Proof of the Hodge decomposition, after Deligne and Illusie*

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1 Hypercohomology

Given a topological space X and a complex $\mathcal{A}^\bullet$ of sheaves of abelian groups on X which is *bounded below* (*i.e.*, $\mathcal{A}^j = 0$ for $j << 0$), define the *hypercohomology* groups $\mathbb{H}^n(X, \mathcal{A}^\bullet)$ as follows:
we can find a complex $\mathcal{I}^\bullet$ of injective sheaves of abelian groups, together with a map of complexes of sheaves $f : \mathcal{A}^\bullet \to \mathcal{I}^\bullet$, such that $\mathcal{A}^j = 0$ for $j \leq j_0 \Rightarrow \mathcal{I}^j = 0$ for $j \leq j_0$ (so that $\mathcal{I}^\bullet$ is bounded below), and f induces isomorphisms on cohomology sheaves (we express this by saying that f is a *quasi-isomorphism*). Define $\mathbb{H}^n(X, \mathcal{A}^\bullet) = H^n(\Gamma(X, \mathcal{I}^\bullet))$, where $\Gamma(X, -)$ is the complex of global sections (this is a complex of abelian groups).

One checks that the groups $H^n(X, \mathcal{A}^\bullet)$ are independent of the choice of the complex $\mathcal{I}^\bullet$ of injectives chosen, and have the following properties.

1. If $\mathcal{A}^\bullet$ consists of a single non-zero sheaf $\mathcal{A}$ in degree j (we then write $\mathcal{A}^\bullet = \mathcal{A}[-j]$), then

$$\mathbb{H}^n(X, \mathcal{A}^\bullet) = \mathbb{H}^n(X, \mathcal{A}[-j]) = H^{n-j}(X, \mathcal{A}),$$

   the standard sheaf cohomology group.

2. If $f : \mathcal{A}^\bullet \to \mathcal{B}^\bullet$ is a quasi-isomorphism of complexes of sheaves (which are bounded below), then f induces isomorphisms on hypercohomology.

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3. Let
$$0 \to \mathcal{A}_1^\bullet \to \mathcal{A}_2^\bullet \to \mathcal{A}_3^\bullet \to 0$$
be an exact sequence of complexes of sheaves, *i.e.*, the sequence
$$0 \to \mathcal{A}_1^j \to \mathcal{A}_2^j \to \mathcal{A}_3^j \to 0$$
is exact for each j. Then there is a long exact sequence of hypercohomology groups, functorial for maps between exact sequences,
$$\cdots \to \mathbb{H}^{i-1}(X, \mathcal{A}_3^\bullet) \to \mathbb{H}^i(X, \mathcal{A}_1^\bullet) \to \mathbb{H}^i(X, \mathcal{A}_2^\bullet) \to \mathbb{H}^i(X, \mathcal{A}_3^\bullet) \to \mathbb{H}^{i+1}(X, \mathcal{A}_1^\bullet) \to \cdots$$

4. There is a spectral sequence (the *first spectral sequence* for hypercohomology)
$$E_1^{p,q} = H^q(X, \mathcal{A}^p) \Rightarrow \mathbb{H}^{p+q}(X, \mathcal{A}^\bullet).$$
The filtration on $\mathbb{H}^n(X, \mathcal{A}^\bullet)$ associated to this spectral sequence is
$$F^p\mathbb{H}^n(X, \mathcal{A}^\bullet) = \operatorname{im}(\mathbb{H}^n(X, F^p\mathcal{A}^\bullet) \to \mathbb{H}^n(X, \mathcal{A}^\bullet));$$
here $F^p\mathcal{A}^\bullet$ is the truncated subcomplex of $\mathcal{A}^\bullet$
$$\cdots \to 0 \to \mathcal{A}^p \to \mathcal{A}^{p+1} \to \mathcal{A}^{p+2} \to \cdots$$
where $\mathcal{A}^p$, the first 'nontrivial' term of the complex $F^p\mathcal{A}^\bullet$, is in degree p. The filtration $\{F^p\mathcal{A}^\bullet\}$ of $\mathcal{A}^\bullet$ is called the *filtration bêtê* ('stupid filtration').

   In particular, we see that if the $\mathcal{A}^j$ are all acyclic (*i.e.*, for each j, $H^i(X, \mathcal{A}^j) = 0\ \forall\ i > 0$), then $\mathbb{H}^n(X, \mathcal{A}^\bullet) = H^n(\Gamma(X, \mathcal{A}^\bullet))$.

5. There is a spectral sequence (the *second spectral sequence* of hypercohomology)
$$E_2^{p,q} = H^p(X, \mathcal{H}^q(\mathcal{A}^\bullet)) \Rightarrow \mathbb{H}^{p+q}(X, \mathcal{A}^\bullet),$$
where $\mathcal{H}^q(\mathcal{A}^\bullet)$ is the q^{th} cohomology sheaf of $\mathcal{A}^\bullet$. The associated filtration on $\mathbb{H}^*$ is
$$F^p\mathbb{H}^n(X, \mathcal{A}^\bullet) = \operatorname{im}(\mathbb{H}^n(X, F_{n-p}\mathcal{A}^\bullet) \to \mathbb{H}^n(X, \mathcal{A}^\bullet))$$
where $F_q\mathcal{A}^\bullet$ is the truncated subcomplex of $\mathcal{A}^\bullet$
$$\cdots \to \mathcal{A}^{q-2} \to \mathcal{A}^{q-1} \to \mathcal{Z}^q \to 0 \to 0 \to \cdots$$

where $\mathcal{Z}^q = \ker(\mathcal{A}^q \to \mathcal{A}^{q+1})$ is in degree q.

If $f : \mathcal{A}^\bullet \to \mathcal{B}^\bullet$ is a quasi-isomorphism, it induces an isomorphism of second hypercohomology spectral sequences.

