with normal crossings, so that we have a log-de Rham complex $\Omega^\bullet_{\overline{X}}(\log Y_I)$ which is a sub-complex (even a sub $\mathcal{O}_{\overline{X}}$-DGA) of $\Omega^\bullet_{\overline{X}}(\log Y)$. For $I = \emptyset$ define this subcomplex to be $\Omega^\bullet_{\overline{X}}$, and define $Y(I) = \overline{X}$. Notice that for subsets $I \neq J$ of the same cardinality, $Y(I)$ and $Y(J)$ have no common irreducible components, since Y is a normal crossing divisor.

We can now define an *increasing filtration* on $\Omega^\bullet_{\overline{X}}(\log Y)$ by subcomplexes:

$$W_n\Omega^\bullet_{\overline{X}}(\log Y) := \left(\sum_{\#I = n} \Omega^\bullet_{\overline{X}}(\log Y_I)\right) \subset \Omega^\bullet_{\overline{X}}(\log Y) \text{ for } 0 \le n \le t$$

(set $W_n = W_0$ for $n < 0$, and $W_n = W_t$ for $n > t$). By examining this locally near a point $x \in \overline{X}$ as above, one sees that

(i) there are natural isomorphisms of complexes

$$\operatorname{gr}_n^W \Omega^\bullet_{\overline{X}}(\log Y) \cong \bigoplus_{\#I=n} \Omega^\bullet_{Y(I)}[-n], \quad 0 \le n \le t$$

(here $[-n]$ indicates a shift of n places to the right)

(ii) for any $n \ge 0$, the natural inclusion

$$W_n\Omega^\bullet_{\overline{X}} \hookrightarrow j_*\Omega^\bullet_X$$

is an isomorphism on cohomology sheaves in degree $\le n$

(iii) for any $n \ge 0$, the complex $W_n\Omega^\bullet_{\overline{X}}(\log Y)$ has vanishing cohomology sheaves in degrees $\ge n$.

In fact (ii) and (iii) follow from (i), since $\Omega^\bullet_{Y(I)}$ is a resolution of the constant sheaf $\mathbb{C}_{Y(I)}$ on the smooth, codimension-n subvariety $Y(I) \subset \overline{X}$ (if $Y(I) = \emptyset$, we take the corresponding de Rham complex to be the 0 complex); thus

$$\operatorname{gr}_n^W \Omega^\bullet_{\overline{X}}(\log Y)$$

is a resolution of $R^n j_*\mathbb{C}_X$ on $\overline{X}$. The isomorphism in (i) is obtained using *iterated residue maps*. Since we have ordered the components $Y_1, \ldots, Y_t$ of Y right in the beginning, we have implicitly chosen an order in which to take repeated residues: if $I = \{i_1, \ldots, i_n\}$, written in increasing order, we may take residues first along Y_{i_1}, then along $Y_{i_1} \cap Y_{i_2}$ (which is a divisor in Y_{i_1}), then along $Y_{i_1} \cap Y_{i_2} \cap Y_{i_3}$, etc. We leave the detailed verification to the reader.

Clearly (ii) and (iii) imply that $\Omega^\bullet_{\overline{X}}(\log Y) \to j_*\Omega^\bullet_X$ is a quasi-isomorphism! □

Remark 5.2. The above proof gives us an identification

$$R^n j_*\mathbb{C}_X \xrightarrow{\cong} 0^{\text{th}} \text{ cohomology sheaf of } \bigoplus_{\#I=n} \Omega^\bullet_{Y(I)} = \bigoplus_{\#I=n} \mathbb{C}_{Y(I)}.$$

It is natural to ask what the subsheaf $R^n j_*\mathbb{Q}_X \subset R^n j_*\mathbb{C}_X$ corresponds to under this identification. A local analysis shows that in fact we have an induced identification

$$R^n j_*\mathbb{Q}_X \xrightarrow{\cong} \frac{1}{(2\pi\sqrt{-1})^n} \bigoplus_{\#I=n} \mathbb{Q}_{Y(I)}.$$

Using our local analytic description of the map j, and the Kunneth formula, this boils down to saying that

$$\frac{1}{2\pi\sqrt{-1}}\frac{dz}{z}$$

represents (in de Rham cohomology) a generator of $\mathbb{Q} = H^1(\Delta^*, \mathbb{Q}) \subset H^1(\Delta^*, \mathbb{C}) = \mathbb{C}$, i.e., that the integral of this particular closed form over a $\mathbb{Q}$-cycle representing a generator for $H_1(\Delta^*, \mathbb{Q})$ (for example a positively oriented circle of radius 1/2) is a non-zero rational number (equal to 1, for this choice of homology generator).

Theorem 5.3. *Let X be a smooth variety over $\mathbb{C}$, $(\overline{X}, Y)$ a smooth compactification with normal crossing boundary divisor $Y = Y_1 + \cdots + Y_t$. Define*

$$F^pH^n(X, \mathbb{C}) = \text{image}\, \mathbb{H}^n(\overline{X}, \Omega^{\ge p}_{\overline{X}}(\log Y)) \to \mathbb{H}^n(\overline{X}, \Omega^\bullet_{\overline{X}}(\log Y)) = H^n(X, \mathbb{C}),$$

$$W_iH^n(X,\mathbb{C}) = \mathrm{image}\,\mathbb{H}^n(\overline{X}, W_{i-n}\Omega^\bullet_{\overline{X}}(\log Y)) \to \mathbb{H}^n(\overline{X}, \Omega^\bullet_{\overline{X}}(\log Y)) = H^n(X,\mathbb{C}),$$

$$W_iH^n(X,\mathbb{Q}) = W_iH^n(X,\mathbb{C}) \cap H^n(X,\mathbb{Q}).$$

Then the data

$$(H^n(X,\mathbb{Z}), \{W_iH^n(X,\mathbb{Q})\}, \{F^pH^n(X,\mathbb{C})\})$$

define a mixed Hodge structure on $H^n(X,\mathbb{Z})$. This MHS is independent of the choice of the normal crossing compactification $(\overline{X}, Y)$. This defines (for each n) a contravariant functor from the category of smooth $\mathbb{C}$-varieties to MHS, in fact to the subcategory of those MHS for which $\mathrm{gr}^i_W = 0$ for $i < n$.

