TOPOLOGICAL AND ALGEBRAIC K-THEORY: AN INTRODUCTION

V. SRINIVAS

These lectures give a brief introduction to two related topics: topological (complex) K-theory and algebraic K-theory[1].

1. Topological K-Theory

Basic references for this section are:

(1) M. F. Atiyah, *K-Theory*, Benjamin (1967).
(2) D. Husemoller, *Fibre Bundles* (Second Ed.), Grad. Texts in Math. 20, Springer-Verlag (1966).

For simplicity, we will restrict the category of topological spaces considered to the category of *finite CW complexes*, unless stated otherwise; in particular, these are compact Hausdorff spaces. Recall that for such spaces, we have the following results from point-set topology:

Theorem 1.1. (a) (Tietze extension theorem) *If X is a compact Hausdorff space, $A \subset X$ a closed subset, and $f : A \to \mathbb{R}^n$ a continuous function, then f can be extended to a continuous function $\widetilde{f} : X \to \mathbb{R}^n$.*

(b) (Partititons of unity) *Let $\{U_i\}_{i=1}^n$ be an open cover of X. Then there exist continuous functions $u_i : X \to [0,1]$ with the following properties: (i) $\overline{\{u_i(x) \neq 0\}} \subset U_i$ for each $i = 1, \ldots, n$ (ii) $u_1(x) + \cdots + u_n(x) = 1$ for all $x \in X$.*

A collection of functions $\{u_i\}$ as in (b) above is called a *continuous partition of unity* on X subordinate to the open covering $\{U_i\}$. If $V_i = \{x \in X \mid u_i(x) > 0\}$, then $\{V_i\}$ is also an open covering of X, such that $\bar{V_i} \subset U_i$; we call such an open cover $\{V_i\}$ a *shrinking* of $\{U_i\}$.

1.1. Vector bundles.

Definition 1.2. Let $k = \mathbb{R}$ or $\mathbb{C}$. A *k-vector bundle* of rank r on a connected[2] topological space X is a space E, together with a continuous map $p : E \to X$, such that "locally on X, E is the product space $X \times k^r$", *i.e.*, there is an open cover $\{U_i\}$ of X and homeomorphisms $\varphi : p^{-1}(U_i) \to U_i \times k^n$ such that

(i) φ_i is compatible with projection to U_i, *i.e.*, $\varphi_i(x) = (p(x), \widetilde{\varphi}_i(x))$ for all $x \in p^{-1}(U_i)$ for some (continuous) function $\widetilde{\varphi}_i : p^{-1}(U_i) \to k^n$

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[1] However, the algebraic K-theory notes were not typeset in the end.

[2] A vector bundle on a non-connected space may have a varying, but locally constant, rank.

(ii) for each $x \in U_i \cap U_j$, the composite homeomorphism

$$A_{ij}(x) : k^n \xrightarrow{\widetilde{\varphi}_i^{-1}} p^{-1}(x) \xrightarrow{\widetilde{\varphi}_j} k^n$$

is a *linear isomorphism* $k^n \to k^n$, *i.e.*, $A_{ij}(x) \in \mathrm{GL}_n(k)$.

Here another such collection of data $\{(V_j, \psi_j)\}$ defines the same vector bundle structure on E if $\{(U_i, \varphi_i)\} \cup \{(V_j, \psi_j)\}$ defines a vector bundle structure on E. Note that for any $x \in U_i$, the homeomorphism $\widetilde{\varphi}_i : p^{-1}(x) \to k^n$ may be used to give $p^{-1}(x)$ the structure of a k-vector space of dimension r. Since A_{ij} are linear isomorphisms, we see that for $x \in U_i \cap U_j$, the vector space structures on $p^{-1}(x)$ induced by $\widetilde{\varphi}_i$ and $\widetilde{\varphi}_j$ agree. We call the k-vector space $p^{-1}(x)$ the *fibre* of $p : E \to X$ over x, and may also denote it by E_x.

Since φ_i are all homeomorphisms, the matrix valued functions $A_{ij} : U_i \cap U_j \to \mathrm{GL}_n(k)$ are continuous, and from the definitions, we verify that they satisfy the condtitions

(a) for all $x \in U_i$, $A_{ii}(x)$ is the identity
(b) for all $x \in U_i \cap U_j$, we have $A_{ji}(x) = g_{ij}(x)^{-1}$, the matrix inverse
(b) for all $x \in U_i \cap U_j \cap U_k$, we have a matrix identity $A_{ik}(x) = A_{jk}(x)A_{ij}(x)$, where the expression on the right denotes the product of matrices.

Conversely, given an open cover $\{U_i\}$ of X and a collection of continuous matrix-valued functions $A_{ij} : U_i \cap U_j \to \mathrm{GL}_r(k)$ satisfying (a), (b), (c) one can define a vector bundle E as follows: let $\sim$ be the equivalence relation on the disjoint union $\coprod_i U_i \times k^n$ generated by $(x, v) \sim (x, A_{ij}(x)v)$ for all $x \in U_i \cap U_j$; let $p : E \to X$ be induced by the projections $U_i \times k^n \to U_i$. One verifies at once that the natural map $U_i \times k^n \to E$ is a homeomorphism onto its image $p^{-1}(U_i)$, whose inverse may be taken as φ_i.

Definition 1.3. If $p : E \to X$, $q : F \to X$ are vector bundles on X, a *morphism of vector bundles* $f : E \to F$ is a continuous map f such that (i) $p = q \circ f : E \to X$, and (ii) for each x, the induced map on fibres $E_x \to F_x$ is a k-linear transformation. If f is also a homeomorphism, then we say that it is an *isomorphism* of vector bundles.

Example 1.4. (*The trivial bundle*) $E = X \times k^n$, $p : E \to X$ is the projection.

Thus, in the definition of a vector bundle, the map φ_i (or $\widetilde{\varphi}_i$) is called a *trivializations* of the bundle $p : E \to X$ over the open set U_i.

Example 1.5. (*Möbius band*) Let $k = \mathbb{R}$, $X = S^1$ (the unit circle in $\mathbb{R}^2$), $M =$ Möbius band , $p : M \to S^1$ is the projection onto the "equator" of M. We may regard S^1 as the identification space of the unit interval $[0, 1]$ modulo the identification of its end points $0, 1$; the identification map $[0, 1] \twoheadrightarrow S^1$ can be taken to be $t \mapsto (\cos 2\pi t, \sin 2\pi t)$. Then M is the identification space of $[0, 1] \times \mathbb{R}$, modulo the identification of $\{0\} \times \mathbb{R}$ with $\{1\} \times \mathbb{R}$ given by $(0, s) \sim (1, -s)$. Since this identification is via a linear isomorphism $\mathbb{R} \to \mathbb{R}$, we see that $p : M \to S^1$ (induced by the second projection $[0, 1] \times \mathbb{R} \to \mathbb{R}$) is an $\mathbb{R}$-vector bundle of rank 1.

The Möbius band $p : M \to S^1$ of Example 1.5 can be seen to be *not* isomorphic to the trivial bundle $S^1 \times \mathbb{R}$. Indeed, for any k-vector bundle $p : E \to X$, there is a continuous mapping $0_E : X \to E$ given by $0_E(x) = 0_{E_x}$, where $0_{E_x} \in E_x$ is the 0-element of the fibre vector space E_x. This map satisfies $p \circ 0_E = 1_X$, the identity map of X. Now if $f : S^1 \times \mathbb{R} \to M$ is an isomorphism of vector bundles, it maps the image of $0_{S^1\times\mathbb{R}}$ homeomorphically to the image of 0_M. Hence it induces a homeomorphism between the complements of these images. But $S^1 \times \mathbb{R} - \mathrm{im}(0_{S^1\times\mathbb{R}}) = S^1 \times (\mathbb{R} - \{0\})$ is disconnected, while $M - \mathrm{im}(0_M)$ is connected (verify!).

Example 1.6. Let $M \subset \mathbb{R}^N$ be an n-dimensional C^∞ differentiable submanifold. For each $x \in M$, we have

$$\begin{aligned} T_xM &= \text{tangent space to } M \text{ at } x \\ &= \{v \in \mathbb{R}^N \mid \text{ the line } \{x + tv \mid t \in \mathbb{R}\} \text{ is tangent to } M\}, \\ TM &= \{(x, v) \in M \times \mathbb{R}^N \mid v \in T_xM\}. \end{aligned}$$

Then we have the following facts:

(i) T_xM is a real vector subspace of $\mathbb{R}^N$ of dimension n, for each $x \in M$

(ii) $p : TM \to M$, $p(x, v) = x$, gives TM the structure of a vector bundle of rank n on M, such that the vector space structure on the fibre $(TM)_x = \{x\} \times T_xM$ is that given on $T_xM \subset \mathbb{R}^N$.

The idea of the proof is as follows. For each $x \in M$, there is a neighbourhood U of x in $\mathbb{R}^N$, and C^∞ functions $f_1, \ldots, f_{N-n}$ on U such that (a) $U \cap M = \{y \in U \mid f_1(y) = \cdots = f_{N-n}(y) = 0\}$, and (b) for any $x \in M \cap U$, the Jacobian matrix $J(f)(x) = \left[\frac{\partial f_j}{\partial x_i}(x)\right]$ has maximal rank $N - n$. Then one sees that $T_xM = \ker J(f)(x)$.

Now after permuting the coordinates, if we write $J(f)(x) = [A, B]$ where the submatrix $A = \left[\frac{\partial f_j}{\partial x_i}(x)\right]_{1\le i,j\le N-n}$ has rank $N - n$, then $\mathbb{R}^n \to \ker J(f)(x)$, $v \mapsto \begin{pmatrix} A^{-1}Bv \\ v \end{pmatrix}$ is a linear isomorphism; the inverse isomorphism gives a trivialization of TM over $U \cap M$.

Remark 1.7. In the above example, one sees that for an open cover $\{U_i\}$ of M by open sets of the type described, the transition functions $A_{ij} : U_i \cap U_j \to \mathrm{GL}_n(\mathbb{R})$ are C^∞ functions. We express this by saying that $p : TM \to M$ is a C^∞ *vector bundle*.

Definition 1.8. A *section* of a k-vector bundle $p : E \to X$ is a continuous map $s : X \to E$ such that $p \circ s : X \to X$ is the identity map. Let $\Gamma(X, E)$ denote the set of sections of E.

For example, the map $0_E : X \to E$ described earlier, given by $0_E(x)0_{E_x}$, is a section, called the *zero section* (or 0-section) of $p : E \to X$. A section of the trivial bundle $s : X \times k^r \to X$ is essentially just a vector valued continuous function $f : X \to k^r$; here f corresponds to the section $x \mapsto (x, f(x))$. Thus a section of an arbirary vector bundle may be viewed, locally on X, as a vector valued function.

The vector space structures on the fibres E_x of a vector bundle $p : E \to X$ determine on $\Gamma(X,E)$ a natural structure of a module over the ring $C_k(X)$ of k-valued continuous functions on X, given by $(a \cdot s)(x) = a(x) \cdot s(x)$ for any $a \in C_k(X)$, $s \in \Gamma(X,E)$; here $f(x) \in k$, and $f(x) \cdot s(x)$ denotes the scalar multiplication for the k-vector space structure on E_x. If $p : E \to X$, $q : F \to X$ are k-vector bundles, and $f : E \to F$ is a morphism of vector bundles, then $s \mapsto f \circ s$ induces a homomorphism of $C_k(X)$-modules $f_* : \Gamma(X,E) \to \Gamma(X,F)$.

Theorem 1.9. (Swan) *There is an equivalence of categories between k-vector bundles on a "good" topological space X and finitely generated projective modules over the ring $C_k(X)$, given by $(p : E \to X) \mapsto \Gamma(X,E)$.* (Here "good" includes the case when X is a compact Hausdorff space.)

Proof. (Sketch) Let $\{U_i\}_{i=1}^n$ be an open cover on which there are trivializations $\varphi_i : p^{-1}(U_i) \to U_i \times k^r$. Let $s_{ij} : U_i \to p^{-1}(U_i)$ be such that $\varphi_i \circ s_{ij} : U_i \to U_i \times k^r$ corresponds to the j-th coordinate function on k^r. Choose a partition of unity (see Theorem 1.1(b)) u_i subordinate to the open cover $\{U_i\}$. Define

$$\widetilde{s}_{ij}(x) = \begin{cases} u_i(x)s_{ij}(x) & \text{if } x \in U_i \\ 0_{E_x} & \text{if } x \notin U_i \end{cases}$$

One checks at once that $\widetilde{s}_{ij}$ is continuous, so that $\widetilde{s}_{ij} \in \Gamma(X,E)$ for all i,j. Now define a morphism of vector bundles

$$\Phi : X \times k^{rn} \to E,$$

$$\Phi(x, \{a_{ij}\}_{1\le i\le n, 1\le j\le r}) = \sum a_{ij}\widetilde{s}_{ij}(x).$$

If $V_i = \{x \in X \mid u_i(x) > 0\}$, so then $\{V_i\}$ is also an open covering of X, shrinking $\{U_i\}$; for $x \in V_i$, the map $k^r \to E_x$, $(a_1,\dots,a_r) \mapsto \sum_{j=1}^r a_j\widetilde{s}_{ij}(x) = u_i(x)\sum_j a_j s_{ij}(x)$, which is an isomorphism of k-vector spaces, since $s_{ij}(x)$ is a basis for E_x, and $u_i(x) \in k$ is a non-zero scalar. Hence $\Phi : X \times k^{rn} \to E$ is a *surjection.*

We claim the surjective bundle morphism Φ is split. Equivalently, there is an injective bundle map $\Psi : E \to X \times k^{rn}$ such that the composition $\Phi \circ \Psi : E \to E$ is the identity. We prove this as follows. If $k = \mathbb{R}$, let $< \, , \, >$ denote a positive definite inner product on k^{rn}; if $k = \mathbb{C}$, let $< \, , \, >$ denote a positive definite Hermitian inner product on k^{rn}. For each $x \in X$, the k-linear surjection $\Phi_x : k^{rn} \cong \{x\} \times k^{rn} \twoheadrightarrow E_x$ induces an isomorphism $\alpha_x : \ker(\Phi_x)^\perp \xrightarrow{\cong} E_x$. Let $\Psi_x : E_x \to \{x\} \times k^{rn}$ be the inverse isomorphism α_x^{-1}, composed with the inclusion $\ker(\Phi_x)^\perp \hookrightarrow \{x\} \times k^{rn}$. We leave it to the reader to check that $\Psi : E \to X \times k^{rn}$, $\Psi(y) = \Psi_{p(y)}(y)$ is continuous, and defines the desired splitting of Φ.

