Some lectures on Algebraic Geometry*

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1 Affine varieties

Let k be an algebraically closed field. We define the *affine n-space over k* to be just the set k^n of ordered n-tuples in k, and we denote it by $\mathbf{A}_k^n$ (by convention we define $\mathbf{A}_k^0$ to be a point). If $x_1, \ldots, x_n$ are the n coordinate functions on $\mathbf{A}_k^n$, any polynomial $f(x_1, \ldots, x_n)$ in $x_1, \ldots, x_n$ with coefficients in k yields a k-valued function $\mathbf{A}_k^n \to k$, which we also denote by f. Let $A(\mathbf{A}_k^n)$ denote the ring of such functions; we call it the *coordinate ring* of $\mathbf{A}_k^n$. Since k is algebraically closed, and in particular is infinite, we easily see that $x_1, \ldots, x_n$ are algebraically independent over k, and $A(\mathbf{A}_k^n)$ is hence a polynomial ring over k in n variables. The coordinate ring of $\mathbf{A}_k^0$ is just k.

Let $S \subset A(\mathbf{A}_k^n)$ be a collection of polynomials. We can associate to it an *algebraic set* (or *affine variety*) in $\mathbf{A}_k^n$ defined by

$$V(S) := \{x \in \mathbf{A}_k^n \mid f(x) = 0 \; \forall \; f \in S\}.$$

This is called the variety defined by the collection S. If

$$I =< S >:= \text{ideal in } k[x_1, \ldots, x_n] \text{ generated by elements of } S,$$

then we clearly have $V(S) = V(I)$. Also, if f is a polynomial such that some power f^m vanishes at a point P, then so does f itself; hence if

$$\sqrt{I} := \{f \in k[x_1, \ldots, x_n] \mid f^m \in I \text{ for some } m > 0\}$$

is the *radical* of I, then $V(I) = V(\sqrt{I})$.

Conversely, let $X \subset \mathbf{A}_k^n$ be an algebraic set. Then we can associate to it

(i) the ideal $I(X)$ of polynomials vanishing on X, and

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(ii) the ring $A(X) \cong k[x_1, \ldots, x_n]/I(X)$ of polynomial functions on X, the *coordinate ring* of X.

Clearly $I(X) = \sqrt{I(X)}$, and $A(X)$ is a reduced ring (has no non-zero nilpotents).

Theorem 1 (Hilbert's Nullstellensatz) *Let $I = \sqrt{I}$ be a radical ideal in the polynomial ring $k[x_1, \ldots, x_n]$. Then $I(V(I)) = I$.*

Corollary 1 *There is a one-one correspondence between three sets of objects:*

(i) algebraic sets in $\mathbf{A}^n_k$

(ii) radical ideals in $k[x_1, \ldots, x_n]$

(iii) pairs $(A, (t_1, \ldots, t_n))$ where A is a reduced k-algebra, and $t_1, \ldots, t_n$ are n elements of A which generate A as a k-algebra.

In particular, if $\mathbf{m} \subset k[x_1, \ldots, x_n]$ is a maximal ideal, it is a non-trivial radical ideal, so that $V(\mathbf{m}) \neq \phi$; if $P = (a_1, \ldots, a_n) \in \mathbf{A}^n_k$ lies in $V(\mathbf{m})$, then $\mathbf{m} \subset I(P) =< x_1 - a_1, \ldots, x_n - a_n >$. Since $\mathbf{m}$ is maximal, we must have equality. Thus all maximal ideals in $k[x_1, \ldots, x_n]$ are of the form $< x_1 - a_1, \ldots, x_n - a_n >$ for suitable $a_1, \ldots, a_n \in k$. If $\mathbf{m}$ is a maximal ideal, the natural map $k \to S(X)/\mathbf{m}$ is an isomorphism; identifying $S(X)/\mathbf{m}$ with k, if $X_i \in S$ has image a_i in $S(X)/\mathbf{m}$, the corresponding point P is clearly $(a_1, \ldots, a_n)$.

Another conclusion is that any radical ideal $I \subset k[x_1, \ldots, x_n]$ is the intersection of all maximal ideals containing it; this just restates that a variety in $\mathbf{A}^n_k$ is determined by its points. The points on an affine variety X are naturally in bijection with the maximal ideals of $A(X)$, *i.e.*, with the maximal ideals of $k[x_1, \ldots, x_n]$ containing $I(X)$. We can the associate to a point $P \in X$ the *local ring of X at P*, as follows: if $\mathbf{m} \subset A(X)$ is the corresponding maximal ideal, then the local ring is $\mathcal{O}_{P,X} := A(X)_{\mathbf{m}}$ is defined to be the localization of $A(X)$ at $\mathbf{m}$.

The set of varieties in a given affine space $\mathbf{A}^n_k$ is closed under finite unions and arbitrary intersections, with the following properties:

(i) if $I \subset J$ are ideals, then $V(J) \subset V(I)$,

(ii) $V(I) \cup V(J) = V(IJ) = V(I \cap J)$ for any pair of ideals I, J, and

(iii) $\cap_\alpha V(I_\alpha) = V(\sum_\alpha I_\alpha)$ for any family of ideals $\{I_\alpha\}$ of $k[x_1, \ldots, x_n]$.

The proofs are easy. To prove the second property, for example, note that $V(I) \cup V(J) \subset V(I \cap J) \subset V(IJ)$, since (use the first property) we have $IJ \subset I \cap J$ and $I \cap J \subset I$, $I \cap J \subset J$. If $x \in \mathbf{A}^n_k - V(I) \cup V(J)$, then there exist $f \in I$, $g \in J$ with $f(x) \neq 0$ and $g(x) \neq 0$; now $fg \in IJ$ with $fg(x) \neq 0$. Thus the varieties in $\mathbf{A}^n_k$ satisfy the axioms for the closed subsets in a topology; the resulting topology on $\mathbf{A}^n_k$ (and the induced subspace topology on any affine variety in $\mathbf{A}^n_k$) is called its *Zariski topology.* A closed subset of a variety is referred to as a *subvariety*; subvarieties of X are clearly in bijection with radical ideals in $A(X)$ (*i.e.*, with radical ideals in $k[x_1, \ldots, x_n]$ which contain $I(X)$).

We can define a suitable notion of *morphisms* between affine varieties. First consider the case of morphisms $f = (f_1, \ldots, f_m) : \mathbf{A}^n_k \to \mathbf{A}^m_k$; we define these to be mappings whose component functions f_i are polynomial functions. Let $x_1, \ldots, x_n$ and $y_1, \ldots, y_m$ be the coordinates on $\mathbf{A}^n_k$ and $\mathbf{A}^m_k$ respectively. For any polynomial function $g(y_1, \ldots, y_m)$, the function

$$f^*(g) := g \circ f$$

is then a polynomial function on $\mathbf{A}^n_k$. If $y_i \circ f = f_i(x_1, \ldots, x_n)$ are the component functions of the mapping f, then clearly

$$f^*(g) = g(f_1(x_1, \ldots, x_n), f_2(x_1, \ldots, x_n), \ldots, f_m(x_1, \ldots, x_n)).$$

Thus $f^* : k[y_1, \ldots, y_m] \to k[x_1, \ldots, x_n]$ is a homomorphism of k-algebras. Conversely, given a k-algebra homomorphism $\varphi : k[y_1, \ldots, y_m] \to k[x_1, \ldots, x_n]$, let $\varphi(y_i) = f_i(x_1, \ldots, x_n)$; then $f = (f_1, \ldots, f_m) : \mathbf{A}^n_k \to \mathbf{A}^m_k$ is a morphism, and $f^* = \varphi$. Note that a morphism is clearly continuous for the Zariski topology.

If $X \subset \mathbf{A}^n_k$, $Y \subset \mathbf{A}^m_k$ are affine varieties, a mapping $f : X \to Y$ is called a *morphism* if it is the restriction of a morphism $\widetilde{f} : \mathbf{A}^n_k \to \mathbf{A}^m_k$ (the morphism $\widetilde{f}$ need not be unique). Since $\widetilde{f}$ extends f, we have $\widetilde{f}(X) \subset Y$, and so $\widetilde{f}^* I(Y) \subset I(X)$, and we have an induced k-algebra homomorphism $f^* : A(Y) \to A(X)$. Conversely, let $\varphi : A(Y) \to A(X)$ be a k-algebra homomorphism. Let $a_1, \ldots, a_n \in A(X)$, $b_1, \ldots, b_m \in A(Y)$ be (the restrictions to X and Y) of the coordinate functions. Then $\varphi(b_i) = f_i(a_1, \ldots, a_n)$ for some polynomials f_i in n variables over k. If

$$\widetilde{f} = (f_1, \ldots, f_m) : \mathbf{A}^n_k \to \mathbf{A}^m_k,$$

and f is its restriction to X, then one checks that $f(X) \subset Y$, and $f^* = \varphi$. Thus morphisms between affine varieties correspond precisely to k-algebra homomorphisms (in the opposite direction) between their coordinate rings. Since we have defined a notion of morphisms, in particular, we have the notion of an *isomorphism*.

Morphisms $f : X \to \mathbf{A}^1_k$ are also called *regular functions* on X. These correspond to k-algebra homomorphisms $k[x_1] \to A(X)$, *i.e.*, to elements of $A(X)$ (a homomorphism $k[x_1] \to A(X)$ is determined by the image of x_1).

Some of the subtleties in the definition of a morphism become apparent from the following examples.

Example 1.1: Let x, y be coordinates on $\mathbf{A}^2_k$. Let $X = \mathbf{A}^1_k$, $Y = V(y^2 - x^3) \subset \mathbf{A}^2_k$, and let $f : X \to Y$, $f(a) = (a^2, a^3)$ for all $a \in \mathbf{A}^1_k$. If t is the coordinate on $\mathbf{A}^1_k$, then $f^*(x) = t^2$, $f^*(y) = t^3$. Hence the component functions of f are polynomials, and so f is a morphism. One sees easily that f is *bijective*, which amounts to the statement that $Y = \{(a^2, a^3) \mid a \in k\} \subset \mathbf{A}^2_k$. Now $A(X) = k[t]$ is the polynomial ring, and $f^*A(Y)$ is the k-subalgebra generated by t^2, t^3. In particular the subalgebra does not contain t, so that f is *not* an isomorphism, though it induces an isomorphism of the quotient fields of $A(Y)$ and $A(X)$. This situation occurs because the variety Y has a *singularity* at the origin.

Example 1.2: Let $f : \mathbf{A}^2_k \to \mathbf{A}^2_k$ be given by $f(a, b) = (a, ab)$. If x, y are the coordinates on the target space, then (i) the fibres of f over the Zariski open set $\{x \neq 0\}$ consist of exactly 1 point (ii) the map $f^* : A(\mathbf{A}^2_k) \to A(\mathbf{A}^2_k)$ induces an isomorphism on quotient fields (iii) the only point of the y-axis $\{x = 0\}$ in the image of f is the origin $(0, 0)$, whose preimage is the y-axis. This is called an affine *blow up* of the origin, since we have 'replaced' the origin of the target $\mathbf{A}^2$ by its preimage, a line, while (more or less) leaving the rest of $\mathbf{A}^2_k$ unchanged.

Note that the image of f contains a non-empty (dense) open subset of $\mathbf{A}^2_k$, but is neither open nor closed. Thus the open mapping theorem is not valid here.

Example 1.3: Let k have characteristic $p > 0$. Then $F : \mathbf{A}^n_k \to \mathbf{A}^n_k$ given by $F(a_1, a_2, \ldots, a_n) = (a_1^p, a_2^p, \ldots, a_n^p)$ is a morphism which is bijective, but not an isomorphism (the map F^* identifies $A(\mathbf{A}^n_k)$ with its subring consisting of p^{th} powers). This is called the (k-linear) *Frobenius morphism* on $\mathbf{A}^n_k$.

Now $k[x_1, \ldots, x_n]$ is a Noetherian ring. Hence any affine variety is the zero set of a finite collection of polynomials. Next, any radical ideal $I \subset k[x_1, \ldots, x_n]$ is uniquely expressible as an irredundant intersection $I = P_1 \cap P_2 \cap \cdots \cap P_r$ of a finite collection of prime ideals ('irredundant' means no P_i is contained in any P_j, or even in the union of the remaining P_j). This is a particular case of the notion of *primary decomposition* for ideals in Noetherian rings.

Define an affine variety X to be *irreducible* if it is impossible to write $X = X_1 \cup X_2$ for subvarieties X_1, X_2 with $X \neq X_1$ and $X \neq X_2$. If such a decomposition exists, we say X is *reducible*. We claim X is irreducible precisely when its ideal $I(X)$ is prime, or equivalently, when $A(X)$ is an integral domain. Indeed, if $I(X)$ is not prime, let $f, g \notin I(X)$ with $fg \in I(X)$. Let $I_1 = I + < f >$, $I_2 = I + < g >$, and set $X_j = V(I_j)$. By the Nullstellensatz, $X_j \subsetneq X$, since $I \subsetneq I_j$ and I is radical. Then $X = X_1 \cup X_2$, since $I(X)^2 \subset I_1 I_2 \subset I(X)$. Hence X is reducible. Conversely, suppose $X = X_1 \cup X_2$ is a non-trivial decomposition. Let $x_1 \in X_2 - X_1$, $x_2 \in X_1 - X_2$. By the Nullstellensatz, there exist $f \in I(X_1)$, $g \in I(X_2)$ with $f(x_1) \neq 0$ and $g(x_2) \neq 0$. In particular neither f nor g is in $I(X)$. But clearly fg vanishes on X.

We deduce also that the irreducible subvarieties of X correspond to the prime ideals in $A(X)$ (*i.e.*, to the prime ideals in $k[x_1, \ldots, x_n]$ containing $I(X)$). Another useful property of irreducible varieties is the following: if X is irreducible, then any non-empty Zariski open subset of X is dense; in particular the intersection of any finite number of non-empty Zariski open subsets of X is non-empty. Thus if X is irreducible, and Y is a proper subvariety, then $X - Y$ has infinitely many points.

Now consider a general affine variety X. If $I(X) = P_1 \cap \cdots \cap P_r$ is its primary decomposition, and $X_i = V(P_i)$, then $X = X_1 \cup \cdots \cup X_r$, such that (i) each X_i is irreducible, and (ii) the decomposition is irredundant. It is easy to see that the decomposition is characterized by these properties. The X_i are called the *irreducible components* of X. The coordinate rings of the irreducible components $A(X_i)$ are just the quotients of $A(X)$ by its minimal primes. For many purposes in algebraic geometry, one reduces the study of general affine varieties to that of irreducible ones; on the level of algebra, this corresponds to reducing the study of finitely generated reduced k-algebras to that of such integral domains.

We may now define the *dimension* of an affine variety. This can be done in several equivalent ways. First, we define the *(Krull) dimension* of a variety X to be the largest d such that there exists a chain $X_0 \subset X_1 \subset \cdots \subset X_d$ of subvarieties of X with X_i irreducible, where all the inclusions are strict. Since irreducible subvarieties in X correspond to prime ideals in the coordinate ring $A(X)$, the dimension of X is just the Krull dimension of $A(X)$, in the sense of commutative algebra. With this definition, it is a standard algebraic fact that $\mathbf{A}_k^n$ has dimension n, *i.e.*, the Krull dimension of the polynomial algebra $k[x_1, \ldots, x_n]$ is n.

From the definition, one sees at once that $\dim X = \sup \dim X_i$ where X_i are the irreducible components of X. If X is irreducible, so that $A(X)$ is a finitely generated k-algebra which is an integral domain, then the quotient field $k(X)$ of $A(X)$ has finite transcendence degree over k; from dimension theory in commutative algebra, this equals the Krull dimension of $A(X)$, *i.e.*, equals the dimension of X. We define a variety X to be equidimensional if all its irreducible components have the same dimension.

If X is a variety, Y a subvariety, we define the codimension of Y in X to be $\text{codim}\,(Y, X) = \dim X - \dim Y$. If Y is irreducible, this also equals the largest n such that there is a strictly increasing chain of subvarieties of X

$$Y \subset X_1 \subset \cdots \subset X_n;$$

This translates into an algebraic fact: for any prime ideal $P \subset A$, we have

$$\text{height}\, P + \dim A/P = \dim A,$$

for any finitely generated[1] k-algebra A.

In this context, we have the following corollary of Krull's Principal Ideal theorem: if A is a Noetherian ring, and $I =< f_1, \ldots, f_r >\subset A$ a proper ideal, then every minimal prime of I has height $\leq r$. Hence if X is an irreducible variety and $A = A(X)$, and $Y = V(I) \neq \phi$, then every irreducible component of Y has codimension $\leq r$. In particular, if $X \subset \mathbf{A}_k^n$ is a non-empty subvariety which is the zero set of a collection of r polynomials, then every irreducible component of X has dimension $\geq n - r$.