Proof. (Sketch!)

From the definitions, and an obvious functoriality property of log de Rham complexes, it follows that if $f : X \to X'$ is a morphism between smooth varieties, which we can extend to $\overline{f} : (\overline{X}, Y) \to (\overline{X'}, Y')$, a morphism between normal crossing compactifications, then the map $f^* : H^n(X',\mathbb{C}) \to H^n(X,\mathbb{C})$ is compatible with both the W and F filtrations, defined as in the statement of the theorem. Thus, assuming these filtrations do define MHS's, $f^* : H^n(X',\mathbb{Z}) \to H^n(X,\mathbb{Z})$ is a morphism of MHS, and further (by corollary 4.4), if f^* is an isomorphism of abelian groups, it is also one of MHS. In particular, since any two normal crossing compactifications of a given smooth variety X are dominated by a third, we deduce that the MHS doesn't depend on the choice of the compactification. The functoriality is then also clear. In the smooth proper case, the W filtration becomes trivial, and we get the usual F filtration, so recover the standard pure HS.

So it suffices to fix a normal crossing compactification $(\overline{X}, Y)$ of a given smooth X, and show that the above definitions do yield a MHS on its cohomology.

First observe that the filtration

$$F^p\Omega^\bullet_{\overline{X}}(\log Y) = \Omega^{\bullet\geq p}_{\overline{X}}(\log Y)$$

induces the Hodge filtration on

$$\mathrm{gr}^W_i\Omega^\bullet_{\overline{X}}(\log Y)[i] = \bigoplus_{\#I=i}\Omega^\bullet_{Y(I)},$$

upto a shift by i. Combined with the compatibility with the $\mathbb{Q}$-structures (Remark 5.2) noted above, this means that

$$(\mathbb{H}^n(\overline{X}, \mathrm{gr}^i_W\Omega^\bullet_{\overline{X}}(\log Y)) \cong \bigoplus_{\#I=i} H^{n-i}(Y(I),\mathbb{Q})(-i)\otimes\mathbb{C}$$

compatibly with the rational structures and Hodge filtrations, i.e., corresponds to an isomorphism of pure $\mathbb{Q}$-Hodge structures of weight $n+i$.

Next, consider the differentials in the (Leray) spectral sequence

$$E_2^{p,q} = H^p(\overline{X}, R^qj_*\mathbb{Q}) \implies H^{p+q}(X,\mathbb{Q}).$$

Tensored with $\mathbb{C}$, the Leray spectral sequence may be viewed as the spectral sequence associated to the W-filtration on $\Omega^\bullet_{\overline{X}}(\log Y)$, from the above discussion.

The E_2 differentials take the form

$$E_2^{p,q} = \bigoplus_{\#I=q} H^p(Y(I), \mathbb{Q}(-q)) \to \bigoplus_{\#J=q-1} H^{p+2}(Y(J), \mathbb{Q}(-(q-1))) = E_2^{p+2,q-1}.$$

Both terms carry pure $\mathbb{Q}$-Hodge structures of the same weight $p+2q$. The term corresponding to $Y(I)$ maps non-trivially to that corresponding to $Y(J)$ only when $J \subset I$; in this case, $Y(I) \subset Y(J)$ is a smooth divisor, and one can show the map

$$H^p(Y(I), \mathbb{Q}(-q)) \to H^{p+2}(Y(J), \mathbb{Q}(-(q-1))$$

is obtained from the Gysin map

$$H^p(Y(I), \mathbb{Q}) \to H^{p+2}(Y(J), \mathbb{Q}(1))$$

by tensoring with $\mathbb{Q}(-q)$, and multiplication by $(-1)^r$, where $I = \{i_1, \ldots, i_p\}$, $J = I \setminus \{i_r\}$. In other words, the E_2 differential is an alternating sum of Gysin maps, upto a Tate twist. In particular, *it is a morphism of pure Hodge structures*, and is thus strictly compatible with the Hodge filtrations. Hence the E_3 terms carry natural pure Hodge structures as well, induced from the filtration $F^p\Omega^{\bullet}_{\overline{X}}(\log Y)$.

Suppose we know that the E_3-differentials are also strictly compatible with the Hodge filtrations. Then they must all vanish, since they go between pure Hodge structures of different weights! Hence the E_4 terms, which then coincide with the E_3-terms, also carry pure Hodge structures. Again, strict compatibility of the E_4-differentials with F would force them to vanish, etc. We would then conclude that $E_3^{p,q} = E_\infty^{p,q}$ carries a pure Hodge structure of weight $p+2q$, induced from the F-filtration on the log-de Rham complex. This means precisely that, on

$$\operatorname{gr}^W_{n+q} H^n(X, \mathbb{C}) = \operatorname{gr}^W_{n+q} \mathbb{H}^n(\overline{X}, \Omega^{\bullet}_{\overline{X}})(\log Y)),$$

F induces a filtration yielding a pure Hodge structure of weight $n+q$. In other words, with our choices of F and W filtrations on cohomology, we have indeed defined a MHS!

