Kleiman's boundedness results[*](SGA 6, Exp. XIII)

1 Regularity and (b)-sheaves[1]

Let X be a projective scheme over an algebraically closed field k, and let $\mathcal{O}_X(1)$ be an ample invertible sheaf on X.

Definition 1.1: A coherent sheaf $\mathcal{F}$ on X is called *m-regular* (with respect to $\mathcal{O}_X(1)$) if

(i) the global sections of $\mathcal{O}_X(1)$ generate it at all points of $\operatorname{supp}(\mathcal{F})$

(ii) $H^q(X, \mathcal{F}(m-q)) = 0$ for all $q > 0$.

Lemma 1.2 *If $0 \to \mathcal{F}(-1) \to \mathcal{F} \to \mathcal{G} \to 0$ is an exact sequence of coherent $\mathcal{O}_X$-modules such that $\mathcal{F}$ is m-regular, then $\mathcal{G}$ is m-regular.*

Proof: This follows trivially from the definitions, and the long exact sequence in cohomology. □

Proposition 1.3 *Let $\mathcal{F}$ be an m-regular sheaf on X. Then for all $n \geq m$,*

(i) $\mathcal{F}$ is n-regular

[*] A loose translation from French of parts of SGA 6,Exp. XIII, Lect. Notes in Math. 225, Springer, 1970.

[1] The numbering in this section agrees with that in Exp. XIII.

(ii) $H^0(\mathcal{F}(n)) \otimes H^0(\mathcal{O}_X(1)) \to H^0(\mathcal{F}(n+1))$ *is surjective*

(iii) $\mathcal{F}(n)$ *is generated by* $H^0(\mathcal{F}(n))$.

Proof: By induction on $s = \dim \operatorname{supp}(\mathcal{F})$; the result is trivial for $s = 0$. If $\sigma \in H^0(\mathcal{O}_X(1))$ generates it at each associated point of $\mathcal{F}$, then multiplication by σ gives an exact sequence

$$0 \to \mathcal{F}(-1) \xrightarrow{\cdot\sigma} \mathcal{F} \to \mathcal{G} \to 0,$$

where $\dim \operatorname{supp}(\mathcal{G}) < \dim \operatorname{supp}(\mathcal{F})$. By induction, the Proposition holds for $\mathcal{G}$, since by lemma 1.2, $\mathcal{G}$ is also m-regular. From the long exact sequence in cohomology, we get an exact sequence

$$H^q(\mathcal{F}(n-q-1)) \to H^q(\mathcal{F}(n-q)) \to H^q(\mathcal{G}(n-q)).$$

Taking $n \geq m$, $q \geq 1$, we get that $H^q(\mathcal{F}(n-q-1)) \twoheadrightarrow H^q(\mathcal{F}(n-q))$. Hence for $n-1 \geq m$, we see that if $\mathcal{F}$ is $(n-1)$-regular, then $\mathcal{F}$ is n-regular. Hence $\mathcal{F}$ is n-regular for all $n \geq m$, giving (i).

For (ii), consider the diagram

$$\begin{array}{ccccccc} & & & H^0(\mathcal{F}(n)) \otimes H^0(\mathcal{O}_X(1)) & \to & H^0(\mathcal{G}(n)) \otimes H^0(\mathcal{O}_X(1)) & \to 0 \\ & 1 \otimes \sigma & \nearrow & \downarrow \alpha_n & & \downarrow \beta_n & \\ 0 \to & H^0(\mathcal{F}(n)) & \to & H^0(\mathcal{F}(n+1)) & \xrightarrow{\varphi_n} & \to H^0(\mathcal{G}(n+1)) & \end{array}$$

Here for $n \geq m$, β_n is surjective, and from the diagram,

$$(\ker \varphi_n) \subset (\operatorname{im} \alpha_n).$$

Hence α_n is surjective.

For (iii), consider the diagram (where A_X denotes the sheaf $A \otimes_k \mathcal{O}_X$)

$$\begin{array}{ccc} H^0(\mathcal{F}(n))_X \otimes H^0(\mathcal{O}_X(1))_X & \xrightarrow{\delta_n} & H^0(\mathcal{F}(n+1))_X \\ \downarrow & & \downarrow \gamma_{n+1} \\ H^0(\mathcal{F}(n))_X \otimes \mathcal{O}_X(1) & \xrightarrow{\gamma_n \otimes 1} & \mathcal{F}(n+1) \end{array}$$

From (ii), δ_n is surjective; hence

$$(\gamma_{n+1} \text{ is surjective}) \Rightarrow (\gamma_n \text{ is surjective}).$$

But γ_n is surjective for $n >> 0$ (Serre). Hence γ_n is surjective for $n \geq m$. □

Proposition 1.4 *Let $0 \to \mathcal{F}(-1) \to \mathcal{F} \to \mathcal{G} \to 0$ be exact, and let $\mathcal{G}$ be m-regular. Then:*

(i) $H^q(X, \mathcal{F}(n)) = 0$ *for* $q \geq 2$, $n \geq m - q$

(ii) $h^1(\mathcal{F}(n-1)) \geq h^1(\mathcal{F}(n))$ *for* $n \geq m - 1$

(iii) $h^1(\mathcal{F}(n)) = 0$ *for* $n \geq m - 1 + h^1(\mathcal{F}(m-1))$.

In particular $\mathcal{F}$ is $(m + h^1(\mathcal{F}(m-1)))$-regular.