This implies that $\Gamma(X,E)$ is a direct summand of $\Gamma(X, X \times k^{rn}) = C_k(X)^{rn}$ as a $C_k(X)$-module. □

1.2. **Operations on vector bundles.** The usual operations on finite dimensional vector spaces which are familiar from linear algebra carry over to similar operations on vector bundles. Examples are given by the direct sum, tensor product, dual, Hom, and the exterior and symmetric powers. These are defined on the trivial bundle $X \times k^n$ through the standard operation on k^n, and may be defined for an arbitrary bundle using local trivializations. We then recover the standard operations on the fibres. Equivalently, applying the equivalence $\Gamma(X, -)$, these correspond to the standard operations on finitely generated projective $C_k(X)$-modules.

Example 1.10. (i) If $E \to X$, $F \to X$ are vector bundles, then $(E \oplus F)_x = E_x \oplus F_x$, where the right side denotes the vector space direct sum of the k-vector spaces E_x and F_x.

(ii) $\mathrm{Hom}(E, F)_x = \mathrm{Hom}_k(E_x, F_x)$, and $\mathrm{Hom}(E, F)$ is the vector bundle whose module of sections $\Gamma(X, \mathrm{Hom}(E, F))$ is the $C_k(X)$-module of $C_k(X)$-module homomorphisms $\Gamma(X, E) \to \Gamma(X, F)$. If $E^\vee$ is the dual k-vector bundle, so that $(E^\vee)_x = (E_x)^\vee$, then for any vector bundle F, we have a natural isomorphism of vector bundles $E^\vee \otimes F \cong \mathrm{Hom}(E, F)$.

(iii) $(E^\vee)^\vee \cong E$ for any vector bundle E.

(iv) $\overset{n}{\vee} E = 0$ for $n > \mathrm{rank} E$.

Definition 1.11. An *inner product* on E is a symmetric bundle morphism $E \otimes E \to X \times k$. If $k = \mathbb{R}$, the inner product is called *positive definite* if the induced inner product $E_x \otimes_{\mathbb{R}} E_x \to \mathbb{R}$ on each fibre is positive definite. In a similar way, we can define the notion of a *Hermitian inner product* on a $\mathbb{C}$-vector bundle.

Remark 1.12. As part of the proof of Theorem 1.9, it has been shown that every k-vector bundle on a compact Hausdorff space is a subbundle of a trivial bundle of finite rank; thus every real vector bundle on such a space carries a positive definite inner product. Similarly every complex vector bundle carries a positive definite Hermitian inner product. Hence any injective bundle morphism $E \to F$ on such a space X is *split*.

Definition 1.13. If $f : Y \to X$ is a continuous map, and $p : E \to X$ is a k-vector bundle of rank r, then the *pullback* $f^*E = Y \times_X E \to Y$ is also a k-vector bundle of rank r. Here $Y \times_X E = \{(y, z) \in Y \times E \mid f(y) = p(z)\}$.

Remark 1.14. The operation of pullback preserves the above operations on vector bundles (direct sums, Hom, duals, tensor, exterior and symmetric products, etc.). On the level of modules, there is a corresponding homomorphism $f^* : \Gamma(X, E) \to \Gamma(Y, f^*E)$; if $f^* : C_k(X) \to C_k(Y)$ is the ring homomorphism given by $g \mapsto g \circ f$, then $\Gamma(X, E) \to \Gamma(Y, E)$ is $C_k(X)$-linear, and the induced $C_k(Y)$-linear map $\Gamma(X, E) \otimes_{C_k(X)} C_k(Y) \to \Gamma(Y, f^*E)$ is an *isomorphism.* Notice that if $f : Y \hookrightarrow X$ is the inclusion of a subset, then $f^*E \cong p^{-1}(Y)$; the homomorphism $f^* : \Gamma(X, E) \to \Gamma(Y, f^*E)$ is then given by restriction of functions.

Proposition 1.15. *If $f, g : Y \to X$ are homotopic maps, then for any bundle $p : E \to X$, the bundles $f^*E \to Y$ and $g^*E \to Y$ are isomorphic.*

Proof. We make use of a simple lemma.

Lemma 1.16. *Let $i : A \to X$ be the inclusion of a closed subset. Then for any vector bundle $p : E \to X$, the restriction map $i^* : \Gamma(X, E) \to \Gamma(A, i^*E)$ is surjective.*

Proof. For the trivial bundle, this follows from the Tietze extension theorem (Theorem 1.1(a)). Since any vector bundle is a direct summand of a trivial bundle, we reduce immediately to the special case. □

Now let $H : Y \times I \to X$ be a homotopy between f and g, where $I = [0, 1]$ is the unit interval. Let $p_1 : Y \times I \to Y$ be the projection. For $t \in I$, let $f_t : Y \to X$ be the map $f_t(y) = H(y, t)$; then $f_0 = f$, and $f_1 = g$.

We claim that for each $t \in I$, there is a neighbourhood V_t of $t \in I$ such that for all $t' \in T$, we have an isomorphism of vector bundles $f_t^*E \cong f_{t'}^*E$. If we grant the claim, a finite number of these open subsets cover I, and it is then clearly possible to find a sequence $t_0 = 0 < t_1 < \cdots < t_n = 1$ in I such that we have isomorphisms $f_{t_i}^*E \cong f_{t_{i+1}}^*E$ for $0 \le i < n$; the comnposition of these isomorphisms is the desired one.

To prove the claim, consider the vector bundle $F = \mathrm{Hom}(H^*E, p_1^*f_t^*E)$ on $Y \times I$. If $i_t : Y \cong Y \times \{t\} \hookrightarrow Y \times I$ is the inclusion, then $i_t^*F \cong \mathrm{Hom}(f_t^*E, f_t^*E)$. By lemma 1.16, the identity endomorphism of f_t^*E extends to a global section $s \in \Gamma(Y, F)$. The subset $\mathrm{Iso}\,(H^*E, p_1^*f_t^*E) \subset \mathrm{Hom}(H^*E, p_1^*f_t^*E) = F$ is an *open* subset, where the fibre over $z \in Y \times I$ of $\mathrm{Iso}\,(H^*E, p_1^*f_t^*E)$ is the set $\mathrm{Iso}\,(H^*E, p_1^*f_t^*E)_z$ of vector space isomorphisms of $(H^*E)_z$ with $(p_1^*f_t^*E)_z$. Hence $s^{-1}(\mathrm{Iso}\,(H^*E, p_1^*f_t^*E))$ is an open subset of $Y \times I$ containing $Y \times \{t\}$; hence it also contains an open subset of the form $Y \times V_t$ for some (relatively) open interval $V_t \subset I$ containing t. Now for $t' \in V_t$, the restriction of s to $Y \times \{t'\}$ gives the desired isomorphism, proving the claim. □

1.3. **Classifying maps to Grassmannians.** For $n \le m$, let $\mathbb{G}_k(n, m)$ denote the Grassmannian of n-dimensional subspaces of k^m. For $k = \mathbb{R}$, fix the standard Euclidean inner product on $\mathbb{R}^m$; for $k = \mathbb{C}$, fix the standard positive definite Hermitian inner product on $\mathbb{C}^m$. In each case, the standard basis vectors form an orthonormal basis. We can then make identifications

$$\mathbb{G}_{\mathbb{R}}(n, m) = \mathbb{O}(m)/\mathbb{O}(n) \times \mathbb{O}(m - n), \;\; \mathbb{G}_{\mathbb{C}}(n, m) = \mathbb{U}(m)/\mathbb{U}(n) \times \mathbb{U}(m - n)$$

as homogeneous spaces for the orthogonal group $\mathbb{O}(m)$ and the unitray group $U(m)$, respectively. We have a tautological k-vector bundle $\nu_{n,m} \to \mathbb{G}_k(n, m)$, whose fibre $(\nu_{n,m})_x$ is $\{x\} \times V$, where $V \subset k^m$ is the subspace of dimension n corresponding to the point $x \in \mathbb{G}_k(n, m)$. The orthogonal projection of $\mathbb{G}_k(n, m) \times k^m$ onto the subbundle $\nu_{n,m}$ gives us m tautological sections $\mathbf{s}_1, \dots, \mathbf{s}_m$ of $\nu_{n,m}$.

If $p : E \to X$ is a vector bundle of rank n, and $s_1, \dots, s_m$ are sections which give rise to a surjective bundle morphism $\psi : X \times k^m \to E$, we say that the sections s_j *generate* E. In this situation, the map $f_\psi : X \to \mathbb{G}_k(n, m)$, given by $x \mapsto [\ker(\psi_x)^\perp]$, is easily seen to be *continuous*. Further, by construction, we see that there is a natural identification $f_\psi^*\nu_{n,m} \cong E$ with $f_\psi^*(\mathbf{s}_i) = s_i$.

Let $\mathbb{G}_k(n) = \varinjlim_m \mathbb{G}_k(n, m)$, induced by $i_{n,m} : \mathbb{G}_k(n, m) \hookrightarrow \mathbb{G}_k(n, m + 1)$, given by $k^m \hookrightarrow k^m \perp k = k^{m+1}$, $v \mapsto (v, 0)$.

Lemma 1.17. *Let $p : E \to X$, ψ are as above, $s \in \Gamma(X, E)$. Suppose $\psi' : X \times k^{m+1} \to E$ is given by $\{s_1, \dots, s_m, s\}$. Then ψ' is homotopic to $i_{n,m} \circ \psi : X \to \mathbb{G}_k(n, m + 1)$.*

Proof. If $p_X : X \times I \to X$ is the projection, then $p_X^* E$ has sections $p_X^*(s_1), \ldots, p_X^*(s_m), tp_X^*(s)$ which yield a bundle surjection $(X \times I) \times k^{m+1} \twoheadrightarrow p_X^* E$. The corresponding map $H : X \times I \to \mathbb{G}_k(n, m+1)$ yields the desired homotopy. □

Theorem 1.18. *The homotopy class of the map* $\widetilde{f}_\psi : X \xrightarrow{\psi} \mathbb{G}_k(n, m) \hookrightarrow \mathbb{G}_k(n)$ *depends only on the vector bundle E; the association* $E \mapsto [\widetilde{f}_\psi]$ *gives a bijection between the sets*

$$\mathbf{Vect}_n(X) = \text{ isomorphism classes of (k-)vector bundles of rank } n \text{ on } X$$

and

$$[X, \mathbb{G}_k(n)] = \text{ homotopy classes of maps } X \to \mathbb{G}_k(n).$$

Proof. If $s_1, \ldots, s_l$ and $t_1, \ldots, t_m$ are two sets of sections generating E, then the corresponding maps $f : X \to \mathbb{G}_k(n, l)$, $g : X \to \mathbb{G}_k(n, m)$ yield homotopic maps into $\mathbb{G}_k(n, l{+}m{+}1)$. Indeed, first consider the the maps $\widetilde{f} : X \to \mathbb{G}_k(n, l{+}m)$, $\widetilde{g} : X \to \mathbb{G}_k(n, l{+}m)$ arising (respectively) from the sets of sections $s_1, \ldots, s_l, 0, \ldots, 0$ (with m zeroes) and $t_1, \ldots, t_m, 0, \ldots, 0$ (with l zeroes); the first map is induced by f, and the second by g. By the above lemma and induction, these two maps $X \to \mathbb{G}_k(n, l+m)$ are respectively homotopic to the maps $F : X \to \mathbb{G}_k(n, l+m)$, $G : X \to \mathbb{G}_k(n, l{+}m)$ corresponding to the sets of sections $s_1, \ldots, s_l, t_1, \ldots, t_m$ and $t_1, \ldots, t_m, s_1, \ldots, s_l$. Now F and G are related by translation by a permutation matrix in $\mathrm{GL}_{l+m}(k)$, for the natural action of $\mathrm{GL}_{l+m}(k)$ on $\mathbb{G}_k(n, l+m)$. For $k = \mathbb{C}$, this permutation matrix is in $U(l+m)$, which is path connected; a path in $U(l+m)$ from the identity element to this permutation yields a homotopy between F and G. If $k = \mathbb{R}$, then the permutation is an element of $\mathbb{O}(l+m)$, which need not be connected. But then the linear map $k^{l+m+1} \to k^{l+m+1}$ which is the given permutation on the first $l+m$ cordinates, and is multiplication by the sign of the permutation on the $l{+}m{+}1$-st coordinate, is an element in the identity component of $\mathbb{O}(l+m+1)$, hence again can be joined to the identity element by a path in $\mathbb{O}(l+m+1)$. Thus the maps $X \to \mathbb{G}_k(n, l+m+1)$ induced by f, g are homotopic. Hence there is a well-defined map $\alpha : \mathbf{Vect}_n(X) \to [X, \mathbb{G}_k(n)]$.