It remains to deal with the technical problem of showing that the differentials above are strictly compatible with F-filtrations (in fact, to properly define the F-filtration on the E_r terms, where $r > 2$, we need to know this strictness on E_s terms for $s < r$; this is OK for $r = 3$, for example, because of the interpretation via Gysin maps). That this holds is the content of the famous "lemma of two filtrations" of Deligne (see [1], Théorème (1.3.16), and also the "cleaned-up" version in [2], Proposition (7.2.5)). □

We now make further remarks, showing how additional results are obtained from the proof above.

As a first by-product, we see that the Leray spectral sequence

$$E_2^{p,q} = H^p(\overline{X}, R^q j_*\mathbb{Q}) \implies H^{p+q}(X, \mathbb{Q})$$

degenerates at E_3. This is a purely topological assertion, though not proved here topologically[3].

[3] I don't know any "purely topological" proof of this degeneration assertion, which (presumably) is not true in general for the Leray spectral sequence with $\mathbb{Z}$-coefficients.

The filtration $W_\bullet \Omega^\bullet_{\overline{X}}(\log Y)$ by subcomplexes determines, in particular, a filtration on each term $\Omega^n_{\overline{X}}(\log Y)$, with

$$\operatorname{gr}^W_i \Omega^n_{\overline{X}}(\log Y) = \bigoplus_{\#I=i} \Omega^{n-i}_{Y(I)}.$$

Re-indexing in a way compatible with the Leray spectral sequence, we obtain an induced spectral sequence

$${}_nE_2^{p,q} = H^{p-q-n}(\overline{X}, \operatorname{gr}^W_q \Omega^n_{\overline{X}}(\log Y)) \implies H^{p-n}(\overline{X}, \Omega^n_{\overline{X}}(\log Y)).$$

Here the notation is chosen to keep track of the index n as well. The E_2-differentials are just gr^n_F of the E_2-differentials of the Leray spectral sequence, and so the E_3 terms of our new spectral sequence satisfy

$$(5.2) \qquad {}_nE_3^{p+q,q} = \operatorname{gr}^n_F \operatorname{gr}^W_q H^p(X, \mathbb{C}).$$

Now

$$\sum_q \dim_{\mathbb{C}} {}_nE_3^{p+q,q} \geq \sum_q \dim_{\mathbb{C}} {}_nE_\infty^{p+q,q} = \dim_{\mathbb{C}} H^{p-n}(\overline{X}, \Omega^n_{\overline{X}}(\log Y)).$$

There is also a spectral sequence induced by the F-filtration on $\Omega^\bullet_{\overline{X}}(\log Y)$ (first spectral sequence for hypercohomology)

$${}_FE_1^{r,s} = H^s(\overline{X}, \Omega^r_{\overline{X}}(\log Y)) \implies H^{r+s}(X, \mathbb{C}),$$

with resulting inequalities

$$(5.3) \qquad \sum_{r+s=p} \dim_{\mathbb{C}} {}_FE_1^{r,s} \geq \sum_{r+s=p} {}_FE_\infty^{r,s} = \dim_{\mathbb{C}} H^p(X, \mathbb{C}).$$

Combining (5.2) and (5.3) we obtain inequalities

$$\sum_{n,q} \operatorname{gr}^n_F \operatorname{gr}^W_q H^p(X, \mathbb{C}) = \sum_{n,q} \dim_{\mathbb{C}} {}_nE_3^{p+q,q} \geq \sum_{n,q} \dim_{\mathbb{C}} {}_nE_\infty^{p+q,q} =$$
$$\sum_n \dim_{\mathbb{C}} {}^n_FE_1^{n,p-n} \geq \sum_n \dim_{\mathbb{C}} {}_FE_\infty^{n,p-n} = \dim_{\mathbb{C}} H^p(X, \mathbb{C}).$$

But the extreme terms are equal, since F, W define a MHS! Hence all the inequalites must be equalities. We deduce that the spectral sequence

$${}_nE_2^{p,q} = H^p(\overline{X}, \operatorname{gr}^W_q \Omega^n_{\overline{X}}(\log Y)) \implies H^{p-n}(\overline{X}, \Omega^n_{\overline{X}}(\log Y))$$

degenerates at E_3, and the "log-Hodge-to-de Rham spectral sequence"

$${}_FE_1^{p,q} = H^q(\overline{X}, \Omega^p_{\overline{X}}(\log Y)) \implies H^{p+q}(X, \mathbb{C})$$

degenerates at E_1.

5.3. **MHS for smooth simplicial schemes.** To construct the MHS on the cohomology of a singular variety, or pair, we need the notions of simplicial schemes and hypercoverings (see [2], Sections 5 and 6). The compatibility with products in Theorem 4.6(iii) requires also a discussion of bisimplicial schemes and spaces, which we omit here (see [2]).

Recall that a *simplicial scheme* $X_\bullet$ is a sequence of schemes $\{X_n\}_{n\geq 0}$, together with a collection of morphisms

(i) (face maps) $\delta_i^n : X_n \to X_{n-1}$, $0 \leq i \leq n$, and $n \geq 1$
(ii) (degeneracy maps) $s_i^n : X_n \to X_{n+1}$, $0 \leq i \leq n$, for all $n \geq 0$

which satisfy certain identities. The idea is to view X_n as a "parameter scheme" for a "family of n-simplices", and the face and degeneracy morphisms are supposed to be induced by obvious operations on simplices (faces are obtained by omitting a vertex, and degeneracies by collapsing along an edge). A good way of keeping track of the identities involved (and to make the above "definition" precise) is to use an indexing category $\mathbf{\Delta}$, whose objects are the sets $\mathbf{n} = \{0, 1, \ldots, n\}$, for each $n \geq 0$, and morphisms are monotonic set maps. Then a simplicial scheme is defined to be a contravariant functor from $\mathbf{\Delta}$ to schemes, where the object $\mathbf{n}$ corresponds to the scheme X_n. The face maps are induced by the obvious injective, monotonic maps $f_i^n : \mathbf{n-1} \to \mathbf{n}$, $0 \leq i \leq n$ (f_i omits the value i), and the degeneracies by the obvious surjective monotonic maps $g_i^n : \mathbf{n+1} \to \mathbf{n}$, $0 \leq i \leq n$ (g_i^n takes the value i twice). Any morphism $\mathbf{m} \to \mathbf{n}$ in $\mathbf{\Delta}$ is clearly a composition of these basic injective and surjective monotonic maps, and one easily works out the identities satisfied by these (for example, this leads to $\delta_i^n \circ s_j^{n-1} = s_j^n \circ \delta_{i+1}^{n+1}$ if $i > j$).