Proof: We have $H^q(\mathcal{F}(n)) \cong H^q(\mathcal{F}(n+1))$ for all $n \geq m-q$, $q \geq 2$. Hence (i) holds by Serre vanishing. For $n \geq m-1$, consider the exact sequence

$$0 \to H^0(\mathcal{F}(n-1)) \to H^0(\mathcal{F}(n)) \xrightarrow{\alpha_n} H^0(\mathcal{G}(n)) \to H^1(\mathcal{F}(n-1)) \to H^1(\mathcal{F}(n)) \to 0$$

(the last map is surjective since $\mathcal{G}$ is m-regular). This gives (ii). Also, consider the diagram

$$\begin{array}{ccc} H^0(\mathcal{F}(n)) \otimes H^0(\mathcal{O}_X(1)) & \xrightarrow{\alpha_n \otimes 1} & H^0(\mathcal{G}(n)) \otimes H^0(\mathcal{O}_X(1)) \\ \downarrow & & \downarrow \beta_n \\ H^0(\mathcal{F}(n+1)) & \xrightarrow{\alpha_{n+1}} & H^0(\mathcal{G}(n+1)) \end{array}$$

Note that β_n is surjective by Proposition 1.4. Hence

$$(\alpha_n \text{ is surjective}) \Rightarrow (\alpha_{n+1} \text{ is surjective}).$$

Hence

$$(H^1(\mathcal{F}(n-1)) = h^1(\mathcal{F}(n))) \Rightarrow$$
$$(H^1(\mathcal{F}(n)) = h^1(\mathcal{F}(n+1)) = \cdots = 0 \text{ (Serre)}).$$

Hence $h^1(\mathcal{F}(n-1)) \neq 0 \Rightarrow h^1(\mathcal{F}(n-1)) > h^1(\mathcal{F}(n))$. Hence for $n \geq m-1$, in at most $h^1(\mathcal{F}(m-1))$ steps, $h^1(\mathcal{F}(n))$ becomes 0. □

Definition 1.5: Let $\mathcal{F}$ be a coherent sheaf on X, $r \geq \dim \operatorname{supp}(\mathcal{F})$ an integer, and let $(b) = (b_0, \ldots, b_r) \in \mathbb{Z}^{\oplus r+1}$. We say that $\mathcal{F}$ is a (b)-*sheaf* if:

(i) $\mathcal{O}_X(1)$ is generated by global sections at all points of $\operatorname{supp}(\mathcal{F})$

(ii) $h^0(\mathcal{F}(-1)) \leq b_0$

(iii) (if $r \geq 1$) there exists $\sigma \in H^0(\mathcal{O}_X(1))$ giving an exact sequence

$$0 \to \mathcal{F}(-1) \xrightarrow{\cdot\sigma} \mathcal{F} \to \mathcal{G} \to 0$$

such that $\mathcal{G}$ is a (b')-sheaf, with $(b') = (b_1, \ldots, b_r) \in \mathbb{Z}^{\oplus r}$.

Proposition 1.6 *Let $\mathcal{F}$ be a (b)-sheaf on X. Then:*

(i) For each "sufficiently general" sequence $(\sigma) = (\sigma_1, \ldots, \sigma_r)$ of sections of $\mathcal{O}_X(1)$, if

$$\mathcal{F}_{\sigma,i} = \textit{restriction of } \mathcal{F} \textit{ to the zero scheme of } \sigma_1 = \cdots = \sigma_i = 0,$$

then $h^0(\mathcal{F}_{\sigma,i}) \leq b_i$ for all $0 \leq i \leq r$.

(ii) Any coherent subsheaf $\mathcal{G} \subset \mathcal{F}$ is a (b)-sheaf.

Proof: Let $S = \mathbb{A}_k^N$ be the affine space whose k-points correspond to sequences (σ), and let T be the open[2] subset of S corresponding to $\mathcal{F}$-regular sequences. For fixed i, the sheaves $\mathcal{F}_{\sigma,i}(-1)$ corresponding to k-points of T are contained in a flat family[3] over T, and by hypothesis, T is non-empty. Now (i) follows from the upper semicontinuity of the function $(\sigma) \mapsto h^0(\mathcal{F}_{\sigma,i}(-1))$, for each i.

Since any "sufficiently general" (σ) is also a $\mathcal{G}$-regular sequence, such that $\mathcal{G}_{\sigma,i} \to \mathcal{F}_{\sigma,i}$ is an inclusion, for each i, we see that (i)⇒(ii). □

Lemma 1.7 *Let $0 \to \mathcal{F}(-1) \to \mathcal{F} \to \mathcal{G} \to 0$ be exact, and let*

$$\chi(\mathcal{F}(n)) = \sum_{i=0}^{r} a_i \binom{n+i}{i}$$

be the Hilbert polynomial of $\mathcal{F}$. Then

$$\chi(\mathcal{G}(n)) = \sum_{i=0}^{r-1} a_{i+1} \binom{n+i}{i}.$$

□

[2] Why is T open?

[3] Presumably because they all have the same Hilbert-Samuel polynomial.

Proposition 1.8 *Let $\mathcal{F}$ be a (b)-sheaf on X, with $s = \dim \operatorname{supp}(\mathcal{F})$, and Hilbert polynomial*

$$\chi(\mathcal{F}(n)) = \sum_{i=0}^{r} a_i \binom{n+i}{i}.$$

Then

(i) for $n \geq -1$, we have

$$h^0(\mathcal{F}(n)) \leq \sum_{i=0}^{s} b_i \binom{n+i}{i}.$$

(ii) $a_s \leq b_s$, and $\mathcal{F}$ is also a $(b_0, \ldots, b_{s-1}, a_s)$-sheaf.

Proof: Induction on s. For $s = 0$, we have $a_0 = h^0(\mathcal{F}) = h^0(\mathcal{F}(-1)) \leq b_0$. If $s \geq 1$, there exists an exact sequence

$$0 \to \mathcal{F}(-1) \to \mathcal{F} \to \mathcal{G} \to 0$$

where $\mathcal{G}$ is a $(b_1, \ldots, b_r)$-sheaf with $\dim \operatorname{supp}(\mathcal{G}) = s - 1$. Further,

$$h^0(\mathcal{F}(n)) - h^0(\mathcal{F}(n-1)) \leq h^0(\mathcal{G}(n)) \leq \sum_{i=0}^{s-1} b_{i+1} \binom{n+i}{i},$$

where the last inequality is by the induction hypothesis. Since $h^0(\mathcal{F}(-1)) \leq b_0$, we deduce (i) by induction on n. Further, $a_s \leq b_s$ and $\mathcal{G}$ is a $(b_1, \ldots, b_{s-1}, a_s)$-sheaf, also by the induction hypothesis. Hence (ii) holds. □

Definition 1.9: The (b)-*polynomials* are defined inductively by

$$\begin{cases} P_{-1} &= 0 \\ P_r(x_0, \ldots, x_r) &= P_{r-1}(x_1, \ldots, x_r) + \sum_{i=0}^{r} x_i \binom{P_{r-1}(x_1, \ldots, x_r) + i - 1}{i} \end{cases}$$

Remark 1.10 Note that $P_r(x_0, \ldots, x_t, 0, 0, \ldots, 0) = P_t(x_0, \ldots, x_t)$.