Conversely, assume given a continuous map $\psi : X \to \mathbb{G}_k(n)$. Since X is compact, we have $\psi(X) \subset \mathbb{G}_k(n, m)$ for some m. Then $E = \psi^* \nu_{n,m}$ is generated by the sections $\psi^*(\mathbf{s}_1), \ldots, \psi^*(\mathbf{s}_m)$, and the corresponding map $X \to \mathbb{G}_k(n, m)$ is just ψ itself. Two homotopic maps yield isomorphic bundles on X, by lemma 1.15. This gives a well-defined map $\beta : [X, \mathbb{G}_k(n)] \to \mathbf{Vect}_n(X)$. It is clear from the definitions that the two maps α, β are inverse to each other. □

Remark 1.19. The bijections α and β are natural (functorial), in the sense that if $f : Y \to X$ is a continuous map, the map $f^* : \mathbf{Vect}_n(X) \to \mathbf{Vect}_n(Y)$, $[E] \mapsto [f^* E]$, corresponds to the map $[X, \mathbb{G}_k(n)] \to [Y, \mathbb{G}_k(n)]$, $(\psi : Y \to \mathbb{G}_k(n)) \mapsto (\psi \circ f : X \to \mathbb{G}_k(n))$. In more abstract language, we say that the space $\mathbb{G}_k(n)$ *represents the functor* $\mathbf{Vect}_n(-)$ (on the category of compact Hausdorff spaces). This is actually a slight abuse of terminology, since the "representing object", namely $\mathbb{G}_k(n)$, is not itself a compact Hausdorff space.

Thus if E is any k-vector bundle of rank n on X, so that we have an element $[E] \in \mathbf{Vect}_n(X)$, then we get an associated ring homomorphism

$$[E]^* : H^*(\mathbb{G}_k(n), A) \to H^*(X, A)$$

on cohomology rings, for any coefficient ring A; images in $H^*(X, A)$ of cohomology classes in $H^*(\mathbb{G}_k(n), A)$ are called the *characteristic classes of E in $H^*(X, A)$*. From the above remark 1.19, it follows that if $\theta(E) \in H^*(X, A)$ is a characteristic class for a vector bundle E on X, and if $f : Y \to X$ is a continuous map, then $\theta(f^*E) = f^*\theta(E)$, where on the right, f^* denotes the ring homomorphism $H^*(X, A) \to H^*(Y, A)$.

In view of the above, it is interesting to compute the cohomology ring $H^*(\mathbb{G}_k(n), A)$ for various rings A; each such computation leads to a "theory of characteristic classes" for vector bundles.

1.4. **Cohomology rings of Grassmannians, and characteristic classes.** We first state a result giving a computation of the cohomology ring of $\mathbb{G}_k(n)$ in the most important cases. Other results can be deduced from these, using the universal coefficient theorem.

Theorem 1.20. (a) *$H^1(\mathbb{G}_{\mathbb{C}}(n), \mathbb{Z}) = \mathbb{Z}[\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_n]$ is a graded polynomial algebra in n variables, where $\mathbf{c}_i \in H^{2i}(\mathbb{G}_k(n), \mathbb{Z})$ is homogeneous of degree $2i$. For any $\mathbb{C}$-vector bundle E of rank n, we call $c_i(E) := [E]^*(\mathbf{c}_i)$ the* ***i-th Chern class*** *of E.*

(b) *$H^*(\mathbb{G}_{\mathbb{R}}(n), \mathbb{Z}[\frac{1}{2}]) = \mathbb{Z}[\frac{1}{2}][\mathbf{p}_1, \ldots, \mathbf{p}_{[\frac{n}{2}]}]$ is a graded polynomial algebra, where $\mathbf{p}_i$ is homogeneous of degree $4i$. For any $\mathbb{R}$-vector bundle of rank n, we call $p_i(E) := [E]^*(\mathbf{p}_i)$ the* ***i-th Pontryagin class*** *of E.*

(c) *$H^*(\mathbb{G}_{\mathbb{R}}(n), \mathbb{Z}/2\mathbb{Z}) = (\mathbb{Z}/2\mathbb{Z})[\mathbf{w}_1, \ldots, \mathbf{w}_n]$ is a graded polynomial algebra, where $\mathbf{w}_i$ is homogeneous of degree i. For any $\mathbb{R}$-vector bundle E of rank n, we call $w_i(E) := [E]^*(\mathbf{w}_i)$ the i-th* Stiefel-Whitney class *of E.*

We now sketch a proof of parts (a) and (c) of this theorem. We begin with the following result; recall that $\mathbb{P}^n_k = \mathbb{G}_k(1, n+1)$ is the projective space of dimension n over k, parametrizing the set of lines in the vector space k^{n+1}.

Theorem 1.21. (1) *For $n \geq 1$, we have $H^*(\mathbb{P}^n_{\mathbb{C}}, \mathbb{Z}) = \mathbb{Z}[x]/(x^{n+1})$, where x is a homogeneous element of degree 2. Further, $H^*(\mathbb{P}^\infty_{\mathbb{C}}, \mathbb{Z}) = H^*(\mathbb{G}_{\mathbb{C}}(1), \mathbb{Z}) = \mathbb{Z}[x]$ is a graded polynomial algebra in 1 variable x of degree 2.*

(2) *For $n \geq 1$, we have $H^*(\mathbb{P}^n_{\mathbb{R}}, \mathbb{Z}/2\mathbb{Z}) = (\mathbb{Z}/2\mathbb{Z})[y]/(y^{n+1})$, where y is a homogeneous element of degree 1. Further, $H^*(\mathbb{P}^\infty_{\mathbb{R}}, \mathbb{Z}/2\mathbb{Z}) = (\mathbb{Z}/2\mathbb{Z})[y]$ is a graded polynomial algebra in 1 variable, with* $\deg y = 1$.

Proof. (Sketch) In both cases, the results for the finite dimensional projective spaces $\mathbb{P}^n$, and the compatibility with natural inclusions $\mathbb{P}^{n-1} \hookrightarrow \mathbb{P}^n$, will imply the result for the infinite projective spaces. So we will only discuss the finite dimensional cases.

Proof of (1): We may consider $\mathbb{P}^n_{\mathbb{C}}$ as the quotient space of $\mathbb{C}^{n+1} - \{0\}$ modulo the diagonal action of the multiplicative $\mathbb{C}^*$, or equivalently, as the quotient of the unit sphere $S^{2n+1} \subset \mathbb{C}^{n+1}$ modulo the action of the unit circle $S^1 \subset \mathbb{C}^*$.

The inclusion $\mathbb{C}^n \hookrightarrow \mathbb{C}^{n+1}$, as the subspace with vanishing last coordinate, induces an inclusion $\mathbb{P}^{n-1}_{\mathbb{C}} \hookrightarrow \mathbb{P}^n_{\mathbb{C}}$; the complement is the homeomorphic image of $\mathbb{C}^n \times \{1\} \subset \mathbb{C}^{n+1} - \{0\}$ under the quotient map $\mathbb{C}^{n+1} - \{0\} \to \mathbb{P}^n_{\mathbb{C}}$. This implies that the quotient space $\mathbb{P}^n_{\mathbb{C}}/\mathbb{P}^{n-1}_{\mathbb{C}}$ (obtained by collapsing $\mathbb{P}^{n-1}_{\mathbb{C}}$ to a point) is homeomorphic to the one-point compactification of $\mathbb{C}^n$, which is S^{2n}. By induction and the long exact sequence for the cohomology of the pair $(\mathbb{P}^n_{\mathbb{C}}, \mathbb{P}^{n-1}_{\mathbb{C}})$, we deduce that

$$H^i(\mathbb{P}^n_{\mathbb{C}}, \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } i = 2j \text{ with } 0 \leq j \leq n, \\ 0 & \text{otherwise.} \end{cases}$$

It remains to show that if x is a generator of $H^2(\mathbb{P}^n_{\mathbb{C}}, \mathbb{Z})$, then x^j is a generator of $H^{2j}(\mathbb{P}^n_{\mathbb{C}}, \mathbb{Z})$ for $2 \leq j \leq n$ (that x satisfies the relation $x^{n+1} = 0$ is clear, since $x^{n+1} \in H^{2n+2}(\mathbb{P}^n_{\mathbb{C}}, \mathbb{Z}) = 0$). This can be deduced by induction on n, and the Poincaré duality theorem, as follows. For $n = 1$ there is nothing to prove. If $i : \mathbb{P}^{n-1}_{\mathbb{C}} \to \mathbb{P}^n_{\mathbb{C}}$ is the inclusion, then the exact sequence for the pair $(\mathbb{P}^n_{\mathbb{C}}, \mathbb{P}^{n-1}_{\mathbb{C}})$ actually implies that $i^* : H^j(\mathbb{P}^n_{\mathbb{C}}\mathbb{Z}) \to H^j(\mathbb{P}^{n-1}_{\mathbb{C}}, \mathbb{Z})$ is an isomorphism for $j < 2n$. Thus i^*x generates $H^2(\mathbb{P}^{n-1}_{\mathbb{C}}, \mathbb{Z})$, and if $(i^*x)^j = i^*(x^j)$ generates $H^{2j}(\mathbb{P}^{n-1}_{\mathbb{C}}, \mathbb{Z})$, then x^j generates $H^{2j}(\mathbb{P}^n_{\mathbb{C}}, \mathbb{Z})$ for $1 \leq j \leq n-1$. If we choose a generator $y \in H^{2n}(\mathbb{P}^n_{\mathbb{C}}, \mathbb{Z}) \cong \mathbb{Z}$ (corresponding to an *orientation* on the compact manifold $\mathbb{P}^n_{\mathbb{C}}$), then since x^{n-1} is a generator of $H^{2n-2}(\mathbb{P}^n_{\mathbb{C}}, \mathbb{Z}) \cong \mathbb{Z}$, Poincaré duality implies that there exists an element $z \in H^2(\mathbb{P}^n_{\mathbb{C}}, \mathbb{Z}) = \mathbb{Z}x$ such that $z \cup x^{n-1} = y$. Since $z = m \cdot x$ for some integer m, we have that $m \cdot x^n = y$. Since y is a generator of $H^{2n}(\mathbb{P}^n_{\mathbb{C}}, \mathbb{Z}) \cong \mathbb{Z}$, we must have $m = \pm 1$, and x^n is also a generator of $H^{2n}(\mathbb{P}^n_{\mathbb{C}}, \mathbb{Z})$.

Proof of (2): This is along similar lines, using the description of $\mathbb{P}^n_{\mathbb{R}}$ as the quotient $\mathbb{R}^{n+1} - \{0\}/\mathbb{R}^* = S^n/(\mathbb{Z}/2\mathbb{Z})$, where the generator of $\mathbb{Z}/2\mathbb{Z}$ acts on S^n by the antipodal map $x \mapsto -x$. Again $\mathbb{P}^{n-1}_{\mathbb{R}} \hookrightarrow \mathbb{P}^n_{\mathbb{R}}$ with quotient space $\mathbb{P}^n_{\mathbb{R}}/\mathbb{P}^{n-1}_{\mathbb{R}}$ homeomorphic to the 1-point compactification of $\mathbb{R}^n$, namely S^n. This gives $H^i(\mathbb{P}^n_{\mathbb{R}}, \mathbb{Z}/2\mathbb{Z}) = 0$ for $i > n$. Further, S^n is simply connected, and the quotient map $S^n \to \mathbb{P}^n_{\mathbb{R}}$ is a covering space; hence $\mathbb{P}^n_{\mathbb{R}}$ has fundamental group $\mathbb{Z}/2\mathbb{Z}$. The long exact sequence of cohomology groups with $\mathbb{Z}/2\mathbb{Z}$-coefficients for the pair $(\mathbb{P}^n_{\mathbb{R}}, \mathbb{P}^{n-1}_{\mathbb{R}})$ implies that if $i : \mathbb{P}^{n-1}_{\mathbb{R}} \to \mathbb{P}^n_{\mathbb{R}}$ is the inclusion, then $i^* : H^j(\mathbb{P}^n_{\mathbb{R}}, \mathbb{Z}/2\mathbb{Z}) \to H^j(\mathbb{P}^{n-1}_{\mathbb{R}}, \mathbb{Z})$ is an isomorphism for $j < n-1$, and yields an exact sequence

$$0 \to H^{n-1}(\mathbb{P}^n_{\mathbb{R}}, \mathbb{Z}/2\mathbb{Z}) \xrightarrow{i^*} H^{n-1}(\mathbb{P}^{n-1}_{\mathbb{R}}, \mathbb{Z}/2\mathbb{Z}) \to H^n(S^n, \mathbb{Z}/2\mathbb{Z}) \to H^n(\mathbb{P}^n_{\mathbb{R}}, \mathbb{Z}/2\mathbb{Z}) \to 0.$$

Here $H^n(S^n, \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$ and $H^n(\mathbb{P}^n_{\mathbb{R}}, \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$ since S^n and $\mathbb{P}^n_{\mathbb{R}}$ are compact connected n-manifolds; this implies that $i^* : H^{n-1}(\mathbb{P}^n_{\mathbb{R}}, \mathbb{Z}/2\mathbb{Z}) \to H^{n-1}(\mathbb{P}^{n-1}_{\mathbb{R}}, \mathbb{Z}/2\mathbb{Z})$ is an isomorphism. Now Poincaré duality and induction imply as before that if $x \in H^1(\mathbb{P}^n_{\mathbb{R}}, \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$ is a generator, then x^j is a generator of $H^j(\mathbb{P}^n_{\mathbb{R}}, \mathbb{Z}/2\mathbb{Z})$ for each $1 \leq j \leq n$.