More generally, a *simplicial object* in any category may be defined to be a contravariant functor from $\mathbf{\Delta}$ to that category. For example, we can define the notion of a simplicial (topological) space $X_\bullet$. This has an associated *geometric realization*, defined as the quotient

$$|X| = \coprod_{n\geq 0} X_n \times \Delta_n / \sim,$$

where Δ_n is the standard n-simplex in $\mathbb{R}^{n+1}$ (we may define Δ_n to be the convex hull of the basis vectors), and $\sim$ is the equivalence relation induced by the obvious inclusions $\Delta_{n-1} \hookrightarrow \Delta_n$ (as faces), and quotient maps $\Delta_{n+1} \twoheadrightarrow \Delta_n$ (collapsing along an edge). More formally, if $\{v_i(n)\}_{i=0}^n$ are the vertices of the standard n-simplex, so that $\Delta_n = \{\sum_i t_i v_i \in \mathbb{R}^{n+1} \mid t_i \geq 0, \sum_i t_i = 1\}$, then for any morphism $f : \mathbf{n} \to \mathbf{m}$ in $\mathbf{\Delta}$, we have an induced map

$$f_* : \Delta_n \to \Delta_m,$$
$$\sum_i t_i v_i(n) \mapsto \sum_i t_i v_{f(i)}(m).$$

If $f^* : X_m \to X_n$ is the map obtained from the structure of $X_\bullet$ as a simplicial space, then for any $(x, s) \in X_m \times \Delta_n$, we identify $(f^*(x), s)$ with $(x, f_*(s))$. This does indeed define an equivalence relation on $\coprod_n X_n \times \Delta_n$, as one easily verifies. Thus, even if we start with a simplicial set X_n, where we regard each X_n

as a discrete topological space, we obtain interesting spaces through geometric realization; in fact, one can show that any CW complex X is homotopy equivalent to the geometric realization of the simplicial set $S_\bullet(X)$, where $S_n(X)$ is the set of singular n-simplices of X.

We may define various topological invariants (e.g., singular cohomology, or homotopy groups) of a simplicial space $X_\bullet$ to be the corresponding invariants of $|X|$. There is an increasing filtration on $|X|$ by closed subsets $|X|_{\le n} =$ image $\coprod_{i\le n} X_i \times \Delta_i$, leading to a spectral sequence

$$E_1^{p,q} = H^q(X_p, A) \implies H^{p+q}(X_\bullet, A) \tag{5.4}$$

for singular cohomology with coefficients in A.

A sheaf on $X_\bullet$ is defined to be a sequence $\{\mathcal{F}_n\}_{n\ge 0}$, where $\mathcal{F}_n$ is a sheaf on X_n, and there are sheaf maps between various $\mathcal{F}_n$ covering the face and degeneracy maps defining the simplicial structure on $X_\bullet$, and satisfying the “obvious” identities. To make this a bit more formal, consider a category $\mathbf{S}$ where objects are pairs $(X, \mathcal{F})$, where $\mathcal{F}$ is a sheaf on a topological space X, and morphisms $f : (X, \mathcal{F}) \to (Y, \mathcal{G})$ are continuous maps $f : X \to Y$ together with “pullback” sheaf maps $f^{-1}\mathcal{G} \to \mathcal{F}$ (or equivalently maps $\mathcal{G} \to f_*\mathcal{F}$). There is an obvious notion of composition, giving a category, with a “forgetful functor” to topological spaces. A sheaf on a simplicial space $X_\bullet : \boldsymbol{\Delta}^{op} \to \mathbf{Top}$ is a “lifting” of $X_\bullet$ to a simplicial object of $\mathbf{S}$.

Examples of simplicial spaces of interest to us are those undelying simpicial schemes over $\mathbb{C}$; on such a simplicial analytic space $X_\bullet$, the de Rham complex $\Omega^\bullet_{X_\bullet}$ of analytic differentials defines a complex of simplicial sheaves.

It turns out that the category of sheaves on a given simplicial space is an abelian category with enough injectives, allowing us to define derived functors of any left exact functor. For example, we may define the global sections functor

$$\Gamma(X_\bullet, \mathcal{F}_\bullet) = \ker\left(\Gamma(X_0, \mathcal{F}_0) \xrightarrow{\delta_1^* - \delta_0^*} \Gamma(X_1, \mathcal{F}_1)\right),$$

where δ_0, δ_1 are the face maps $X_1 \to X_0$. The derived functors give the sheaf cohomology groups for the simplicial scheme $H^n(X_\bullet, \mathcal{F}_\bullet)$. It turns out that there is a “component spectral sequence”

$$E_1^{p,q} = H^q(X_p, \mathcal{F}_p) \implies H^{p+q}(X_\bullet, \mathcal{F}_\bullet). \tag{5.5}$$

In particular, on a simplicial analytic space $X_\bullet$, the cohomology groups $H^n(X_\bullet, A_{X_\bullet})$ of the constant simplicial sheaf defined by an abelian group A, coincide with the singular cohomolgy of the geometric realization, and the spectral sequence (5.5) reduces to (5.4).