Theorem 1.11 *Let $\mathcal{F}$ be a (b)-sheaf on X, with Hilbert polynomial*

$$\chi(\mathcal{F}(n)) = \sum_{i=0}^{r} a_i \binom{n+i}{i}.$$

Let $(c) = (c_0, \ldots, c_r)$ be a sequence of integers such that $c_i \geq b_i - a_i$. Let $m = P_r(c_0, \ldots, c_r)$. Then $m \geq 0$, and $\mathcal{F}$ is m-regular. In particular, if $s = \dim \operatorname{supp}(\mathcal{F})$, then $\mathcal{F}$ is $P_{s-1}(c_0, \ldots, c_{s-1})$-regular.

Proof: Induction on r. If $r = 0$, then $m = 0$, and $\mathcal{F}$ is certainly 0-regular (since $s \leq r$). If $r \geq 1$, and

$$0 \to \mathcal{F}(-1) \to \mathcal{F} \to \mathcal{G} \to 0$$

is exact, then $\mathcal{G}$ is a $(b_1, \ldots, b_r)$-sheaf. Hence by induction, if $n = P_{r-1}(c_1, \ldots, c_r)$, then $n \geq 0$, and $\mathcal{G}$ is n-regular. Then $\mathcal{F}$ is $[n + h^1(\mathcal{F}(n-1))]$-regular, and $h^q(\mathcal{F}(n-1)) = 0$ for $q \geq 2$. Now

$$h^1(\mathcal{F}(n-1)) = h^0(\mathcal{F}(n-1)) - \chi(\mathcal{F}(n-1)) \leq \sum_{i=0}^{r} (b_i - a_i) \binom{n+i-1}{i},$$

where the last inequality follows from Proposition 1.8(i), and because $b_i \geq 0$. Hence $\mathcal{F}$ is also $\left[n + \sum_{i=0}^{r} c_i \binom{n+i-1}{i}\right]$-regular.

The final assertion results from Proposition 1.8(ii) (which implies that we may take $c_s = 0$), and Remark 1.10. □

2 Boundedness

Let S be a Noetherian scheme, X an S-scheme of finite type. Let $\mathbb{F}$ be a *family of classes of coherent sheaves on the fibres of X/S*, that is to say, for each point $s \in S$ and each extension K of $k(s)$, we are given a coherent sheaf $\mathcal{F}_K$ on X_K, where $\mathcal{F}_K$ and $\mathcal{F}_{K'}$ determine the same class if there exist $k(s)$-homomorphisms of K, K' into some extension K'' of $k(s)$ such that $\mathcal{F}_{K''} = \mathcal{F}_K \otimes_K K''$ and $\mathcal{F}'_{K''} = \mathcal{F}_{K'} \otimes_{K'} K''$ are isomorphic on $X_{K''}$.

We say that the family $\mathbb{F}$ is *bounded* (or *limited*) *by a coherent sheaf $\mathcal{F}$ on* $X_T = X \times_S T$, where T is of finite type over S, if $\mathbb{F}$ is contained in the family of classes of coherent sheaves $\mathcal{F}_{k(t)}$ with $t \in T$. We say that $\mathbb{F}$ is *bounded* if there exists such a pair $T, \mathcal{F}$.

Suppose X/S is also projective with a (relatively) ample invertible sheaf $\mathcal{O}_X(1)$. We call $\mathbb{F}$ a (b)-*family* for a sequence of integers $(b) = (b_0, \ldots, b_r)$ if each class in $\mathbb{F}$ is representable by an $\mathcal{F}_K$, with K algebraically closed, which is a (b)-sheaf.

Theorem 2.1 *Let S be a Noetherian scheme, X a projective S-scheme with an ample invertible sheaf $\mathcal{O}_X(1)$, such that for any $s \in S$, the induced invertible sheaf $\mathcal{O}_{X_s}(1)$ is generated by $H^0(X_s, \mathcal{O}_{X_s}(1))$. Let $\mathbb{F}$ be a family of classes of coherent sheaves on fibres of X/S. Then the following conditions are equivalent.*

(i) $\mathbb{F}$ is bounded. If in addition, each $\mathcal{F}_K \in \mathbb{F}$ is locally free of rank p, then $\mathbb{F}$ is bounded by a locally free sheaf $\mathcal{F}$ of rank p on X_T, for some T.

(ii) The set of Hilbert polynomials $\chi(\mathcal{F}_K(n))$, for $\mathcal{F}_K \in \mathbb{F}$, is finite, and there exists a sequence of integers (b) such that $\mathbb{F}$ is a (b)-family.

(iii) The set of Hilbert polynomials $\chi(\mathcal{F}_K(n))$, for $\mathcal{F}_K \in \mathbb{F}$, is finite, and there exists an integer m such that each $\mathcal{F}_K \in \mathbb{F}$ is m-regular.

(iv) The set of Hilbert polynomials $\chi(\mathcal{F}_K(n))$, for $\mathcal{F}_K \in \mathbb{F}$, is finite, and $\mathbb{F}$ is contained in the family of quotents of sheaves of the form $\mathcal{E}_K$, where $\mathcal{E}$ is a coherent sheaf on some X_T. Further, one may take $T = S$ and $\mathcal{E} = \mathcal{O}_X(-m)^{\oplus M}$, for some $m, M \geq 0$.

(v) $\mathbb{F}$ is contained in the family of classes of cokernels of homomorphisms $\mathcal{E}'_K \to \mathcal{E}_K$, where $\mathcal{E}, \mathcal{E}'$ are coherent sheaves on some X_T. Further, one may take $T = S$ and $\mathcal{E}, \mathcal{E}'$ of the form $\mathcal{O}_X(-m)^{\oplus M}$, $\mathcal{O}_X(-m')^{\oplus M'}$.

(Exp. XIII, 1.13)

Proof: (i)⇒(ii): suppose $\mathbb{F}$ is bounded by a sheaf $\mathcal{F}$ on X_T. Applying the theorem of generic flatness, and replacing T by a finite disjoint union of locally closed subschemes, we may assume that $\mathcal{F}$ is flat over T. Then the number of Hilbert polynomials $\chi(\mathcal{F}_{k(t)}(n))$ is at most the number of connected components of T. It is an easy lemma that if $t \in T$, and $\mathcal{F}_{k(t)}$ is locally free of rank p, then the same is true of $\mathcal{F}$ over a neighbourhood of

$t \in T$; thus, further subdividing T, we may assume that $\mathcal{F}$ is locally free of rank p, if we are given that each $\mathcal{F}_K$ is locally free of rank p.