□

Remark 1.22. As a consequence, we may define the following characteristic classes associated to *line bundles* (*i.e.*, vector bundles of rank 1), as follows. If $p : L \to X$ is a complex line bundle, and $[L] : X \to \mathbb{P}^\infty_{\mathbb{C}}$ is a classifying map, then define $[L]^*(x) = c_1(L)$, where $x \in H^2(\mathbb{P}^\infty_{\mathbb{C}}, \mathbb{Z})$ is the following generator: the inclusion $i : \mathbb{P}^1_{\mathbb{C}} \hookrightarrow \mathbb{P}^\infty_{\mathbb{C}}$ induces an isomorphism $i^* : H^2(\mathbb{P}^\infty_{\mathbb{C}}, \mathbb{Z}) \to H^2(\mathbb{P}^1_{\mathbb{C}}, \mathbb{Z})$; now $\mathbb{P}^1_{\mathbb{C}}$ is homeomorphic to the 2-sphere S^2, and $H^2(S^2, \mathbb{Z})$ has a standard generator y (corresponding to te standard orientation of S^2), and we take x to be the generator of $H^2(\mathbb{P}^\infty_{\mathbb{C}}, \mathbb{Z})$ to be the generator such that $i^*x = y$. Similarly, $H^1(\mathbb{P}^\infty_{\mathbb{R}}, \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$ has a unique generator z; for any real line bundle $L \to X$, define $w_1(L) = [L]^*(z)$ for a classifying map $[L] : X \to \mathbb{P}^\infty_{\mathbb{R}}$.

We claim that for any two complex line bundles L_1, L_2, we have $c_1(L_1 \otimes L_2) = c_1(L_1) + c_1(L_2)$. Using suitable classifying maps for L_1 and L_2, we reduce to proving this in the following special case: $X = \mathbb{P}^m_{\mathbb{C}} \times \mathbb{P}^n_{\mathbb{C}}$, $L_1 = p_1^*\nu_{1,m+1}$, $L_2 = \nu_{1,n+1}$ where p_i, $i = 1, 2$ are the two projections, and $\nu_{1,r+1} \to \mathbb{P}^r_{\mathbb{C}} = \mathbb{G}_{\mathbb{C}}(1, r+1)$ denotes the tautological line bundle, for any r. Now the natural map

$$H^2(\mathbb{P}^m_{\mathbb{C}}, \mathbb{Z}) \oplus H^2(\mathbb{P}^n_{\mathbb{C}}, \mathbb{Z}) \stackrel{p_1^*+p_2^*}{\longrightarrow} H^2(\mathbb{P}^m_{\mathbb{C}} \times \mathbb{P}^n_{\mathbb{C}}, \mathbb{Z})$$

is an isomorphism, from the Künneth formula. For any point $(P, Q) \in \mathbb{P}^m_{\mathbb{C}} \times \mathbb{P}^n_{\mathbb{C}}$, if $i_Q : \mathbb{P}^m_{\mathbb{C}} \to \mathbb{P}^m_{\mathbb{C}} \times \mathbb{P}^n_{\mathbb{C}}$ is given by $t \mapsto (t, Q)$, and if $i_P : \mathbb{P}^n_{\mathbb{C}} \to \mathbb{P}^m_{\mathbb{C}} \times \mathbb{P}^n_{\mathbb{C}}$ is given by $s \mapsto (P, s)$, then i_P, i_Q are inclusions such that $p_1 \circ i_Q = 1_{\mathbb{P}^m}$, $p_2 \circ i_P = 1_{\mathbb{P}^n}$, while the other two composites $p_1 \circ i_P$ and $p_2 \circ i_Q$ are constant maps, and hence induce 0 on cohomology. Thus

$$i_P^* : H^2(\mathbb{P}^m_{\mathbb{C}} \times \mathbb{P}^n_{\mathbb{C}}, \mathbb{Z}) \to H^2(\mathbb{P}^n_{\mathbb{C}}, \mathbb{Z}),$$
$$i_Q^* : H^2(\mathbb{P}^m_{\mathbb{C}} \times \mathbb{P}^n_{\mathbb{C}}, \mathbb{Z}) \to H^2(\mathbb{P}^m_{\mathbb{C}}, \mathbb{Z})$$

are the two projections corresponding to the isomorphism $p_1^* + p_2^*$ considered above. Now $i_Q^*(L_1 \otimes L_2) \cong \nu_{1,m+1}$ and $i_P^*(L_1 \otimes L_2) \cong \nu_{1,n+1}$; hence $c_1(L_1 \otimes L_2)$ is the unique element of $H^2(\mathbb{P}^m_{\mathbb{C}} \times \mathbb{P}^n_{\mathbb{C}}, \mathbb{Z})$ which projects to $c_1(\nu_{1,m+1})$ and $c_1(\nu_{1,n+1})$; this element is clearly $p_1^*(c_1(\nu_{1,m+1})) + p_2^*(c_1(\nu_{1,n+1}))$, which is just $c_1(L_1) + c_1(L_2)$.

By an analogous argument, we also have $w_1(L_1 \otimes L_2) = w_1(L_1) + w_1(L_2)$ for any real line bundles L_1, L_2 on X.

Recall that if $p : E \to X$ is a k-vector bundle of rank r, then we may form the associated *projective bundle* $\pi : \mathbb{P}(E) \to X$, where $\mathbb{P}(E)$ is the quotient space $(E - 0_E(X))/k^*$ for the action of k^* by scalar multiplication on each fibre E_x. Thus the fibre over $x \in X$ of $\mathbb{P}(E) \to X$ is the projective space $\mathbb{P}(E_x) \cong \mathbb{P}^{r-1}_k$, which is the space of lines (1-dimensional k-vector subspaces) in E_x. There is an associated tautological line bundle on $\mathbb{P}(E)$, which restricts on each fibre $\mathbb{P}(E_x) \cong \mathbb{P}^{r-1}_k$ to the tautological line bundle $\nu_{1,r}$; it is a subbundle of π^*E. Following the notation in algebraic geometry, we denote the tautological line bundle on $\mathbb{P}(E)$ by $\mathcal{O}_{\mathbb{P}(E)}(-1)$.

In the next theorem, the reader should keep in mind that (i) though the graded cohomology rings with $\mathbb{Z}$-coefficients may be non-commutative, homogeneous elements of even degree are central (ii) cohomology rings with $\mathbb{Z}/2\mathbb{Z}$ are always commutative. This follows from the general commutation formula $x \cup y = (-1)^{pq} y \cup x$, for homogeneous elements x, y of degrees p, q respectively.

Theorem 1.23. (Leray-Hirsch)

1) *Let* $p : E \to X$ *be a* complex *vector bundle of rank* n, *and* $\pi : \mathbb{P}(E) \to X$ *the associated* $\mathbb{P}^{n-1}_{\mathbb{C}}$*-bundle, with tautological line subbundle* $\mathcal{O}_{\mathbb{P}(E)}(-1) \hookrightarrow \pi^*E$. *Let* $\xi = c_1(\mathcal{O}_{\mathbb{P}(E)}(-1)) \in H^2(\mathbb{P}(E), \mathbb{Z})$. *Then the homomorphism on*

cohomology rings $\pi^* : H^*(X,\mathbb{Z}) \to H^*(\mathbb{P}(E),\mathbb{Z})$ *makes* $H^*(\mathbb{P}(E),\mathbb{Z})$ *into a free module over* $H^*(X,\mathbb{Z})$ *with basis* $1, \xi, \ldots, \xi^{n-1}$.

2) *Let* $p : E \to X$ *be a* real *vector bundle of rank* n*, and* $\pi : \mathbb{P}(E) \to X$ *the associated* $\mathbb{P}^{n-1}_{\mathbb{R}}$*-bundle, with tautological line subbundle* $\mathcal{O}_{\mathbb{P}(E)}(-1) \hookrightarrow \pi^*E$*. Let* $\xi = w_1(\mathcal{O}_{\mathbb{P}(E)}(-1)) \in H^1(\mathbb{P}(E),\mathbb{Z}/2\mathbb{Z})$*. Then the homomorphism on cohomology rings* $\pi^* : H^*(X,\mathbb{Z}/2\mathbb{Z}) \to H^*(\mathbb{P}(E),\mathbb{Z}/2\mathbb{Z})$ *makes* $H^*(\mathbb{P}(E),\mathbb{Z}/2\mathbb{Z})$ *into a free module over* $H^*(X,\mathbb{Z})$ *with basis* $1, \xi, \ldots, \xi^{n-1}$.

Proof. (Sketch) We consider below the case of a complex vector bundle; the real case is similar.

If $E \to X$ is the trivial bundle, then the result is true by the *Künneth formula* for the cohomology of a product space, and from the formula for the cohomology of a complex projective space (Theorem 1.21).

For any open subset $W \subset X$, we have maps

$$\oplus_{i=0}^{n-1} H^{j-2i}(W,\mathbb{Z}) \xrightarrow{\Phi^j_W} H^j(\mathbb{P}(E\mid_W,\mathbb{Z}),$$

$$(\alpha_0, \ldots \alpha_{n-1}) \mapsto \sum_{i=0}^{n-1} \pi_W^*(\alpha_i) \cup (\xi_W)^i,$$

where $p_W : E\mid_W \to W$ is the restriction of the vector bundle E to the open set W, and $\pi_W : \mathbb{P}(W\mid_E) \to W$ is the corresponding projective bundle; $\xi_W \in H^2(\mathbb{P}(W\mid_E),\mathbb{Z})$ is c_1 of the corresponding tautological bundle, and is hence just the restriction to $\mathbb{P}(E\mid_W) \subset \mathbb{P}(E)$ of ξ.

From the Mayer-Vietoris exact sequence in cohomology and the 5-lemma, we see that if U, V are open subsets of X such that Φ^j_U, Φ^j_V and $\Phi^j_{U\cap V}$ are isomorphisms for all j, then $\Phi^j_{U\cup V}$ is also an isomorphism for all j. Now cover X by open subsets $V_1, \ldots, V_r$ such that $E\mid_{V_i}$ is trivial for all i, and set $U_i = V_1 \cup V_2 \cup \cdots \cup V_i$. By induction on i, we then see that $\Phi^j_{U_i}$ is an isomorphism for all i, j; in particular, taking $i = r$, so that $U_i = X$, we have the theorem. □

Corollary 1.24. 1) *For any complex vector bundle* E *of rank* n *on* X*, there is a unique relation*

$$\xi^n + \pi^*(\alpha_1)\xi^{n-1} + \pi^*(\alpha_2)\xi^{n-2} + \cdots + \pi^*(\alpha^n) = 0$$

in $H^*(\mathbb{P}(E),\mathbb{Z})$*, with* $\alpha_i \in H^{2i}(X,\mathbb{Z})$.

2) *For any real vector bundle* E *of rank* n *on* X*, there is a unique relation*

$$\xi^n + \pi^*(\alpha_1)\xi^{n-1} + \pi^*(\alpha_2)\xi^{n-2} + \cdots + \pi^*(\alpha^n) = 0$$

in $H^*(\mathbb{P}(E),\mathbb{Z}/2\mathbb{Z})$*, with* $\alpha_i \in H^i(X,\mathbb{Z}/2\mathbb{Z})$.

Remark 1.25. Though theorem 1.23 and corollary 1.24 are proved above when X is compact, they are valid without this hypothesis; a proof can be given using the Leray-Serre spectral sequence, for example.

Definition 1.26. 1) For any complex vector bundle E of rank n on X, define its i-th *Chern class* to be $c_i(E) = \alpha_i$, where α_i is as in 1) of the above corollary 1.24.

2) For any real vector bundle E of rank n on X, define its i-th *Stiefel-Whitney class* to be $w_i(E) = \alpha_i$, where α_i is as in 2) of the above corollary.

Remark 1.27. (a) The new definition of c_1 of a complex line bundle agrees with the old one, since for a line bundle L, we have $\mathbb{P}(L) = X$, and the tautological line bundle on $\mathbb{P}(L)$ is L itself. Similarly there is no ambiguity in defining $w_1(L)$ for a real line bundle L.
(b) Since, as noted earlier (remark 1.25), the Leray-Hirsch theorem is valid more generally, the definitons of the Chern classes and Stiefel-Whitney classes make sense for bundles on more general base spaces X, for example, on arbitrary CW complexes.
(c) If E is a complex vector bundle of rank n on X, and $f : Y \to X$ is a splitting map for E, with $f^*E = L_1 \oplus \cdots \oplus L_n$, then $x_i = c_1(L_i) \in H^2(Y, \mathbb{Z})$ are called *Chern roots* for E.