For an arbitrary simplicial sheaf $\mathcal{F}_\bullet$ on $X_\bullet$, one may compute the cohomology groups $H^i(X_\bullet, \mathcal{F}_\bullet)$ using resolutions $\mathcal{F}_\bullet \to \mathcal{F}^i_\bullet$ by simplicial sheaves such that $\mathcal{F}^i_n$ is acyclic on X_n for each i, n, as follows: we then have a natural double complex

$$C^{p,q} = \Gamma(X_p, \mathcal{F}^q_p),$$

with p-differential induced by the face maps from the simplical structure, and q-differential by the maps $\mathcal{F}^q_\bullet \to \mathcal{F}^{q+1}_\bullet$ of simplicial sheaves. The associated total complex computes the cohomology groups $H^n(X_\bullet, \mathcal{F}_\bullet)$, in such a way that, filtering by columns (i.e., by the total complexes of the sub double complexes $F^iC^{\bullet,\bullet} = C^{\bullet\geq i,\bullet}$), the resulting spectral sequence becomes the component spectral sequence (5.5).

One way such a resolution arises is from the Godement resolutions of the sheaves $\mathcal{F}_p$; functoriality properties of this resolution imply that the terms $\mathcal{G}^q(\mathcal{F}_p)$ determine a simplicial sheaf $\mathcal{G}^q_\bullet$, for each q, and thus a resolution of $\mathcal{F}_\bullet$ by these simplicial sheaves.

The above discussion extends to defining, and computing, the hypercohomology of a bounded-below complex of simplicial sheaves. Thus, for a simplicial complex manifold $X_\bullet$ (a simplicial object in the categroy of complex manifolds), the hypercohomology group $\mathbb{H}^n(X_\bullet, \Omega^\bullet_{X_\bullet})$ of the de Rham complex is naturally identified with $H^n(X_\bullet, \mathbb{C}_{X_\bullet})$, which in turn may be identified with the singular cohomology group $H^n(|X|, \mathbb{C})$.

An interesting (though simple) example of a simplicial space is obtained from an open covering $\mathcal{U} = \{U_i\}_{i\in I}$ of a space X, by the "Čech construction". Let $X_0 = \coprod_i U_i$, with the obvious map $X_0 \to X$, and take $X_n = (X_0/X)^{n+1} = X_0 \times_X X_0 \times_X \cdots \times_X X_0$ to be the $n+1$-fold self fibre product of X_0 over X. Then $X_\bullet$ is a simplicial space in a natural way, with faces given by projections, and degeneracies by diagonals. Here we have a more concrete description

$$X_n = \coprod_{(i_0,\ldots,i_n)\in I^{n+1}} U_{i_0} \cap \cdots \cap U_{i_n}.$$

Now the face maps are given by inclusions of open sets on each piece of the above decomposition, and degeneracies are similarly interpreted via redundant intersections. If $\mathcal{F}$ is any sheaf on X, we obtain a simplicial sheaf $\mathcal{F}_\bullet$ by defining $\mathcal{F}_n$ to be the pull-back of $\mathcal{F}$ under the structure map $X_n \to X$. Conversely, gven a simplicial sheaf on $X_\bullet$, we obtain a sheaf on X by patching the given sheaves $\mathcal{F}_0 \mid U_i$ via the isomorphisms determined by the "face maps" $U_i \cap U_j \to U_i$ and $U_i \cap U_j \to U_j$ (the existence of degeneracies forces the corresponding sheaf maps to be sheaf isomorphisms!). The condition that $\mathcal{F}_\bullet$ is a simplicial sheaf means that the local $\mathcal{F}_0 |_{U_i}$ do patch together in a consistent way. Hence $\mathcal{F}_\bullet$ is obtained from a sheaf on X as before, i.e., the category of simplicial sheaves on $X_\bullet$ is equivalent to the category of sheaves on X, in this case. Now one sees that there is a canonical isomorphism $\Gamma(X, \mathcal{F}) = \Gamma(X_\bullet, \mathcal{F}_\bullet)$, from the sheaf axioms. This identifies $H^n(X_\bullet, \mathcal{F}_\bullet)$ with $H^n(X, \mathcal{F})$ and the component spectral sequence becomes the Čech spectral sequence

$$E_1^{p,q} = \coprod_{(i_0,\ldots,i_p)\in I^{p+1}} H^q(U_{i_0} \cap \cdots \cap U_{i_p}, \mathcal{F}) \implies H^{p+q}(X, \mathcal{F}).$$

This leads to the ideas of a *hypercovering*, and of *cohomological descent*, where one interprets sheaf cohomology $H^n(X, \mathcal{F})$ via cohomology of the "pull-back" simplicial sheaf $\mathcal{F}_\bullet$ on "suitable" simplicial spaces $X_\bullet$. We first consider simplicial spaces (or schemes) $X_\bullet$ which are *augmented over* X, i.e., such that $X_\bullet$ is a simplicial object in the category of spaces (or schemes) over X (which is a space, or scheme, as the case may be).

Suppose given partial data defining a simplicial space X_i for $0 \leq i \leq n$ augmented over X, in that we are given the necessary face and degeneracy maps involving only these X_i, and they satisfy the required identities (expressible involving only these maps).

To "extend" the data to define X_{n+1}, we need to define a map from a space X_{n+1} to a certain subspace Z_n of a product of $n+2$ copies of X_n (we also need to define suitable degeneracy maps). In the example corresponding to an open covering of the space X, X_{n+1} is in fact a disjoint union of open subsets giving an open covering of the space Z_n. In general, we instead assume $X_{n+1} \to Z_n$ is a "covering" is a suitable sense (perhaps generalizing the notion of an "open covering"); a simplicial space $X_\bullet$ augmented over X such that $X_0 \to X$, and each of the maps $X_{n+1} \to Z_n$, are "coverings", is called a "hypercovering" (associated to the given notion of "covering") of X.