Now by further subdividing T, one may assume that there is a sequence (σ) of sections $\sigma_1, \ldots, \sigma_r \in H^0(X_T, \mathcal{O}_{X_T}(1))$ which is a regular sequence on $\mathcal{F}$. Now (ii) follows from the semicontinuity of the function $t \mapsto h^0(\mathcal{F}_{t,i}(-1))$, for $t \in T$, for each $0 \le i \le r$ (here $\mathcal{F}_{t,i}$ denotes the restriction of $\mathcal{F}_t$ to the common zero-scheme of $\sigma_1, \ldots, \sigma_i$).

The implication (ii)$\Rightarrow$(iii) follows immediately from Theorem 1.11. The implication (iii)$\Rightarrow$(iv) follows from Proposition 1.3(iii), if we take $M = \max \chi(\mathcal{F}_K(m))$ and $\mathcal{E} = \mathcal{O}_X(-m)^{\oplus M}$.

Suppose $\mathbb{F}$ satisfies (iv); then for each $\mathcal{F}_K \in \mathbb{F}$, there exists an exact sequence

$$0 \to \mathcal{F}'_K \to \mathcal{E}_K \to \mathcal{F}_K \to 0,$$

and the set of Hilbert polynomials $\chi(\mathcal{F}_K(n))$ is finite. By hypothesis, the family of classes $\mathcal{E}_K$ is bounded; hence by (i)$\Rightarrow$(ii), the set of Hilbert polynomials $\chi(\mathcal{E}_K(n))$ is finite, and there exists a sequence of integers (b) such that each $\mathcal{E}_K$ (with K algebraically closed) is a (b)-sheaf. Hence the set of Hilbert polynomials $\chi(\mathcal{F}'_K(n))$ is finite, and by Proposition 1.6, each $\mathcal{F}'_K$ (with K algebraically closed) is a (b)-sheaf. Applying (ii)$\Rightarrow$(iv) to the family of classes $\mathcal{F}'_K$ yields (v).

Now suppose $\mathbb{F}$ satisfies (v); we prove $\mathbb{F}$ is bounded. First, we reduce to the case when $\mathcal{E}$ (respectively $\mathcal{E}'$) is of the form $\mathcal{O}_X(-m)^{\oplus M}$. Indeed, by (i)$\Rightarrow$(iv), we can find a surjection $\mathcal{L} = \mathcal{O}_{X_T}(-m)^{\oplus M} \twoheadrightarrow \mathcal{E}$. Subdividing T, we may assume $\mathcal{E}$ is flat over T, and the formation of

$$0 \to \mathcal{I} \xrightarrow{u} \mathcal{L} \to \mathcal{E} \to 0$$

commutes with restriction to the fibres. By (i)$\Rightarrow$(iii) and (i)$\Rightarrow$(iv), we can find $m_1 >> 0$ such that $\operatorname{Ext}^1(\mathcal{O}_{X_K}(-m_1), \mathcal{I}_K) = H^1(\mathcal{I}_K(m_1))$ vanishes, and there exist surjections

$$\mathcal{L}_1 = \mathcal{O}_{X_T}(-m_1)^{\oplus M_1} \twoheadrightarrow \mathcal{E}', \quad \mathcal{L}_2 = \mathcal{O}_{X_T}(-m_1)^{\oplus M_2} \twoheadrightarrow \mathcal{I}.$$

Then the maps

$$\operatorname{Hom}(\mathcal{L}_{1,K}, \mathcal{L}_K) \to \operatorname{Hom}(\mathcal{L}_{1,K}, \mathcal{E}_K)$$

are surjective. Let $\beta : \mathcal{E}'_K \to \mathcal{E}_K$ be a homomorphism, and let γ be the composition $\mathcal{L}_{1,K} \to \mathcal{E}'_K \to \mathcal{E}_K$; then γ lifts to a homomorphism $\delta : \mathcal{L}_{1,K} \to$

$\mathcal{L}_K$, such that

$$(\delta, u_K \circ \alpha) : \mathcal{L}_{1,K} \oplus \mathcal{L}_{2,K} \to \mathcal{L}_K, \quad \beta : \mathcal{E}'_K \to \mathcal{E}_K$$

have the same cokernel. Hence $\mathbb{F}$ is contained in the family of classes of cokernels of homomorphisms of the form

$$\mathcal{O}_{X_K}(-m_1)^{\oplus(M_1+M_2)} \to \mathcal{O}_{X_K}(-m)^{\oplus M}.$$

Subdividing T again, we may suppose that

(a) $f_T : X_T \to T$ is flat

(b) $\mathcal{H}om_{\mathcal{O}_{X_T}}(\mathcal{E}', \mathcal{E})$ is flat over T

(c) each of the sheaves $R^q(f_T)_*\mathcal{H}om_{\mathcal{O}_{X_T}}(\mathcal{E}', \mathcal{E})$ is flat over T

(d) the formation of $\mathcal{G} = (f_T)_*\mathcal{H}om_{\mathcal{O}_{X_T}}(\mathcal{E}', \mathcal{E})$ commutes with arbitrary base changes $T' \to T$.