Another corollary of the Leray-Hirsch theroem is the following.

Corollary 1.28. (Splitting principle)

1) *Let $p : E \to X$ be a complex vector bundle. Then there exists a continuous map $f : P \to X$ such that (a) f^*E is a direct sum of complex line bundles on P (b) $f^* : H^*(X, \mathbb{Z}) \to H^*(P, \mathbb{Z})$ is injective.*
2) *Let $p : E \to X$ be a real vector bundle. Then there exists a continuous map $f : P \to X$ such that (a) f^*E is a direct sum of real line bundles on P (b) $f^* : H^*(X, \mathbb{Z}/2\mathbb{Z}) \to H^*(P, \mathbb{Z}/2\mathbb{Z})$ is injective.*

Proof. The proof in the real and complex cases is similar, so we consider the latter. We first reduce easily to the case when X is connected. Then we work by induction on the rank of E, where we may take $P = X$ if E has rank 1. In general, if rank$E = n > 1$, note that $\pi : \mathbb{P}(E) \to X$ satisfies the condition that $\pi^* : H^*(X, \mathbb{Z}) \to H^(\mathbb{P}(E), \mathbb{Z})$ is injective, and there is a line subbundle $\mathcal{O}_{\mathbb{P}(E)}(-1) \subset \pi^*E$. Choosing a Hermitian metric on E, we may write $\pi^*E = \mathcal{O}_{\mathbb{P}(E)}(-1) \oplus F$, where $q : F \to \mathbb{P}(E)$ has rank $n - 1$. Now by induction, there is a map $g : P \to \mathbb{P}(E)$ such that g^*F is a direct sum of complex line bundles, and g^* is injective on cohomology rings. Hence $f = g \circ \pi$ satisfies the desired conditions. □

We call a map $f : P \to X$ as in corollary 1.28 a *splitting map* for the vector bundle E. It is easy to see that if $E_1, \ldots, E_r$ are vector bundles, then there exists a continuous map $f : P \to X$ which is simultaneously a splitting map for each of the bundles E_i (for example, if $f_1 : P_1 \to X$ is a splitting map for E_1, and $f_2 : P_2 \to P_1$ is a splitting map for $f_1^*E_2$, then $f_2 \circ f_1 : P_2 \to X$ is simultaneously a splitting map for E_1 as well as E_2).

Lemma 1.29.

1) *Let $L_1, \ldots, L_n$ be $\mathbb{C}$-line bundles, such that there exists a nowhere-vanishing section $s \in \Gamma(X, L_1 \oplus \cdots \oplus L_n)$ (i.e., $s(x) \neq 0_{E_x}$ for any $x \in X$, where $E = L_1 \oplus \cdots \oplus L_n$). Then*

$$c_1(L_1) \cup \cdots \cup c_1(L_n) = 0$$

in $H^*(X,\mathbb{Z})$.

2) *Let* $L_1,\ldots,L_n$ *be* $\mathbb{R}$*-line bundles, such that there exists a nowhere-vanishing section* $s \in \Gamma(X, L_1 \oplus \cdots \oplus L_n)$ *(i.e.,* $s(x) \neq 0_{E_x}$ *for any* $x \in X$*, where* $E = L_1 \oplus \cdots \oplus L_n$*)). Then*

$$w_1(L_1) \cup \cdots \cup w_1(L_n) = 0$$

in $H^*(X,\mathbb{Z}/2\mathbb{Z})$.

Proof. We consider the complex case, since the real case is similar. Let $s_i \in \Gamma(X, L_i)$ be the component of s in L_i, and $U_i = \{x \in X \mid s_i(x) \neq 0_{(L_i)_x}\}$ be the locus where s_i does not vanish. Then we are given that $\{U_i\}_{i=1}^n$ is an open cover of X. Now for each i, we have that $c_1(L_i) \mapsto 0$ under $H^2(X,\mathbb{Z}) \to H^2(U_i,\mathbb{Z})$, since we have a trivialization $L_i \mid_{U_i} \cong U_i \times \mathbb{C}$ (using the section s_i), and c_1 of the trivial line bundle vanishes[3]. Hence we can find relative cohomology classes $\widetilde{c_1}(L_i) \in H^2(X, U_i; \mathbb{Z})$ such that $\widetilde{c_1}(L_i) \mapsto c_1(L_i)$. Then the cup product

$$\widetilde{c_1}(L_1) \cup \cdots \cup \widetilde{c_1}(L_n) \in H^{2n}(X, U_1 \ldots, U_n; \mathbb{Z})$$

maps to $c_1(L) \cup \cdots \cup c_1(L_n)$ under the natural map

$$H^{2n}(X, U_1 \ldots, U_n; \mathbb{Z}) \to H^{2n}(X,\mathbb{Z}).$$

But $U_1 \cup \cdots \cup U_n = X$, so that $H^{2n}(X, U_1 \ldots, U_n; \mathbb{Z}) = 0$. □

Corollary 1.30. 1) *If* $p : E \to X$ *is a complex vector bundle of rank* n*, and* $f : P \to X$ *a splitting map for* E*, with* $f^*E = L_1 \oplus \cdots \oplus L_n$*, then*

$$f^*(c_i(E)) = \text{ i-th elementary symmetric function in } c_1(L_1), \ldots, c_1(L_n).$$

2) *If* $p : E \to X$ *is a real vector bundle of rank* n*, and* $f : P \to X$ *a splitting map for* E*, with* $f^*E = L_1 \oplus \cdots \oplus L_n$*, then*

$$f^*(w_i(E)) = \text{ i-th elementary symmetric function in } w_1(L_1), \ldots, w_1(L_n).$$

Proof. As usual, we consider the case of complex vector bundles, and leave the (very similar) case of real bundles to the reader.

Let Q be the fibre product

$$\begin{array}{ccc} Q = P \times_X \mathbb{P}(E) & \xrightarrow{g} & \mathbb{P}(E) \\ \eta \downarrow & & \downarrow \pi \\ P & \xrightarrow{f} & X \end{array}$$

Then the inclusion of the tautological line subbundle $L = \mathcal{O}_{\mathbb{P}(E)}(-1) \subset \pi^*E$ induces an inclusion of a line subbundle

$$g^*L \hookrightarrow g^*\pi^*E = \eta^* f^* E = \eta^*(L_1 \oplus \cdots \oplus L_n).$$

Thus we have an inclusion of a trivial line subbundle

$$Q \times \mathbb{C} \hookrightarrow (\eta^*(L_1) \otimes g^*(L^\vee)) \oplus \cdots \oplus (\eta^*(L_n) \otimes g^*(L^\vee)),$$

[3] A constant classifying map induces 0 on cohomology in positive degrees.

where $L^\vee$ denotes the dual line bundle. The inclusion of a trivial line bundle is equivalent to giving a section which does not vanish anywhere, and so by lemma 1.29, we have an identity in $H^*(Q,\mathbb{Z})$

$$\prod_{i=1}^{n}(\eta^*(c_1(L_i)) + g^*\xi) = 0.$$

Thus we have a relation

$$g^*(\xi^n)+\eta^*s_1(c_1(L_1),\dots,c_1(L_n))g^*(\xi^{n-1})+\eta^*s_2(c_1(L_1),\dots,c_1(L_n))g^*(\xi^{n-2})+\cdots+ \eta^*s_n(c_1(L_1),\dots,c_1(L_n)) = 0,$$

where s_i denotes the i-th elementary symmetric polynomial (note that the classes $c_1(L_i)$ are in the centre of the cohomology ring, and so it makes sense to evaluate a polynomial on the $c_1(L_i)$). We also have a relation

$$g^*(\xi^n) + g^*\pi^*(c_1(E))g^*(\xi^{n-1}) + g^*\pi^*(c_2(E))g^*(\xi^{n-2}) + \cdots + g^*\pi^*(c_n(E)) = 0,$$

which we may rewrite as

$$g^*(\xi^n) + \eta^*f^*(c_1(E))g^*(\xi^{n-1}) + \eta^*f^*(c_2(E))g^*(\xi^{n-2}) + \cdots + \eta^*f^*(c_n(E)) = 0.$$

Since $Q = \mathbb{P}(f^*E)$ is a projective bundle over P, the elements $g^*(\xi^j)$, $0 \le j \le n-1$ are linearly independent over $H^*(P,\mathbb{Z})$, and so the above two monic relations satisfied by $g^*(\xi^n)$ must coincide. Thus, comparing coefficients, and using the injectivity on cohomology of η^*, we get that

$$f^*c_i(E) = s_i(c_1(L_1),\dots,c_1(L_n)).$$

□

Corollary 1.31. (Whitney sum formula)

1) *Let $E \to X$, $F \to X$ be two complex vector bundles. Then we have a formula*
$$\sum_{i\ge0} c_i(E\oplus F) = (\sum_{i\ge0} c_i(E))(\sum_{i\ge0} c_i(F))$$
in $H^*(X,\mathbb{Z})$.
2) *Let $E \to X$, $F \to X$ be two real vector bundles. Then we have a formula*
$$\sum_{i\ge0} w_i(E\oplus F) = (\sum_{i\ge0} w_i(E))(\sum_{i\ge0} w_i(F))$$
in $H^*(X,\mathbb{Z}/2\mathbb{Z})$.

Proof. Notice that if E is a complex vector bundle which is a direct sum of line bundles, $E \cong L_1 \oplus \cdots \oplus L_n$, then from corollary 1.30, we have an expression

$$\sum_{i\ge0} c_i(E) = \prod_{i=1}^{n}(1 + c_1(L_i)).$$

We now prove 1): we reduce first easily to the case when X is connected; then, by the splitting principle, we reduce to considering the case when E and F are both direct sums of complex line bundles, say $E \cong L_1 \oplus \cdots \oplus L_r$, $F \cong M_1 \oplus \cdots \oplus M_s$; then we have that

$$E \oplus F \cong L_1 \oplus \oplus L_r \oplus M_1 \oplus \cdots M_s$$

is also a direct sum of line bundles, and so we have formulas

$$\sum_{i\geq 0} c_i(E) = \prod_{i=1}^{r}(1 + c_1(L_i)),$$

$$\sum_{i\geq 0} c_i(F) = \prod_{j=1}^{s}(1 + c_1(M_j)),$$

$$\sum_{i\geq 0} c_i(E \oplus F) = \left(\prod_{i=1}^{r}(1 + c_1(L_i))\right)\left(\prod_{j=1}^{s}(1 + c - 1(M_j))\right);$$

from these formulas, the desired formula in 1) is obvious. The proof of 2) is very similar. □

Example 1.32. Show that the Chern classes of a tensor product $E \otimes F$ of two complex vector bundles are given by 'universal' polynomials with integer coefficients in the Chern classes of E and F.

We now give a proof of Theorem 1.20(a), assuming that the classification of vector bundles via homotopy classes of maps to an infinite Grassmannian, the formula for the cohomology ring of a projective bundle, the resulting formalism of Chern classes, and the splitting principle, are all valid even when the base space X is a "sufficiently good" non-compact Hausdorff space; this can be rigorously justified, but we do not do this here. We will need that the above results hold even when X is an infinite dimensional CW-complex, with finitely many cells of any given dimension. The idea is that if X is such a space, and X_n is its n-skeleton, then X_n is a compact Hausdorff space, $\cup_{n\geq 0} X_n = X$, and any compact subset of X lies in some X_n; further, a map $X \to Y$ is continuous if and only if its restriction to each X_n is continuous. Thus, using the theory developed above, applied to each of the "finite dimensional approximations" X_n, one can extend its validity to such spaces X as well.

Another way to make our arguments rigorous is to use the fact that for any $i \geq 0$, the natural maps

$$H^i(X_{n+1}, A) \to H^i(X_n, A),$$

and hence also

$$H^i(X, A) \to H^i(X_n, A),$$

are isomorphisms for $n > i$. Thus, any conclusions regarding cohomology of any infinite dimensional CW complex X as above can be obtained by considering the cohomology groups of the finite dimensional approximations X_n. This approach avoids the need for constructing classifying maps for vector bundles on such an infinite dimensional space X.