We say $X \subset \mathbf{A}_k^n$ is a *complete intersection* if $I(X)$ is generated by $n - \dim X$ elements. We say X is a *set theoretc complete intersection* if there exists a set of $n - \dim X$ polynomials whose zero set is precisely X, *i.e.*, there is an ideal I generated by $n - \dim X$ elements with $\sqrt{I} = I(X)$. From Krull's theorem, a set theoretic complete intersection is equidimensional, *i.e.*, all its irreducible components have the same dimension. As we see in the example below, there are equidimensional varieties which are not complete intersections. It is in general very hard to decide if an affine variety

[1] This does not hold for arbitrary Noetherian rings A, however.

is a set theoretic complete intersection. For example, a famous conjecture asserts that every curve (purely 1-dimensional variety) in $\mathbf{A}^3_k$ is s set theoretic complete intersection; the celebrated theorem of Cowsik and Nori asserts that this is true if k has characteristic $p > 0$.

Example 1.4: Let $X \subset \mathbf{A}^3_k$ be the image of the morphism $f : \mathbf{A}^1_k \to \mathbf{A}^3_k$, $f(t) = (t^3, t^4, t^5)$. If x, y, z are the coordinates on $\mathbf{A}^3_k$, the ideal $I(X) = (y^2 - xz, x^3 - yz, z^2 - x^2y)$, and $I(X)$ cannot be generated by 2 elements. However (exercise for the reader!) there exist $f, g \in I(X)$ such that $\sqrt{< f, g >} = I(X)$, *i.e.*, X is a set theoretic complete intersection.

There is another way in which we may understand the dimension of an affine variety. This is through the Noether normalization theorem.

Theorem 2 (Noether normalization) *Let A be a finitely generated k-algebra of dimension d. Then there exist d elements $y_1, \ldots, y_d \in A$ such that*

(i) $y_1, \ldots, y_d$ are homogeneous linear polynomials in the x_i, and are algebraically independent over k

(ii) if B is the k-subalgebra of A generated by the y_j, then A is a finite B-module; in particular, A is integral over B

(iii) if A is an integral domain, then its quotient field K is a fintie separable extension of L, the quotient field of B; if $[K : L] = r$, then there is a non-zero element $b \in B$ such that $A[1/b]$ is a free $B[1/b]$-module of rank r, and for any maximal ideal $\mathbf{m} \subset B[1/b]$, the ideal $\mathbf{m}A[1/b]$ is an intersection of r distinct maximal ideals.

Proof: (Sketch) We first show that B can be chosen so that A is a finite B-module, and if A is an integral domain, then the extension of quotient fields is separable.

Let $x_1, \ldots, x_n$ be generators for A as a k-algebra. If they are algebraically independent, then A is a polynomial ring, and so (as noted earlier) $d = n$ and we are done. If not, there is a non-trivial polynomial relation $f(x_1, \ldots, x_n) = 0$ with coefficients in k. If A is an integral domain, we may assume that the polynomial f is irreducible; in particular, since k is algebraically closed, we may assume f is separable in at least 1 of the variables.

One shows that there is a non-empty, Zariski open subset $U \subset \mathrm{M}_n(k)$, the $n \times n$ matrices, such that

(i) $U \subset \mathrm{GL}_n(k)$;

(ii) for $\sigma \in U$, if we set $t_i = \sigma^{-1}(x_i)$, so that $x_i = \sigma(t_i)$, then the polynomial $f(x_1, \dots, x_n) = f(\sigma(t_1), \dots, \sigma(t_n)) = g(t_1, \dots, t_n)$ is *monic* in t_n

(iii) in (ii), if A is a domain, then t_n is also separable over the quotient field of the subalgebra $k[t_1, \dots, t_{n-1}]$.

To prove that such an open set U exists, one takes σ to be a matrix of indeterminates, and if $\deg f = m$, one computes the coefficient of t_n^m of g. This is shown to be a non-zero polynomial h in the entries of σ, so that the condition that the product $h \det(\sigma)$ is non-zero defines a non-empty Zariski open set U_0 in $\mathrm{M}_n(k)$. In a similar way, the separability condition also determines a non-empty Zariski open subset $U_1 \subset \mathrm{M}_n(k)$ — if k has characteristic $p > 0$, and f has a non-zero coefficient of the monomial $x_i^r M'$, where M' involves only the other $n-1$ variables, and p does not divide r, then the open set U_1 can be defined by the condition that in

$$g = t_n^m + a_1(t_1, \dots, t_{n-1})t_n^{m-1} + \cdots,$$

the polynomial $a_r(t_1, \dots, t_{n-1})$ (the coefficient of t_n^r) is not identically 0. Now take $U = U_0 \cap U_1$.

Now replace A by the k-subalgebra A' generated by $t_1, \dots, t_{n-1}$; by induction on n, the result holds for A'. By construction, $A = A'[t_n]$ where t_n is integral over A', so A is a finite A'-module. Further, if A is a domain, then t_n is separable over the quotient field of A'. If $B \subset A'$ is a polynomial subalgebra over which A' is a finite module, then clearly A is also a finite B-module; if also the quotient field of A' is separable over that of B, then so is the quotient field of A.

Now assume A is a domain, and fix $B \subset A$ such that A is a finite B-module, and the quotient field K of A is separable over L, the quotient field of B. By the primitive element theorem, $K = L(t)$ for some element $t \in A$. Since A is a finite B-module, we can find a non-zero element $b_1 \in B$ such that $A[1/b_1] = B[1/b_1][t]$. Now t is a root of a monic polynomial with coefficients in B; since B is a unique factorization domain (it is a polynomial ring over a field), the Gauss lemma implies that the monic irreducible polynomial $p(T)$ of degree $r = [K : L]$ for t over L satisfies $p(T) \in B[T]$. In particular, $A[1/b_1]$ is a free $B[1/b_1]$-module with basis $1, t, \dots, t^{r-1}$. Since $K = L(t)$ is separable over L, $p(T)$ is a separable polynomial, and so $N_{K/L}(p'(t))$ is a non-zero element $b_2 \in B$. Now take $b = b_1 b_2$. Then $A[1/b] = B[1/b][t]$ where t is a root of the monic irreducible separable polynomial $p(T)$ over

$B[1/b]$ (separability means the norm of $p'(t)$ is a unit in $B[1/b]$). Hence for any maximal ideal $\mathbf{m}$ of $B[1/b]$, the algebra $C = A[1/b]/\mathbf{m}A[1/b]$ over the field $B[1/b]/\mathbf{m} = k$ has a presentation $C = k[T]/(\overline{p}(T))$, where $\overline{p}(T) \in k[T]$ is a separable monic polynomial of degree r. Since k is algebraically closed, this means $\overline{p}(T)$ is a product of r distinct monic linear factors, *i.e.*, $\mathbf{m}A[1/b]$ is the intersection of r distinct maximal ideals. $\square$

The category of affine varieties has a direct product. First, we identify $\mathbf{A}^n_k \times \mathbf{A}^m_k$ with $\mathbf{A}^{n+m}_k$ in the obvious way (both are identified with k^{m+n}). Let $x_1, \ldots, x_n$ be the coordinates on $\mathbf{A}^n_k$, and $y_1, \ldots, y_m$ those on $\mathbf{A}^m_k$. Then the $m+n$ functions $x_1, \ldots, x_n, y_1, \ldots, y_m$ are the coordinates on $\mathbf{A}^{n+m}_k$. If $X \subset \mathbf{A}^n_k$ and $Y \subset \mathbf{A}^m_k$ are subvarietes, with ideals $I(X) \subset k[x_1, \ldots, x_n]$ and $I(Y) \subset k[y_1, \ldots, y_m]$, then $X \times Y$ is the subset $V(I(X) \cup I(Y)) \subset \mathbf{A}^{n+m}_k$, where $I(X)$, $I(Y)$ are both regarded as subsets of $k[x_1, \ldots, x_n, y_1, \ldots, y_m]$. With this identification, one can show that in fact $< I(X), I(Y) >$ is a radical ideal in $k[x_1, \ldots, x_n, y_1, \ldots, y_m]$, and so $A(X) \otimes_k A(Y) \xrightarrow{\cong} A(X \times Y)$ as k-algebras (the isomorphism is induced by $x_i \otimes 1 \mapsto x_i$, $1 \otimes y_j \mapsto y_j$). Now $X \times Y$ is indeed the direct product in the category of affine varieties, *i.e.*, to give a morphism from an affine variety Z to $X \times Y$ is precisely to give a pair of morphisms $Z \to X$ and $Z \to Y$.

There is one subtlety in the definition of the product: the Zariski topology on $X \times Y$ is in general *not* the product topology; it is usually finer (*i.e.*, open sets in the product topology are also Zariski open, but there are usually more Zariski open sets in $X \times Y$). A simple example to illustrate this is the variety $\mathbf{A}^2_k = \mathbf{A}^1_k \times \mathbf{A}^1_k$; the basic open sets in the product topology are complements of finite unions of vertical and horizontal lines, and the Zariski open set $\mathbf{A}^2_k - V(xy-1)$ (the complement of the hyperbola with equation $xy = 1$) is not a union of such sets, *i.e.*, is not open in the product topology. Since the product variety is indeed the direct product in the category of affine varieties, this 'strange' topology on the product does not cause any difficulties, and is in fact forced on us.

We next discuss the important notion of non-singularity. We do this using the Zariski tangent space. This is defined as follows. If $X \subset \mathbf{A}^n_k$ is an affine variety, and P a point of X, let $\mathbf{m} \subset k[x_1, \ldots, x_n]$ be its maximal ideal in the polynomial ring, and $\overline{\mathbf{m}} \subset A(X)$ its image in $A(X)$. There is then a surjective k-linear map $\mathbf{m}/\mathbf{m}^2 \to \overline{\mathbf{m}}/\overline{\mathbf{m}}^2$. If $W \subset \mathbf{m}$ is the vector space of linear polynomials in $k[x_1, \ldots, x_n]$ vanishing at P, then the natural map $W \to \mathbf{m}/\mathbf{m}^2$ is an isomorphism. Let $W_0 = \ker(W \to \overline{\mathbf{m}}/\overline{\mathbf{m}}^2)$. Thus W_0 consists of the linear polynomials vanishing at P 'to order at least 2 along

X'; geometrically, we should then expect that for any non-zero $f \in W_0$, the hyperplane $\{f = 0\}$ in $\mathbf{A}^n_k$ should be tangent to X along P.

With this as motivation, we define the *Zariski tangent space* $T_{P,X}$ to X at P to be the linear subvariety of $\mathbf{A}^n_k$ defined by

$$T_{P,X} = V(W_0) = \{x \in \mathbf{A}^n_k \mid f(x) = 0 \, \forall \, f \in W_0\}.$$

This is clearly an affine linear subspace (a coset of a vector subspace) of $\mathbf{A}^n_k$ which passes through P.

We may regard $k[x_1, \ldots, x_n]$ as the symmetric algebra over k of W, *i.e.*, a basis $y_1, \ldots, y_n$ for W gives a new set of variables in the polynomial ring, related to the old ones by an affine linear transformation (*i.e.*, we have $y_i = \sum a_{ij} x_j + c_j$ with $[a_{ij}] \in \mathrm{GL}_n(k)$ and $c_j \in k$). Now $T_{P,X}$ has coordinate ring isomorphic to the symmetric algebra of W/W_0, *i.e.*, to the symmetric algebra of $\overline{\mathbf{m}}/\overline{\mathbf{m}}^2$. Thus points of $T_{P,X}$ are naturally in bijection with elements of the dual vector space

$$(\overline{\mathbf{m}}/\overline{\mathbf{m}}^2)^* = \mathrm{Hom}_k(\overline{\mathbf{m}}/\overline{\mathbf{m}}^2, k).$$

Hence this dual space is also known as the Zariski tangent space (this is an intrinsic definition which depends only on the coordinate ring and the maximal ideal, and not the chosen set of n generators of $A(X)$, *i.e.*, the given 'embedding' of X in $\mathbf{A}^n_k$).

From the discussion above, we see that

$$\dim T_{P,X} = \dim_k \overline{\mathbf{m}}/\overline{\mathbf{m}}^2.$$

From commutative algebra,

$$\dim_k \overline{\mathbf{m}}/\overline{\mathbf{m}}^2 \geq (\text{Krull}) \dim A(X)_{\overline{\mathbf{m}}} = \dim \mathcal{O}_{P,X}.$$

We define a point P in X to be *non-singular* if equality holds, or equivalently, if the local ring $\mathcal{O}_{P,X}$ is a *regular local ring* in the sense of commutative alg bra. A third equivalent characterization is that the completion

$$\widehat{\mathcal{O}}_{P,X} = \varprojlim_s \mathcal{O}_{P,X}/\overline{\mathbf{m}}^s$$

of $\mathcal{O}_{P,X}$ is isomorphic to the ring $k[[x_1, \ldots, x_d]]$ of formal power series over k in $d = \dim \mathcal{O}_{P,X}$ variables. In particular, since a regular local ring is an integral domain, P must lie on a unique irreducible component of X.

Now

$$\overline{\mathbf{m}}/\overline{\mathbf{m}}^2 \cong \mathbf{m}/(\mathbf{m}^2 + I(X)) = \text{coker}\,(I(X) \to \mathbf{m}/\mathbf{m}^2).$$

Let $P = (a_1, \ldots, a_n)$, so that $\mathbf{m} =< x_1 - a_1, \ldots, x_n - a_n >$. Identifying $\mathbf{m}/\mathbf{m}^2$ with V, which has a basis $x_1 - a_1, \ldots, x_n - a_n$, the map $I(X) \to \mathbf{m}/\mathbf{m}^2$ is identified with the map (given by Taylor expansion to order 1)

$$f \mapsto \left(\frac{\partial f}{\partial x_1}(a_1, \ldots, a_n), \ldots, \frac{\partial f}{\partial x_n}(a_1, \ldots, a_n)\right).$$

Hence if $f_1, \ldots, f_m$ are generators for I, then

$$\dim T_{P,X} = n - \text{rank}\,\left[\frac{\partial f_j}{\partial x_i}(P)\right].$$

This leads to the *Jacobian criterion* for non-singularity: if X is irreducible and $\dim X = d$, then $P \in X$ is a non-singular point if and only if there exist $n - d$ elements $f_1, \ldots, f_{n-d}$ in $I(X)$ such that

$$\left[\frac{\partial f_j}{\partial x_i}(P)\right]$$

has maximal rank (equal to $n-d$). One immediate consequence of the above analysis is that the set of nonsingular points in X is Zariski open (it is a finite union of sets defined by the non-vanishing of determinants of matrices of polynomials).

Now affine space of any dimension is non-singular. From Noether normalization as proved above, applied to $A = A(X)$, the points of an irreducible affine variety X corresponding to maximal ideals of $A[1/b]$ are non-singular points of X. This is because for any maximal ideal $\mathbf{m}$ of $B[1/b]$, if $\mathbf{m}A[1/b] = \mathbf{m}_1 \cap \cdots \cap \mathbf{m}_r$, then (with $d = \dim A = \dim X$)

$$rd = \dim_k \mathbf{m}/\mathbf{m^2} = \dim_k \mathbf{m}A[1/b]/\mathbf{m}^2 A[1/b] = \sum \dim_k \mathbf{m}_i/\mathbf{m}_i^2$$

where each term on the right is at least d; hence all the terms in the sum equal d. In particular $A_{\mathbf{m}_i}$ is a regular local ring of dimension d for each i. Thus the set of non-singular points of an irreducible affine variety is a *non-empty* Zariski open set.

Non-singular affine varieties have several good properties which distinguish them in the class of all varieties. For example, if X is connected and non-singular, then it is irreducible. Further, if $f : X \to Y$ is a bijective morphism between irreducible affine varieties, such that Y is non-singular, and f^* induces an isomorphism of the quotient fields of $A(Y)$ and $A(X)$, then it is in fact an isomorphism. This result is, however, not so easy to prove; it is a (relatively simple) form of 'Zariski's Main Theorem'.

Another subtle propert of non-singular varieties is that their local rings $\mathcal{O}_{P,X}$ are all unique factorization domains (UFDs). This follows from the theorem of Auslander-Buchsbaum in commutative algebra, that a regular local ring is a unique factorization domain. In particular, one can deduce that the local rings $\mathcal{O}_{P,X}$, and hence (if X is irreducible) the coordinate ring $A(X)$, is integrally closed in its quotient field. Finally, if $k = \mathbf{C}$, the field of complex numbers, then a non-singular subvariety $X \subset \mathbf{A}^n_{\mathbf{C}}$ can also be regarded as a complex submanifold of $\mathbf{C}^n$, so that techniques from the theories of differential and complex manifolds can be applied to the study of X; sometimes analogous algebraic notions can be defined, which would then make sense for arbitrary non-singular affine varieties. One such notion we will encounter later is that of an *algebraic differential form.*

We can also associate to X its *tangent variety*

$$T_X := \{(x, y) \in \mathbf{A}^n_k \times \mathbf{A}^n_k \mid y \in T_{x,X}\}.$$

If $x_1, \ldots, x_n$ ands $y_1, \ldots, y_n$ are the coordinates on the two factors $\mathbf{A}^n_k$, then T_X is the subvariety of $\mathbf{A}^n_k \times \mathbf{A}^n_k = \mathbf{A}^{2n}_k$ defined by $V(I(X) \cup S)$, where

$$S = \{\sum_{i=1}^{n} (y_i - x_i) \frac{\partial f(x_1, \ldots, x_n)}{\partial x_i} \mid f \in I(X)\}.$$

Hence T_X is an affine variety in $\mathbf{A}^{2n}_k = \mathbf{A}^n_k \times \mathbf{A}^n_k$, such that the first projection gives a surjective morphism $\pi_X : T_X \to X$. The fibre $\pi_X^{-1}(P)$ is identified with $T_{P,X}$, and the diagonal embedding of X in $\mathbf{A}^n_k \times \mathbf{A}^n_k$ gives an algebraic section of the morphism π_X. If X is non-singular, then π_X is an example of an *algebraic vector bundle*, the *tangent bundle* of X.