Now taking $R = \mathbb{V}(\mathcal{G}^\vee)$, the scheme X_R supports a canonically defined "universal" exact sequence

$$\mathcal{E}'_R \to \mathcal{E}_R \to \mathcal{F} \to 0,$$

and $\mathcal{F}$ bounds the family $\mathbb{F}$. □

3 (b)-sheaves on $P = \mathbb{P}^N_k$

Let k be an algebraically closed field, and $P = \mathbb{P}^N_k$. Let $\mathcal{F}$ be a coherent sheaf on P, with Hilbert polynomial

$$\chi(\mathcal{F}(n)) = \sum_{i=0}^{r} a_i \binom{n+i}{i},$$

and let $(c) = (c_0, \ldots, c_r)$ be a sequence of integers with $c_i \geq a_i$ for all i. Let

$$a_{m,i} = \sum_{j=0}^{r-i} a_{j+i} \binom{m-1+j}{j}, \quad c_{m,i} = \sum_{j=0}^{r-i} c_{j+i} \binom{m-1+j}{j}.$$

We see easily by induction on m that there is an identity between polynomials in x

$$\sum_{i=0}^{r} a_{m,i}\binom{x+i}{i} = \sum_{j=0}^{r} a_j\binom{x+m+j}{j}. \qquad (1)$$

Lemma 3.1 *Under the above conditions, suppose there exists an exact sequence*

$$0 \to \mathcal{I} \to \mathcal{O}_P^{\oplus M} \to \mathcal{F}(m) \to 0,$$

with $m \geq 0$. Then $\mathcal{I}$ is p-regular with $p = P_r(c_{m,0}, \ldots, c_{m,r})$, where P_r is the r-th (b)-polynomial (definition 1.5).
(Exp XIII,(6.2))

Proof: $\mathcal{O}_P$ is a $(0,0,\ldots,0,1)$-sheaf, so $\mathcal{I}$ is a $(0,\ldots,0,M)$-sheaf (since $\mathcal{I}$ is a subsheaf of such a sheaf — apply Proposition 1.6). Also, we have a formula

$$\chi(\mathcal{I}(n)) = M\chi(\mathcal{O}_P(n)) - \chi(\mathcal{F}(m+n)) = M\binom{n+N}{N} - \sum_{i=0}^{r} a_{m,i}\binom{n+i}{i},$$

where the last equality is using the formula (1). The lemma now follows from Theorem 1.11. □

Proposition 3.2 *Suppose $\mathcal{F}$ is a coherent (b)-sheaf on $P = \mathbb{P}_k^N$ (k algebraically closed), with Hilbert polynomial*

$$\chi(\mathcal{F}(n)) = \sum_{i=0}^{r} a_i\binom{n+i}{i}.$$

Then there exist universal polynomials in a_i, b_i and N bounding m, M, m_1 and M_1 such that there exist exact sequences

$$\mathcal{O}_P(-m_1)^{\oplus M_1} \to \mathcal{O}_P(-m)^{\oplus M} \to \mathcal{F} \to 0.$$

Further, the polynomial bounding[4] *M_1 does not involve N.*
(Exp. XIII, (6.3))

[4]This seems to be a typo, it should presumably be m_1.

Proof: Let $m = P_{r-1}(c_0, \ldots, c_{r-1})$ with $c_i = b_i - a_i$, and let $M = \sum_i a_i \binom{m+i}{i}$. Then from Theorem 1.11 and Proposition 1.3, there is an exact sequence

$$0 \to \mathcal{I} \to \mathcal{O}_P^{\oplus M} \to \mathcal{F}(m) \to 0.$$

Now take $m_1 = m + p$ with $p = P_r(a_{m,0}, \ldots, a_{m,r})$ and

$$M_1 = M\binom{p+N}{N} - \sum a_i \binom{m_1+i}{i} \quad (= \chi(\mathcal{I}(p)))\,.$$

□

Theorem 3.3 *Let $\mathcal{F}$ be a coherent (b)-sheaf on $P = \mathbb{P}_k^N$ (k algebraically closed) with Hilbert polynomial*

$$\chi(\mathcal{F}(n)) = \sum_{i=0}^{r} a_i \binom{n+i}{i}.$$

Suppose $\mathcal{F}$ is a quotient of $\mathcal{O}_P(-m)^{\oplus M}$ for some $m > 0$. Let $b_i = P_{r-i}(c_{m,i}, \ldots, c_{m,r})$ for $i = 0, \ldots, r$, where P_s is the s-th (b)-polynomial, and $(c) = (c_0, \ldots, c_r)$ is a sequence of integers with $c_i \geq a_i$. Let $b = b_1 + m - 1$, and $B = \sum_{i=1}^{r} a_i \binom{b+i}{i}$. Then:

(i) $\mathcal{F}$ *is b-regular*

(ii) $-B \leq a_0 = h^0(\mathcal{F}(b)) - B$

(iii) $\mathcal{F}$ *is a $(b_0, \ldots, b_r)$-sheaf.*

(Exp. XIII, (6.4))

Proof: Choosing a general section of $\mathcal{O}_P(1)$, one obtains a commutative diagram with exact rows and columns (with $Y \cong \mathbb{P}_k^{N-1}$)

$$\begin{array}{ccccc}
 & 0 & & 0 & & 0 \\
 & \downarrow & & \downarrow & & \downarrow \\
0 \to & \mathcal{I}(-1) & \to & \mathcal{I} & \to & \mathcal{J} & \to 0 \\
 & \downarrow & & \downarrow & & \downarrow \\
0 \to & \mathcal{O}_P(-1)^{\oplus M} & \to & \mathcal{O}_P^{\oplus M} & \to & \mathcal{O}_Y^{\oplus M} & \to 0 \\
 & \downarrow & & \downarrow & & \downarrow \\
0 \to & \mathcal{F}(m-1) & \to & \mathcal{F}(m) & \to & \mathcal{G}(m) & \to 0 \\
 & \downarrow & & \downarrow & & \downarrow \\
 & 0 & & 0 & & 0
\end{array}$$

Since (lemma 1.7)

$$\chi(\mathcal{G}(n)) = \sum_{i=0}^{r-1} a_{i+1} \binom{n+i}{i},$$

$\mathcal{J}$ is b_1-regular, by lemma 3.1; hence $H^q(\mathcal{I}(p)) = 0$ for $q \geq 2$ and $p \geq b_1 - q$ (Proposition 1.4). The exact sequence

$$H^q(\mathcal{O}_P(p)^{\oplus M}) \to H^q(\mathcal{F}(m+p)) \to H^{q+1}(\mathcal{I}(p))$$

implies that $\mathcal{F}(m)$ is $(b_1 - 1)$-regular. This gives (i).