Consider the space $X = (\mathbb{P}^{\infty}_{\mathbb{C}})^n = \mathbb{P}^{\infty}_{\mathbb{C}} \times \mathbb{P}^{\infty}_{\mathbb{C}} \times \cdots \times \mathbb{P}^{\infty}_{\mathbb{C}}$. If $p_i : X \to \mathbb{P}^{\infty}_{\mathbb{C}}$ is the i-th projection, then there is a vector bundle $E = p_1^*\nu_{1,\infty} \oplus p_2^*\nu_{1,\infty} \oplus \cdots \oplus p_n^*\nu_{1,\infty}$

on X of rank n. Let $f : X \to \mathbb{G}_{\mathbb{C}}(n)$ be a classifying map for this bundle. We claim that if $\nu_{n,\infty}$ is the tautological bundle on $\mathbb{G}_{\mathbb{C}}(n)$, and $g : P \to \mathbb{G}_{\mathbb{C}}(n)$ is a splitting map for $\nu_{n,\infty}$, then there is a continuous map $h : P \to X$ giving a diagram (commutative up to homotopy)

$$\begin{array}{ccc} P & \xrightarrow{h} & X \\ & \searrow g & \downarrow f \\ & & \mathbb{G}_{\mathbb{C}}(n) \end{array}$$

(This is from the universal properties of $\mathbb{G}_{\mathbb{C}}(n)$ and $\mathbb{P}^{\infty}_{\mathbb{C}} = \mathbb{G}_{\mathbb{C}}(1)$.) Hence the natural map on cohomology $f : H^*(\mathbb{G}_{\mathbb{C}}(n), \mathbb{Z}) \to H^*(X, \mathbb{Z})$ is *injective.* If $\sigma : X \to X$ is any permutation of the factors, then there is a natural isomorphism $\sigma^* E \cong E$; hence $f \circ \sigma$ must be homotopic to f, and so $f^* = \sigma^* \circ f^*$ on $H^*(\mathbb{G}_{\mathbb{C}}(n), \mathbb{Z})$. This means that the subring

$$f^*(H^*(\mathbb{G}_{\mathbb{C}}(n), \mathbb{Z})) \subset H^*(X, \mathbb{Z}) \cong \otimes_{i=1}^n H^*(\mathbb{P}^{\infty}_{\mathbb{C}}, \mathbb{Z})) = \mathbb{Z}[t_1, \ldots, t_n]$$

(where $t_i = p_i^*(x)$, for the generator $x \in H^2(\mathbb{P}^{\infty}_{\mathbb{C}}, \mathbb{Z})$) is contained in the ring of invariants for the permutation group S_n on n symbols, acting by permuting the variables t_i. Hence if $s_i(t_1, \ldots, t_n)$ denotes the i-th elementary symmetric polynomial, then

$$f^*(H^*(\mathbb{G}_{\mathbb{C}}(n), \mathbb{Z})) \subset \mathbb{Z}[s_1(t_1, \ldots, t_n), \ldots, s_n(t_1, \ldots, t_n)].$$

But by corollary 1.30, $s_i(t_1, \ldots, t_n) = c_i(E) = f^*(c_i(\nu_{n,\infty}))$. Since f^* is injective, we deduce that $H^*(\mathbb{G}_{\mathbb{C}}(n), \mathbb{Z})$ is the polynomial algebra in the n (algebraically independent) elements $c_1(\nu_{n,\infty}), \ldots, c_n(\nu_{n,\infty})$.

In a similar way, we may formally deduce the structure of $H^*(\mathbb{G}_{\mathbb{R}}(n), \mathbb{Z}/2\mathbb{Z})$ (*i.e.*, theorem 1.20(c)) from the theory of Steifel-Whitney classes applied to an analogous bundle on $X = (\mathbb{P}^{\infty}_{\mathbb{R}})^n$, and the splitting principle applied to the universal bundle on the infinite Grassmanian $\mathbb{G}_{\mathbb{R}}(n)$.

1.5. **The Grothendieck group of vector bundles.** Let

$$\mathbf{Vect}\,(X) = \coprod_{n \geq 0} \mathbf{Vect}_n(X),$$

where $\mathbf{Vect}_n(X)$ is the set of isomorphism classes of complex vector bundles of rank n. The direct sum and tensor product of vector bundles makes this a "commutative semi-ring", *i.e.*, there are two commutative, associative binary operations $+$ and $\cdot$, with identity elements, such that $\cdot$ is distributive over $+$, $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$. Thus $(\mathbf{Vect}\,(X), +)$ is a commutative monoid.

From the Whitney sum formula (corollary 1.31), the assignment $[E] \mapsto c(E) = 1 + c_1(E) + c_2(E) + \cdots$ gives a homomorphism from the monoid $(\mathbf{Vect}\,(X), +)$ to the group of units of the commutative ring

$$H^{even}(X, \mathbb{Z}) = \oplus_{i \geq 0} H^{2i}(X, \mathbb{Z}).$$

This motivates the Grothendieck construction of $K(X)$.

Definition 1.33. If M is a commutative monoid, the *Grothendieck group* is an abelian group $K(M)$, together with a homomorphism of monoids $u : M \to K(M)$, such that for any homomorphism of monoids $f : M \to A$, for an abelian group A, there exists a unique group homomorphism $\widetilde{f} : K(M) \to A$ such that $f = \widetilde{f} \circ u$.

Such a pair $(K(M), u)$ is unique up to unique isomorphism, since it is specified by a universal mapping property; an explicit construction is given by

$$K(M) = \frac{\text{Free abelian group on elements of } M}{\text{Subgroup generated by classes } (a+b)-(a)-(b) \text{, for all } a, b \in M}$$

with $u : M \to K(M)$ being given by $a \mapsto [a]$; here (a) denotes the class of a in the free abelian group and $[a]$ its image in the quotient group. The universal property is trivially verified. Note further that if M is a commutative semiring, then $K(M)$ is a commutative ring, with multiplication induced by $[a][b] = [a \cdot b]$.

Definition 1.34. For any topological space, define its *Grothendieck ring* $K(X)$ to be $K(\mathbf{Vect}\,(X))$. If $\mathbf{Vect}^{\mathbb{R}}(X)$ is the semiring of isomorphism classes of real vector bundles, define $KO(X)$ to be $K(\mathbf{Vect}^{\mathbb{R}}(X))$.

We will not comment further about $KO(X)$, except to remark that part of the theory of $K(X)$ has a parallel for $KO(X)$, but certain specific key results (like Bott Periodicity) take rather different forms for the two theories, so that the final conclusions are different. As such, from now onwards, "vector bundle" will mean "complex vector bundle" unless specified otherwise.

For compact Hausdorff spaces X, it is easy to see from the definition that for 2 complex vector bundles E, F, we have $[E] = [F]$ in $K(X)$ if and only if $E \oplus (X \times \mathbb{C}^n) \cong F \oplus (X \times \mathbb{C}^n)$ for some $n \geq 0$, where $X \times \mathbb{C}^n$ is the trivial bundle of rank n. In a similar fashion, if E, F are real vector bundles, then $[E] = [F]$ in $KO(X)$ if and only if $E \oplus (X \times \mathbb{R}^n) \cong F \oplus (X \times \mathbb{R}^n)$. We express either condition by saying that E and F are *stably equivalent*.

Example 1.35. Consider the $n-1$-sphere S^{n-1} as a submanifold of $\mathbb{R}^n$. Then its tangent bundle TS^{n-1} is a subbundle of $S^{n-1} \times \mathbb{R}^n = \varepsilon^n_{S^{n-1}}$, the trivial bundle of rank n, where the fibre T_xS^{n-1} consists of the orthogonal complement of $\mathbb{R}x \subset \mathbb{R}^n$. We may also regard the trival bundle $S^{n-1} \times \mathbb{R} = \varepsilon_{S^{n-1}}$ as a sub-bundle of $\varepsilon^n_{S^{n-1}}$ via the mapping $(x, t) \mapsto (x, tx)$. Then clearly the induced map

$$TS^{n-1} \oplus \varepsilon_{S^{n-1}} \to \varepsilon^n_{S^{n-1}}$$

is an isomorphism. Hence $[TS^{n-1}] = (n-1)[\varepsilon_{S^{n-1}}]$ in $KO(S^{n-1})$. However, for n odd, it is known that S^{n-1} has no non-vanishing vector fields, *i.e.*, TS^{n-1} has no subbundle isomorphic to $\varepsilon_{S^{n-1}}$. Thus TS^{n-1} and $\varepsilon^{n-1}_{S^{n-1}}$ are stably isomorphic, but non-isomorphic.

Similarly there are examples of stably trivial non-trivial complex vector bundles, but their construction (and proof of non-triviality) is a little more involved.

Theorem 1.36. (The Chern character) *There is a unique functorial ring homomorphism*

$$ch_X : K(X) \to H^{\text{even}}(X, \mathbb{Q}),$$

where X is a finite CW complex, such that for any line bundle L,

$$ch_X([L]) = e^{c_1(L)} = \sum_{n\geq 0} \frac{c_1(L)^n}{n!}.$$

For arbitrary X, the homomorphism makes sense provided we define

$$H^{\text{even}}(X, \mathbb{Q}) = \prod_{n\geq 0} H^{2n}(X, \mathbb{Q}).$$

Proof. First notice that if X is finite CW, then $H^i(X, \mathbb{Q}) = 0$ for $i > \dim X$, so the formula for $ch_X(L)$ makes sense.

If $E = L_1 \oplus \cdots \oplus L_n$, and $c_1(L_i) = x_i \in H^2(X, \mathbb{Q})$, then since ch_X is to be a ring homomorphism, we must have

$$ch_X(E) = \sum_{i=1}^n ch_X(L_i) = \sum_{i=1}^n e^{x_i}$$

$$= n + (\sum_i x_i) + \frac{\sum_i x_i^2}{2!} + \cdots + \frac{\sum_i x_i^m}{m!} + \cdots ,$$

where we observe that

$$\frac{\sum_i x_i^m}{m!}$$

is uniquely expressible as a polynomial with rational coefficient in the elementary symmeric polynomials in $x_1, \ldots, x_n$, *i.e.*, is a polynomial with rational coefficients in the Chern classes of E. By the splitting principle, we must now have

$$ch_X(E) = (\text{rank} E) + c_1(E) + \frac{c_1(E)^2 - c_2(E)}{2!} + \cdots + P_m(c_1(E), \ldots, c_m(E)) + \cdots$$

for a certain polynomial in $c_1(E), \ldots, c_n(E)$ which is 'isobaric' (or 'weighted homogeneous'), *i.e.*, , is homogeneous of degree m, where we define $c_i(E)$ to have degree i. In fact $P_m(t_1, \ldots, t_n)$ is the unique polynomial in n variables such that if $s_1, \ldots, s_n$ are the elementary symmetric polynomials in variables $x_1, \ldots, x_n$, then

$$P_m(s_1, \ldots, s_n) = \frac{x_1^m + \cdots + x_n^m}{m!} .$$

If we now take this formula as the definition of $ch_X(E)$, it is easy to check using the splitting principle that if E, F are vector bundles with Chern roots $x_1, \ldots, x_n$ and $y_1, \ldots, y_m$, then $E \oplus F$ has Chern roots $x_1, \ldots, x_n, y_1, \ldots, y_m$, and $E \otimes F$ has Chern roots $x_1 + y_1, x_1 + y_2, \ldots, x_n + y_m$. Hence

$$ch_X(E) = \sum_i e^{x_i}, \quad ch_X(F) = \sum_j e^{y_j},$$

$$ch_X(E \oplus F) = \sum_i e^{x_i} + \sum_j e^{y_j} = ch_X(E) + c_X(F),$$

$$ch_X(E \otimes F) = \sum_{i,j} e^{x_i + y_j} = (\sum_i e^{x_i})(\sum_j e^{y_j}) = ch_X(E) ch_X(F).$$

Hence ch_X is a ring homomorphism. □

Theorem 1.37. *If X is a finite CW complex, the induced map*

$$ch_X : K(X) \otimes \mathbb{Q} \to H^{\text{even}}(X, \mathbb{Q})$$

is an isomorphism.

We will later sketch a proof of this important result using the *Atiyah-Hirzebruch spectral sequence.*

1.6. **Relative K-groups.** If X is a space with a base point $x \in X$, define the *reduced* K-group group $\widetilde{K}(X) = \ker(K(X) \to K(\{x\}) = \mathbb{Z}$. It is a ring without identity, whose elements have the form $[E] - n[\varepsilon_X]$, where $n = \dim E_x$ is the rank of E at x; here

$$[E] - n[\varepsilon_X] = [F] - m[\varepsilon_X] \iff E \oplus \varepsilon_X^{n+N} \cong F \oplus \varepsilon_X^{m+N}$$

for some $N \geq 0$.

Consider the map $a_n(X) : \mathbf{Vect}_n(X) \to \mathbf{Vect}_{n+1}(X)$ given by $E \mapsto E \oplus \varepsilon_X$. This corresponds to the inclusion $\alpha_n : \mathbb{G}_{\mathbb{C}}(n) \hookrightarrow \mathbb{G}_{\mathbb{C}}(n+1)$ induced by $\mathbb{G}_{\mathbb{C}}(n,m) \hookrightarrow \mathbb{G}_{\mathbb{C}}(n+1, m+1)$, which in turn is induced by $\mathbb{U}(m) \hookrightarrow \mathbb{U}(m+1)$ given by

$$A \mapsto \begin{bmatrix} A & 0 \\ 0 & 1 \end{bmatrix}.$$

There are commutative diagrams

$$\begin{array}{ccc} \mathbf{Vect}_n(X) & \xrightarrow{a_n(X)} & \mathbf{Vect}_{n+1}(X) \\ \cong\downarrow & & \downarrow\cong \\ [X, \mathbb{G}_{\mathbb{C}}(n)] & \xrightarrow{\alpha_n} & [X, \mathbb{G}_{\mathbb{C}}(n+1)] \end{array}$$

and

$$\begin{array}{ccc} \mathbf{Vect}_n(X) & \overset{a_n(X)}{\to} & \mathbf{Vect}_{n+1}(X) \\ b_n \searrow & & \swarrow b_{n+1} \\ & \widetilde{K}(X) & \end{array}$$

where $b_n(E) = [E] - n[\varepsilon_X]$, which identifies $\varinjlim_n \mathbf{Vect}_n(X)$ with $\widetilde{K}(X)$.