6. Let $\mathcal{U} = \{U_i\}_{i \in I}$ be a Leray cover of X for each term $\mathcal{A}^j$ of a (bounded below) complex $\mathcal{A}^\bullet$ (*i.e.*, $H^q(U_{i_1} \cap \cdots \cap U_{i_n}, \mathcal{A}^j) = 0$ for all $q > 0$, for all n, j). Then the Čech complexes $\check{C}^p(\mathcal{U}, \mathcal{A}^\bullet)_{p \geq 0}$ yield a double complex in a natural way. There are natural isomorphisms

$$\mathbb{H}^n(X, \mathcal{A}^\bullet) \to H^n(Tot(\check{C}(\mathcal{U}, \mathcal{A}^q)))$$

(where Tot denotes the total complex of a double complex), which are compatible with an isomorphism of the first hypercohomology spectral sequence (see 4. above) with one of the spectral sequences for the double complex (obtained by first forming the cohomology "in the p-direction").

[Idea of proof: Let $\mathcal{A}^\bullet \to \mathcal{I}^\bullet$ be a quasi-isomorphism, where $\mathcal{I}^\bullet$ is a complex of injective sheaves on X. Then for each q,

$$\psi_q : \check{C}^q(\mathcal{U}, \mathcal{A}^\bullet) \to \check{C}^q(\mathcal{U}, \mathcal{I}^\bullet)$$

is a quasi-isomorphism, since both sides compute the groups

$$\prod_{i_0 < i_1 < \cdots < i_q} \mathbb{H}^*(U_{i_0} \cap \cdots \cap U_{i_q}, \mathcal{A}^\bullet),$$

as $\mathcal{U}$ is a Leray cover for each $\mathcal{A}^j$ as well as for $\mathcal{I}^j$ (since $\mathcal{I}^j \mid_U$ is an injective sheaf for each open set $U \subset X$). The maps ψ_q thus induce a quasi-isomorphism

$$\psi : Tot(\check{C}^\bullet(\mathcal{U}, \mathcal{A}^\bullet)) \to Tot(\check{C}(^\bullet(\mathcal{U}, \mathcal{I}^\bullet)).$$

The natural maps $\Gamma(X, \mathcal{I}^p) \to \check{C}^0(\mathcal{U}, \mathcal{I}^p)$ define a map of complexes

$$\varphi : \Gamma(X, \mathcal{I}^\bullet) \to Tot(\check{C}^\bullet(\mathcal{U}, \mathcal{I}^\bullet)),$$

which is a quasi-isomorphism, since the complexes

$$0 \to \Gamma(X, \mathcal{I}^p) \to \check{C}^0(\mathcal{U}, \mathcal{I}^p) \to \check{C}^1(\mathcal{U}, \mathcal{I}^p) \to \cdots$$

are exact for all p. Thus, φ induces an isomorphism

$$\mathbb{H}^n(X, \mathcal{A}^\bullet) \to H^n(Tot(\check{\mathrm{C}}^\bullet(\mathcal{U}, \mathcal{I}^\bullet))),$$

and hence using ψ, an isomorphism

$$\mathbb{H}^n(X, \mathcal{A}^\bullet) \to H^n(Tot(\check{\mathrm{C}}^\bullet(\mathcal{U}, \mathcal{A}^\bullet))).$$

The filtration $\{F^p\mathcal{A}^\bullet\}$ on $\mathcal{A}^\bullet$ induces a filtration

$$F^p(Tot(\check{\mathrm{C}}^\bullet(\mathcal{U}, \mathcal{A}^\bullet))) = Tot(\check{\mathrm{C}}^\bullet(\mathcal{U}, F^p\mathcal{A}^\bullet))$$

which is the filtration used to define one of the spectral sequences of the double complex, viewed as a particular case of the spectral sequence of a filtered complex.]

7. (*Base change*) Let $\pi : X \to \operatorname{Spec} R$ be a proper morphism of schemes, where R is a Noetherian ring, and $\mathcal{A}^\bullet$ is a *bounded* complex of sheaves such that

   (a) each $\mathcal{A}^j$ is a coherent $\mathcal{O}_X$-module which is R-flat, and

   (b) the differentials $\mathcal{A}^j \to \mathcal{A}^{j+1}$ are R-linear for all j.

   Then $\mathbb{H}^n(X, \mathcal{A}^\bullet)$ is a finite R-module for each n, and the two hypercohomology spectral sequences are spectral sequences of finite R-modules. The function on $\operatorname{Spec} R$ given by

$$f_n : x \mapsto \dim_{k(x)} \mathbb{H}^n(\pi^{-1}(x), \mathcal{A}^\bullet \mid_{\pi^{-1}(x)})$$

   is upper semi-continuous, for each n. The function

$$x \mapsto \sum_n (-1)^n \dim_{k(x)} \mathbb{H}^n(\pi^{-1}(x), \mathcal{A}^\bullet \mid_{\pi^{-1}(x)})$$

   is locally constant. Further, if R is an integral domain, and all the hypercohomology modules $\mathbb{H}^n(X, \mathcal{A}^\bullet)$ are projective R-modules, then the above dimension functions f_n are all constant; in fact, if S is any R-algebra, $X_S = X \times_{\operatorname{Spec} R} \operatorname{Spec} S$, $\mathcal{A}^j_S = \mathcal{A}^j \otimes_R S$, then

$$\mathbb{H}^n(X_S, \mathcal{A}^\bullet_S) = \mathbb{H}^n(X, \mathcal{A}^\bullet) \otimes_R S$$

(the constancy of the functions f_n follows from the special cases $S = \operatorname{Spec} k(x)$).
[Idea of proof: Let $\mathcal{U} = \{U_i\}$ be an affine open cover of X. Then it is a Leray cover for each of the $\mathcal{A}^j$, as is $\mathcal{U}_S = \{(U_i)_S\}$ for each $\mathcal{A}^j_S$, for any R-algebra S. Further, we have

$$\check{\mathrm{C}}^{\bullet}(\mathcal{U}_S, \mathcal{A}^{\bullet}_S) = \check{\mathrm{C}}^{\bullet}(\mathcal{U}, \mathcal{A}^{\bullet}) \otimes_R S.$$