We end this section with the notions of *rational functions* and *rational maps.* First note that there is a basis of open sets for the Zariski topology on $\mathbf{A}^n_k$ consisting of the open sets

$$D(f) := \mathbf{A}^n_k - V(f) = \{x \in \mathbf{A}^n_k \mid f(x) \neq 0\}. \tag{1}$$

Consider the variety $X_f \subset \mathbf{A}^{n+1}_k$ defined by $f(x_1, \ldots, x_n)x_{n+1} - 1 = 0$ (*i.e.*, $X_f = V(f(x_1, \ldots, x_n)x_{n+1} - 1)$). The coordinate ring of X_f is isomorphic to the localization of $A(X)$ with respect to powers of f. Then projection onto the first n coordinates yields a morphism $\varphi : X_f \to \mathbf{A}_k^n$, which is gives a homeomorphism of X_f with $D(f)$, such that

(i) $\varphi^* : A(\mathbf{A}_k^n) \to A(X_f)$ is identified with the localisation map $k[x_1, \ldots, x_n] \to k[x_1, \ldots, x_n, f^{-1}]$

(ii) for any $P \in X_f$ with image $Q \in D(f)$, the local rings $\mathcal{O}_{P,X_f}$ and $\mathcal{O}_{P,\mathbf{A}^n}$ are isomorphic, via φ^*.

Thus we may regard $D(f)$ as an affine variety in its own right, by identifying it with X_f. Hence it also makes sense to consider regular functions on $D(f)$; now restrictions of regular functions from $D(f)$ to $D(fg)$ are regular on $D(fg)$.

Now let X be an *irreducible* affine variety. A *rational map* from X to an affine variety Y is a morphism $f : U \to Y$, for some non-empty affine open subset $U \subset X$, such that (U, f) and (V, g) determine the same rational map if $f - g$ vanishes on a non-empty open set in $U \cap V$ (more formally, a rational map is an equivalence class of pairs (U, f) as above). In particular, a *rational function* on X is a rational map $X \to \mathbf{A}_k^1$. One sees easily that rational functions are identified with elements of the quotient field of $A(X)$.

A rational map (or rational function) is said to be *regular at a point* P if there is a morphism (U, f) representing it, with $P \in U$. The local ring $\mathcal{O}_{P,X}$ acquires the following more function theoretic interpretation: it is the ring of germs of rational functions on X which are regular at P (recall that a *germ* of a function at P is a pair (U, f) consisting of an open set U containing P, and a function f on U; again (U, f) is equivalent to (V, g) if f and g agree on a neighbourhood of P in $U \cap V$).

The interpretation of local rings in terms of germs of regular functions allows us to reexamine the notion of a morphism between affine varieties. Clearly if $f : X \to Y$ is a morphism between affine varieties, then f^* (*i.e.*, composition with f) induces homomorphisms $f^* : \mathcal{O}_{f(P),Y} \to \mathcal{O}_{P,X}$ for all $P \in X$. Conversely, suppose given a continuous map (for the Zariski topology) $f : X \to Y$, such that for each $P \in X$, the association $(g, V) \mapsto (g \circ f, f^{-1}(V))$ on germs of functions yields a homomorphism $f^* : \mathcal{O}_{f(P),Y} \to \mathcal{O}_{P,X}$. Then one can prove that f is in fact a morphism (see [H], Theorem 3.2). This boils down to the following statement in commutative algebra: let A be a finitely generated reduced k-algebra; then A is the intersection (in its total quotient ring) of its localizations at all its maximal ideals.

2 Projective and quasi-projective varieties

We begin by defining *projective n-space* over the (algebraically closed) field k to be a quotient of $k^{n+1} - \{0\}$ by an equivalence relation,

$$\mathbf{P}^n_k := (k^{n+1} - \{0\})/\sim,$$

where $(a_0, \ldots, a_n) \sim (\lambda a_0, \lambda a_1, \ldots, \lambda a_n)$ for all $\lambda \in k^* = k - \{0\}$. We denote the equivalence class of $(a_0, a_1, \ldots, a_n)$ by $(a_0 : a_1 : \cdots : a_n)$. If $P = (a_0 : a_1 : \cdots : a_n)$ is a point of $\mathbf{P}^n_k$, then the a_j are called *homogeneous coordinates* for P.

Let $S = k[X_0, \ldots, X_n]$ be the polynomial ring in $n+1$ variables. We consider this as a graded k-algebra such that each variable X_i is homogeneous of degree 1. Thus $S = \oplus_{d \geq 0} S_d$, where S_d denotes the k-vector space of homogeneous polynomials of degree d. If $f \in S$ is homogeneous of degree d, then for any $P = (a_0 : a_1 : \cdots : a_n) \in \mathbf{P}^n_k$, the condition that $f(a_0, a_1, \ldots, a_n) = 0$ depends only on the point P, and not on the specific set of homogenous coordinates chosen to represent P. We express this by writing '$f(P) = 0$'. With this convention, if T is a set of *homogeneous* polynomials in $k[X_0, \ldots, X_n]$, we define the associated *projective variety* X in $\mathbf{P}^n_k$ to be

$$X = V(T) = \{P \in \mathbf{P}^n_k \mid f(P) = 0 \, \forall \, P \in T\}.$$

The same set T also defines a subvariety $C(X)$ of $\mathbf{A}^{n+1}_k$, which is conical (*i.e.*, if x lies in the subvariety, so does λx, for all $\lambda \in k$). We call $C(X)$ the *affine cone* over $X \subset \mathbf{P}^n_k$.

Recall that an ideal $I \subset k[X_0, \ldots, X_n]$ is called a *homogeneous ideal* if I is generated by homogeneous elements; equivalently, $I \subset S$ is a graded submodule of S. The Hilbert basis theorem then implies that I is generated by a finite set of homogeneous elements. For any set T of homogeneous polynomials, the ideal $< T >$ generated by them is homogeneous, and $V(T) = V(< T >) \subset \mathbf{P}^n_k$. Also, the radical of a homogeneous ideal is homogeneous, so that we again have $V(I) = V(\sqrt{I})$ for any homogeneous ideal I.

Note that the ideal $S_+ = \oplus_{d>0} S_d$ generated by all homogeneous elements of positive degree defines the *empty set* in $\mathbf{P}^n_k$, even though S_+ is a non-trivial maximal ideal; we call it the *irrelevant maximal ideal* of S. Further, for any homogeneous ideal I, we have either $I = S$ or $I \subset S_+$, and

$$V(I) = \phi \Leftrightarrow \text{ either } \sqrt{I} = S \text{ or } \sqrt{I} = S_+ = \oplus_{d>0} S_d.$$

As in the case of affine varieties in $\mathbf{A}^n_k$, the projective varieties in $\mathbf{P}^n_k$ are closed under finite unions and arbitrary intersections, and so form the closed subsets for a topology on $\mathbf{P}^n_k$ (giving rise to a topology on any projective variety in $\mathbf{P}^n_k$). We call this the *Zariski topology* on $\mathbf{P}^n_k$. We also define the notion of a *quasi-projective variety* in $\mathbf{P}^n_k$; this is a subset of the form $X - Y$ where $X, Y \subset \mathbf{P}^n_k$ are projective varieties. *From now on 'variety' will mean 'quasi-projective variety'*, unless specified otherwise.

If $X \subset \mathbf{P}^n_k$ is a projective variety, then its ideal $I(X) \subset S$ is the homogeneous ideal generated by all homogeneous polynomials vanishing on X. The *homogeneous coordinate ring* $S(X)$ of X is defined to be the quotient graded ring $S/I(X)$. The Nullstellensatz takes on the following form: for any homogeneous ideal I, if $f \in S_d$ with $d > 0$ and $f \in I(V(I))$, then $f \in \sqrt{I}$. This can be deduced from the usual Nullstellensatz applied to the affine cone $C(X)$. The homogeneous Nullstellensatz implies that if $I = \sqrt{I}$ is a graded radical ideal in S with $V(I) \neq \phi$, then $I(V(I)) = I$. We obtain a 1-1 correspondence between non-empty projective varieties in $\mathbf{P}^n_k$ and radical homogeneous ideals $I \subset S_+$ with $I \neq S_+$. Further, for any set $Y \subset \mathbf{P}^n_k$, its Zariski closure (*i.e.*, closure in the Zariski topology) is $V(I(Y))$, where $I(Y)$ is the ideal generated by homogeneous polynomials vanishing on Y. Finally, there is a 1-1 correspondence between closed subsets of a projective variety X and radical homogeneous ideals $I \subsetneq S(X)_+$, where $S(X)_+ = \oplus_{d>0} S(X)_d$ is the ideal generated by homogeneous elements of positive degree in $S(X)$.

Projective space $\mathbf{P}^n_k$, and hence any quasi-projective variety, is a *Noetherian topological space*, *i.e.*, any strictly descending chain of closed subsets stops after a finite number of steps. This follows from the ascending chain condition for graded ideals. A Noetherian space may also be characterized by the property that any collection of closed subsets has a minimal element. In a formal way, this implies that any Noetherian topological space X can be uniquely written as an irredundant union $X = X_1 \cup X_2 \cup \cdots \cup X_n$ where $X_i \subset X$ are irreducible, closed subsets, as follows. The collection of all closed subsets of X which do not have a finite decomposition into irreducible closed subsets has a minimal element X_0, which must clearly be reducible; if $X_0 = X' \cup X''$, where X', X'' are proper closed subsets of X_0, then minimality of X_0 implies that X and X' are each finite unions of irreducible closed subsets; but then so is $X_0 = X' \cup X''$. So X does admit a finite decomposition into irreducible closed subsets, hence also an irredundant one. Uniqueness is easily proved by induction on the number of sets in a given decomposition.

In particular, any quasi-projective variety X is uniquely expressed as a finite, irredundant unions $X = X_1 \cup X_2 \cup \cdots \cup X_r$ of irreducible varietes of the same type, which we call the *irreducible components* of X.

In algebraic terms, if $I(X)$ is the homogeneous ideal of a projective variety, then in its primary decomposition $I(X) = P_1 \cap P_2 \cap \cdots \cap P_r$, the primes P_i are also homogeneous, and determine projective subvarieties $X_1, \ldots, X_r$ of X; then $X = X_1 \cup X_2 \cup \cdots X_r$ is the decomposition of X into irreducible components; in particular, X is irreducible $\Leftrightarrow I(X)$ is a (homogeneous) prime ideal. Thus points in X correspond to homogeneous prime ideals $\mathbf{m} \subsetneq S(X)_+$, which are maximal among such prime ideals. Note that $S(X)/\mathbf{m}$ must then be a graded integral domain of Krull dimension 1, generated as a k-algebra by its homogeneous elements of degree 1; this forces $S(X)/\mathbf{m} = k[t]$, a polynomial ring, where t is any non-zero homogeneous element of degree 1. Now under the composite

$$S \twoheadrightarrow S/I(X) = S(X) \twoheadrightarrow S(X)/\mathbf{m},$$

if X_i maps to $a_i t$, then one verifies easily that $P = (a_0 : a_1 : \cdots : a_n)$. Conversely, if $\varphi : S(X) \twoheadrightarrow k[t]$ is a graded surjection of k-algebras, then $\varphi(X_i) = a_i t$, so that $\ker \varphi$ is the graded ideal corresponding to the point $P = (a_0 : \cdots : a_n)$. Hence there is a 1-1 correspondence between points of X and equivalence classes of graded surrjective homomorphisms $S(X) \to k[t]$, where the equivalence is upto composition with an automorphism $t \mapsto ct$ with $c \in k^*$.

For $X = \mathbf{P}^n_k$, such graded homomorphisms are of course equivalent to linear functionals on S_1, vector space of homogeneous elements of degree 1 in S; the homomorphism corresponding to a functional is the induced map on symmetric algebras. This gives a 'coordinate free' description of the points of $\mathbf{P}^n_k$ as non-zero linear functionals on S_1 upto scalar multiples.

The *dimension* of a quasi-projective variety X is now defined as for affine varieties: it is the largest d such that there is a strictly increasing chain of irreducible closed subsets $X_0 \subsetneq X_1 \subsetneq \cdots \subsetneq X_d$ of X. One can show using a little commutative algebra that for any quasi-projective variety Y, we have $\dim Y = \dim \overline{Y}$, where $\overline{Y}$ is the projective closure of Y (see [H], Prop. 1.10 and Exercise 2.10).

Krull's principal ideal theorem has the following consequence for subvarieties of $\mathbf{P}^n_k$ — if $F_1, \ldots, F_r$ are homogeneous polynomials of positive degree, then each irreducible component of $V(< F_1, \ldots, F_r >)$ has dimension $\geq n - r$. Unlike in the case of affine varieties, this variety X is always non-empty, provided $r \leq n$, since we may apply the affine Krull theorem to the cone $C(X)$ in $\mathbf{A}_k^{n+1}$ (note that $C(X)$ is always non-empty, since it contains the origin $0 \in k^{n+1} = \mathbf{A}_k^{n+1}$).

If

$$U_i = \{(a_0 : \cdots : a_n) \mid a_i \neq 0\},$$

then there is a natural bijection $\varphi_i : k^n \to U_i$ given by

$$(b_1, \ldots, b_n w) \mapsto (b_1 : \cdots : b_i : 1 : b_{i+1} : \cdots : b_n).$$

In fact this is easily seen to be a homeomorphism with respect to the respective Zariski topologies ([H], Prop. 2.2). This amounts to the assertion that if $Y \subset \mathbf{A}_k^n$, then $Y = \varphi^{-1}(\overline{\varphi(Y)})$, where $\overline{\varphi(Y)}$ is the Zariski closure in $\mathbf{P}_k^n$ of $\varphi(Y)$. This is true because if $f(x_1, \ldots, x_n)$ is a polynomial of degree d, so that

$$f = \sum_{\nu_1 + \cdots + \nu_n \leq d} a_{\nu_1 \nu_2 \cdots \nu_n} x_1^{\nu_1} x_2^{\nu_2} \cdots x_n^{\nu_n},$$

and if

$$F(X_0, X_1, \ldots, X_n) = X_i^d f(X_0/X_i, X_1/X_i, \ldots, X_{i-1}/X_i, X_{i+1}/X_i, \ldots, X_n/X_i) = \sum_{\nu_1 + \cdots + \nu_n \leq d} a_{\nu_1 \nu_2 \cdots \nu_n} X_0^{\nu_1} X_1^{\nu_2} \cdots X_{i-1}^{\nu_i} X_i^{d - \nu_1 - \nu_2 - \ldots - \nu_n} X_{i+1}^{\nu_{i+1}} \cdots X_n^{\nu_n}$$

is the (unique) homogeneous polynomial of degree d with

$$F(x_1, x_2, \ldots, x_{i-1}, 1, x_i, \ldots, x_n) = f(x_1, \ldots, x_n),$$

then

(i) $V(F) = \overline{\varphi(V(f))}$ is the Zariski closure of $\varphi(V(f))$

(ii) $\varphi^{-1}(V(F)) = V(f)$.

We may thus identify U_i with the affine space $\mathbf{A}_k^n$. Hence $\mathbf{P}_k^n$, and hence also any quasi-projective variety, has a basis of Zariski open sets consisting of affine varieties. This is because (as seen in §1) any affine variety has a basis of Zariski open sets which are each affine varieties in a natural way.

This local structure of quasi-projective varieties allows us to transfer 'locally defined' notions for affine varieties to quasi-projective varieties. The first important one is that of *morphisms*. A continuous map $f : X \to Y$ between quasi-projective varieties is called a morphism if for any affine open subsets $U \subset X$ and $V \subset Y$ such that $f(U) \subset V$, the restriction $f : U \to V$ is a morphism of affine varieties. Morphisms to $\mathbf{A}_k^1$ are called *regular functions* on X, and form a k-algebra in a natural way under pointwise addition and multiplication. We denote this k-algebra by $\mathcal{O}(X)$. If $f \in \mathcal{O}(X)$, let $D(f) = X - V(f)$.

Similarly, a *rational map* $X \to Y$ from an irreducible quasi-projective variety X is an equivalence class of pairs (U, f) where U is a non-empty open set in X, and $f : U \to Y$ is a regular function on U; here (U, f) and (V, g) are equivalent if $f = g$ on $U \cap V$.

A rational map $f : X \to \mathbf{A}^1_k$ is called a *rationa function* on X. The rational functions on X form a field, denoted $K(X)$, which we refer to as the (rational) function field of X. Examples of rational functions are given by (the restriction to X of) ratios F/G of homogeneous polynomials of the same degree, where $G \notin I(X)$. If $U \subset X$ is a non-empty open subset, clearly $K(U) = K(X)$; since $K(\mathbf{A}^n_k)$ is just the quotient field of the polynomial ring in n variables, we deduce that $K(\mathbf{P}^n_k)$ is also isomorphic to the field of rational functions over k in n variables.