Now (i) implies that

$$0 \leq h^0(\mathcal{F}(b)) = \chi(\mathcal{F}(b)) = \sum_{i=0}^{r} a_i \binom{b+i}{i}.$$

This implies (ii). Then by the formula (1),

$$\chi(\mathcal{F}(b)) = \sum_{i=0}^{r} a_{m,i} \binom{b_1 - 1 + i}{i} \leq b_1 + \sum_{i=0}^{r} c_{m,i} \binom{b_1 - 1 + i}{i}$$

$$= P_{r-1}(c_{m,1}, \ldots, c_{m,r}) + \sum_{i=0}^{r} c_{m,i} \binom{P_{r-1}(c_{m,1}, \ldots, c_{m,r}) + i - 1}{i}$$

$$= P_r(c_{m,0}, c_{m,1}, \ldots, c_{m,r}) \text{by } (1.5)$$

$$= b_0 \text{ (by definition of } b_0).$$

Hence $h^0(\mathcal{F}(-1)) \leq h^0(\mathcal{F}(b)) \leq b_0$. By induction on r, we may assume $\mathcal{G}$ is a $(b_1, \ldots, b_r)$-sheaf. Hence $\mathcal{F}$ is a $(b_0, \ldots, b_r)$-sheaf. □

Lemma 3.4 *Suppose $\mathcal{F}$ is a coherent sheaf on P, and has no subsheaf supported at closed points.*

(i) If $h^0(\mathcal{F}) \geq 1$, then $h^0(\mathcal{F}(-1)) \leq h^0(\mathcal{F}) - 1$.

(ii) $H^0(\mathcal{F}(-n)) = 0$ for $n \geq h^0(\mathcal{F})$.

(iii) Suppose there exists an exact sequence

$$0 \to \mathcal{F}(-1) \to \mathcal{F} \to \mathcal{G} \to 0$$

and an integer $n_0 \geq 0$ such that $H^0(\mathcal{G}(-n_0)) = 0$. Then $h^0(\mathcal{F}(-n_0)) = 0$, and $h^0(\mathcal{F}) \leq n_0 h^0(\mathcal{G})$.

(Exp. XIII, (6.5))

Proof: Suppose $0 \neq \sigma \in H^0(\mathcal{F})$. Then $\sigma \cdot \mathcal{O}_X = \mathcal{F}' \hookrightarrow \mathcal{F}$ is a subsheaf with $s = \dim \operatorname{supp}(\mathcal{F}') \geq 1$. This gives rise to a a diagram with exact rows and columns

$$\begin{array}{ccccc} & 0 & & 0 & 0 \\ & \downarrow & & \downarrow & \downarrow \\ 0 \to & \mathcal{F}'(-1) & \xrightarrow{\cdot x} & \mathcal{F}' & \to \mathcal{G}' \to 0 \quad (\text{for a suitable } x \in H^0(\mathcal{O}_X(1))) \\ & \downarrow & & \downarrow & \downarrow \\ 0 \to & \mathcal{F}(-1) & \xrightarrow{\cdot x} & \mathcal{F} & \to \mathcal{G} \to 0 \end{array}$$

Since $s \geq 1$, $\mathcal{G}' \neq 0$. Hence $\mathcal{F}' \not\subset \mathcal{F}(-1)$, and so $\sigma \notin H^0(\mathcal{F}(-1)) \subset H^0(\mathcal{F})$. This proves (i); now (ii) follows immediately. For (iii), note that

$$0 \leq h^0(\mathcal{F}(n)) - h^0(\mathcal{F}(n-1)) \leq h^0(\mathcal{G}(n)) \tag{2}$$

If $n \geq n_0$, we get that $h^0(\mathcal{F}(-n_0)) = h^0(\mathcal{F}(-n))$, since $h^0(\mathcal{G}(-n_0)) = 0$. But by (ii), $h^0(\mathcal{F}(-n)) = 0$ for $n >> 0$. Hence $h^0(\mathcal{F}(-n)) = 0$ for all $n \geq n_0$. Also, for $p \geq 1$, we get from (2) that

$$h^0(\mathcal{F}(p-n_0)) \leq h^0(\mathcal{G}(1-n_0)) + \cdots + h^0(\mathcal{G}(p-n_0)) \leq ph^0(\mathcal{G}(p-n_0)).$$

This gives (iii). □

Let $\mathcal{F}$ be a coherent sheaf on P. For each integer $q \geq 1$, let $\mathcal{N}_q$ be the largest subsheaf of $\mathcal{F}$ with $\dim \operatorname{supp}(\mathcal{N}_q) < q$, and set $\mathcal{F}_q = \mathcal{F}/\mathcal{N}_q$.

Proposition 3.5 *If $\mathcal{F}$ is a coherent (b)-sheaf on $\mathbb{P}$, then $\mathcal{F}_q$ is a $(b^q_{q-1}, b^{q-1}_{q-1}, \ldots, b^2_{q-1}, b_{q-1}, b_q, \ldots, b_r)$-sheaf.*
(Exp. XIII, (6.6)

Proof: A "general" section of $\mathcal{O}_X(1)$ gives rise to a commutative diagram with exact rows and columns

$$\begin{array}{ccccc} & 0 & & 0 & 0 \\ & \downarrow & & \downarrow & \downarrow \\ 0 \to & \mathcal{N}_q(-1) & \to & \mathcal{N}_q & \to \mathcal{G}' \to 0 \\ & \downarrow & & \downarrow & \downarrow \\ 0 \to & \mathcal{F}(-1) & \to & \mathcal{F} & \to \mathcal{G} \to 0 \\ & \downarrow & & \downarrow & \downarrow \\ 0 \to & \mathcal{F}_q(-1) & \to & \mathcal{F}_q & \to \mathcal{G}'' \to 0 \\ & \downarrow & & \downarrow & \downarrow \\ & 0 & & 0 & 0 \end{array}$$

where $\mathcal{G}$ is a $(b_1, \ldots, b_r)$-sheaf, and $\mathcal{G}'' = \mathcal{G}_{q-1}$ (this is "a question of depth").

Suppose $q = 1$. Then $\dim \operatorname{supp}(\mathcal{N}_q) = 0$, so that $H^1(\mathcal{N}_q(-1)) = 0$ and $\mathcal{G}' = 0$. Thus $h^0(\mathcal{F}_q(-1)) \le h^0(\mathcal{F}(-1)) \le b_0$, and $\mathcal{G} \cong \mathcal{G}''$. Hence $\mathcal{F}_1$ is a $(b_0, \ldots, b_r)$-sheaf (since $\mathcal{G}''$ is a $(b_1, \ldots, b_r)$-sheaf). Further, $h^0(\mathcal{F}_1(-b_0)) = 0$.