Now $\check{\mathrm{C}}^{p}(\mathcal{U}, \mathcal{A}^q)$ is a *flat* R-module for each p, q so that $Tot(\check{\mathrm{C}}^{\bullet}(\mathcal{U}, \mathcal{A}^{\bullet}))$ is a finite complex of flat R-modules, whose cohomology groups are finite R-modules. By lemma 1, pg. 47 of Ch. 2 in Mumford's *Abelian Varieties*, we can find an R-linear quasi-isomorphism

$$K^{\bullet} \to Tot(\check{\mathrm{C}}^{\bullet}(\mathcal{U}, \mathcal{A}^{\bullet}))$$

where $K^{\bullet}$ is a finite complex of finitely generated projective R-modules. As in Mumford (*loc. cit.*) pg. 49, lemma 2, the natural map of complexes

$$K^{\bullet} \otimes_R S \to Tot(\check{\mathrm{C}}^{\bullet}(\mathcal{U}, \mathcal{A}^{\bullet})) \otimes_R S \cong Tot(\check{\mathrm{C}}^{\bullet}(\mathcal{U}_S, \mathcal{A}^{\bullet}_S))$$

is a quasi-isomorphism for any R-algebra S. But semicontinuity, etc. are all easy to prove for the dimension functions of $K^{\bullet}$.]

2 The Hodge spectral sequence and algebraic de Rham cohomology

If $X/\mathbb{C}$ is a compact Kähler manifold, the singular cohomology groups with complex coefficients $H^*(X, \mathbb{C})$ have a Hodge decomposition, obtained using the theory of harmonic forms,

$$H^n(X, \mathbb{C}) \cong \bigoplus_{p+q=n} H^p(X, \Omega^q_X);$$

further $\dim H^p(X, \Omega^q_X) = \dim H^q(X, \Omega^p_X)$, a fact often referred to as *Hodge symmetry.* Here, Ω^p_X is the sheaf of holomorphic p-forms on X.

This can be given another formulation as follows. There is a decreasing filtration

$$\{F^p H^n(X, \mathbb{C})\}_{p \geq 0}$$

on $H^n(X,\mathbb{C})$, called the *Hodge filtration*, which satisfies the following properties:

(i) $F^pH^n(X,\mathbb{C}) \subset H^n(X,\mathbb{C})$ is a $\mathbb{C}$-subspace, and $F^pH^n(X,\mathbb{C}) = 0$ for $p > n$.

(ii) $F^pH^n(X,\mathbb{C}) \cap \overline{F^{n-p+1}H^n(X,\mathbb{C})} = 0$ for all p; here the overbar denotes the complex conjugate, where conjugation is defined using the isomorphism $H^n(X,\mathbb{C}) \cong H^n(X,\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}$.

(iii) there are isomorphisms

$$F^pH^n(X,\mathbb{C})\cap\overline{F^{n-p}H^n(X,\mathbb{C})} \cong F^pH^n(X,\mathbb{C})/F^{p+1}H^n(X,\mathbb{C}) \cong H^{n-p}(X,\Omega_X^p).$$

The Hodge filtration can be interpreted as the filtration associated to a spectral sequence. From the holomorphic Poincaré lemma, the *holomorphic de Rham complex* gives a resolution of the constant sheaf $\mathbb{C}$ (on any complex manifold X)

$$0 \to \mathbb{C} \to \mathcal{O}_X \xrightarrow{d} \Omega_X^1 \xrightarrow{d} \Omega_X^2 \xrightarrow{d} \cdots \xrightarrow{d} \Omega_X^N \to 0,$$

where d denotes the exterior derivative, and $N = \dim X$. Thus we may identify $H^n(X,\mathbb{C})$ with the n^{th} hypercohomology group of the holomorphic de Rham complex, $\mathbb{H}^n(X,\Omega_X^\bullet)$, where $\Omega_X^\bullet$ is the complex

$$0 \to \mathcal{O}_X \xrightarrow{d} \Omega_X^1 \xrightarrow{d} \Omega_X^2 \xrightarrow{d} \cdots \xrightarrow{d} \Omega_X^N \to 0.$$

The first hypercohomology spectral sequence

$$E_1^{p,q} = H^q(X,\Omega_X^p) \Rightarrow \mathbb{H}^n(X,\Omega_X^\bullet)$$

is associated to the *filtration bêtê* $\{F^p\Omega_X^\bullet\}$ of $\Omega_X^\bullet$ by the truncated subcomplexes

$$F^p\Omega_X^\bullet = \quad (0 \to \cdots \to 0 \to \Omega_X^p \xrightarrow{d} \Omega_X^{p+1} \xrightarrow{d} \cdots \xrightarrow{d} \Omega_X^N \to 0).$$

The associated filtration on $H^n(X,\mathbb{C}) \cong \mathbb{H}^n(X,\Omega_X^\bullet)$ is called the *Hodge filtration*; further, if X is compact, so that the cohomology groups $H^q(X,\Omega_X^p)$ are finite dimensional $\mathbb{C}$-vector spaces, we have inequalities

$$\sum_{p+q=n} \dim_{\mathbb{C}} H^q(X,\Omega_X^p) = \sum_{p+q=n} \dim_{\mathbb{C}} E_1^{p,q} \geq \sum_{p+q=n} \dim_{\mathbb{C}} E_\infty^{p,q} = \dim_{\mathbb{C}} H^n(X,\mathbb{C}).$$

If X is a compact Kähler manifold, so that $H^n(X,\mathbb{C})$ has a Hodge decomposition, then the above inequality must be an equality. Hence

$$\dim_{\mathbb{C}} E_1^{p,q} = \dim_{\mathbb{C}} E_\infty^{p,q} \ \forall\, p,q$$

i.e., the spectral sequence must degenerate at E_1. Conversely, if the spectral sequence degenerates at E_1, then we have isomorphisms

$$F^pH^n(X,\mathbb{C})/F^{p+1}H^n(X,\mathbb{C}) \cong E_\infty^{p,n-p} \cong E_1^{p,q} \cong H^{n-p}(X,\Omega_X^p).$$