In general, for any irreducible variety X, the dimension of X equals the transcendence degree over k of its rational function field $K(X)$. To prove this, we may replace X by an affine open subset, without changing the function field; for an irreducible affine variety X, the function field is the quotient field of its coordinate ring $A(X)$. Now dimension theory in commutative algebra implies that the transcendence degree of the quotient field of $A(X)$ equals its Krull dimension.

We mention one good property of morphisms between quasi-projective varieties. Let $f : X \to Y$ be a morphism between irreducible varieties with Zariski dense image (for example, f may be onto); we say f is a *dominant* morphism. Then there is a non-empty Zariski open set $U \subset Y$ such that for any $P \in U$, we have $\dim f^{-1}(P) = \dim X - \dim Y$. A proof is sketched in [H], II, Ex. 3.22. The idea is to first reduce to the case when X and Y are affine, and then consider a factorization $f = g \circ h$, where (i) $g : X \to Y \times \mathbf{A}^r_k$, with $r = \dim X - \dim Y$, (ii) the morphism g is dominant, and (iii) h is the projection to Y; further, (iv) $K(X)$ is a finite algebraic extension of $K(Y \times \mathbf{A}^r_k)$.

We do this on the level of algebra — if $A(X) = A$, $A(Y) = B$, then $f^* : B \to A$ is injective, since f is dominant. The quotient fields $K(X)$ of A and $K(Y)$ of B respectively have transcendence degrees $\dim X$ and $\dim Y$ over k; hence $K(X)$ has transcendence degree r over $K(Y)$, and we can find a polynomial subalgebra $B[T_1, \ldots, T_r]$ of A such that $[K(X) : K(Y)(T_1, \ldots, T_r)] < \infty$.

Hence it suffices to prove that if $f : X \to Y$ is dominant, and $\dim X =$

$\dim Y$, then for a non-empty Zariski open set $U \subset Y$, and any $P \in U$, the fibre $f^{-1}(P)$ is finite. Since $K(X)$ is finite algebraic over $K(Y)$, and $A(X)$ is a finitely generated $A(Y)$-algebra, we can find a non-zero $b \in A(Y)$ such that $A(X)[1/b]$ is integral and finitely generated over $A(Y)[1/b]$, *i.e.*, $A(X)[1/b]$ is a finite $A(Y)[1/b]$-module. Then for any maximal ideal $\mathbf{m} \subset A(Y)[1/b]$, the $A(Y)[1/b]/\mathbf{m} = k$ vector space $A(X)[1/b]/\mathbf{m}A(X)[1/b]$ is finite dimensional, so that there are only finitely many maximal ideals of $A(X)[1/b]$ lying over $\mathbf{m}$. Thus if $P \in D(b) \subset Y$, the fibre $f^{-1}(P)$ is finite.

We can define the *local ring* $\mathcal{O}_{P,X}$ at a point P of a quasi-projective variety X as the ring of germs of regular functions at P (such a germ is, as before, an equivalence class (U, f) where U is an open neighbourhood of P, and f is a regular function on U). Since this is true for affine varieties, we see that the notion of a morphism $f : X \to Y$ is local on X: in fact f is a morphism $\Leftrightarrow f$ is continuous, and composition of germs with f yields homomorphisms $f^*\mathcal{O}_{f(P),Y} \to \mathcal{O}_{P,X}$ for all $P \in X$.

We can read off some invariants of a projective variety directly from its homogeneous coordinate ring, as follows. We make use of the following notations. Let A be a graded ring, and $T \subset A$ a multiplicative set consisting only of homogeneous elements. The localized ring $T^{-1}A$ has a natural $\mathbf{Z}$-grading, whose homogeneous elements are of the form f/g, where $f \in A$, $g \in T$ are homogeneous elements; we define $\deg f/g = \deg f - \deg g$. Now let $A_{(T)}$ denote the elements of degree 0 in the $\mathbf{Z}$-graded localized ring $T^{-1}A$. If I is a homogeneous prime ideal of A, let $A_{(I)} = A_{(T)}$ where T is the set of homogeneous elements of $A - I$ (this is indeed a multiplicatively closed set of homogeneeous elements in A). If T is the multiplicative set of powers of a homogeneous element f, we write $A_{(f)}$ for $A_{(T)}$. Note that if T contains a homogeneous element of degree 1, or even a power of such an element, then

$$T^{-1}A = A_{(T)}[t, t^{-1}]$$

is a Laurent polynomial ring in 1 variable, as a graded ring. Here t is the image of any homogeneous element of degree 1 of A such that a power of t lies in T.

Theorem 3 *Let $X \subset \mathbf{P}^n_k$ be a projective variety, with homogeneous coordinate ring $S(X)$.*

(i) For any point $P \in X$, let $\mathbf{m} \subset S(X)_+$ be the homogeneous prime ideal corresponding to P. Then $\mathcal{O}_{P,X}$ is naturally isomorphic to $S(X)_{(\mathbf{m})}$.

(ii) *Let $f \in S(X)$ be homogeneous of degree > 0, and let $Y = V(f) \subset X$ be the closed subvariety defined by the vanishing of f (i.e., Y corresponds to the graded radical ideal $\sqrt{fS(X)}$). Then $X - Y$ is affine, with coordinate ring naturally isomorphic to $S(X)_{(T)}$.*

(iii) $\dim S(X) = \dim X + 1$*; if $X \neq \phi$, then the irreducible components of X are in bijection with the minimal primes of $S(X)$ (which are homogeneous).*

Suppose further that X is irreducible. Then

(iv) *$\mathcal{O}(X) = k$, i.e., any regular function on X is constant, and*

(v) *$K(X) = S(X)_{((0))}$, the elements of degree 0 in the localization of $S(X)$ with respect to the multiplicative set of homogeneous elements of $S(X) - (0)$.*

This is contained in [H] (I, Theorem 3.4 and Ex. 2.10), except for (ii), which is proved in the special case when f is one of the variables. The general case of (ii) is proved (in a more general form, in the context of schemes) in [H], II, Prop. 2.5. The idea is as follows. If f has degree d, then $S(X)_{(f)}$ consists of all elements g/f^r where $g \in S(X)_{rd}$, for some $r \geq 0$. Let $P = (a_0 : a_1 : \cdots : a_n) \in X - Y$. Since $\deg g = \deg f^r$, the ratio $g/f^d(a_0, a_1, \ldots, a_n)$ depends only on P, and yields a well defined function $\widetilde{g/f^d} : X - Y \to k$. One checks that this is a regular function, and that the resulting map $S(X)_{(f)} \to \mathcal{O}(X - Y)$ is an isomorphism. This can be done by covering X by the opens sets $X \cap U_i$, which are affine (by the special case of (ii) for linear f); now one appeals to the analogous result for affine varieties (seen in §1, (1)).

In particular, we see that for any homogeneous polynomial $f \in S$, the Zariski open set in $\mathbf{P}^n_k$ defined by

$$D_+(f) := \mathbf{P}^n_k - V(f) = \{P \in \mathbf{P}^n_k \mid f(P) \neq 0\} \tag{2}$$

is affine, with coordinate ring $S_{(f)}$, the subring of homogeneous elements of degree 0 in the localization $S_f = k[X_0, \ldots, X_n, 1/f]$.

The set $\mathbf{P}^n_k \times \mathbf{P}^m_k$ has a natural structure as a projective subvariety in $\mathbf{P}^{nm+n+m}_k$, given by the image of the mapping

$$((a_0 : a_1 : \cdots : a_n), (b_0 : b_1 : \cdots : b_m)) \mapsto (a_0 b_0 : a_0 b_1 : \cdots : a_i b_j : \cdots : a_n b_m) \in \mathbf{P}^{nm+n+m}$$

(the dimension of the target projective space is $(n+1)(m+1)-1$). This is called the *Segre embedding* of $\mathbf{P}^n_k \times \mathbf{P}^m_k$. Its image is indeed a projective subvariety; if we let $Z_{ij},\ 0 \leq i \leq n, 0 \leq j \leq m$ be the variables corresponding to homogeneous coordinates on $\mathbf{P}^{mn+m+n}_k$, then the image of the Segre embedding is the variety defined by the homogeneous polynomial equations

$$X_{ij}X_{kl} - X_{il}X_{kj} = 0,\ \forall\, 0 \leq i,k \leq n, 0 \leq j,l \leq m.$$

These are equivalent to the assertion that the matrix $[X_{ij}]$ has rank ≤ 1. Clearly, the homogeneous coordinates of any point P in the image of the Segre embedding do satisfy these quadratic equations. Conversely, given any non-zero solution $[c_{ij}]$ of these equations, the assertion that the rank of $[c_{ij}]$ equals 1 implies that there exist (unique upto scalars) $(a_0,\ldots,a_n) \in k^{n+1}$, $(b_0,\ldots,b_m) \in k^{m+1}$ in k such that $c_{ij} = a_i b_j$. Then P is the image of the (unique) point $((a_0 : \cdots : a_n),(b_0 : \cdots : b_m))$ under the Segre embedding. If $S = \oplus_{d\geq 0} S_d$ and $S' = \oplus_{d \geq 0} S'_d$ are the (graded) homogeneous coordinate rings of $\mathbf{P}^n_k$ and $\mathbf{P}^m_k$ respectively, then there is a natural identification of the homogeneous coordinate ring

$$S(\mathbf{P}^n_k \times \mathbf{P}^m_k) = k[Z_{ij}]_{i,j} / < X_{ij}X_{kl} - X_{il}X_{kj} \mid \ \forall i,j,k,l >$$

with the graded k-algebra

$$\oplus_{d\geq 0} S_d \otimes_k S'_d.$$

Here the image of the variable Z_{ij} is $X_i \otimes X'_j$, where $X_i \in S_1$ and $X'_j \in S'_1$ give the homogeneous coordinates on $\mathbf{P}^n_k$, $\mathbf{P}^m_k$ respectively.

Note that though we call this the Segre embedding, in our treatment, it is only a bijection, which we then use to put the structure of a variety on $\mathbf{P}^n_k \times \mathbf{P}^m_k$. In the theories of 'abstract varieties' (which we do not discuss in this course) or schemes (which are discussed later, and in principle include 'abstract varieties'), there is a direct product, so that $\mathbf{P}^n_k \times \mathbf{P}^m_k$ has a structure as a variety; in either of these contexts, one has to show that the Segre embedding is in fact an isomorphism, in the appropriate category. This is true, and will be discussed later for schemes.

One very important property which characterizes projective varieties among arbitrary ones is that they are complete. This is an algebraic analogue of the compactness (with respect to the Euclidean topology) of projective varieties over the complex number field. A variety X is called *complete* if for any variety Y, the projection $X \times Y \to Y$ is a closed map (*i.e.*, the image of a closed subset of $X \times Y$ is closed in Y). If X is complete, the image of any morphism $f : X \to Z$ is closed, since it is the image of the graph of f under the projection $X \times Z \to Z$. This imples that any regular function on a connected complete variety X is constant: if $f : X \to \mathbf{A}^1_k$ is a regular function, and $i : \mathbf{A}^1_k \to \mathbf{P}^1_k$ is the inclusion, then f and $i \circ f$ each must have closed image, which is possible only if this image is a single point. Next, if X is a complete quasi-projective variety in $\mathbf{P}^n_k$, the inclusion $X \hookrightarrow \mathbf{P}^n_k$ has closed image, so X is in fact projective. Clearly a closed subvariety of a complete variety is complete. So it remains to prove $\mathbf{P}^n_k$ is complete. To do this, it suffices to show that for any m, the projection $\mathbf{P}^n_k \times \mathbf{P}^m_k \to \mathbf{P}^m_k$ is closed. This can be proved via *elimination theory*; see [H], Theorem 5.7 A for the statement needed. Another proof will be given later in this course via valuation theory, in the context of schemes.

Example 2.1: The simplest non-trivial example of a Segre embedding is that of $\mathbf{P}^1_k \times \mathbf{P}^1_k$. The matrix $[Z_{ij}]$ reduces to a 2×2 matrix, and the system of quadratic equations reduces to a single equation. Renaming the coordinates on $\mathbf{P}^3_k$ as x, y, z, w, the map may be described by the formulas

$$x = X_0X_0', y = X_0X_1', z = X_1X_0', w = X_1X_1'.$$

The quadratic relation becomes $xw = yz$, *i.e.*, $\mathbf{P}^1_k \times \mathbf{P}^1_k$ is identified with the quadric surface in $\mathbf{P}^3_k$ defined by $xw - yz = 0$.

As in the case of affine varieties, the Segre embedding now allows us to define a product for arbitrary quasi-projective varieties. Again, the Zariski topology on the product is finer than the product topology, but the product variety is indeed the direct product in the category of quasi-projective varieties (and morphisms between these). If X and Y are affine, then the product defined earlier agrees with the present one.

We next define the notion of a non-singular point of a quasi-projective variety X; it is most convenient to define P to be *non-singular* if $\mathcal{O}_{P,X}$ is a regular local ring. This is consistent with the definition for affine varieties given earlier. If X is irreducible and projective of dimension d in $\mathbf{P}^n_k$, then there is a 'homogeneous' Jacobian criterion for a point P to be non-singular: there must exist homogeneous polynomials $F_1, \ldots, F_{n-d}$ in $I(X)$ such that the Jacobian matrix

$$J = \left[\frac{\partial F_j}{\partial X_i}(a_0, a_1, \ldots, a_n)\right]$$

has (maximal) rank $n-d$, where $P = (a_0 : a_1 : a_2 : \cdots : a_n)$. This condition does not depend on the particular homogeneous coordinates chosen for P.

The homogeneous Jacobian criterion can be deduced from the one for affine varieties (see [H], I, Ex. 5.8), using the Euler relation

$$\sum_i X_i \frac{\partial F}{\partial X_i} = (\deg F)F$$

for a homogeneous polynomial F.

One can define the *projective tangent space* $\mathbf{T}_{P,X}$ to a projective variety X at a point $P = (a_0 : a_1 : \cdots : a_n)$ as the linear projective subvariety of $\mathbf{P}^n_k$ defined by

$$\sum_{i=0}^{n} \frac{\partial F_j}{\partial X_i}(a_0, \ldots, a_n)X_i = 0, \; \forall\, 1 \leq j \leq r,$$

where $F_1, \ldots, F_r$ are homogeneous generators for the ideal $I(X)$. This is easily seen to be independent of the choice of generators, and if $P \in U_i = D_+(X_i) \cong \mathbf{A}^n_k$, then $\mathbf{T}_{P,X}$ is the closure in $\mathbf{P}^n_k$ of the Zariski tangent space $T_{P,X\cap U_i} \subset U_i$. Thus the projective tangent space has the same dimension as the Zariski tangent space; if X is irreducible and non-singular of dimension d, then this also equals the dimension of either tangent space. Finally, we can define the projective tangent variety $\mathbf{T}_X$ of X as a subvariety of $\mathbf{P}^n_k \times \mathbf{P}^n_k$, just as in the affine case; it is the subvariety of $\mathbf{P}^n_k \times \mathbf{P}^n_k$ whose image under the first projection is X, and whose fibre over $P \in X$ is the projective tangent space to X at P. If X is non-singular of dimension d, this gives an example of a *projective bundle* of fibre dimension d (*i.e.*, a $\mathbf{P}^d$*-bundle*) over X.

We now examine in some more detail the structure of morphisms. A first simplifying remark is the following. Let X, Y be varieties, with X irreducible, and $i : Z \hookrightarrow Y$ the inclusion of an irreducible locally closed subset (hence Z is an irreducible variety also, in a natural way). Let $f : X \to Z$ be a mapping of sets. Then f is a morphism $\Leftrightarrow$ the induced map $i \circ f : X \to Y$ is a morphism. Thus the important case to consider is of morphisms $f : X \to \mathbf{P}^m_k$, where $X \subset \mathbf{P}^n_k$ is locally closed. One way to describe such morphisms is via *linear systems.*

There are several ways of thinking about linear systems. We give here a concrete description, though a fuller understanding of linear systems comes from the formulation in terms of invertible sheaves. The idea is that if $X_0, \ldots, X_n$ give homogeneous coordinates on $\mathbf{P}^n_k$ and if $F_0, \ldots, F_m$ are homogeneous polynomials of a fixed degree d, then for any point $P = (a_0 : a_1 : \cdots : a_n) \in \mathbf{P}^n_k$, we have two possibilities:

(i) either $F_0(P) = F_1(P) = \cdots = F_m(P) = 0$, or

(ii) the point $(F_0(a_0, \ldots, a_n) : F_1(a_0, \ldots, a_n) : \cdots : F_m(a_0, \ldots, a_n)) \in \mathbf{P}^m_k$ depends only on P.