Even where "covering" means "disjoint union of open subsets in an open covering" we obtain a notion more general than the Čech construction. This special case is useful: for example, any smooth variety over $\mathbb{C}$ has a basis for the Zariski topology consisting of open subsets U whose underlying topological spaces are Eilenberg-MacLane $K(\pi_U, 1)$-spaces associated to free groups π_U (these are sometimes called Artin's "good neighbourhoods"). So we can, in principle, "reduce" the computation of singular cohomology to computing group cohomology of free groups, and a spectral sequence. This is one route to proving the comparison theorem between étale and singular cohomology with finite coefficients, for example (the same spectral sequence will compute both of them). One can also use this procedure to define (following Artin and Mazur) the étale homotopy type of a scheme, where we now let "covering" mean "étale covering".

For the application to Hodge theory, we instead need to assume "covering" to mean "proper, surjective map"; the corresponding hyercoverings $X_\bullet \to X$ are called *proper hypercoverings* of X. Let $f : U \to X$ be a "covering" in this sense. We have induced projections $p_i : U \times_X U \to U$, $i = 1, 2$ and a structure map $g : U \times_X U \to X$, all of which are "coverings". Choose any proper surjective map $h : V \to U \times_X U$ (again this is a "covering"). Then one observes that for any sheaf $\mathcal{F}$ on X, the sequence

$$0 \to \mathcal{F} \to f_* f^{-1}\mathcal{F} \stackrel{h^*p_2^* - h^*p_1^*}{\longrightarrow} g_* h_* h^{-1} g^{-1}\mathcal{F}$$

is exact. This means that the sheaf axioms are also satisfied for such "coverings". One can show further, using the proper base change theorem, that if $U \to X$ is any proper surjective map, and $U_\bullet$ is the X-augmented simplicial space determined by the Čech construction, i.e., with $U_n = (U/X)^{n+1}$, then for any sheaf $\mathcal{F}$ on X, the natural map $H^i(X, \mathcal{F}) \to H^i(U_\bullet, \mathcal{F}_\bullet)$ are isomorphisms, where $\mathcal{F}_\bullet$ is the pull-back to $U_\bullet$ of $\mathcal{F}$ (this is expressed by saying that "proper surjective maps have universal cohomological descent"). In a formal way, this implies that for any proper hypercovering $X_\bullet \to X$, and any sheaf $\mathcal{F}$ on X, with pull-back $\mathcal{F}_\bullet$ on $X_\bullet$, the natural map $H^i(X, \mathcal{F}) \to H^i(X_\bullet, \mathcal{F}_\bullet)$ is an isomorphism.

Now we are ready to return to Hodge theory. If X is a proper variety over $\mathbb{C}$, let $X_0 \to X$ be a proper surjective map from a smooth $\mathbb{C}$-variety X_0 (e.g., take X_0 to be a resolution of singularities). Next, let $X_1 = X_0 \coprod Y_1$, where $Y_1 \to (X_0 \times_X X_0)_{\rm red}$ is a resolution of singularities; then there is an obvious inclusion $X_0 \hookrightarrow X_1$ (a "degeneracy map") are two "face maps" $X_1 \to X_0$, which are the identity on the component X_0, and are induced on Y_1 by the two projections $X_0 \times_X X_0 \to X_0$. This gives us a "1-truncated" part of a smooth proper simplicial $\mathbb{C}$-scheme. In a similar fashion, one can use a disjoint union trick to extend this to a smooth proper simplicial scheme $X_\bullet$ (i.e., where each X_n is proper and smooth over $\mathbb{C}$), augmented over X, where by construction, we would have arranged also that the proper hypercovering condition is satisfied: the maps $X_{n+1} \to Z_n$ are all proper and surjective (see [2], Section 6.2 for more details). Then we have a chain of isomorphisms

$$H^n(X, \mathbb{C}) \cong H^n(X_\bullet, \mathbb{C}_{X_\bullet}) \cong \mathbb{H}^n(X_\bullet, \Omega^\bullet_{X_\bullet}).$$

Using this, we may define a *Hodge filtration*

$$F^p H^n(X, \mathbb{C}) = \text{image}\, \mathbb{H}^n(X_\bullet, \Omega^{\bullet \geq p}_{X_\bullet}) \to H^n(X, \mathbb{C}).$$

The component spectral sequence

$$E_1^{p,q} = H^q(X_p, \mathbb{C}) \implies H^{p+q}(X_\bullet, \mathbb{C}_{X_\bullet}) \cong H^{p+q}(X, \mathbb{C})$$

induces a filtration $\{L^i H^n(X, \mathbb{C})\}_{i \geq 0}$ such that

$$E_\infty^{p,q} = \text{gr}\,^p_L H^{p+q}(X, \mathbb{C}).$$

Define $W_i H^n(X, \mathbb{C}) = L^{n-i} H^n(X, \mathbb{C})$ (thus $W_n H^n(X, \mathbb{C}) = H^n(X, \mathbb{C})$, and $W_{-1} H^n(X, \mathbb{C}) = 0$, and $\text{gr}\,^W_i H^n(X, \mathbb{C}) = \text{gr}\,^{n-i}_L H^n(X, \mathbb{C})$).