Let X be a smooth, proper variety over a field k. Let $\Omega^\bullet_{X/k}$ be the complex of algebraic differential forms on X *i.e.*, $\Omega^1_{X/k}$ is the module of Kähler differentials of X/k, $\Omega^p_{X/k} = \overset{p}{\wedge}_{\mathcal{O}_X} \Omega^1_{X/k}$, and the exterior derivative $d : \Omega^p_{X/k} \to \Omega^{p+1}_{X/k}$ is defined by the same formula as in the analytic case. We define the *algebraic de Rham cohomology* groups of X/k to be

$$H^n_{DR}(X/k) = \mathbb{H}^n(X, \Omega^\bullet_{X/k}).$$

These groups are finite dimensional k-vector spaces, and have a Hodge filtration $\{F^pH^n_{DR}(X/k)\}$ by k-subspaces, together with a spectral sequence (of finite dimensional k-vector spaces)

$$E_1^{p,q} = H^q(X,\Omega^p_{X/k}) \Rightarrow H^{p+q}_{DR}(X/k).$$

If $k = \mathbb{C}$, let X_{an} denote the complex manifold corresponding to the algebraic variety X. Then there is a natural map

$$\rho_n : H^n_{DR}(X/\mathbb{C}) \to \mathbb{H}^n(X_{an}, \Omega^\bullet_{X_{an}})$$

which is compatible with the respective Hodge filtrations, and a morphism of hypercohomology spectral sequences, which on E_1 terms is the natural map

$$E_1^{p,q}(X) = H^q(X, \Omega^p_{X/\mathbb{C}}) \to H^q(X_{an}, \Omega^p_{X_{an}}) = E_1^{p,q}(X_{an}).$$

By GAGA, this is an isomorphism for each p and q, and hence yields an isomorphism of spectral sequences. Hence ρ_n is an isomorphism of filtered $\mathbb{C}$-vector spaces.

If X is projective, then X_{an} is a compact Kähler manifold. From Hodge theory, the spectral sequence for X_{an} degenerates at E_1; hence so does the spectral sequence for the algebraic de Rham cohomology. By a standard argument using Chow's lemma, we deduce that both spectral sequences degenerate for any smooth complete variety X (an analogous argument shows that the Hodge decomposition is valid for any *birationally Kähler* compact complex manifold X, by reduction to the Kähler case).

If $k \subset \mathbb{C}$ is any subfield, and X is a smooth complete k-variety, then

$$H^n_{DR}(X/k) \otimes_k \mathbb{C} \cong H^n_{DR}(X_{\mathbb{C}}/\mathbb{C}),$$

and there is an isomorphism of spectral sequences (of finite dimensional $\mathbb{C}$-vector spaces)

$$E_r^{p,q}(X) \otimes_k \mathbb{C} \cong E_r^{p,q}(X_{\mathbb{C}}).$$

Hence the spectral sequence for algebraic de Rham cohomology degenerates for any smooth proper variety X over a subfield k of $\mathbb{C}$. From this, we immediately deduce the result in the case when k is an arbitrary field of characteristic 0.

For any holomorphic map $f : Y \to X$ between complex manifolds, there is an induced morphism of Hodge spectral sequences, since there is a map between their holomorphic de Rham complexes. Hence the map $f^* : H^*(X, \mathbb{C}) \to H^*(Y, \mathbb{C})$ on cohomology respects the Hodge filtrations. The map on E_1 terms $H^q(X, \Omega^p_X) \to H^q(Y, \Omega^p_Y)$ is the natural one. If $f : Y \to X$ is a morphism between smooth proper varieties over k, then a similar statement holds for the map

$$f^* : H^*_{DR}(X/k) \to H^*_{DR}(Y/k).$$

In particular, suppose X is a smooth projective variety over $\mathbb{C}$, and $f : Y \hookrightarrow X$ is a smooth hyperplane section; then the Lefschetz hyperplane theorem states that $f^* : H^n(X, \mathbb{C}) \to H^n(Y, \mathbb{C})$ is an isomorphism for $n < \dim Y$ and an injection for $n = \dim Y$. Thus

$$H^n_{DR}(X/\mathbb{C}) \to H^n_{DR}(Y/\mathbb{C})$$

is an isomorphism for $n < \dim Y$ and an injection for $n = \dim Y$. Hence

$$f^* : H^q(X, \Omega^p_{X/\mathbb{C}}) \to H^q(Y, \Omega^p_{Y/\mathbb{C}})$$

is an isomorphism for $p + q < \dim Y$, and an injection for $p + q = \dim Y$. From the cohomology sequences of the exact sheaf sequences

$$0 \to \Omega^p_{X/\mathbb{C}}(-Y) \to \Omega^p_{X/\mathbb{C}} \to \Omega^p_{X/\mathbb{C}} \otimes \mathcal{O}_Y \to 0,$$

$$0 \to \Omega^{p-1}_{Y/\mathbb{C}}(-Y) \to \Omega^p_{X/\mathbb{C}} \otimes \mathcal{O}_Y \to \Omega^p_{Y/\mathbb{C}} \to 0,$$

we deduce (by induction on $\dim X$) that

$$H^q(X, \Omega^p_{X/\mathbb{C}}(-Y)) = 0 \text{ for } p + q < \dim X.$$

More generally, if $\mathcal{L}$ is any ample line bundle on X, the vanishing theorem of Kodaira-Akizuki-Nakano states that

$$H^q(X, \Omega^p_{X/\mathbb{C}} \otimes \mathcal{L}^{-1}) = 0 \text{ for } p + q < \dim X;$$

as C. P. Ramanujam observed (*c.f.* the C. P. Ramanujam Memorial volume), this follows from the special case considered above using the "branched covering trick".