If $q \ge 2$, we may suppose by induction that $\mathcal{G}_{q-1}$ is a $(b_{q-1}^{q-1}, \ldots, b_{q-1}^2, b_{q-1}, \ldots, b_r)$-sheaf, and that $H^0(\mathcal{G}_{q-1}(-b_{q-1})) = 0$. Then by lemma 3.4, it follows that $H^0(\mathcal{F}_q(-b_{q-1})) = 0$, and $h^0(\mathcal{F}_q(-1)) \le b_{q-1}^q$. □

Theorem 3.6 *There exist 2 sequences of polynomials $\{A_i(x_0, \ldots, x_i; y)\}$ and $\{A_i^{(q)}(x_0, \ldots, x_q; y)\}$ with the following properties. Let $\mathcal{F}$ be a coherent sheaf on $P = \mathbb{P}_k^N$ with Hilbert polynomial*

$$\chi(\mathcal{F}(n)) = \sum_{i=0}^{r} a_i \binom{n+i}{i},$$

and let $(c) = (c_0, \ldots, c_r)$ be a sequence of integers with $a_i \le c_i$. Assume $\mathcal{F}$ is a quotient of $\mathcal{O}_P(-m)^{\oplus M}$ with $m \ge 0$.

(i) If $\mathcal{F}$ is a $(b_0, \ldots, b_r)$-sheaf, then for $i = 0, \ldots, r$ we have

$$\mid a_i \mid \le A_{r-i}(b_i, \ldots, b_r; m).$$

(ii) If

$$\chi(\mathcal{F}_q(n)) = \sum_{i=0}^{r} a_i^{(q)} \binom{n+i}{i}$$

is the Hilbert polynomial of $\mathcal{F}_q$, then for $i = 0, \ldots, q-1$ we have

$$\mid a_i^{(q)} \mid \le A_{r-i}^{(q)}(c_{q-1}, \ldots, c_r; m).$$

(Note that $a_{q-1}^{(q)} \le a_{q-1}$, and $a_q^{(q)} = a_q, \ldots, a_r^{(q)} = a_r$.)

(Exp. XIII, (6.7))

Proof: Let $0 \to \mathcal{F}(-1) \to \mathcal{F} \to \mathcal{G} \to 0$ be an exact sequence such that $\mathcal{G}$ is a $(b_1, \ldots, b_r)$-sheaf. We reason by induction on r, and suppose that $A_0, \ldots, A_{r-1}$ have already been defined with the above properties, by lemma 1.7. Now Proposition 1.8 and Theorem 3.3(ii) imply the existence of a polynomial A_r with

$$| a_0 | \leq A_r(b_0, \ldots, b_r; m).$$

This proves (i).

Now (ii) follows from Theorem 3.3(iii), Proposition 3.5 and the above assertion (i) applied with $\mathcal{F}_q$ in place of $\mathcal{F}$. □

Corollary 3.7 (Grothendieck) *Let X be projective over a Noetherian scheme S, and $\mathcal{O}_X(1)$ very ample for X/S. Let $\mathbb{F}$ be a family of classes of coherent sheaves on the fibres of X/S. Suppose:*

(a) there exists a coherent sheaf $\mathcal{E}$ on X such that $\mathcal{F}$ is contained in the family of classes of quotients of $\mathcal{E}_K$ (notation as in §2)*.*

$(b)_q$ for the Hilbert polynomials $\chi(\mathcal{F}_K(n))$ of the $\mathcal{F}_K \in \mathbb{F}$, the coefficients in degrees $\geq q-1$ are bounded.

Then the $\mathcal{F}_F \in \mathbb{F}$ form a bounded family. In addition, the coefficients of $\chi(\mathcal{F}_K(n))$ in degrees $\geq q-2$ are bounded below.
(Exp. XIII, (6.8))

Proof: We may evidently suppose $X = \mathbb{P}^N_S$, $\mathcal{E} = \mathcal{O}_X(-m)^{\oplus M}$. The first assertion results from Theorem 3.6(ii) and Theorem 2.1; the second follows by induction using an exact sequence

$$0 \to \mathcal{F}(-1) \to \mathcal{F} \to \mathcal{G} \to 0$$

and Theorem 3.6(ii). □

Definition 3.8: Let k be a field. A *special positive k-cycle of dimension r* is a projective k-scheme X, with a very ample invertible sheaf $\mathcal{O}_X(1)$, which is a union of closed subschemes X_j, each of dimension r, where X_j is obtained by a base-change $\operatorname{Spec} k \to \operatorname{Spec} k_j$ from an integral k_j-scheme X'_j, together with $\mathcal{O}_{X'_j}(1)$. We call the coefficient a_r of the Hilbert polynomial $\chi(\mathcal{O}_X(n)) = \sum_{i=0}^{r} a_i \binom{n+i}{i}$ the *degree* of X.
(Exp. XIII, (6.9))

Lemma 3.9 *Let k be an algebraically closed field and X a special positive k-cycle of dimension r and degree d. Then $\mathcal{O}_X$ is a $(0, 0, \ldots, 0, d)$-sheaf.*
(Exp. XIII, (6.10))

Proof: Let $X \subset \mathbb{P}^N_k$ (using the invertible sheaf $\mathcal{O}_X(1)$). Replacing k by the algebraic closure of a pure transcendental extension, we may assume that there exist $r(N+1)$ elements of k which are algebraically independent over each of the subfields $kj \subset k$ (involved in the definition 3.8). Then intersection with the corresponding r "generic" hyperplanes yields special positive k-cycles of degree d (and dimensions ranging from r to 0). We now reason by induction; it suffices to note that for $r = 0$, we have $h^0(\mathcal{O}_X(-1)) = d$, and for $r \geq 1$, we have $H^0(\mathcal{O}_X(-1)) = 0$ (for the latter point, use the inclusion $\mathcal{O}_X \hookrightarrow \prod \mathcal{O}_{X_j}$ to reduce to the case when X is an integral scheme, in which case $H^0(\mathcal{O}_X)$ is a field). □

Corollary 3.10 *(i) Let r, d be integers, $c_i = A_{r-i}(0, \ldots, 0, d; 0)$ for $i = 0, \ldots, r$* (notation as in Theorem 3.6(i))*, and let $p = P_r(c_0, \ldots, c_r)$* (*c.f.* definition 1.5)*. Let k be an algebraically closed field, and X a special positive k-cycle of dimension $\leq r$ and degree $\leq d$, with Hilbert polynomial $\chi(\mathcal{O}_X(n)) = \sum_{i=0}^{r} a_i \binom{n+i}{i}$. Then for $0 \leq i \leq r$, we have*

$$| a_i | \leq c_i.$$

Further, X can be embedded in $\mathbb{P}^N_k$ with $N = d(r+1) - 1$, and defined there by (at most) $\binom{N+p}{p}$ equations of degree p.