Let us compute

$$H^n(X, \mathbb{C}) = \mathbb{H}^n(X_\bullet, \Omega^\bullet_{X_\bullet})$$

using a double complex, as follows. First consider the total complex associated to the double complex of Godement resolutions of the sheaves $\Omega^p_{X_q}$, for a given q, as p varies. This gives a complex $\mathcal{G}^\bullet_q$ of flasque sheaves on X_q, quasi-isomorphic to the de Rham complex $\Omega^\bullet_{X_q}$. The filtration $\Omega^{\bullet \geq p}_{X_q}$ naturally determines a filtration $F^p \mathcal{G}^\bullet_q$ by subcomplexes, induced by suitable sub double complexes, such that $\Omega^{\bullet \geq p}_{X_q}$ is quasi-isomorphic to $F^p \mathcal{G}^\bullet_q$. These filtered complexes $\mathcal{G}^\bullet_q$, as q varies, determine a filtered complex of simplicial sheaves $\mathcal{G}^\bullet_\bullet$, whose "terms" are all flasque sheaves. Hence we obtain a double complex of global sections, whose total complex $K^\bullet$ computes $H^n(X, \mathbb{C})$. The component spectral sequence corresponds to one filtration of this total complex by subcomplexes, $L^i K^\bullet$, determining the L^i filtration on cohomology $H^n(X, \mathbb{C})$, while $F^p \mathcal{G}^\bullet_\bullet$ determines another filtration on $C^\bullet$ by subcomplexes $F^p C^\bullet$, corresponding to the Hodge filtration on the cohomology $H^n(X, \mathbb{C})$, as defined above.

The E_1 terms of the component spectral sequence support pure Hodge structures, $E_1^{p,q} = H^q(X_p, \mathbb{C})$. The E_1 differentials are homomorphisms

$$H^q(X_p, \mathbb{C}) \cong E_1^{p,q} = H^{p+q}\mathrm{gr}\,_L^p C^\bullet \to \mathrm{gr}\,_L^{p+1} C^\bullet = E_1^{p+1,q} = H^q(X_{p+1}, \mathbb{C})$$

which are alternating sums of maps induced by the face maps, i.e., correspond to morphisms of pure Hodge structures of weight q, determined by the F-filtrations on $\mathrm{gr}\,_L^* C^\bullet$, with $* = p, p+1$. Thus $E_2^{p,q}$ carries a natural pure Hodge structure of weight q, which by construction comes from the F-filtration on $C^\bullet$. Now, if we assume that the E_2-differentials are strictly compatible with F, then these differentials must vanish, since they are maps between Hodge structures of different wieghts, and so the E_3 terms carry pure Hodge structures, etc., leading ultimately to the degeneration of the compnent spectral sequence at E_2, as well as the assertion that F and W do define a MHS on $H^n(X_\bullet, \mathbb{Z}_{X_\bullet}) = H^n(X, \mathbb{Z})$. This MHS would then be independent of choices, since one can show that any two proper smooth hypercoverings of X are dominated by a third one, and that an X-morphism $f : X_\bullet \to X'_\bullet$ between two proper smooth hypercoverings of X induces an isomorphism $f^* : H^n(X_\bullet, \mathbb{C}) \to H^n(X'_\bullet, \mathbb{C})$ which is compatible with the respective F and L (and so also W) filtrations (for the F-filtration, this is because there is a morphism between simplicial holomorphic de Rham complexes, compatible with truncations; on the other hand, for any morphism of complexes of sheaves $f^{-1}\mathcal{F}'^{\bullet}_{\bullet} \to \mathcal{F}_\bullet^{\bullet}$, the induced map $\mathbb{H}^n(X', \mathcal{F}'^{\bullet}_{\bullet}) \to \mathbb{H}^n(X, \mathcal{F}_\bullet^{\bullet})$ is compatible with a morphism between component spectral sequences, and L-filtrations).

Again, Deligne's "lemma of two filtrations" provides a proof that this degeneration does indeed take place, and in fact this works to put a MHS on $H^n(X_\bullet, \mathbb{Z}_{X_\bullet})$ for an arbitrary smooth and proper simplicial $\mathbb{C}$-scheme $X_\bullet$, not necessarily obtained as a hypercovering of some given proper $\mathbb{C}$-variety X. If $X_\bullet \to X'_\bullet$ is any prophism between smooth proper simplicial $\mathbb{C}$-schemes, the induced map $H^n(X'_\bullet, \mathbb{C}) \to H^n(X, \mathbb{C})$ is compatible with F and L filtrations, and so is a morphism of MHS. In particular, if $X \to X'$ is a morphism between proper C-schemes, it is possible to find a compatible morphism $X_\bullet \to X'_\bullet$ between suitably chosen smooth proper hypercoverings $X_\bullet \to X$ and $X'_\bullet \to X'$; this implies that $f^* : H^n(X', \mathbb{Z}) \to H^n(X, \mathbb{Z})$ is a morphism of MHS.

As earlier in the case of the MHS on the cohomology of a smooth variety, we can deduce also (from the proofs) that

(i) the "Hodge to de Rham spectral sequence"

$$E_1^{p,q} = H^q(X_\bullet, \Omega^p_{X_\bullet}) \implies H^{p+q}(X_\bullet, \mathbb{C})$$

degenerates at E_1

(ii) the component spectral sequences

$$E_1^{p,q} = H^q(X_p, \Omega^i_{X_p}) \implies H^{p+q}(X_\bullet, \Omega^i_{X_\bullet})$$

degenerate at E_2, for all i.