3 The main theorems

If X/A is a smooth and proper scheme over a Noetherian ring A, we define the algebraic de Rham cohomology groups

$$H^n_{DR}(X/A) = \mathbb{H}^n(X, \Omega^\bullet_{X/A})$$

where $\Omega^\bullet_{X/A}$ is the de Rham complex of relative algebraic differentials of X/A (*i.e.*, $\Omega^1_{X/A}$ is the sheaf of relative Kähler differentials, and $\Omega^i_{X/A} = \overset{i}{\wedge}_{\mathcal{O}_X} \Omega^1_{X/A}$). Then

(i) $H^n_{DR}(X/A)$ is a finite A-module for each n

(ii) if $A = \mathbb{C}$, then $H^n_{DR}(X/\mathbb{C}) \cong H^n(X_{an}, \mathbb{C})$

(iii) there is a spectral sequence (of finite A-modules)

$$E_1^{p,q} = H^q(X, \Omega^p_{X/A}) \Rightarrow H^{p+q}_{DR}(X/A)$$

(iv) if $H^q(X, \Omega^p_{X/A})$ and $H^n_{DR}(X/A)$ are projective A-modules for all p, q, n, then for any $t \in \operatorname{Spec} A$, if X_t denotes the scheme theoretic fibre of $X \to \operatorname{Spec} A$ over t, and $k(t)$ is the residue field at t, then

$$H^q(X_t, \Omega^p_{X_t/k(t)}) \cong H^q(X, \Omega^p_{X/A}) \otimes_A k(t), \quad \forall\, p, q$$

and

$$H^n_{DR}(X_t/k(t)) \cong H^n_{DR}(X/A) \otimes_A k(t) \;\; \forall\, n.$$

The main results in the paper of Deligne and Illusie are:

Theorem 1 *Let X be a smooth proper variety of dimension d over a perfect field k of characteristic $p > d$. Suppose that X lifts to a scheme $\widetilde{X}$ which is smooth and proper over $W_2(k)$ (the ring of Witt vectors of length 2 over k; if $W(k)$ is the ring of Witt vectors over k, then $W_2(k) = W(k)/p^2W(k)$). Then the spectral sequence*

$$E_1^{i,j} = H^j(X, \Omega^i_{X/k}) \Rightarrow H^{i+j}_{DR}(X/k)$$

degenerates at E_1.

Theorem 2 *Let X/k be as in Theorem 1, and let $\mathcal{L} \in \operatorname{Pic} X$ be an ample line bundle. Then $H^i(X, \Omega^j_{X/k} \otimes \mathcal{L}^{-1}) = 0$ for all $i + j < d = \dim X$.*

Corollary 3 *Let $X/\mathbb{C}$ be a smooth and proper variety. Then*

(i) $E_1^{p,q} = H^q(X, \Omega^p_{X/\mathbb{C}}) \Rightarrow H^{p+q}_{DR}(X/\mathbb{C})$ *degenerates at E_1.*

(ii) *if $\mathcal{L} \in \operatorname{Pic} X$ is ample, then*

$$H^q(X, \Omega^p_{X/\mathbb{C}} \otimes \mathcal{L}^{-1}) = 0 \ \ \forall \ \ p + q < \dim X.$$

Proof: X and $\mathcal{L}$ defined over a finitely generated $\mathbb{Z}$-subalgebra $A \subset \mathbb{C}$. Let X_A be an A-scheme with $X = (X_A) \times_{\operatorname{Spec} A} \operatorname{Spec} \mathbb{C}$, and let $\mathcal{L}_A \in \operatorname{Pic} X_A$ with $(\mathcal{L}_A)_{\mathbb{C}} = \mathcal{L}$. If k is the quotient field of A, it suffices to prove the degeneration of the first spectral sequence for $H^*_{DR}(X_k/k)$, and to prove the Kodaira-Akizuki-Nakano vanishing theorem for $\mathcal{L}_k$ on X_k.

By replacing A by a localisation $A[1/s]$, we may assume that $X_A \to \operatorname{Spec} A$ is smooth and proper, $\operatorname{Spec} A \to \operatorname{Spec} \mathbb{Z}$ is smooth, and

$$H^*_{DR}(X_A/A), \quad H^i(X_A, \Omega^j_{X_A/A}), \quad H^i(X_A, \Omega^j_{X_A/A} \otimes \mathcal{L}_A^{-1})$$

are projective A-modules. Then by base change (see 7. of §1), it suffices to prove the results for the variety $X_{k(x)}$ and $\mathcal{L}_A \otimes k(x)$, for a point $x \in X_A$ (since the various dimension functions are constant on $\operatorname{Spec} A$). Next, we can find a closed point $x \in \operatorname{Spec} A$, such that if $(R, \mathcal{M})$ is the local ring of $\operatorname{Spec} A$ at x, then R is regular, and the residue characteristic $p \notin \mathcal{M}^2$. If $p, x_1, \ldots, x_r$ is a regular system of parameters, then $S = R/(x_1, \ldots, x_r)$ is a regular local ring of dimension 1 (*i.e.*, a discrete valuation ring) whose maximal ideal is generated by the prime number p, and whose residue field is a finite field $\mathbb{F}$. Hence the completion $\widehat{S}$ is the ring of p-adic integers in the corresponding unramified extension of $\mathbb{Q}_p$, which is the ring of Witt vectors $W(\mathbb{F})$. Hence, by Theorems 1 and 2, we see that the claimed results hold for $X_{\mathbb{F}}$ and the line bundle $\mathcal{L}\mid_{X_{\mathbb{F}}}$. □

4 The Cartier operator

Let k be a perfect field of characteristic p, X/k a smooth variety, $F : X \to X$ the absolute Frobenius morphism (identity on points, and the p^{th} power homomorphism on functions). If $\mathcal{G}$ is a coherent $\mathcal{O}_X$-module, its direct image $F_*\mathcal{G}$ is the coherent $\mathcal{O}_X$-module with the same underlying sheaf of abelian groups, and new $\mathcal{O}_X$-module structure given by

$$f \cdot s = f^p s, \ \ \forall\, f \in \Gamma(\mathcal{O}_X), \ \ s \in \Gamma(\mathcal{G}).$$