Hence there is a well defined map

$$\varphi : \mathbf{P}^n_k - V(< F_0, \ldots, F_m >) \to \mathbf{P}^m_k.$$

It is easy to see that this is a morphism, by restricting to the affine open sets $D_+(F_j)$ for $0 \leq j \leq m$ — if $Y_0, \ldots, Y_m$ give the homogeneous coordinates on $\mathbf{P}^m_k$, then $\varphi : D_+(F_j)) \to D_+(Y_j) \cong \mathbf{A}^m_k$ is the map

$$P \mapsto (\frac{F_0}{F_j}(P), \ldots, \frac{F_{j-1}}{F_j}(P), \frac{F_{j+1}}{F_j}(P), \ldots, \frac{F_m}{F_j}(P));$$

the component functions F_i/F_j are regular functions on $D_+(F_j)$. Now for any subvariety $X \subset \mathbf{P}^n_k$, we may restrict φ to $X - V(< F_0, \ldots, F_m >)$, which is a non-empty (=dense) open subset of X provided some $F_j \notin I(X)$, *i.e.*, $X \not\subset V(< F_0, \ldots, F_m >)$. In this case, $\varphi \mid_X : X \to \mathbf{P}^m_k$ is a well-defined *rational map.*

However, it may happen that the rational map may extend (uniquely) to a morphism on a larger open subset of X than the 'obvious' one

$$X - V(< F_0, \ldots, F_m >);$$

it may even extend to all of X. For instance, for any non-zero homogeneous polynomial F, the rational maps $\mathbf{P}^n_k \to \mathbf{P}^m_k$ given by $(F_0 : F_1 : \cdots : F_m)$ and $(FF_0 : FF_1 : \cdots : FF_m)$ are clearly the same, though the latter is apparently defined only on the open set

$$\mathbf{P}^n_k - (V(F) \cup V(< F_0, \ldots, F_m >)).$$

In general, the same phenomenon can occur *locally* on X: one may cancel common factors 'defined locally on X' (the 'invariant' way to view this is in terms of invertible sheaves, as mentioned earlier; these will be dealt with later in the course). Here we just give some examples of this phenomenon.

Example 2.2: Let $n > 1$, and let $O = (0 : 0 : \cdots : 0 : 1) \in \mathbf{P}^n_k$. Then we have a well defined morphism $\mathbf{P}^n_k - O \to \mathbf{P}^{n-1}_k$, given by

$$(a_0 : a_1 : \cdots : a_{n-1} : a_n) \mapsto (a_0 : a_1 : \cdots : a_{n-1}),$$

called *projection* from the point O. More generally, the assignment

$$(a_0 : \cdots : a_n) \mapsto (a_0 : \cdots : a_{n-r})$$

is a morphism $\mathbf{P}^n_k - L \to \mathbf{P}^{n-r}_k$; it is called *projection from the linear subspace* $L \cong \mathbf{P}^{r-1}_k$ defined by $X_0 = \cdots = X_{n-r} = 0$.

Projection from a linear subspace L has the following geometric description. Choose a linear subspace $M \subset \mathbf{P}^n_k$ which is disjoint from L, such that $\dim L + \dim M = n - 1$ (on the level of cones, $C(L)$ and $C(M)$ form complementary linear subspaces of k^{n+1}). Then the projection from L is an isomorphism restricted to M, so we may consider projection form L as a morphism $\mathbf{P}^n_k - L \to M$. The image of a point P is determined geometrically as follows: the smallest linear subspace containing P and L (the *linear span* of P and L) intersects M in a unique point, which is defined to be the image of P under the projection.

For any pair of linear subspaces L, M as above ($L \cap M = \phi$, and $\dim L + \dim M = n - 1$)) we also call the resulting morphism $\mathbf{P}^n_k - L \to M$ a projection. Note that the geometric description of a projection is 'coordinate free', since we do not need to fix homogeneous coordinates on L, M to define the projection.

Example 2.3: Suppose $X = V(x^2 - yz) \subset \mathbf{P}^2_k$ is a conic, and $O = (0 : 0 : 1)$; then $O \in X$. The projection $(a : b : c) \mapsto (a : b)$ gives a morphism $f : X - \{O\} \to \mathbf{P}^1_k$. Now consider the projection $(a : b : c) \to (c : a)$. This is defined on $\mathbf{P}^2_k - \{O'\}$, where $O' = (0 : 1 : 0) \neq O$. Thus we have a well defined morphism $g : X - \{O'\} \to \mathbf{P}^1_k$. But both f and g agree with the rational map $h : X \to \mathbf{P}^1_k$ given by $(a : b : c) \mapsto (ac : bc)$, since along X, we have $a^2 = bc$. Hence $f = g = h$ is a morphism on all of X.

Example 2.4: Let $X \subset \mathbf{P}^3_k$ be the quadric surface $X = V(xy - zw)$, where x, y, z, w give homogeneous coordinates on $\mathbf{P}^3_k$. As seen earlier, $X \cong \mathbf{P}^1_k \times \mathbf{P}^1_k$ is the image of the Segre embedding. Consider the rational map $f : X \to \mathbf{P}^1_k$ given by $f(a : b : c : d) = (a : c)$. This is a morphism on the complement of $x = z = 0$. Since $ab = cd$ for $P = (a : b : c : d) \in X$, we have equalities between rational maps

$$(a : c) = (ab : bc) = (cd : bc) = (d : b).$$

Now $(a : b : c : d) \mapsto (d : b)$ is obviously a morphism on the complement of $y = w = 0$. Hence f is a morphism on the complement in X of $x = y =$

$z = w = 0$, *i.e.*, $f : X \to \mathbf{P}^1_k$ is a morphism. In fact, if we trace through the definitons, we see that under the identification of X with $\mathbf{P}^1_k \times \mathbf{P}^1_k$, the morphism f is just projection onto one of the factors of $\mathbf{P}^1_k$.

Example 2.5: (d-tuple embedding, or Veronese embedding)

Let $M_0, \ldots, M_N$ be a listing of the distinct monomials of degree d in $n+1$ variables $X_0, \ldots, X_n$. Then

$$P \mapsto (M_0(P) : M_1(P) : \cdots : M_N(P))$$

is a well defined morphism $\mathbf{P}^n_k \to \mathbf{P}^N_k$, called the *$d$-tuple (or Veronese) embedding*. We will show that this is an isomorphism from $\mathbf{P}^n_k$ onto a projective variety in $\mathbf{P}^N_k$, *i.e.*, is indeed an embedding. In particular, if $n = 2$, $d = 2$, then we compute that $N = 5$; the resulting surface in $\mathbf{P}^5_k$ is called the Veronese surface.

The dimension of the k-vector space of homogeneous polynomials over k of degree d in the variables $X_0, \ldots, X_n$ is the binomial coefficient $\binom{n+d}{d}$. Let $Y_0, \ldots, Y_N$ give the homogeneous coordinates on $\mathbf{P}^N_k$, so that the d-tuple embedding of $\varphi_{n,d} : \mathbf{P}^n_k \to \mathbf{P}^N_k$ (where $N = \binom{n+d}{d} - 1$) is given by

$$P = (a_0 : a_1 : \cdots : a_n) \mapsto$$
$$(M_0(a_0, a_1, \ldots, a_n) : M_1(a_0, a_1, \ldots, a_n) : \cdots : M_N(a_0, a_1, \ldots, a_n)).$$

Let $R \subset S(\mathbf{P}^n_k) = k[X_0, X_1, \ldots, X_n]$ be the k-subalgebra generated by all monomials of degree d. Thus $R = \oplus_{s \geq 0} S(\mathbf{P}^n_k)_{sd}$, the subalgebra of polynomials all of whose terms have degree divisible by d. We may redefine the grading in the ring R by defining $R_s = S(\mathbf{P}^n_k)_{sd}$; then R is generated by its homogeneous elements of degree 1. The graded k-algebra R is naturally expressed as a graded quotient of $k[Y_0, \ldots, Y_N]$, by mapping Y_i to the corresponding monomial M_i. Let $\psi : k[Y_0, \ldots, Y_N] \twoheadrightarrow R$. The kernel of ψ is clearly a homogeneous prime ideal, defining an irreducible projective variety $Z \subset \mathbf{P}^N_k$. We claim that the d-tuple embedding $\varphi_{n,d}$ gives an isomorphism $\mathbf{P}^n_k \to Z$.

First note that if $P = (a_0 : a_1 : \cdots : a_n) \in \mathbf{P}^n_k$, then the corresponding homogeneous prime ideal in $S(\mathbf{P}^n_k)$ is

$$I(P) = < a_i X_j - a_j X_i \mid 0 \leq i < j \leq n > .$$

Consider the surjective, graded k-algebra homomorphism $\theta_P : S(\mathbf{P}^n_k) \to k[t]$ given by $X_i \mapsto a_i t$. We see at once that its kernel is $I(P)$. Restriction to R, and composing with ψ, yields a graded k-linear surjection

$k[Y_0,\ldots,Y_N]\twoheadrightarrow k[t^d]$, provided we redefine the grading in $k[t^d]$ so that t^d now has degree 1. The kernel is the homogeneous ideal of a point $Q \in Z$. Since $\theta_P(Y_i) = M_i(a_0,\ldots,a_n)t^d$, by construction, clearly we have $Q = \varphi_{n,d}(P)$. In particular $\varphi_{n,d}$ factors through Z.

Conversely, any point of Z determines a graded k-linear surjection $\eta : R \to k[t^d]$. If we know that this is the restriction of a unique k-algebra surjection $\theta : S(\mathbf{P}^n_k) \to k[t]$, then we would have shown $\varphi_{n,d}$ is bijective. Now there is some monomial M of degree d (an element of degree 1 in R) such that $\eta(M) \neq 0$ in $k[t^d]$; then $\eta(M^d) \neq 0$, so $\eta(X_i^d) \neq 0$ for some i. After composing with an automorphism of $k[t^d]$ of the form $t^d \mapsto ct^d$ for some $c \in k^*$, we may assume $\eta(X_i^d) = t^d$. Now if $\eta(X_i^{d-1}X_j) = a_j t^d$, define $\theta : k[X_0,\ldots,X_n] \to k[t]$ by $\theta(X_j) = a_j t^d$ (set $a_i = 1$). For any monomial $X_0^{\nu_0}X_1^{\nu_1}\cdots X_n^{\nu_n}$ with $\sum \nu_j = d$, we have a relation in R

$$(X_0^{\nu_0}X_1^{\nu_1}\cdots X_n^{\nu_n})(X_i^d)^{d-1} = \prod_{j=0}^{n}(X_jX_i^{d-1})^{\nu_j},$$

so that

$$\begin{aligned}\theta(X_0^{\nu_0}X_1^{\nu_1}\cdots X_n^{\nu_n}) &= \frac{\prod_{j=0}^n \theta(X_jX_i^{d-1})^{\nu_j}}{\theta(X_i^d)} \\ &= \frac{\prod_{j=0}^n \eta(X_jX_i^{d-1})^{\nu_j}}{\eta(X_i^d)} \\ &= \eta(X_0^{\nu_0}X_1^{\nu_1}\cdots X_n^{\nu_n}).\end{aligned}$$

Hence θ restricts to η, as desired.

To complete the proof that $\varphi_{n,d}$ is an isomorphism onto Z, it suffices to check that for a covering of Z by affine open sets V_j, the set $\varphi_{n,d}^{-1}(V_j) = V_j'$ is affine, and $\varphi_{n,d} : V_j' \to V_j$ is an isomorphism of affine varieties. In fact, let $M_0,\ldots,M_n$ be the monomials $M_j = X_j^d$, and $Y_0,\ldots,Y_n$ the corresponding variables. Then Z is covered by the open sets $V_j = D_+(Y_j) \cap Z = D_+(M_j)$, since the radical in R of the ideal $< M_0,\ldots,M_n >$ is just $R_+ = \oplus_{s>0} R_s$ — in fact for any monomial M of degree d in the X_i, we have that

$$M^d \in < M_0,\ldots,M_n > \subset R.$$

The sets V_j are affine open subvarieties of Z, from Theorem 3, and the coordinate ring of V_j is $A(V_j) = R_{(M_j)}$. Now $\varphi_{n,d}^{-1}(V_j) = D_+(X_j) = U_j$ is the standard affine open cover of $\mathbf{P}^n_k$, and its coordinate ring is $A(U_j) = S(\mathbf{P}^n_k)_{(X_j)} = k[X_0/X_j,\ldots,X_{j-1}/X_j,X_{j+1}/X_j,\ldots,X_n/X_j]$, which is a polynomial ring. The inclusion $R \hookrightarrow S(\mathbf{P}^n_k)$ induces a homomorphism of localizations $A(V_j) = R_{(M_j)} \to S(\mathbf{P}^n_k)_{(X_j)} = A(U_j)$, since inverting X_j in the polynomial ring $S(\mathbf{P}^n_k) = k[X_0, \ldots, X_n]$ is equivalent to inverting $M_j = X_j^d$. The corresponding morphism $U_j \to V_j$ is just $\varphi_{n,d}$ (exercise for the reader!). So it suffices to show that $R_{(M_j)} \to S(\mathbf{P}^n_k)_{(X_j)}$ is an isomorphism. This is clear, since it is an inclusion (both are subrings of the quotient field of $k[X_0, \ldots, X_n]$) and is surjective, since for F homogeneous of degree r, if we choose s with $sd \geq r$, then $F/X_j^r = FX_j^{sd-r}/(X_j^d)^s$, and $FX_j^{sd-r} \in R$.

One important property of the d-tuple embedding is the following: the hyperplane sections (intersections with hyperplanes) of $\varphi_{n,d}(\mathbf{P}^n_k)$ are precisely the hypersurfaces of degree d in $\mathbf{P}^n_k$ (*i.e.*, the subvarieties $V(F) \subset \mathbf{P}^n_k$, where F is a homogeneous polynomial of degree d).

Example 2.6: (d-tuple embedding of a projective variety)

Let $X \subset \mathbf{P}^n_k$ be a projective variety, and $d > 1$. Then $\varphi_{n,d}(X)$ is an isomorphic copy of X in $\mathbf{P}^N_k$, where $N = \binom{n+d}{d} - 1$. The homogeneous coordinate ring of X in this new projective embedding is $\oplus_{s \geq 0} S(X)_{sd}$, where $S(X)$ is the homogeneous coordinate ring for the original embedding.

Now suppose $X \subset \mathbf{P}^n_k$, and let $F_0, \ldots, F_m$ be homogeneous polynomials of degree $d > 1$ in $k[X_0, \ldots, X_n] = S(\mathbf{P}^n_k)$. If $V(< F_0, \ldots, F_m >) \cap X = \phi$, then the F_i certainly determine a morphism $X \to \mathbf{P}^m_k$. However this morphism *does not* in general correspond to a homomorphism of graded rings $S(\mathbf{P}^n_k) \to S(X)$, as seen for the d-tuple embedding. So unlike the case of affine varieties, the homogeneous coordinate rings do not directly determine the structure of morphisms between projective varieties. However, again as in the case of the d-tuple embedding, the morphism $f : X \to \mathbf{P}^m_k$ defined by the F_i *does* correspond to a graded homomorphism $S(\mathbf{P}^m_k) \to \oplus_{s \geq 0} S(X)_{sd}$. In geometric language, the morphisms which correspond to graded homomorphisms between homogeneous coordinate rings are those defined by linear functions, *i.e.*, are essentially projections from linear subspaces composed with linear embeddings; more general morphisms are composites of such 'linear' morphisms with a suitable d-tuple embedding. Again, this situation will be more clearly understood later in terms of invertible sheaves.

As with the d-tuple embedding of $\mathbf{P}^n_k$, we see that the hyperplane sections (intersections with hyperplanes) in the d-tuple embedding of X are precisely the hypersurface sections of degree d of X in the original embedding (*i.e.*, the subvarieties $V(F)$, where $F \in S(X)_d$ is homogeneous of degree d). Thus any general result (for example, *Bertini's theorem*, discussed in §3) which is valid for hyperplane sections of any projective variety (or any projective variety in some class, like irreducible, non-singular varieties) is automatically valid also for hypersurface sections of arbitrary degree.

Example 2.7:

If $f : \mathbf{P}^n_k \to \mathbf{P}^m_k$ is a non-constant morphism, then $m \geq n$ and f has finite fibres. See [H], II, Ex. 7.3; this is an easy consequence of the description of linear systems in terms of invertible sheaves.

Instead of describing linear systems on an irreducible projective variety as ordered $m+1$-tuples of homogeneous polynomials of the same degree, we can consider them as $m+1$-tuples of rational functions $(f_0, \ldots, f_m)$; if U is a non-empty open set where all the f_i are regular, and not all 0, then $P \mapsto (f_0(P) : f_1(P) : \cdots : f_m(P))$ is a morphism $U \to \mathbf{P}^m_k$. If $F_0, \ldots, F_m$ are homogeneous of the same degree, and $F_j \notin I(X)$, then let $f_i = F_i/F_j$; now the rational maps determined by $(f_0, \ldots, f_m)$ and $(F_0, \ldots, F_m)$ are the same. Further, we may replace $(f_0, \ldots, f_m)$ by $(ff_0, \ldots, ff_m)$, where $f \in k(X)^*$ is a non-zero rational function on X, without changing the rational map $X \to \mathbf{P}^m_k$.