Hence if $\mathbb{G}_{\mathbb{C}} = \varinjlim_n \mathbb{G}_{\mathbb{C}}(n)$, then there exists a natural bijection

$$\widetilde{K}(X) \cong [X, \mathbb{G}_{\mathbb{C}}].$$

The following alternative description of a "classifying space" for K-theory is proved in an appendix to Atiyah's book.

Theorem 1.38. *Let H be a separable complex Hilbert space, $\mathcal{A}(H)$ the Banach algebra of bounded linear operators on H, and $\mathcal{F} \subset \mathcal{A}(H)$ the subset of Fredholm operators (these are elements $T \in \mathcal{A}(H)$ with $\ker T$ and $\operatorname{coker} T$ finite dimensional). Then for any compact Hausdorff space X, there is a natural bijection*

$$K(X) \cong [X, \mathcal{F}].$$

Lemma 1.39. *Let $A \subset X$ be a closed subcomplex of a finite CW complex.*

(a) *There exists an exact sequence*

$$\widetilde{K}(X/A) \to K(X) \to K(A).$$

(b) *If we define* $\widetilde{K}^{-n}(X) = \widetilde{K}(\Sigma^n X)$*,* $K^n(X,A) = \widetilde{K}(\Sigma^n(X/A))$*, then there is a functorial long exact sequence*

$$\cdots K^{-n}(X,A) \to K^{-n}(X) \to K^{-n}(A) \to K1-n(X,A) \to \cdots$$
$$\to K^0(X,A) \to K^0(X) \to K^0(A).$$

If we define $K^{-n}(X) = \widetilde{K}^{-n}(X^+)$*, where* X^+ *is the disjoint union of* X *with a base point (denoted* $+$*), then we obtain a similar exact sequence with* K^{-n} *in place of* $\widetilde{K}^{-n}$*.*

Proof. Since $\widetilde{K}(X) = [X, \mathbb{G}_{\mathbb{C}}]$, the lemma follows from the Puppe (co)exact sequence (see Spanier, *Algebraic Topology*)

$$A \xrightarrow{f} X \to C(f) \to \Sigma A \xrightarrow{\Sigma f} \Sigma X \to \Sigma(C(f)) \to \Sigma^2 A \to \cdots$$

and the homotopy equivalence $C(f) \simeq X/A$. □

1.7. **Bott Periodicity and the Atiyah-Hirzebruch spectral sequence.** If X and Y are spaces with base points x, y respectively, let $X \vee Y$ be the subspace $X \times \{y\} \cup \{x\} \times Y \subset X \times Y$. Define the *smash product* of X and Y by $X \wedge Y = X \times Y / X \vee Y$ (with the evident base point).

Example 1.40. For any pointed pair (X, x), we have

$$\begin{aligned} X \wedge S^1 &= (\text{reduced suspension of } X) \\ &\cong \Sigma X / \Sigma\{x\} = \Sigma X / I, \end{aligned}$$

so that $\widetilde{K}(X \wedge S^1) = \widetilde{K}(\Sigma X)$.

Example 1.41. $S^n \wedge S^1 \cong S^{n+1}$.

Lemma 1.42. *The sequence*

$$\begin{array}{ccccc} 0 \to \widetilde{K}(X \wedge Y) & \to \widetilde{K}(X \times Y) \to & \widetilde{K}(X \vee Y) \to 0 \\ & & \downarrow\cong \\ & & \widetilde{K}(X) \oplus \widetilde{K}(Y) \end{array}$$

is split exact.

Proof. Since $X \times \{y\}$ is a retract of $X \times Y$, the sequence

$$0 \to \widetilde{K}(X \times Y / X \times \{y\}) \to \widetilde{K}(X \times Y) \to \widetilde{K}(X) \to 0$$

is split exact. Next, $\{x\} \times Y$, regarded as a (closed) subset of $X \times Y / X \times \{y\}$, is a retract, and so the sequence

$$0 \to \widetilde{K}(X \wedge Y) \to \widetilde{K}(X \times Y / X \times \{y\}) \to \widetilde{K}(Y) \to 0$$

is split exact. Combining these two sequences we obtain the lemma. □

Note that the composition

$$K(X) \otimes K(Y) \to K(X \times Y) \to K(X \vee Y) \cong K(X) \oplus K(Y),$$
$$[E] \otimes [F] \mapsto [p_1^* E \otimes p_2^* F],$$

induced by tensor product of pullbacks of bundles from the two factors, is trivial on $\widetilde{K}(X) \otimes \widetilde{K}(Y)$. Hence from the above lemma, we have a well-defined pairing

$$\widetilde{K}(X) \otimes \widetilde{K}(Y) \to \widetilde{K}(X \wedge Y).$$

Taking $Y = S^2$, we have a pairing

$$\beta_X : \widetilde{K}(X) \otimes \widetilde{K}(S^2) \to \widetilde{K}(\Sigma^2 X) = \widetilde{K}^{-2}(X),$$

for any X.

Theorem 1.43. (Bott Periodicity) *β_X is an isomorphism for any X.*

Now $\widetilde{K}(S^2) \cong \varinjlim_n \pi_1(\mathrm{GL}_n(\mathbb{C})) = \mathbb{Z}$, since $Vect_n(S^2) \cong \pi_1(\mathrm{GL}_n(\mathbb{C}))$ (write S^2 as the union of its upper and lower hemispheres, which are contractible). Hence we obtain:

Corollary 1.44. *$\widetilde{K}(X) \cong \widetilde{K}^{-2}(X)$.*

Motivated by the above corollary, define

$$K^n(X) = \begin{cases} \widetilde{K}(X^+) & \text{if } n \text{ is even} \\ \widetilde{K}(\Sigma(X^+)) & \text{if } n \text{ is odd,} \end{cases}$$

and for any pair of spaces (X, A),

$$K^n(X, A) = \begin{cases} \widetilde{K}(X/A) & \text{if } n \text{ is even} \\ \widetilde{K}(\Sigma(X/A)) & \text{if } n \text{ is odd,} \end{cases}$$

Then there exists a functorial, doubly infinite long exact sequence for pairs

$$\cdots \to K^n(X, A) \to K^n(X) \to K^n(A) \to K^{n+1}(X, A) \to \cdots$$

Theorem 1.45. (Atiyah-Hirzebruch Spectral Sequence) *For any finite CW complex X, there is a spectral sequence, which is functorial in X,*

$$E_2^{p,q} = H^p(X, \mathbb{Z}) \otimes_{\mathbb{Z}} K^q(\text{point}) \implies K^{p+q}(X).$$

Proof. (Sketch) Filter X by skeleta X_n; notice that

$$K^i(X_n, X_{n-1}) \cong C^n(X) \otimes_{\mathbb{Z}} \widetilde{K}^i(S^n),$$

where $C^n(X)$ is the group of cellular n-cochains on X. We may rewrite this as $C^n(X) \otimes_{\mathbb{Z}} K^{i-n}(\text{point})$. The collection of long exact sequences for the pairs (X_n, X_{n-1}) yield an exact couple, and hence a spectral sequence

$$E_1^{p,q} = K^{p+q}(X_p, X_{p-1}) \implies K^{p+q}(X),$$

which we may rewrite as

$$E_1^{p,q} = C^p(X) \otimes_{\mathbb{Z}} K^q(\text{point}) \implies K^{p+q}(X).$$

The E_1 differentials are thus maps

$$E_1^{p,q} \to E_1^{p+1,q},$$
$$K^q(\text{point}) \otimes C^p(X) \to K^q(\text{point}) \otimes C^{p+1}(X),$$

which we claim to be of the form $1 \otimes d^p$, where $d^p : C^p(X) \to C^{p+1}(X)$ is the cellular differential; this reduces to showing that if $f : S^q \to S^q$ has degree d, then the induced map $f^* : \widetilde{K}(S^q) \to \widetilde{K}(S^q)$ is multiplication by d. The claim implies the formula for the E_2 terms. □

Note that the construction of the Atiyah-Hirzebruch spectral sequence is in fact formal, and works to give a similar spectral sequence for any "generalized cohomology theory", determined by a sequence of functors on CW-pairs, and homotopy classes of maps between them, satisfying all of the Eilenberg-Steenrod axioms, except the dimesnion axiom, which specifies the cohomology of a point.

One somewhat artifical example of such a cohomology theory is as follows. Define

$$\hat{H}^i(X,A) = \begin{cases} H^{\text{even}}(X,A;\mathbb{Q}), & \text{if } i \text{ is even} \\ H^{\text{odd}}(X,A;\mathbb{Q}), & \text{if } i \text{ is odd.} \end{cases}$$

The usual exact sequence of a pair gives an exact sequence

$$\cdots \to \hat{H}^i(X,A) \to \hat{H}^i(X) \to \hat{H}^i(A) \to \hat{H}^{i+1}(X,A) \to \cdots$$

The other desired properties (functoriality, homotopy invariance, etc.) follow from the corresponding properties of usual cohomology. From the definition of the cohomology theory $\hat{H}^*$, it is clear that the corresponding spectral sequence

$$E_2^{p,q} = H^p(X,\mathbb{Z}) \otimes \hat{H}^q(\text{point}) \implies \hat{H}^{p+q}(X)$$

degenerates at E_2.

One verifies that the Chern character yields natural transformations $K^i(X) \otimes \mathbb{Q} \to \hat{H}^i(X)$ and $K^i(X,A) \to \hat{H}^i(X,A)$, for all i, compatible with the respective long exact sequences of pairs. This leads to a *morphism of spectral sequences*

$$E_r^{p,q}(K^* \otimes \mathbb{Q}) = E_r^{p,q}(K^*) \otimes \mathbb{Q} \to E_r^{p,q}(\hat{H}^*).$$

One can show that this is an isomorphism on E_2-terms, since

$$ch : K(S^n) \otimes \mathbb{Q} \to H^{even}(S^n, \mathbb{Q})$$

is an isomorphism, for all n (use Bott periodicity to reduce to the cases $n = 1, 2$, where it is clear). Hence the above morphism of spectral sequences is an *isomorphism*. In particular,

(i) the Atiyah-Hirzebruch spectral sequence *degenerates at E_2 after $\otimes\mathbb{Q}$*
(ii) we obtain isomorphisms $K^n(X)\otimes\mathbb{Q} \cong \hat{H}^n(X)$, and hence also $K^n(X,A)\otimes \mathbb{Q} \cong \hat{H}^n(X,A)$, for all CW pairs (X,A).

Now suppose X is a finite CW complex whose cohomology groups $H^i(X,\mathbb{Z})$ are torsion-free abelain groups. Then we claim the Atiyah-Hirzebruch spectral sequence degeneates at E_2, even without $\otimes\mathbb{Q}$. Indeed, there is a natural transformation of spectral sequences

$$E_r^{p,q}(K) \to E_r^{p,q}(K \otimes \mathbb{Q}),$$

compatible with the natural map $K^n(X) \to K^n(X) \otimes \mathbb{Q}$. The map on E_2 terms is injective, since they are torsion-free abelian groups, by hypothesis, and the E_2 differentials vanish $\otimes\mathbb{Q}$, as seen above. Hence the E_2 differentials of the Atiyah-Hirzebruch spectral sequence vanish, and its E_3 terms (being isomorphic to the corresponding E_2 terms) are torsion-free as well. Now argue inductively, showing that the E_r terms are torsion-free, and the E_r differentials vanish, for all r.

We deduce that if $H^*(X, \mathbb{Z})$ is torsion-free, then the Atiyah-Hirzebruch spectral sequence degenerates at E_2. Thus there is a finite, decreasing filtration $\{F^iK(X)\}_{i\geq 0}$ on $K(X)$ such that $F^iK(X)/F^{i+1}K(X) \cong H^{2i}(X, \mathbb{Z})$. In particular, $K(X)$ is also torsion-free.

1.8. **Adams operations.** Let $K(X)[[t]]$ denote the formal power series ring in the indeterminate t over the commutative coefficient ring $K(X)$. Define a homomorphism

$$\lambda_t : K(X) \to K(X)[[t]]^*,$$
$$\lambda_t[E] = \sum_{i\geq 0}[\bigwedge^i E]t^i,$$

where $K(X)[[t]]^*$ denotes the multiplicative group of units of $K(X)[[t]]$. Here $\bigwedge^i E$ denotes the i-th exterior power of the vector bundle E; from the formula

$$\bigwedge^n(E \oplus F) \cong \oplus_{i+j=n}(\bigwedge^i E) \otimes (\bigwedge^j F),$$

we see that $\lambda_t(E \oplus F) = \lambda_t(E)\lambda_t(F)$, and so the above formula for λ_t on vector bundles does induce a well-defined homomorphism on $K(X)$. For each $i \geq 0$, we let $\lambda_i : K(X) \to K(X)$ denote the coefficient of t^i in $\lambda_t : K(X) \to K(X)[[t]]^*$; it is a map of sets.

Note that λ_t takes values in the subgroup of units of the form $1 + t\alpha(t)$, where $\alpha(t) \in K(X)[[t]]$. Hence we can define another formal series

$$\psi_t(x) = -t\frac{d}{dt}\log \lambda_{-t}(x) \in K[[t]]$$

for all $x \in K(X)$; here $\lambda_{-t}(x)$ has the obvious meaning. Now $\psi_t : K(X) \to K(X)[[t]]$ is an additive homomorphism. Let $\psi_i : K(X) \to K(X)$ be the additive homomorphism determined by the coefficient of t^i in ψ_t, for each $i \geq 1$; the map ψ_i is called the i-th *Adams operation* on $K(X)$.