Then the direct image $F_*\Omega^{\bullet}_{X/k}$ of the de Rham complex is a complex of coherent $\mathcal{O}_X$-modules and $\mathcal{O}_X$-linear maps, so that if

$$\mathcal{Z}^j_X = \ker(F_*\Omega^j \xrightarrow{d} F_*\Omega^{j+1}_{X/k},$$

$$\mathcal{B}^j_X = \operatorname{im}(F_*\Omega^{j-1}_{X/k} \xrightarrow{d} F_*\Omega^j_{X/k}),$$

$$\mathcal{H}^j_X = \mathcal{Z}^j_X/\mathcal{B}^j_X,$$

then

$$\mathcal{B}^j_X \subset \mathcal{Z}^j_X \subset F_*\Omega^j_{X/k}$$

are $\mathcal{O}_X$-submodules, and $\mathcal{H}^j_X$ is a coherent $\mathcal{O}_X$-module. We then have the two hypercohomology spectral sequences

$$E_1^{i,j} = H^j(X, \Omega^i_{X/k}) \Rightarrow H^{i+j}_{DR}(X/k),$$

and

$$E_2^{i,j} = H^i(X, \mathcal{H}^j_X) \Rightarrow H^{i+j}_{DR}(X/k).$$

These are spectral sequences of finite dimensional k-vector spaces.

Theorem 0 *(Cartier) If X/k is a smooth variety, there are $\mathcal{O}_X$-linear maps*

$$C = C_j : \mathcal{Z}^j_X \to \Omega^j_{X/k},$$

inducing isomorphisms $\mathcal{H}^j_X \cong \Omega^j_{X/k}$. These maps satisfy

(i) $C_1(d\,f) = 0$

(ii) $C_1(\frac{d\,f}{f}) = \frac{d\,f}{f}$

(iii) $C_j(f^p\omega) = fC_j(\omega)$ *for all* $\omega \in \mathcal{Z}^j_X$, $f \in \mathcal{O}_X$.

(iv) $C_i(\omega) \wedge C_j(\omega') = C_{i+j}(\omega)$ *for* $\omega \in \mathcal{Z}^i_X$, $\omega' \in \mathcal{Z}^j_X$.

Proof: (Deligne) We will define instead a map $C_1^{-1} : \Omega^1_{X/k} \to \mathcal{H}^1_X$, and set $C_i^{-1} =$ "$\overset{i}{\wedge}\, C_1^{-1}$". To define C_1^{-1}, we note that $\varphi : \mathcal{O}_X \to \mathcal{H}^1_X$ given by $\varphi(f) = f^{p-1}d\,f\,(\mathrm{mod}\,\mathcal{B}^1_X)$ is in fact a derivation (and hence induces the desired map C_1^{-1}). Indeed, we verify that

(i) $\varphi(f_1 f_2) = f_1^p\varphi(f_2) + f_2^p\varphi(f_1)\,(\mathrm{mod}\,\mathcal{B}^1_X)$

(ii) $\varphi(f_1 + f_2) = (f_1 + f_2)^{p-1}(d\,f_1 + d\,f_2)\,(\mathrm{mod}\,\mathcal{B}^1_X)$,
while

$$\varphi(f_1) + \varphi(f_2) = f_1^{p-1}d\,f_1 + f_2^{p-1}d\,f_2\,(\mathrm{mod}\,\mathcal{B}^1_X)$$

$$= (f_1 + f_2)^{p-1}d(f_1 + f_2) - d(\sum_{i-1}^{p-1}\frac{1}{p}\left(\begin{array}{c} p \\ i \end{array}\right)f_1^{i-1}f_2^{p-i}\,(\mathrm{mod}\,\mathcal{B}^1_X)$$

$$= (f_1 + f_2)^{p-1}d(f_1 + f_2)\,(\mathrm{mod}\,\mathcal{B}^1_X).$$

Now one verifies easily by a local computation that the maps C_i^{-1} are all isomorphisms if X is smooth. □

5 Proofs of Theorems 1 and 2

Let X be a smooth variety over a perfect field k of characteristic p. Consider the exact sequence of locally free $\mathcal{O}_X$-modules

$$0 \to \mathcal{B}^1_X \to \mathcal{Z}^1_X \to \Omega^1_{X/k} \to 0$$

where $\mathcal{B}_X^1$ is the sheaf of locally exact 1-forms on X, so that there is another exact sequence

$$0 \to \mathcal{O}_X \to F_*\mathcal{O}_X \stackrel{d}{\to} \mathcal{B}_X^1 \to 0$$

(d is the exterior derivative; the exactness of the second sheaf sequence means, in particular, that $df = 0 \Rightarrow f = g^p$ for $f \in \Gamma(\mathcal{O}_X)$). The first extension corresponds to a class

$$\xi_1 \in \mathrm{Ext}^1_{\mathcal{O}_X}(\Omega^1_{X/k}, \mathcal{B}^1_X) = H^1(X, \mathcal{T}_X \otimes \mathcal{B}^1_X).$$

Here $\mathcal{T}_X$ is the tangent sheaf of X. The surjection

$$d : F_*\mathcal{O}_X \to \mathcal{B}^1_X$$

induces a map

$$f : \mathrm{Ext}^1_{\mathcal{O}_X}(\Omega^1_{X/k}, F_*\mathcal{O}_X) \to \mathrm{Ext}^1_{\mathcal{O}_X}(\Omega^1_{X/k}, \mathcal{B}^1_X).$$

Lemma 1 *If X lifts to a scheme flat over $W_2(k)$, then*

$$\xi_1 = \alpha(\zeta)$$

for some $\zeta \in \mathrm{Ext}^1_X(\Omega^1_{X/k}, F_*\mathcal{O}_X)$. Hence we have a pushout diagram of exact sequences

$$\begin{array}{ccccccccc} 0 \to & F_*\mathcal{O}_X & \to \mathcal{V} \to & \Omega^1_{X/k} & \to 0 \\ & d \downarrow & \downarrow & \| & \\ 0 \to & \mathcal{B}^1_X & \to Z^1_X \to & \Omega^1_{X/k} & \to 0 \end{array}$$

for some locally free $\mathcal{O}_X$-module $\mathcal{V}$.