We can now also put a (functorial) MHS on the cohomology of a pair (X, Y), where X is proper over $\mathbb{C}$, and Y is a closed (reduced) subscheme. This is because it is possible to find smooth proper hypercoverings $X_\bullet \to X$ and $Y_\bullet \to Y$, such that there is a compatible morphism of simplicial X-schemes $f : Y_\bullet \to X_\bullet$. It is then possible (see [2], Section 6.3) to define a new simplicial scheme, the *cone over* f, denoted $C(f)$, with

$$C(f)_n = X_n \coprod Y_{n-1} \coprod \cdots \coprod Y_0 \coprod \{0\}$$

(where $\{0\}$ is a point, regarded as a final object in $\mathbb{C}$-schemes), with suitable face and degeneracy maps. In particular, this is also a smooth proper simplicial C-scheme, so that its cohomology carries a MHS. One can then identify $H^n(C(f), A)$ with $H^n(X, Y, A)$ for any abelian group A, in a natural way. The long exact sequence in cohomology of the pair (X, Y) is also identified with an exact sequence constructed from the simplicial schemes $X_\bullet$, $Y_\bullet$, $C(f)$ and $\Sigma(Y_\bullet)$. Here $\Sigma(Y_\bullet)$ denotes the simplicial suspension, which is the cone of the map $Y_\bullet \to \{0\}_\bullet$, where $\{0\}_\bullet$ is the final object in simplicial schemes, i.e., $\{0\}_n = \{0\}$ is a singleton set for each n, with all face and degeneracy maps equal to the identity (the geometric realization of the corresponding simplicial space is a point). The sequence of morphisms of simplicial schemes

$$Y_\bullet \to X_\bullet \to C(f) \to \Sigma(Y_\bullet)$$

gives a sequence of MHS

$$H^n(\Sigma(Y_\bullet), \mathbb{Z}) \to H^n(C(f), \mathbb{Z}) \to H^n(X_\bullet, \mathbb{Z}) \to H^n(Y_\bullet, \mathbb{Z})$$

which is identified with

$$H^{n+1}(Y, \mathbb{Z}) \to H^n(X, Y, \mathbb{Z}) \to H^n(X, \mathbb{Z}) \to H^n(Y, \mathbb{Z}).$$

Thus the long exact sequence in cohomology of the pair (X, Y) is an exact sequence of MHS.

To put a MHS on $H^n(X, \mathbb{Z})$ for an arbitrary $\mathbb{C}$-scheme X, or for pairs, there is an extra technical point. We'll very briefly discuss the case of a single scheme X, noting that pairs are dealt with by a variation of the above remarks on cones. If $X_\bullet$ is a smooth proper simplicial $\mathbb{C}$-scheme, define a *normal crossing divisor* $D_\bullet$ in $X_\bullet$ to be a sequence of normal crossing divisors D_n on X_n, such that if $U_n = X_n \setminus D_n$, then U_n determine a simplicial "open subscheme" $U_\bullet$ of $X_\bullet$, i.e., the face and degeneracy operations for $X_\bullet$ restrict suitably to define a simplicial structure on $U_\bullet$. Under these conditions, one can define a simplicial log de Rham complex $\Omega^\bullet_{X_\bullet}(\log Y_\bullet)$. Now we may find $(X_\bullet, Y_\bullet)$ so that $U_\bullet$ is augmented to our given $\mathbb{C}$-scheme X, and is a smooth proper hypercovering of X. Then we obtain isomorphims

$$H^n(X, \mathbb{C}) = H^n(U_\bullet, \mathbb{C}) = \mathbb{H}^n(X_\bullet, \Omega^\bullet_{X_\bullet}(\log Y_\bullet)).$$

The filtration by subcomplexes $\Omega^{\bullet \geq p}_{X_\bullet}(\log Y_\bullet))$ defines a Hodge filtration $F^p H^n(X, \mathbb{C})$. The definition of the weight filtration is a bit more tricky, since it has to simultaneously "take into account" the component spectral sequence filtration, as well as the "weight filtrations" on the log de Rham complex $W_\bullet \Omega^\bullet_{X_q}(\log Y_q)$ of the components X_q, for each q.

One proceeds as follows. First use Godement resolutions to construct a complex of vector spaces $C^\bullet_q$ for each q, which computes the hypercohomology groups

$\mathbb{H}^*(X_q, \Omega^\bullet_{X_q}(\log Y_q))$, such that there are filtrations $F^pC^\bullet_q$ and $W_iC^\bullet_q$ which correspond to the filtrations of $\Omega^\bullet_{X_q}(\log Y_q)$ by $\Omega^{\bullet \geq p}_{X_q}(\log Y_q)$ and $W_i\Omega^\bullet_{X_q}(\log Y_q)$, which are used to define the MHS on $H^n(X_q, \mathbb{Z})$ in Theorem 5.3. The face maps of the simplicial scheme determine maps of complexes $C^\bullet_q \to C^\bullet_{q+1}$ which are compatible with both the F^p and W_i filtrations. Taking alternating sums of face maps, we obtain a double complex, whose total complex $C^\bullet$ computes the cohomology groups $H^*(U_\bullet, \mathbb{C})$, that is to say, the groups $H^*(X, \mathbb{C})$. This total complex has terms

$$C^n = \oplus_{q=0}^n C_q^{n-q}.$$

Define

$$\widetilde{W}_i C^n = \oplus_{q=0}^n W_{i+n-q} C_q^{n-q}.$$

One verifies that this does define a filtration of $C^\bullet$ by subcomplxes, and hence induces a filtration $\widetilde{W}_i H^n(X, \mathbb{C})$ on cohomology. Define $W_iH^n(X, \mathbb{C}) = \widetilde{W}_{i-n}H^n(X, \mathbb{C})$. Deligne shows (see [2], Théorème (8.1.15) and Proposition (8.1.20)) that this choice of weight filtration, together with the Hodge filtration as defined above, does indeed determine a MHS on $H^n(U_\bullet, \mathbb{Z}) = H^n(X, \mathbb{Z})$, such that the component spectral sequence

$$E_1^{p,q} = H^q(U_p, \mathbb{Z}) \implies H^{p+q}(U_\bullet, \mathbb{Z}) = H^{p+q}(X, \mathbb{Z})$$

becomes a spectral sequence of MHS. One also obtains the degeneration at E_1 of the "Hodge to de Rham" spectral sequence

$$E_1^{p,q} = H^q(X_\bullet, \Omega^p_{X_\bullet}(\log Y_\bullet)) \implies H^{p+q}(X, \mathbb{C}).$$

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