To complete the discussion, one can prove that for any irreducible variety $X \subset \mathbf{P}^n_k$, every morphism $X \to \mathbf{P}^m_k$ is obtained by extending to all of X the rational map determined by $m+1$ homogeneous polynomials of the same degree; in particular, it is described by a linear system in either sense mentioned above. This is an easy consequence of Theorem 3 above. Indeed, let $f : X \to \mathbf{P}^n_k$ be a morphism, whose image intersects $U_j = D_+(Y_j)$ (where $Y_0, \ldots, Y_m$ give the homogeneous coordinates in $\mathbf{P}^m_k$). Then $f^{-1}(U_j) = U$ is a non-empty open subset of X, and $f : U \to U_j \cong \mathbf{A}^m_k$ is a morphism. Thus $f\mid_U = (f_1, \ldots, f_m)$ where $f_i \in \mathcal{O}(U) \subset k(X)$. Clearly f is then equal to the rational map $X \to \mathbf{P}^m_k$ given by $P \mapsto (f_1(P) : \cdots : f_{j-1}(P) : 1 : f_j(P) : \cdots : f_m(P))$. Since $k(X) = S(X)_{((0))}$, we can find homogeneous elements $F_0, \ldots, F_m \in S(X)_d$, for some $d > 0$, such that

$$f_i = \begin{cases} F_{i+1}/F_j & \text{for all } i \leq j, \\ F_i/F_j & \text{for all } i > j \end{cases}$$

(that is, F_j is a 'common denominator' for $f_1, \ldots, f_m$). Now clearly $f = (F_0 : F_1 : \cdots : F_m)$.

We conclude this section with a discussion of the *degree* of a projective variety. If $X \subset \mathbf{P}^2_k$ is a curve (*i.e.*, all irreducible components of X have dimension 1), then $I(X) = (F)$ for a homogeneous polynomial $F(x, y, z)$, unique upto a constant factor. If this polynomial has degree d, and is not divisible by z, then for 'general' $a, b \in k$, the equation $F(x, ax+b, 1) = 0$ in x has d distinct roots. We may interpret this to mean that a 'general' line intersects X in d points. This property of X gives an interpretation of the integer d without directly refer to the defining polynomial F.

In general, there is no single defining polynomial, nor even any preferred set of defining polynomials, for a projective variety. However, we can still associate a certain positive integer, which we call its degree, to any projective variety X. Geometrically, the degree is thought of as follows: if $X \subset \mathbf{P}^n_k$ has dimension m, then

(i) a 'general' linear subspace $L \cong \mathbf{P}^{n-m}_k$ should intersect X 'transversally' in a finite set of points

(ii) the number of points in such a general linear intersection is independent of the 'general' L chosen, and is an invariant of the embedding of X.

This directly generalizes the example of curves in $\mathbf{P}^2_k$. However, one then needs to make precise the notions of 'general' and 'transversal intersection', and to prove the property (ii). This can be done. More generally, the degree should be a special case of the 'intersection number' of two projective varieties of complementary dimension in a given projective space. This should have several 'intuitively obvious' properties. For example, if X_t is a 'continuously varying' family of projective varieties of a given dimension r in $\mathbf{P}^n_k$, and Y is a given variety of complementary dimension $n-r$, the intersection number $(X_t \cdot Y)$ of X_t and Y should be independent of t. The theory of intersection multiplicities arises out of these ideas, but there are formidable technical difficulties in carrying this out with full rigour.

For technical reasons, it turns out to be better to define the degree in another way (similarly, the desired intersection theory is also set up in an indirect way; this is pursued later in this course).

If $R = \oplus_{s\geq 0} R_s$ is a graded k-algebra, with $R_0 = k$, such that $\dim_k R_1 = n$ is finite, and $R = R_0[R_1]$ (*i.e.*, R is generated as a k-algebra by elements of degree 1), then R is a graded quotient of a polynomial ring $k[X_0, \ldots, X_n]$. Hence

$$H(R, s) := \dim_k R_s$$

is finite, for each $s \geq 0$; in fact

$$0 \leq H(R, s) \leq H(k[X_0, \ldots, X_n], s) = \binom{n+s}{s}.$$

The function $H(R, s)$ is called the *Hilbert-Samuel function* of the graded k-algebra R. Similarly, if M is a finitely generated graded R-module, then we can define its Hilbert-Samuel function to be

$$H(M,s) := \dim_k M_s,$$

where $M = \oplus_{s\in\mathbf{Z}} M_s$ (we allow M to be non-zero in negative degrees, but $M_s = 0$ for all sufficiently negative s, since M is finitely generated). Clearly it suffices to discuss Hilbert-Samuel functions of finitely generated graded modules over a polynomial ring, since in the above situation, we may regard M as a $k[X_0,\ldots,X_n]$-module.

Now one has the following general result on Hilbert-Samuel functions.

Lemma 1 *Let M be a finitely generated graded module over the polynomial ring $k[X_0,\ldots,X_n]$. Let*

$$r+1 = \dim M := \dim k[X_0,\ldots,X_n]/\mathrm{Ann}\,(M)$$

(so that $r = \dim V(\mathrm{Ann}\,(M)) \subset \mathbf{P}^n_k$). Then there exist (unique) integers $c_0, c_1, \ldots, c_r$ with $c_0 > 0$, such that

$$H(M,s) = c_0\binom{s}{r} + c_1\binom{s}{r-1} + \cdots + c_r\binom{s}{0}\ \forall\, s >> 0.$$

In particular, for all sufficiently large s, $H(M,s)$ equals a polynomial in s with rational coefficients, which is called the Hilbert-Samuel polynomial *of M.*

A proof is given in [H], I, Theorem 7.5. It is by induction on r, and relies on the facts that

(i) the primary decomposition of a finitely generated graded $k[X_0,\ldots,X_n]$-module is given by graded submodules

(ii) any finitely generated graded $k[X_0,\ldots,X_n]$-module has a finite filtration by modules of the form $(k[X_0,\ldots,X_n]/P)(t)$, where P is a graded prime ideal, and (t) indicates that the grading is shifted by the integer t (thus

$$M(t)_s := M_{s+t},$$

for any graded module M)

(iii) if $0 \to M' \to M \to M'' \to 0$ is an exact sequence of finitely generated graded modules over $k[X_0,\ldots,X_n]$, then $H(M,s) = H(M',s) + H(M'',s)$. .

Applying the above lemma to $M = S(X)$ for a projective variety $X \subset \mathbf{P}^n_k$ of dimension r, we get that

$$H(S(X), s) = d(X)\frac{s^r}{r!} + \text{ lower degree terms in } s,$$

where $d(X)$ is a positive integer. We define $d(X)$ to be the *degree* of X in $\mathbf{P}^n_k$.

Since the d-tuple embedding of a projective variety X has coordinate ring $\oplus_{s\geq 0} S(X)_{sd}$, the above lemma implies that the degree of X in its d-tuple embedding is $d^r d(X)$, where $d(X)$ is the degree in the original embedding. This is consistent with our idea of intersection numbers, since we can find a continuously varying family of varieties in $\mathbf{P}^n_k$, one member of which is a given hypersurface of degree d, and another member of which consists of d general hyperplanes (if F, $G = G_1 G_2 \cdots G_d$ are the respective equations, consider the family in $\mathbf{P}^n \times \mathbf{A}^1$ given by $tF + (1-t)G = 0$, where t is the coordinate on $\mathbf{A}^1_k$). Now intersection (in the d-tuple embedding) with a linear space of codimension r is the intersection (in the original space) with r hypersurfaces of degree d, each of which we 'replace' by d hyperplanes, all mutually in general position. This leads us to intersecting X with d^r different linear spaces in $\mathbf{P}^n_k$, each of codimension r. Each such intersection should contribute $d(X)$ points, so all of them should contribute $d^r d(X)$ points. As the reader can imagine, it seems difficult to make such reasoning rigorous, though it seems plausible. The statement about degrees defined via Hilbert-Samuel functions is, however, easy to prove, as we have seen.

In a similar fashion, we would expect that if $X = X_1 \cup X_2$ where $\dim X_1 = \dim X_2 > \dim X_1 \cap X_2$, then $d(X) = d(X_1) + d(X_2)$. This is easily deduced from the exact sequence of graded $S = k[X_0, \ldots, X_n]$-modules

$$0 \to S/I(X) \to S/I(X_1) \oplus S/I(X_2) \to S/(I(X_1) + I(X_2)) \to 0,$$

after noting that $\dim S/(I(X_1) + I(X_2)) = \dim S/\sqrt{(I(X_1) + I(X_2))} = 1 + \dim(X_1 \cap X_2)$ (thus $H(S/(I(X_1) + I(X_2)), s)$ is a polynomial of degree $< \dim X = \dim X_i$, for all $s >> 0$).

Example 2.8: Let $X \subset \mathbf{P}^n_k$ with $I(X) = < F >$, where $F \in k[X_0, \ldots, X_n]$ is homogeneous of degree d. Then $d(X) = d$.

To prove this, consider the exact sequence of graded $S = k[X_0, \ldots, X_n]$-modules

$$0 \to S(-d) \xrightarrow{\cdot F} S \to S(X) \to 0,$$

where $S(-d)$ is S with its grading shifted (so that $S(-d)_t = S_{t-d}$), and $\cdot F$ denotes multiplication by F. Hence

$$H(S(X), s) = \binom{n+s}{n} - \binom{n+s-d}{n} \ \forall\, s \geq d.$$

The right side is of the form

$$\left(\frac{s^n}{n!} + a_{n-1}s^{n-1} + \text{lower degree terms in } s\right) - \left(\frac{(s-d)^n}{n!} + a_{n-1}(s-d)^{n-1} + \text{lower degree terms in } (s-d)\right),$$

where a_{n-1} is a rational constant. Thus

$$H(S(X), s) = d\frac{s^{n-1}}{(n-1)!} + \text{lower degree terms in } s.$$

The degree of a projective variety X has another interpretation, in terms of projections.

Theorem 4 (Projective Noether normalization) *Let $X \subset \mathbf{P}^n_k$ be an irreducible projective variety with* $\dim X = r$. *Then we can find a linear subspace $L \subset \mathbf{P}^n_k$ of dimension $n - r - 1$ such that*

- *(i) $L \cap X = \phi$, and the projection $\mathbf{P}^n_k - L \to \mathbf{P}^r_k$ restricts to a morphism $f : X \to \mathbf{P}^r_k$ with finite fibres*
- *(ii) the field extension $k(X)/f^*k(\mathbf{P}^r_k)$ is separable algebraic, and $[k(X) : f^*k(\mathbf{P}^r_k)] = d(X)$*
- *(iii) there is a non-empty Zariski open subset $U \subset \mathbf{P}^r_k$ such that for all $Q \in U$, the fibre $f^{-1}(Q) \subset X$ has $d(X)$ points.*

(In fact the 'general' linear subvariety $L \cong \mathbf{P}^{n-r-1}_k \subset \mathbf{P}^n_k$ will have the stated properties).

Proof: We first use the affine version of Noether normalization, proved in §1. We have $\dim S(X) = r + 1$; let $x_i \in S(X)$ be the image of the variable X_i. Then according to affine Noether normalization, we can find $r + 1$ homogeneous linear polynomials $y_0, \ldots, y_r$ in the x_i such that if $B =$

$k[y_0, \ldots, y_r]$, then B is a polynomial ring, and $A = S(X)$ is a finite B-module, and the quotient field of A is separable algebraic over that of B. Then A is in fact a finite *graded* B-module, where we give the polynomial ring the usual grading (y_j are homogeneous of degree 1). Thus we can consider B to be the homogeneous coordinate ring of $\mathbf{P}^r_k$. For each $0 \leq j \leq r$, choose a linear homogeneous polynomial Y_j in $X_0, \ldots, X_n$ such that $Y_j \mapsto y_j \in S(X)$. Then the homomorphism $B \to A$ can be considered as induced by the rational map $f = (Y_0 : Y_1 : \cdots : Y_r) : X \to \mathbf{P}^r_k$. By construction f is a morphism on $X - V(< Y_0, \ldots, Y_r >)$.

In particular, if $B_+ = \oplus_{s>0} B_s$ is the irrelevant maximal ideal of B, then

$$S(X)/B_+S(X) = S(X)/ < y_0, \ldots, y_r >$$

is a finite dimensional graded $B/B_+ = k$ vector space. Hence $\sqrt{B_+S(X)} = S(X)_+$, and so $V(< Y_0, \ldots, Y_r >) \cap X = \phi$. Hence $f : X \to \mathbf{P}^r_k$ is a morphism on all of X.

Next, consider the field extension $K(X)$ of $K(\mathbf{P}^r_k)$. From Theorem 3, we have $K(X) = A_{((0))}$, and $K(\mathbf{P}^r_k) = B_{((0))}$. The localization of B with respect to the set of homogeneous elements of B_+ is clearly just $B_{((0))}[t, t^{-1}] = K(\mathbf{P}^r_k)[t, t^{-1}]$, where $t \in B_1$ is any non-zero homogeneous element of degree 1 (if $F \in B_s$ is homogeneous of degree s, then $F = (F/t^s)t^s$ with $(F/t^s) \in B_{((0))}$).

Now the localization of A with respect to the homogeneous elements of A_+ equals the localization with respect to the homogeneous elements of B_+, since $\sqrt{B_+A} = A_+$, and B is a finite A-module. This localization of A is thus equal to $A_{((0))}[t, t^{-1}]$, where $t \in B_1$ is the element chosen earlier.

Hence the quotient field of B is $K(\mathbf{P}^r_k)(t)$, while that of A is $K(X)(t)$; the degree of the quotient field of A over that of B is thus $[K(X) : K(\mathbf{P}^r_k)] = d$, say. We will show below that $d = d(X)$. Assuming this, the affine Noether normalization lemma will then imply that (i) $K(X)(t)$ is separable over $K(\mathbf{P}^r_k)(t)$, *i.e.*, $K(X)$ is separable over $K(\mathbf{P}^r_k)$, (ii) all fibres of f are finite (since A is a finite B-module), and (iii) the 'general' fibre of f has cardinality $d(X)$.

Since A is a finite graded B-module, we can define the Hilbert-Samuel function of A as a B-module, which equals that of its Hilbert-Samuel function as a graded A-module. Since $[K(X) : K(\mathbf{P}^r_k)] = [K(X)(t) : K(\mathbf{P}^r_k)(t)] = d$, the $K(\mathbf{P}^r_k)[t, t^{-1}]$-module $K(X)[t, t^{-1}]$ is $\mathbf{Z}$-graded and free of rank d, generated by the images of homogeneous elments of A. Hence there is a $K(\mathbf{P}^r_k)$-basis consisting of elements $F_1, \ldots, F_d \in A_m$, for some m. The F_i

determine a map of graded B-modules

$$B(-m)^{\oplus d} \xrightarrow{\psi} A,$$

where $B(-m)_s = B_{m-s}$ is B with its grading shifetd, and $\psi(b_1, \ldots, b_d) = \sum_i b_i F_i$. Since ψ is an isomorphism when tensored with $K(\mathbf{P}^r_k)$, the kernel and cokernel of ψ are annihilated by some non-zero homogeneous element of B_+. We have a formula relating Hilbert functions

$$H(S(X), s) = H(A, s) = H(B(-m)^{\oplus d}, s) - H(\ker \psi, s) + H(\operatorname{coker} \psi, s).$$

From lemma 1, all the Hilbert functions equal polynomials in s for $s >> 0$; $H(S(X), s)$ and $H(B(-m)^{\oplus d}, s) = dH(B, s-m)$ are polynomials in s of degree r, while $H(\ker \psi, s)$ and $H(\operatorname{coker} \psi, s)$ are each of strictly samller degree in s (the precise degrees depend on the dimensions of the subvarieties of $\mathbf{P}^r_k$ defined by the annihilators of $\ker \psi$ and $\operatorname{coker} \psi$ in B). In particular, the leading coefficients of $H(S(X), s)$ and $H(B(-m)^{\oplus d}, s)$ are equal, *i.e.*, $d(X) = d$. □

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3 Geometry of projective varieties-I

In this section we discuss some aspects of the extrinsic geometry of projective varieties. We begin with projective plane curves.

A *curve* is a variety all of whose irreducible components have dimension 1. A *projective plane curve* is a curve which is a projective variety in $\mathbf{P}^2_k$. In this section, 'curve' will mean 'projective plane curve' unless specified otherwise.

Let X, Y, Z be variables giving the homogeneous coordinates on $\mathbf{P}^2_k$. If $C \subset \mathbf{P}^2_k$ is a curve, then $I(C) \subset k[X, Y, Z]$ is a radical homogeneous ideal which is purely of height 1, hence is a principal ideal generated by a homogeneous polynomial $F(X, Y, Z)$, unique upto a constant. The *degree* of C is defined to be the degree of the polynomial F; as seen in §2, Example 2.8, this agrees with the definition in terms of Hilbert-Samuel polynomials.

A *line* is a projective plane curve of degree 1, *i.e.*, is defined by a homogeneous linear polynomial. Any two lines in $\mathbf{P}^2_k$ are isomorphic, via an automorphism of $\mathbf{P}^2_k$ given by a homogeneous linear change of variables (in fact these are the only automorphisms of $\mathbf{P}^2_k$, as we will see later). A *conic* is a curve of degree 2. It can either be irreducible, or a union of two distinct (intersecting) lines. Two conics are isomorphic via an automorphism of $\mathbf{P}^2_k$ if either both are reducible, or both are irreducible. The classification of curves of degree ≥ 3, however, gets progressively more complicated as the degree increases. For example, there are infinitely many isomorphism classes of irreducible non-singular curves of any degree $d \geq 3$.