We defer the proof of the lemma.

Proof of Theorem 1: In the diagram in the statement of lemma 1, the top row may be regarded as a quasi-isomorphism (which we denote ψ_1) between the 2-term complex (which we denote $\mathcal{C}_1$)

$$F_*\mathcal{O}_X \to \mathcal{V}$$

and

$$\Omega^1_{X/k}[-1].$$

The pushout diagram induces a map of complexes

$$\varphi_1 : \mathcal{C}_1 \to F_*\Omega^\bullet_{X/k},$$

which induces an isomorphism on first cohomology sheaves. This is done as follows: if $\beta : \mathcal{V} \to \mathcal{Z}^1_X$ is the vertical map in the pushout diagram, and $\gamma : \mathcal{V} \to F_*\Omega^1_{X/k}$ is obtained from β by composing with the inclusion $\mathcal{Z}^1_X \subset F_*\Omega^1_{X/k}$, then we have a diagram whose vertical maps give a map of complexes ($d = \dim X$)

$$\begin{array}{ccccccccc} 0 \to & F_*\mathcal{O}_X & \to \mathcal{V} \to & 0 & \to \cdots \to & 0 & \to 0 \\ & \| & \gamma \downarrow & \downarrow & & \downarrow & \downarrow \\ 0 \to & F_*\mathcal{O}_X & \to F_*\Omega^1_{X/k} \to & F_*\Omega^2_{X/k} & \to \cdots \to & F_*\Omega^d_{X/k} & \to 0 \end{array}$$

The complex $\mathcal{C}_1^{\otimes i}$ is clearly quasi-isomorphic to the complex $(\Omega^1_{X/k})^{\otimes i}[-i]$; the multiplicative structure on the de Rham complex yields a map "$\varphi_1^{\otimes i}$" of complexes

$$\mathcal{C}_1^{\otimes i} \to F_*\Omega^\bullet_{X/k},$$

which yields the exterior product map

$$\wedge^i : (\Omega^1_{X/k})^{\otimes i} \to \Omega^i_{X/k}$$

on i^{th} cohomology sheaves. If $i < p$, the characteristic, then as in [DI], we see that the map $\wedge^i$ is a *split surjection*; hence we can find an injective $\mathcal{O}_X$-linear map

$$h_i : \Omega^i_{X/k} \to (\Omega^1_{X/k})^{\otimes i}$$

which splits the surjection $\wedge^i$. Let $\mathcal{C}_i$ be the complex of locally free $\mathcal{O}_X$-modules obtained from the pullback diagram with exact rows

$$\begin{array}{ccccccccc} 0 \to & (\mathcal{C}_i)^0 & \cdots & (\mathcal{C}_i)^{i-1} & \to & (\mathcal{C}_i)^i & \to & \Omega^i_{X/k} & \to 0 \\ & \| & & \| & & \downarrow & & h_i \downarrow & \\ 0 \to & (\mathcal{C}_1^{\otimes i})^0 & \cdots & (\mathcal{C}_1^{\otimes i})^{i-1} & \to & (\mathcal{C}_1^{\otimes i})^i & \to & (\Omega^1_{X/k})^{\otimes i} & \to 0 \end{array}$$

Thus, there is a quasi-isomorphism

$$\psi_i : \mathcal{C}_i \to \Omega^i_{X/k}[-i],$$

and "$\varphi_1^{\otimes i}$" induces a map of complexes

$$\varphi_i : \mathcal{C}_i \to F_*\Omega^\bullet_{X/k}$$

which yields an isomorphism on the i^{th} cohomology sheaves.

Hence, if $d < p$, the map

$$\Phi = \sum_{i=0}^{d} \varphi_i : \oplus_{i=0}^{d}\mathcal{C}_i \to F_*\Omega^\bullet_{X/k}$$

is a quasi-isomorphism (define $\mathcal{C}_0$ to be $\mathcal{O}_X$ in degree 0, and φ_0 to be the map induced by the inclusion $\mathcal{O}_X \to F_*\mathcal{O}_X$).

The map Φ induces isomorphisms (i) on hypercohomology, and (ii) on the second hypercohomology spectral sequences. Further, the second spectral sequence for

$$\oplus_{i=0}^{d}\mathcal{C}_i$$

is the direct sum of the spectral sequences for each of the $\mathcal{C}_i$, while the second spectral sequence for $\mathcal{C}_i$ degenerates at E_2 (since $\mathcal{C}_i$ has only one non-trivial cohomology sheaf). Thus, the second spectral sequence for $\oplus\mathcal{C}_i$ degenerates at E_2; hence the second spectral sequence for $F_*\Omega^\bullet{}_{X/k}$ also degenerates at E_2.

We thus obtain an equality between dimensions

$$\sum_{i+j=n} \dim_k H^i(X, \Omega^j_{X/k}) = \sum_{i+j=n} \dim_k E_2^{i,j} = \sum_{i+j=n} \dim_k E_\infty^{i,j}$$

$$= \dim_k \mathbb{H}^n(X, F_*\Omega^\bullet_{X/k}) = \dim_k H^n_{DR}(X/k),$$

where the last equality holds because k is perfect (so that the dimensions over k and 'F_*k' are equal). But the equality of the two extreme terms is equivalent to the degeneration of the first hypercohomology spectral sequence at E_1, as observed earlier.

□

Proof of Theorem 2: Observe that $H^j(X, \Omega^i \otimes \mathcal{L}^{-p^n}) = 0$ for any ample $\mathcal{L} \in \operatorname{Pic} X$, for $n >> 0$, by Serre duality and the Serre vanishing theorem. Hence by induction on n, it suffices to prove that

$$H^j(X, \Omega^i_{X/k} \otimes \mathcal{L}^{-p}) = 0 \ \forall \ i+j < d \Rightarrow H^j(X, \Omega^i_{X/k} \otimes \mathcal{L}^{-1}) = 0 \ \forall \ i+j < d.$$

But

$$\Phi \otimes 1 : \oplus_{i=0}^{d} \mathcal{C}_i \otimes \mathcal{L}^{-1} \to F_* \Omega^{\bullet} \otimes \mathcal{L}^{-1}$$

is a quasi-isomorphism, and hence induces isomorphisms on second spectral sequences. Thus, the second spectral sequence for $F_* \Omega^{\bullet}_{X/k} \otimes \mathcal{L}^{-1}$ degenerates at E_2.