Proposition 1.46. *The Adams operations have the following properties.*

(1) $\psi_i : K(X) \to K(X)$ *is a ring homomorphism, for each* $i \geq 1$.
(2) $\psi_i(x) = x^i$ *if* $x = [L]$ *is the class of a line bundle.*
(3) $\psi_i \circ \psi_j = \psi_{ij} = \psi_j \circ \psi_i$.
(4) $\psi_i(x) = s_i(\lambda^1(x), \ldots, \lambda^i(x))$ *for a certain universal polynomial* $s_i(t_1, \ldots, t_i) \in \mathbb{Z}[t_1, \ldots, t_i]$.
(5) *If* p *is a prime number, then* $\psi_p : K(X) \otimes \mathbb{Z}/p\mathbb{Z} \to K(X) \otimes \mathbb{Z}/p\mathbb{Z}$ *is* $x \mapsto x^p$.
(6) *If the image of* $x \in K(X)$ *under the Chern character is* $ch_X(x) = x_0 + x_2 + x_4 + \cdots$ *with* $x_i \in H^i(X, \mathbb{Q})$, *then* $ch_X(\psi_k(x)) = x_0 + kx_2 + k^2x_4 + \cdots$.

(7) *ψ_k is diagonalizable on $K(X) \otimes \mathbb{Q}$, with eigenvalues k^i; the k^i-eigenspace is independent of $k \geq 2$, and is identified with $H^{2i}(X, \mathbb{Q})$.*

Proof. First note that if $x = [L]$ is the class of a line bundle, then $\lambda^i(x) = 0$ for $i > 1$. Hence $\lambda_{-t}(x) = 1 - xt$, and so

$$\psi_t(x) = -t\frac{d}{dt}\log(1 - xt) = \sum_{i \geq 1} x^i t^i.$$

This gives (2) above. Now (1) and (3) follow from the special case of line bundles, by the splitting principle.

For (4), note that if $x = [L_1 \oplus \cdots \oplus L_n]$ for line bundles L_i, then $\psi_i(x) = [L_1]^i + \cdots + [L_n]^i$. On the other hand, $\lambda^i(x)$ is the i-th elementary symmetric polynomial in $x_1 = [L_1], \ldots, x_n = [L_n]$. Let $X_1, \ldots, X_N$ be indeterminates, with $N \geq i$, and let $\sigma_i(X_1, \ldots, X_N)$ denote the j-th elementary symmetric polynomial in $X_1, \ldots, X_N$. Let $s_i(t_1, \ldots, t_i) \in \mathbb{Z}[t_1, \ldots, t_i]$ be the unique polynomial such that $s_i(\sigma_1(X_1, \ldots, X_N), \ldots, \sigma_i(X_1, \ldots, X_N)) = X_1^i + \cdots + X_N^i$, the i-th Newton symmetric polynomial. This polynomial s_i is in fact independent of $N \geq i$. Clearly we have $s_i(\lambda^1(x), \ldots, \lambda^i(x)) = \psi_i(x)$, for x as above. By the splitting principle, the formula holds for $x = [E]$, for any vector bundle E.

The formula (5) follows from (4), for example, since $s_p(\sigma_1, \ldots, \sigma_p) \cong \sigma_1^p (\bmod\, p)$. For (6), note that for $x = [L]$, we have $ch_X(x) = e^{c_1(x)}$, and $ch_X(\psi_k(x)) = ch_X(x^k) = e^{c_1(x^k)} = e^{kc_1(x)}$. Hence (6) holds for $x = [L]$. The splitting principle implies (6) in general. Clearly (7) follows from (6). □

1.9. **The Hopf Invariant.** Let $\alpha \in \pi_{4n-1}(S^{2n})$, and let X be the CW complex

$$X = S^{2n} \cup_\alpha D^{4n}$$

obtained by attaching $\partial D^{4n} = S^{4n-1}$ to S^{2n} using α. Then

$$H^0(X, \mathbb{Z}) \cong H^{2n}(X, \mathbb{Z}) \cong H^{4n}(X, \mathbb{Z}) \cong \mathbb{Z}.$$

Let $a \in H^{2n}(X, \mathbb{Z})$ and $b \in H^{4n}(X, \mathbb{Z})$ be the generators corresponding to the standard orientations of S^{2n} and S^{4n}. Then $a^2 = a \cup a = H(\alpha)b$ for some integer $H(\alpha) \in \mathbb{Z}$, called the *Hopf invariant* of α. This construction in fact gives rise to a homomorphism $H : \pi_{4n-1}(S^{2n}) \to \mathbb{Z}$. The Hopf invariant of α is $\pm 1 \iff$ the cohomology ring $H^*(X, \mathbb{Z}) = \mathbb{Z}[a]/(a^3)$, where $a \in H^{2n}(X, \mathbb{Z})$ is a generator.

Note that if we make a similar construction with an element $\alpha \in \pi_{2n-1}(S^n)$ with n odd, then with the earlier notation, we have $a \cup a = 0$ for any generator $a \in H^n(X, \mathbb{Z})$, since $a \cup a = (-1)^n a \cup a$. Hence the "Hopf invariant" of any $\alpha \in \pi_{2n-1}(S^n)$ vanishes, if n is odd.

Example 1.47. For the Hopf fibrations $S^3 \to S^2 = \mathbb{CP}^1$, $S^7 \to S^4 = \mathbb{HP}^1$ and $S^{15} \to S^8 = \mathbb{OP}^1$, one computes that the Hopf invariant is 1, *i.e.*, for $n = 1, 2, 4$.

Question: Are there any other values of n for which there exists $\alpha \in \pi_{4n-1}(S^{2n})$ with $H(\alpha) = \pm 1$?

This question has a *negative* answer. Suppose X is the space obtained form $\alpha \in \pi_{4n-1}(S^{2n})$. Since the integral cohomology of X is torsion-free, the Atiyah-Hirzebruch spectral sequence degenerates at E_2. Thus we have a filtration $K(X) \supset$

$\widetilde{K}(X) = F^nK(X) \supset F^{2n}K(X)$, where the Chern character induces isomorphisms

$$ch_X : F^nK(X)/F^{2n}K(X) \xrightarrow{\cong} H^{2n}(X,\mathbb{Z}) = \mathbb{Z}\cdots a \subset H^{2n}(X,\mathbb{Q}),$$

$$ch_X : F^{2n}K(X) \xrightarrow{\cong} H^{4n}(X,\mathbb{Z}) = \mathbb{Z}a^2 \subset H^{4n}(X,\mathbb{Q}).$$

Let $a' \in F^nK(X)$, $b' \in F^{2n}K(X)$ such that $ch_X(a') = a + \lambda a^2$ with $\lambda \in \mathbb{Q}$, and $ch_X(b') = b = a^2$. Since $ch_X(a'^2) = (a+\lambda a^2)^2 = a^2$, we have $ch_X(b' - a'^2) = 0$, and so $b' = a'^2$.

Now $ch_X(\psi_2(a')) = 2^n a + \lambda 2^{2n} a^2$. Hence $\psi_2(a') = 2^n a' + \lambda_2 a'^2$ for some $\lambda_2 \in \mathbb{Z}$. Since $\psi_2(a') \equiv a'^2 \pmod{2K(X)}$, the integer λ_2 must be *odd*. Similarly, we see that $\psi_3(a') = 3^n a' + \lambda_3 a'^2$ for some $\lambda_3 \in \mathbb{Z}$. Since $\psi_2\psi_3 = \psi_3\psi_2$, we get

$$\psi_3(2^n a' + \lambda_2 a'^2) = \psi_2(3^n a' + \lambda_3 a'^2),$$

$$i.e.,\ 2^n(3^n a' + \lambda_3 a'^2) + 3^{2n}\lambda_2 a'^2 = 3^n(2^n a' + \lambda_2 a'^2) + 2^{2n}\lambda_3 a'^2$$

$$\implies \quad (2^{2n} - 2^n)\lambda - 3 = (3^2 - 3^n)\lambda_2.$$

It is an easy number theoretic exercise to show that if $2^n \mid (3^n - 1)$, then $n = 1, 2$ or 4.

Example 1.48. Let $f : S^{n-1} \times S^{n-1} \to S^{n-1}$ be such that $f\mid_{S^{n-1}\times\{x_0\}}$ and $f\mid_{\{x_0\}\times S^{n-1}}$ each have degree ± 1, for any base point x_0 (*i.e.*, $f \mid S^{n-1} \vee S^{n-1}$ has bidegree $(\pm1, \pm1)$). Such a map arises, for example, if there exists a continuous non-singular product $\mu : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ ("non-singular" means that the induced maps $\mu : \{x\} \times \mathbb{R}^n \to \mathbb{R}^n$, $\mu : \mathbb{R}^n \times \{x\} \to \mathbb{R}^n$ are homeomorphisms preserving the origin, for any non-zero $x \in \mathbb{R}^n$). The map f is obtained by restricting μ to the product of the unit spheres, and composing with the obvious retraction $\mathbb{R}^n - \{0\} \to S^{n-1}$.

Extend f to continuous maps $f_+ : D^n \times S^{n-1} \to D^n_+$, $f_- : S^{n-1} \times D^n \to D^n_-$, which preserve the respective boundaries, where $D^n_\pm$ are the two hemispheres in S^n, and such that both maps $f_\pm$ restrict to f on the respective boundaries. We may then glue these maps to obtain a map

$$\widetilde{f} : \partial(D^n \times D^n) = (D^n \times S^{n-1}) \cup_{S^{n-1}\times S^{n-1}} (S^{n-1} \times D^n) \to D^n_+ \cup_{S^{n-1}} D^n_- = S^n.$$

Identifying $\partial D^n \times D^n \cong \partial D^{2n} = S^{2n-1}$, we may view $\widetilde{f}$ as a map $S^{2n-1} \to S^n$. One can show that n must necessarily be even, and $H(\widetilde{f}) = \pm 1$ (see Steenrod and Epstein, *Cohomology Operations*, Princeton Univ. Press, Chapter 1, Lemma 5.3). Hence $n = 2, 4$ or 8.

Example 1.49. S^n is parallelizable $\iff n = 1, 3$ or 7.

In fact if S^n is parallelizable, there exist n continous functions $V_i : S^n \to \mathbb{R}^{n+1}$, $i = 1, \ldots, n$ such that $x, V_1(x), \ldots, V_n(x)$ are linearly independent, for each x, and $V_i(x)$ is orthogonal to x for each i. Using the Gram-Schmidt process, we may assume $x, V_1(x), \ldots, V_n(x)$ is an orthonormal basis of $\mathbb{R}^{n+1}$ for each x. Let $M(x) \in \mathbf{O}(n+1)$ be the corresponding orthogonal transformation on $\mathbb{R}^{n+1}$. Then $S^n \times S^n \to S^n$, $(x, y) \mapsto M(x) \cdot y$ is a map of bidegree $(\pm1, \pm1)$.

Example 1.50. Suppose $\mathbb{R}^n$ has a (non-zero) vector product (or "cross product"), *i.e.*, there is a continuous map $\nu : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ such that (i) $\nu(x, y)$ is orthogonal to both x and y (ii) $\parallel \nu(x, y) \parallel^2 = \parallel x \parallel^2 \parallel y \parallel^2 - < x, y >^2$, where $< x, y >$ is the standard scalar product in $\mathbb{R}^n$, and $\parallel \quad \parallel$ is the corresponding Euclidean norm. Then $n = 3$ or 7.

Indeed, let $\nu : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ be such a vector product. Write $\mathbb{R}^{n+1} = \mathbb{R} \times \mathbb{R}^n$, and define a product

$$\mu : \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \to \mathbb{R}^{n+1},$$

$$\mu((a, x), (b, y)) = (ab- < x, y >, ay + bx + \nu(x, y)).$$

One checks that

$$\parallel \mu(X, Y) \parallel^2 = \parallel X \parallel^2 \parallel Y \parallel^2.$$

Hence μ is a non-singular multiplication on $\mathbb{R}^{n+1}$. Hence $n + 1 = 1, 2, 4$ or 8. However one cannot have $n = 0$ or 1.

Note that for $n = 3$, one has the standard cross product in $\mathbb{R}^3$, while for $n = 7$ there is a similar cross product defined using multiplication of Cayley numbers.

Example 1.51. Suppose S^n has an almost complex structure. Then $n = 2$ or 6.

Indeed, if $TS^n \subset S^n \times \mathbb{R}^{n+1}$ is the tangent bundle, consisting of pairs (x, y) with $< x, y >= 0$, then an almost complex structure is a continuous map $J : TS^n \to TS^n$ such that $J(x, y) = (x, J_x(y))$, for some linear transformation $(\mathbb{R}x)^\perp \to (\mathbb{R}x)^\perp$, with $J_x^2 = -(\text{identity})$. One can further assume J_x to be an orthogonal transformation (after modifying the original choice, if necessary). In particular, $J_x(y)$ is orthogonal to both x and y, and has the same length (Euclidean norm) as y. Using this, it is easy to define a non-trivial vector product $\nu : \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$, with $J_x(y) = \nu(x, y)$.