From the Jacobian criterion, a point $P = (a : b : c)$ on a curve $C = \{F = 0\}$ in $\mathbf{P}^2_k$ is a non-singular point precisely when one of

$$\frac{\partial F}{\partial X}(a,b,c), \frac{\partial F}{\partial Y}(a,b,c), \frac{\partial F}{\partial Z}(a,b,c)$$

is non-zero. In this case the projective tangent variety to C at P is a line in $\mathbf{P}^2_k$, with equation

$$X\frac{\partial F}{\partial X}(a,b,c) + Y\frac{\partial F}{\partial Y}(a,b,c) + Z\frac{\partial F}{\partial Z}(a,b,c) = 0.$$

We denote the projective tangent line to C at P by $\mathbf{T}_{P,C}$. At a singular point, the projective tangent space is of course the whole plane.

For any point P on a curve C, we have a surjection $\mathcal{O}_{P,\mathbf{P}^2} \twoheadrightarrow \mathcal{O}_{P,C}$, whose kernel (which we denote $I_{P,C}$) is a radical ideal which is purely of height 1, and is hence principal, say generated by f. If $\mathbf{m}$ is the maximal ideal of $\mathcal{O}_{P,\mathbf{P}^2}$, then there is a unique $r > 0$ such that $f \in \mathbf{m}^r - \mathbf{m}^{r+1}$ (this is because $\cap_{r>0}\mathbf{m}^r = 0$); the integer r does not depend on the choice of f. This integer r is called the *multiplicity* of P on C, and is denoted by $m_P(C)$. This is computed in practice as follows: after a linear change of variables, we may assume $P = (0 : 0 : 1)$. Then we can uniquely write

$$F(x,y,1) = F_r(x,y) + F_{r+1}(x,y) + \cdots$$

where F_s is homogeneous of degree s, and $F_r \neq 0$. Now $P \in U_2 = D_+(Z) \subset \mathbf{P}^2_k$, and $\mathcal{O}_{P,\mathbf{P}^2} = k[x,y]_{(x,y)}$, where we have affine coordinates $x = X/Z, y = Y/Z$ on U_2. Now $\mathbf{m} = (x,y)$, so $\mathbf{m}^r/\mathbf{m}^{r+1}$ is identified with the space of homogeneous polynomials in x, y of degree r. Also, the coordinate ring of the affine curve $C \cap U_2$ is

$$A(C \cap U_2) = k[x,y]/ < F(x,y,1) > .$$

Hence we may take $f = F(x,y,1)$. Thus $F(x,y,1) \in \mathbf{m}^r - \mathbf{m}^{r+1}$, where $F_r(x,y)$ is the first non-zero homogeneous term in the expansion of F. If $\overline{\mathbf{m}}$ is the maximal ideal of $\mathcal{O}_{P,C}$, then

$$\overline{\mathbf{m}}/\overline{\mathbf{m}}^2 \cong \mathbf{m}/(\mathbf{m}^2 + f\mathcal{O}_{P,\mathbf{P}^2}).$$

Hence P is a non-singular point precisely when $m_P(C) = 1$. The terms $F_i(x, y)$ are the terms in the Taylor expansion of $F(x, y, 1)$ at P. So the multiplicity is the order (number of derivatives) of the first non-vanishing term of the Taylor expansion of $F(x, y, 1)$ at P. The multiplicity is a first measure of how singular a curve is at a point; for example, proofs of the theorem of *resolution of singularities*[2] for curves usually work by induction on the maximum multiplicity of the singular points.

A basic fact about projective plane curves is that any two of them must intersect, unlike the case of affine curves in $\mathbf{A}^2_k$; thus there are no parallel lines in $\mathbf{P}^2_k$. In fact a stronger result is true: if C and D are non-singular curves of degrees m and n respectively which meet transversally (*i.e.*, with distinct tangent lines at each point of intersection), then Bezout's theorem states that $C \cap D$ consists of exactly mn points. More generally, Bezout's theorem states that if C and D have no common irreducible components (equivalently $C \cap D$ is finite), then 'counted properly' there are again mn points of intersection. This is made precise via the notion of intersection multiplicity, which we now introduce.

Suppose C, D are curves with no common irreducible component passing through $P \in C \cap D$. Let $f, g \in \mathcal{O}_{P,\mathbf{P}^2}$ generate the ideals of C, D respectively. Since $\mathcal{O}_{P,\mathbf{P}^2}$ is a UFD, the elements f, g are relatively prime, and form a regular sequence in $\mathcal{O}_{P,\mathbf{P}^2}$, such that $\sqrt{< f, g >}$ is the maximal ideal of $\mathcal{O}_{P,\mathbf{P}^2}$. Define

$$I(P; C, D) = \dim_k \mathcal{O}_{P,\mathbf{P}^2} / < f, g >$$

to be the *intersection multiplicity of* C *and* D *at* P. Note that

(i) $I(P; C, D) > 0$, and

(ii) $I(P; C, D) = 1 \Leftrightarrow\ < f, g >= \mathbf{m}$, the maximal ideal of $\mathcal{O}_{P,\mathbf{P}^2}$.

But $< f, g >= \mathbf{m} \ \Leftrightarrow f, g$ are linearly independent modulo $\mathbf{m}^2$, *i.e.*, $\mathcal{O}_{P,C}$ and $\mathcal{O}_{P,D}$ are regular, and C, D have distinct tangents at P. Thus a transversal intersection of non-singular curves has intersection multiplicity 1.

Theorem 5 (Bezout) *Let C, D be projective plane curves such that $C \cap D$ is a finite set (i.e., C, D have no common irreducible components). Then*

$$\sum_{P \in C \cap D} I(P; C, D) = (\deg C)(\deg D).$$

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[2]This states that for any curve C, there is a morphism $f : \widetilde{C} \to C$ from a non-singular projective curve $\widetilde{C}$, such that (i) f is surjective with finite fibres, and (ii) if $U \subset C$ is the open set of non-singular points, then $f^{-1}(U) \to U$ is an isomorphism.

Proof: Let $I(C) =< F >$, $I(D) =< G >$ where F, G are homogeneous polynomials of degrees $m = \deg C$ and $n = \deg D$ respectively. If $S = k[X, Y, Z]$, then there is an exact sequence of graded S-modules

$$0 \to S(-mn) \xrightarrow{\alpha} S(-m) \oplus S(-n) \xrightarrow{\beta} S \to S/ < F, G > \to 0.$$

Here $\beta(A, B) = AF + BG$, and $\alpha(A) = (-AG, AF)$. The exactness is equivalent to the statement that F, G form a regular sequence in S, which holds because S is a UFD, and F, G have no common prime factor (since C, D have no common irredcuible component). Since the Hilbert-Samuel function of S is known, we can use the exact sequence to compute the Hilbert-Samuel function of the graded ring $R = S/ < F, G >$; carrying out the computation, we find that

$$H(R, s) = mn \; \forall \; s >> 0.$$

After a linear change of variable, we may assume without loss of generality that $C \cap D \cap V(Z) = \phi$ (*i.e.*, no point of intersection of C, D lies on the line $Z = 0$). Then the image z of Z in R satisfies $\sqrt{zR} = R_+$ by the homogeneous Nullstellensatz, since $V(Z) \cap C \cap D = \phi$. We claim that multiplication by z yields an isomorphism $R_s \to R_{s+1}$ for all $s >> 0$. One way to prove this is as follows: since R_s, R_{s+1} are both k-vector spaces of the same dimension mn, it suffices to show multiplication by z is surjective; but R/zR is an Artinian graded ring, so is 0 in large enough degrees. This implies that the localized ring $R[1/z] = R_{(z)}[z, z^{-1}]$ is a Laurent polynomial ring over the Artinian ring $R_{(z)}$, with $\dim_k R_{(z)} = mn$ (in fact $R_s \to R_{(z)} z^s$ is an isomorphism of k-vector spaces, for all large s).

Now

$$\begin{aligned} R[1/z] = & \; S[1/Z]/ < F, G > \\ = & \; S_{(Z)}[Z, Z^{-1}]/ < F(X/Z, Y/Z, 1), G(X/Z, Y/Z, 1) > \\ = & \; (k[x, y]/ < f, g >)[z, z^{-1}], \end{aligned}$$

where $x = X/Z, y = Y/Z$ give the affine coordinates on $D_+(Z) \cong \mathbf{A}^2_k$, and $f = F(x, y, 1)$, $g = G(x, y, 1)$. Thus $R_{(z)} = k[x, y]/ < f, g >$.

If $P_1, \ldots, P_r \in \mathbf{A}^2_k$ are the points of intersection of C and D, with maximal ideals $\mathbf{m}_1, \ldots, \mathbf{m}_r \subset k[x, y]$, then $< f, g >= J_1 \cap J_2 \cap \cdots \cap J_r$, where

$$J_\nu = k[x, y] \cap (< f, g > k[x, y]_{\mathbf{m}_\nu}$$

is the primary component of $< f, g >$ for the prime ideal $\mathbf{m}_\nu$. Since the ideals J_ν are pairwise co-maximal, the Chinese remainder theorem gives

$$R_{(z)} = (k[x,y]/J_1) \times \cdots \times (k[x,y]/J_r). \tag{3}$$

Also, from the definition of intersection multiplicity,

$$\dim_k k[x,y]/J_\nu = I(P_\nu; C, D),$$

since $k[x,y]/J_\nu$ is local, and so is unchanged upon localization at $\mathbf{m}$. Hence Bezout's theorem follows from (3) and the formula $\dim_k R_{(z)} = mn$. □

The conclusion of Bezout's theorem may be restated in the language of zero-cycles. We define a *zero-cycle* on a variety X to be an element of the free abelian group on the points of X; we write a zero-cycle as a formal linear combination of points $\sum_i n_i P_i$, where P_i are points. The *degree* of a zero-cycle $\delta = \sum_i n_i P_i$ is defined to be the integer $\sum_i n_i$, and is denoted by $\deg \delta$. We define the *intersection cycle* of two curves C, D with no common component by

$$(C \cdot D) = \sum_{P \in C \cap D} I(P; C, D)P.$$

Then Bezout's theorem states that

$$\deg(C \cdot D) = (\deg C)(\deg D).$$

Later we will interpret this equation in terms of the multiplication in the *Chow ring* of algebraic cycles on $\mathbf{P}^2_k$.

In the proof of Bezout's theorem, we saw that multiplication by z on R is an isomorphism in sufficiently large degrees. In fact $S = k[X,Y,Z]$ is Cohen-Macaulay and graded, and F, G, Z are homogeneous such that $\sqrt{< F, G, Z >} = S_+$, *i.e.*, they form a homogeneous system of parameters; hence F, G, Z form a regular sequence on S, *i.e.*, z is a non zero-divisor on R. Another way to see this is as follows (see [F], page 113). Since $F(X,Y,0)$ and $G(X,Y,0)$ are homogeneous, each factorizes into homogeneous linear factors, which correspond to points of $C \cap \{Z = 0\}$ and $D \cap \{Z = 0\}$, respectively; in particular, $F(X,Y,0)$ and $G(X,Y,0)$ are relatively prime in $k[X,Y]$. Now if $ZH = AF + BY$, then $A(X,Y,0)F(X,Y,0) = -B(X,Y,0)G(X,Y,0)$; hence $A(X,Y,0) = E(X,Y)G(X,Y,0)$ and $B(X,Y,0) = -E(X,Y)F(X,Y,0)$, so that $A - E(X,Y)G = ZA'$ and $B + E(X,Y)F = ZB'$. Then

$$ZH = AF + BG = (A - E(X,Y)G)F + (B + E(X,Y)F)G = Z(A'F + B'G),$$

so that $H = A'F + B'G$. Hence the image h of H in R is annihilated by z $\Leftrightarrow h = 0$ in R, *i.e.*, z is a non zero-divisor in R.

We use this to prove another interesting result on plane curves, Max Noether's theorem. Loosely speaking, it states that if $C = \{F = 0\}$, $D = \{G = 0\}$ are curves with no common component, and $C_0 = \{H = 0\}$ is a curve passing through all points of $C \cap D$ (including the 'infinitely near' ones'), then $H = AF + BG$ for some homogeneous polynomials A, B. If C, D intersect transversally at non-singular points, the condition is just that $C \cap D \subset C_0$. If some intersections have intersection multiplicity > 1, a more subtle condition is needed at P. One way of stating Noether's theorem is as follows; the hypothesis (4) is clearly necessary for the conclusion of the theorem to hold.

Theorem 6 (Noether's $AF + BG$ theorem) *Let C, D be plane curves with no common component, and $I(C) =< F >$, $I(D) =< G >$. Suppose C_0 is a plane curve, with $I(C_0) =< H >$, such that for each point $P \in C \cap D$, we have*

$$I_{P,C_0} \subset I_{P,C} + I_{P,D} \subset \mathcal{O}_{P,\mathbf{P}^2}. \tag{4}$$

Then $H = AF + BG$ for some homogeneous polynomials A, B.

Proof: Assume without loss of generality that $C \cap D \cap \{Z = 0\} = \phi$. Then as seen above, $z \in R = S/ < F, G >$ is a non zero-divisor. Let f, g, h respectively denote the images of $F(x, y, 1), G(x, y, 1), H(x, y, 1)$ in $S_{(Z)} = k[x, y]$. Since z is a non-zero-divisor on R, to check if H maps to 0 in R, it is sufficient to check that h maps to 0 in $k[x, y]/ < f, g >$. From (3), with the notation as there, we see that $h \in < f, g > \Leftrightarrow h \in J_\nu$ for each $\nu = 1, 2, \ldots, r$. But

$$(h \in J_\nu = k[x, y] \cap (< f, g > k[x, y]_{\mathbf{m}_\nu}) \Leftrightarrow (h \in < f, g > k[x, y]_{\mathbf{m}_\nu}),$$

and the last condition is just (4). □

Corollary 2 *Suppose each point of intersection of C and D is non-singular on D. Then $H = AF + BG \Leftrightarrow I(P; C_0, D) \geq I(P; C, D)$ for all $P \in C \cap D$.*

Proof: $\mathcal{O}_{P,C}$ is a regular local ring of dimension 1, *i.e.*, a discrete valuation ring (d.v.r.). Hence for any curve C_0 through P, if f_0 is a generator for the ideal of C' in $\mathcal{O}_{P,\mathbf{P}^2}$, and $\overline{f_0}$ is its image in $\mathcal{O}_{P,C}$, we see that $I(P; C, C')$ is the valuation of $\overline{f_0}$. In a d.v.r., an ideal $I =< a >$ is contained in $J =< b >$ precisely when the valuation of b is $\leq$ that of a. Hence the

Noether conditions (4) are equivalent to an inequality between valuations, *i.e.*, between intersection multiplicities. $\square$

Some applications (taken from [F]) are the following.

Example 3.1: A non-singular point P on a curve C is called an *inflection point* (or just *flex*), if $I(P; \mathbf{T}_{P,C}, C) \geq 3$. Here we assume C is not a line, so that $\mathbf{T}_{P,C}$ is not a component of C. Let $I(C) =< F > \subset S = k[X, Y, Z]$. One can show that if k has characteristic 0, then P is a flex $\Leftrightarrow H(P) = 0$, where H is the determinant

$$H = \begin{vmatrix} \frac{\partial^2 F}{\partial X^2} & \frac{\partial^2 F}{\partial X \partial Y} & \frac{\partial^2 F}{\partial X \partial Z} \\ \frac{\partial^2 F}{\partial Y \partial X} & \frac{\partial^2 F}{\partial Y^2} & \frac{\partial^2 F}{\partial Y \partial Z} \\ \frac{\partial^2 F}{\partial Z \partial X} & \frac{\partial^2 F}{\partial Z \partial Y} & \frac{\partial^2 F}{\partial Z^2} \end{vmatrix}$$

is the *Hessian* of F. This follows from a local calculation in affine coordinates. Clearly $\deg H = (d-2)^3$, where $d = \deg C$. Now Bezout's theorem implies that C has $d(d-2)^3$ flexes, 'counted with multiplicity'. See [F], Ex. 5.23.

Example 3.2: Let C, D be cubics with $(C \cdot C') = P_1 + \cdots + P_9$ (where some P_i may be repeated). Suppose D is a conic, such that $(C \cdot D) = P_1 + \cdots + P_6$, where $P_i \in C$ are distinct and non-singular. Then P_7, P_8, P_9 lie on a line. Indeed, $I(P_i; C', C) \geq I(P_i; D, C)$ for $1 \leq i \leq 6$, so if $I(C) =< F >$, $I(D) =< G >$ and $I(C') =< H >$, then $H = AF + BG$, where we must have $\deg B = 1$. Now $V(B)$ is the desired line.

We deduce *Pascal's theorem*: if a hexagon is inscribed in a conic, the opposite sides intersect in collinear points. Take C to be three of the sides, C' the other three sides (so that the intersections of opposite sides are interpreted as some of the intersections of C and C'). Take D to be the conic.