However,

$$F_* \Omega^i_{X/k} \otimes \mathcal{L}^{-1} \cong F_*(\Omega^i_{X/k} \otimes F^* \mathcal{L}^{-1}) \cong F_*(\Omega^i \otimes \mathcal{L}^{-p}),$$

and

$$H^j(X, F_*(\Omega^i \otimes \mathcal{L}^{-p})) \cong H^j(X, \Omega^i_{X/k} \otimes \mathcal{L}^{-p}) = 0 \;\; \forall \;\; i + j < d,$$

by assumption. From the first spectral sequence, we then see that

$$\mathbb{H}^n(X, F_* \Omega^{\bullet} \otimes \mathcal{L}^{-1}) = 0 \;\; \forall \;\; n < d.$$

Since the second spectral sequence degenerates at E_2, we must have

$$H^i(X, \Omega^j_{X/k} \otimes \mathcal{L}^{-1}) = E_2^{j,i} = 0 \;\; \forall \;\; i + j < d.$$

This is the inductive step.

□

Proof of lemma 1: Let $U = \operatorname{Spec} A$ be an affine open set in X, $\widetilde{U} = \operatorname{Spec} \widetilde{A} \subset \widetilde{X}$ the corresponding open subset. Then $\widetilde{A}$ is a smooth $W_2(k)$-algebra. Since $\ker(\widetilde{A} \to A) = p\widetilde{A}$ is nilpotent, the p^{th} power homomorphism $A \to A$ lifts to a homomorphism $\varphi_U : \widetilde{A} \to \widetilde{A}$, covering the "Frobenius" map $W_2(k) \to W_2(k)$ (the ring automorphism of $W_2(k)$ which induces the Frobenius on the residue field). Then we may write $\varphi_U(a) = a^p + p\theta_U(\overline{a})$, where $\theta_U : A \to F_*A$ is an A-linear homomorphism, and $\overline{a}$ is the image of $a \in \widetilde{A}$ in A; the function θ_U satisfies

$$\theta_U(ab) = a^p \theta_U(b) + b^p \theta_U(a) \;\; \forall \;\; a, b \in A,$$

$$\theta_U(a + b) = \theta_U(a) + \theta_U(b) - \sum_{i=1}^{p-1} (\left(\begin{array}{c} p \\ i \end{array} \right) / p) a^i b^{p-i}.$$

These formulae imply that

$$a \mapsto a^{p-1} d\, a + d\, \theta_U(a)$$

defines a (k-linear) derivation $A \to \mathcal{Z}^1_A = \Gamma(U, \mathcal{Z}^1_X)$, such that the induced map $\Omega^1_{U/k} \to \mathcal{Z}^1_U$ splits the Cartier operator $C_1 : \mathcal{Z}^1_U \to \Omega^1_{U/k}$ (since the above derivation lifts the derivation $\mathcal{O}_U \to \mathcal{Z}^1_U \pmod{\mathcal{B}^1_U}$ used in defining the Cartier operator).

Now if $\{U_\alpha\}$ is an affine open cover of X, then $\xi_1 \in \mathrm{Ext}^{\,1}_{\mathcal{O}_X}(\Omega^1_{X/k}, \mathcal{B}^1_X)$ is represented by the Čech cocycle

$$h_{\alpha\beta} \in \Gamma(U_\alpha \cap U_\beta, \mathcal{H}om\,(\Omega^1_{X/k}, \mathcal{B}^1_X)) = \Gamma(U_\alpha \cap U_\beta, \mathcal{D}er(\mathcal{O}_X, \mathcal{B}^1_X)),$$

$$h_{\alpha\beta}(a) = d\,\theta_{U_\alpha}(a) - d\,\theta_{U_\beta}(a),$$

where $\mathcal{D}er$ denotes the sheaf of derivations. But then

$$\widetilde{h}_{\alpha\beta} \in \Gamma(U_\alpha \cap U_\beta, \mathcal{D}er(\mathcal{O}_X, F_*\mathcal{O}_X)),$$

$$\widetilde{h}_{\alpha\beta}(a) = \theta_{U_\alpha}(a) - \theta_{U_\beta}(a),$$

is also a 1-cocycle, giving the desired preimage under f of ξ_1.

□

Exercise 5.1:

1. Show that the Cartier operator yields a map on differential forms on a non-singular variety X over k with *logarithmic poles* along a divisor D with normal crossings, yielding isomorphisms of cohomology sheaves

$$\mathcal{H}^j(F_*\{\Omega^\bullet_{X/k}(\log D)\}) \xrightarrow{C} \Omega^j_{X/k}(\log D),$$

for each $0 \leq j \leq \dim X$.

2. Show that if X is non-singular of dimension $d < p = \mathrm{char}\, k$, and the pair (X, D) has a flat lift to $\mathrm{Spec}\, W_2(k)$, then the logarithmic de Rham complex decomposes in the derived category as

$$F_*\{\Omega^\bullet_{X/k}(\log D)\} \cong \bigoplus_{j=0}^{\dim X} \Omega^j_{X/k}(\log D)[-j].$$

3. Deduce that if X is non-singular and proper over a field k of characteristic 0, and D a divisor in X with normal crossings, then the spectral sequence

$$E_1^{r,s} = H^s(X, \Omega^r_{X/k}) \Rightarrow \mathbb{H}^{r+s}(X, \Omega^\bullet_{X/k})$$

degenerates at E_1.

(Compare with the proofs in P. Deligne, *Theorié de Hodge II, III*, Publ. Math. IHES, (1971) 5-57 and (1974) 5-77.)

References

[DI] P. Deligne and L. Illusie, *Rélevements modulo p^2 et décomposition du complexe de de Rham*, Invent. Math. 89 (1987) 247-270.