(ii) (Chow) *Let S be a Noetherian scheme. For K varying over the algebraically closed extension fields of $k(s)$, $s \in S$, the special positive K-cycles of bounded dimension and degree form a "bounded family" (where the cycles X are considered as subschemes of a fixed $\mathbb{P}^N_S$, in an "evident abstract sense").*

(Exp. XIII, (6.11))

Proof: In (i), the first assertion follows from Theorem 3.6(i); the second assertion follows from lemma 3.9 and Proposition 1.8(i), together with lemma 3.1 and Proposition 1.3, applied to the exact sequence

$$0 \to \mathcal{I} \to \mathcal{O}_P \to \mathcal{O}_X \to 0, \quad P = \mathbb{P}^N_k.$$

(ii) follows from (i) and Theorem 2.1. □

Remark 3.11 (Exp. XIII, (6.12))

Let k be an algebraically closed field, X a projective k-scheme of dimension r, with no embedded associated prime cycle. If one admits nilpotent elements in an arbitrary fashion, one cannot bound the coefficients of the Hilbert polynomial $\chi(\mathcal{O}_X(n)) = \sum_{i=0}^{r} a_i \binom{n+i}{i}$ solely in terms of a_r.

For example, let Z be a smooth projective curve of degree d in $\mathbb{P}^3_k$. For each $n >> 0$, there exists a smooth surface Y of degree n in $\mathbb{P}^3_k$ with $Z \subset Y$. Then Z is a Cartier divisor on Y; let $\mathcal{O}_{X_n} = \mathcal{O}_Y/\mathcal{O}_Y(-2Z)$. From the exact sequence

$$0 \to \mathcal{O}_{X_n}(-Z) \to \mathcal{O}_{X_n} \to \mathcal{O}_Z \to 0,$$

and the formula $(Z \cdot Z)_Y = 2p_a(Z) - 2 - (n-4)d$, we have that $a_0 = \chi(\mathcal{O}_{X_n}(-1)) \to \infty$ as $n \to \infty$, while $a_1 = 2d$.

Corollary 3.12 *Let $\mathcal{F}$ be a coherent sheaf on $P = \mathbb{P}^N_k$, where k is an algebraically closed field, such that there exists an exact sequence*

$$\mathcal{O}_P(-m_1)^{\oplus M_1} \to \mathcal{O}_P(-m)^{\oplus M} \xrightarrow{\alpha} \mathcal{F} \to 0$$

with $m_1 \geq m \geq 0$. Let

$$\chi(\mathcal{F}(n)) = \sum_{i=0}^{r} a_i \binom{n+i}{i}$$

be the Hilbert polynomial of $\mathcal{F}$. Then for $0 \leq i \leq N$, we have

$$|\, a_i \,| \leq A_{N-i}(0, \ldots, 0, M; m_1) + M\binom{m}{N-i}.$$

(Exp. XIII, (6.13))

Proof: If $\mathcal{I} = \ker \alpha$, then $\mathcal{I}$ is a $(0, \ldots, 0, m)$-sheaf (being a subsheaf of one), and has Hilbert polynomial

$$\chi(\mathcal{I}(n)) = \sum_{i=0}^{N} \left[M\binom{-m-1+N-i}{N-i} - a_i \right] \cdot \binom{n+i}{i}.$$

The corollary follows from Theorem 3.6(i) and the identity $\binom{-m-1+Q}{Q} = (-1)^Q \binom{m}{Q}$. $\square$

Corollary 3.13 (Hermann) *For each $N \geq 0$, there exists a polynomial $R_N(x)$ such that for any field k_0, an any ideal $I \subset k_0[T_1, \ldots, T_N]$ in the polynomial ring such that I is generated by elements of degree $\leq m$, the radical $\sqrt{I}$ is generated by elements of degree $\leq R_N(m)$.*
(Exp. XIII, (6.14))

Proof: Let k be the algebraic closure of k_0. Introduce an auxilliary variable T_0, and consider the subscheme Y (respectively X) of $P = \mathbb{P}_k^N$ defined by the homogenization of I (respectively $\sqrt{I}$). Then for the Hilbert polynomial

$$\chi(\mathcal{O}_Y(n)) = \sum_{i=0}^{N} b_i \binom{n+i}{i},$$

we have $| b_i | \leq A_{N-i}(0, \ldots, 0, 1; m)$, by corollary 3.12.

Let $X = \cup X^q$ be the decomposition according to dimension (*i.e.*, X^q is the union of the q-dimensional irreducible components). Then we claim $\deg(X^q) \leq e_q = P_{N-q}(c_q, \ldots, c_N)$. Indeed, intersecting X (respectively Y) by a "general" linear subspace of codimension q, we may assume $q = 0$. Then evidently

$$\deg(X^q) = h^0(\mathcal{O}_{X^q}) \leq h^0(\mathcal{O}_{Y^q}) \leq e_q,$$

where the last inequality is by Theorem 3.3(iii).

Consequently, $\mathcal{O}_{X^q}$ is a $(0, \ldots, 0, e_q)$-sheaf, by lemma 3.9. From the injection $\mathcal{O}_X \to \prod \mathcal{O}_{X^q}$, it follows that $\mathcal{O}_X$ is an $(e_0, \ldots, e_q)$-sheaf. Hence the coefficients of the Hilbert polynomial $\chi(\mathcal{O}_X(n)) = \sum_{i=0}^{r} a_i \binom{n+i}{i}$ satisfy the estimate

$$| a_i | \leq f_i = A_{N-i}(e_i, \ldots, e_N; 0),$$

by Theorem 3.6. Then the ideal $\sqrt{I}$ of X is $R_N(m)$-regular, with $R_N(m) = P_N(f_0, \ldots, f_N)$; now we are done, by Proposition 1.3. $\square$