A particular case is *Pappus theorem*: if L, M are two lines, $P_1, P_2, P_3 \in L - M$ and $Q_1, Q_2, Q_3 \in M - L$ distinct points, and L_{ij} the line though P_i and Q_j, then the points $L_{12} \cap L_{21}$, $L_{13} \cap L_{31} and L_{23} \cap L_{32}$ are collinear. Here we interpret $L \cup M$ as a conic, and the union of the L_{ij} as a hexagon.

Example 3.3: We first obtain one more corollary of the $AF + BG$ theorem. Let C be an irreducible non-singular cubic, and D another cubic, with $(C \cdot D) = P_1 + \cdots + P_9$. Suppose C' is another cubic such that $(C \cdot C') = P_1 + \cdots + P_8 + P$. Then $P = P_9$.

Indeed, suppose L is a line through P_9 which does not contain P. Let $(L \cdot C) = P_9 + Q + R$. Then $((L \cup C') \cdot C) = (D \cdot C) + P + Q + R$. Hence if $I(L \cup C') = < H >$, $I(C) = < F >$, $I < D > = G$, then $H = AF + BG$. Since $\deg H = 4$, A and B are homogeneous linear. If M is the line $\{B = 0\}$, then $(M \cdot C) = P + Q + R$. Hence $L = M$ (both pass though Q, R, if $Q \neq R$, and both are tangent to C at the same point $Q = R$, otherwise). This implies $P = P_9$.

This can be used to put a group law on the points of a non-singular cubic curve C. Fix a point $O \in C$, which will be the additive identity. For any pair $P, Q \in C$, let $\varphi(P, Q)$ be the point R such that $P + Q + R$ is the intersection cycle of C with the line through P and Q. If $P = Q$, we take the line through P and Q to mean the tangent line.

Now define $P * Q = \varphi(O, \varphi(P, Q))$. It is fairly easy to check that $*$ is a commutative binary operation, such that O is a 2-sided identity, and $*$ has a 2-sided inverse. The tricky point is to check that $*$ is associative. This will use the above corollary to the $AF + BG$ theorem.

Suppose $P, Q, R \in C$. Let L_1, M_1, L_2 be lines such that $(L_1 \cdot C) = P + Q + S'$, $(M_1 \cdot C) = O + S' + S$, and $(L_2 \cdot C) = S + R + T'$. Then the line through T' and O determines $(P * Q) * R$.

Let M_2, L_3, M_3 be lines such that $(M_2 \cdot C) = Q + R + U'$, $(L_3 \cdot C) = O + U + U'$ and $(M_3 \cdot C) = P + U + T''$. Then the line through T'' and O determines $P * (Q * R)$. So we need to prove $T' = T''$. This follows by taking $L_1 \cup L_2 \cup L_3 = D$, and $M_1 \cup M_2 \cup M_3 = C'$, in the above corollary of the $AF + BG$ theorem.

We next introduce the *dual curve* of a projective plane curve. To do this, we first note that *the set of hyperplanes in* $\mathbf{P}^n_k$ *is itself a projective space of dimension* n. The identification is as follows: if $H \subset \mathbf{P}^n_k$ is a hyperplane, the ideal $I(H) = < F >$ for some linear homogeneous polynomial F, which is unique upto a constant multiple.

Let V denote the vector space k^{n+1}. Then $\mathbf{P}^n_k = V - \{0\}/\sim$, where $v \sim w \Leftrightarrow v = \lambda w$ for some $\lambda \in k^*$. Of course this construction makes sense for any finite dimensional k-vector space V; the corresponding quotient space is denoted by $\mathbf{P}(V)$, the projective space associated to V. If $\dim_k V = n+1$, and we choose a basis of V, then we obtain a bijection of $\mathbf{P}(V)$ with $\mathbf{P}^n_k$, which we can use to regard $\mathbf{P}(V)$ as a variety; if we choose a different basis, the variety structure on $\mathbf{P}(V)$ (*i.e.*, its Zariski topology, the rings of regular functions on open sets, and its local rings) remains unchanged.

The homogeneous coordinate ring of $\mathbf{P}(V)$ is now naturally identified with the symmetric algebra $S(V^*)$, where V^* is the dual vector space to V. This is clear, if we think of the homogeneous coordinate ring as the affine coordinate ring of the cone, *i.e.*, of the original vector space V; now the coordinate functions are linear functions $V \to k$, that is, are linear functionals on V. Thus polynomials in the coordinate functions are naturally elements of the symmetric algebra of V^*.

In particular, a hyperplane in $\mathbf{P}(V)$ determines a non-zero linear homogeneous polynomial, well defined upto scalar multiples; this can be considered as a well defined element of the projective space $\mathbf{P}(V^*)$. Conversely a point of $\mathbf{P}(V^*)$ corresponds to a linear homogeneous polynomial, whose zero-set in $\mathbf{P}(V)$ is a hyperplane. We call $\mathbf{P}(V^*)$ the *dual projective space* to $\mathbf{P}(V)$. If $\mathbf{P}$ is a projective space over some vector space, we denote its dual projective space by $\check{\mathbf{P}}$.

If we fix a basis of V, identifying it with k^{n+1}, the dual basis for V^* identifies $\mathbf{P}(V^*)$ also with $\mathbf{P}^n_k$. Now if H is a hyperplane, with equation

$$a_0X_0 + \cdots + a_nX_n = 0,$$

then the corresponding point of $\mathbf{P}(V^*) = \check{\mathbf{P}}(V)$ is $(a_0 : a_1 : \cdots : a_n)$.

The relationship between $\mathbf{P}(V)$ and $\mathbf{P}(V^*)$ is encapsulated in the *incidence variety*, which is the projective variety

$$I(V) = \{(P, [H]) \in \mathbf{P}(V) \times \mathbf{P}(V^*) \mid P \in H\}.$$

Let $p : I(V) \to \mathbf{P}(V)$ and $q : I(V) \to \mathbf{P}(V^*)$ denote the two projections. If $\mathbf{P} \subset \mathbf{P}(V)$ is any projective linear subspace, then

$$\mathbf{P}^* = q(p^{-1}(\mathbf{P})) = \{[H] \in \mathbf{P}(V^*) \mid \mathbf{P} \subset H\}$$

is called the *dual linear subspace* to $\mathbf{P}$. Then $\dim \mathbf{P} + \dim \mathbf{P}^* = \dim \mathbf{P}(V) - 1$; thus the dual of a point is a hyperplane, the dual of a line is of codimension 2, etc.

Note that the set of hypersurfaces in $\mathbf{P}^n_k$ of a fixed degree d are in bijection with the hyperplanes in $\mathbf{P}^N_k$, where $N = \binom{n+d}{d} - 1$, via the d-tuple embedding; hence hypersurfaces of degree d in $\mathbf{P}^n_k$ are parametrized by the points of the dual projective space $\check{\mathbf{P}}^N_k$ (in more invariant terms, the parameter space is $\mathbf{P}(S^d(V^*))$, where $S^d(V^*)$ is the dth symmetric power of $V^* = k^{n+1}$).

Returning to our plane curve $C \subset \mathbf{P}^2$, suppose $P \in C$ is a non-singular point. Then the projective tangent line $\mathbf{T}_{P,C}$ to C at P gives a point of the dual projective plane $[H] \in \check{\mathbf{P}}^2_k$. If $F = 0$ defines C, then the tangent at $(a : b : c)$ is

$$\frac{\partial F}{\partial X}(a,b,c)X + \frac{\partial F}{\partial Y}(a,b,c)Y + \frac{\partial F}{\partial Z}(a,b,c)Z = 0,$$

so that

$$[H] = (\frac{\partial F}{\partial X}(a,b,c) : \frac{\partial F}{\partial Y}(a,b,c) : \frac{\partial F}{\partial Z}(a,b,c)).$$

Thus the association

$$D(C) : P \mapsto [\mathbf{T}_{P,C}] \in \check{\mathbf{P}}^2_k$$

is a rational map, given by

$$D(C)(P) = (\frac{\partial F}{\partial X}(P) : \frac{\partial F}{\partial Y}(P) : \frac{\partial F}{\partial Z}(P)).$$

In particular, if C is non-singular, then the 3 partial derivative polynomials have no common zeroes in $\mathbf{P}^2_k$, so that they define a morphism $D(C) : \mathbf{P}^2_k \to \check{\mathbf{P}}^2_k$, the *dual map* associated to C. The image of C under this morphism is another projective plane curve C^*, the *dual curve* to C. The restricted map $C \to C^*$ is also called the dual map of C. Even if C is singular, let $U \subset C$ be the open set of smooth points; then define C^* to be the Zariski closure of $D(C)(U)$. The terminology 'dual curve' is justified, because $D(C)(U)$ is a point $\Leftrightarrow C$ is a line; so if $\deg C \geq 2$, then C^* is also a curve in $\check{\mathbf{P}}^2_k$.

We claim that if C is irreducible and $\deg C \geq 2$, and if k has characteristic 0, then the rational map $D(C) : C \to C^*$ is a birational isomorphism (*i.e.*, has a rational inverse). In fact, one can prove that $D(C^*) \circ D(C)$ is the identity rational map on C. This imples that for all but a finite set of non-singular points of C, the tangent line to C at P is not tangent to C at any other point of intersection with C, *i.e.*, C has only a finite number of *bitangents*. Next, the *degree* of C^* is the number of points of intersection of C^* with a general line in $\check{\mathbf{P}}^2_k$, *i.e.*, the number of intersections of C with a general curve of degree $d-1$ of the form

$$a\frac{\partial F}{\partial X} + b\frac{\partial F}{\partial Y} + c\frac{\partial F}{\partial Z} = 0,$$

where $a, b, c \in k$. Hence by Bezout's theorem, we see that $\deg C^* = d(d-1)$.

In fact, if C is non-singular and k has characteristic 0, then in general (*i.e.*, for C in a nonempty Zariski open set in the projective space of curves of degree d), the only singularities of C^* are those where the local equation for C is of the form

$$xy + (\text{higher order terms}) = 0$$

(this is called a *node* of C) or of the form

$$x^2 + y^3 + (\text{higher order terms}) = 0$$

(this is called an ordinary *cusp* of C). Note that in each cae the multiplicity of the singular point (the origin, in the system of coordinates) is 2. Then the number of nodes of C^* is the number of bitangent lines (lines tangent to C at 2 points), C has no tritangent lines (lines tangent at 3 points), and the number of cusps of C^* equals the number of flexes of C. The above facts about C^* are to be found in [F], and in [H], IV, Ex. 2.3.

One final remark about the dual curve is the following: let $P_1, \ldots, P_r$ denote the singular points of an irreducible curve C, and for $1 \leq i \leq r$, let $L_i = P_i^*$ be the line in $\check{\mathbf{P}}^2_k$ parametrizing lines in $\mathbf{P}^2_k$ passing through P_i. Then points of $U = \check{\mathbf{P}}^2_k - (C^* \cup L_1 \cup \cdots \cup L_r)$ correspond to lines $H \subset \mathbf{P}^2_k$ such that $H \cap C$ is not tangent to C at any of its points of intersection, *i.e.*, $H \cap C$ consists of $d = \deg C$ distinct points. Note that U is a non-emty Zariski open set of lines in $\mathbf{P}^2_k$; thus we see that the 'general' line in $\mathbf{P}^2_k$ intersects C transversally in d points. This assertion is the first case of *Bertini's theorem*, which we discuss below.

We will first generalize the discussion of the duals to the case of irreducible varieties of arbitrary dimension. If X is an irreducible projective variety of dimension d, and $P \in X$ a non-singular point, then the projective tangent space $\mathbf{T}_{P,X}$ is a linear subvariety of $\mathbf{P}^n_k$ of dimension d. So we can consider 2 types of duals, when X is non-singular:

(i) the variety of hyperplanes H in $\mathbf{P}^n_k$ such that $\mathbf{T}_{P,X} \subset H$ for some $P \in X$ — this is a variety in $\check{\mathbf{P}}^n_k$;

(ii) the variety of tangent spaces $\mathbf{T}_{P,X}$, considered as a subvariety of a parameter variety for all linear subvarieties $L \subset \mathbf{P}^n_k$ with $\dim L = d$.

In the second case, we need to first put a natural structure of a variety on the collection of all linear subvarieties of dimension d in $\mathbf{P}^n_k$, *i.e.*, on the collection of all $d+1$-dimensional linear subspaces $W \subset V = k^{n+1}$.

In the first type of dual, we see that if $\mathbf{T}_X \subset \mathbf{P}^n_k \times \mathbf{P}^n_k$ is the projective tangent variety, we have

$$J_X = \{(P, [H]) \in X \times \check{\mathbf{P}}^n_k \mid \mathbf{T}_{P,X} \subset H\}$$

is a subvariety, whose image $X^* \subset \check{\mathbf{P}}^n_k$ under the second projection is called the dual variety to X in $\check{\mathbf{P}}^n_k$. To see that J_X is indeed a subvariety, note that if

$$p_1 : \mathbf{P}^n_k \times \mathbf{P}^n_k \times \check{\mathbf{P}}^n_k \to \mathbf{P}^n_k,$$
$$p_2 : \mathbf{P}^n_k \times \mathbf{P}^n_k \times \check{\mathbf{P}}^n_k \to \mathbf{P}^n_k,$$
$$p_3 : \mathbf{P}^n_k \times \mathbf{P}^n_k \times \check{\mathbf{P}}^n_k \to \check{\mathbf{P}}^n_k$$

are the projections onto the 3 factors, then

$$J_X = p_1 \times p_2 \left((p_1 \times p_2)^{-1}(\mathbf{T}_X) \cap (p_2 \times p_3)^{-1}(I) \right),$$

where $I \subset \mathbf{P}^n_k \times \check{\mathbf{P}}^n_k$ is the incidence variety. Now J_X is Zariski closed because $p_1 \times p_2$ is a closed map (since $\check{\mathbf{P}}^n_k$ is complete).

One can of course also choose homogeneous generators $F_1, \ldots, F_s$ for $I(X) \subset S = k[X_0, \ldots, X_n]$, and using the Jacobian matrix of the F_j, write down equations defining J, in order to prove J is a closed subvariety.

The fibre of the first projection $p : J \to X$ over a point P is just $\mathbf{T}^*_{P,X}$, the dual linear space to $\mathbf{T}_{P,X}$; in particular, p is surjective, and all its fibres have dimension $n - d - 1$ (since for any linear subvariety $L \subset \mathbf{P}^n_k$, we have $\dim L + \dim L^* = n - 1$). Hence

$$\dim J_X = \dim X + (n - d - 1) = n - 1.$$

Now consider the second projection $q : J_X \to \check{\mathbf{P}}^n_k$, whose image is X^*. Then $\dim X^* \leq \dim J = n - 1$. In particular, $X^* \subsetneq \check{\mathbf{P}}^n_k$.

Again as for plane curves, there is a 'double duality' theorem, that if k has characteristic 0, then $(X^*)^* = X$. In fact one can show that the transpose of J_X (obtained by interchanging the factors in $X \times X^*$) is J_{X^*}, the analogue of J_X associated to the projective variety X^*. As with curves, in order to do this, we have to extend the notions of J_X and the dual to irreducible, possibly singular varieties, by taking the Zariski closure of the variety defined as before over the non-singular points. A proof that $(X^*)^* = X$ over $\mathbf{C}$ can be found in [La]. A local analysis of singularities and the dual, and related topics, can be found in [Lo]. Another source for duals from an algebraic point of view, with an analysis of the situation in characteristic $p > 0$ as well, is [SGA 7 II].

Note that if X is a non-singular hypersurface, then $\mathbf{T}^*_{P,X}$ is a point, so that the first projection $J_X \to X$ is an isomorphism. Hence J_X is the graph of a morphism $X \to X^*$, which we call the *dual morphism* of the hypersurface X. As for curves, one can show it is given by the linear system of partial derivatives of the defining polynomial of X, which extends to a morphism on all of $\mathbf{P}^n_k$. This implies it has finite fibres (see Example 2.7). Since the fibres of $J_X \to X^*$ over non-singular points are just projective spaces (since they are isomorphic to fibres of $J_{X^*} \to X^*$), these fibres must be points, *i.e.*, $X \to X^*$ is an isomorphism over the open set of non-singular points of X^*.

Theorem 7 (Bertini's theorem) *Let $X \subset \mathbf{P}^n_k$ be an irreducible projective variety. Then for all hyperplanes $[H] \notin X^* \subset \check{\mathbf{P}}^n_k$, the variety $X \cap H$ is non-singular of dimension $d-1$. In particular, the 'general' hyperplane section of X is non-singular.*

Proof: Let $P \in X$; since X is non-singular, $\mathcal{O}_{P,X}$ is a regular local ring of dimension d. It suffices to show that if H is a hyperplane such that $\mathbf{T}_{P,X} \not\subset H$, then $\mathcal{O}_{P,X\cap H}$ is a regular local ring of dimension $d-1$.

Let $F \in S = k[X_0, \ldots, X_n]$ be a linear homogeneous polynomial defining H. We may assume after a linear change of coordinates that $P = (1:0:0:\cdots:0)$; now $x_i = X_i/X_0$, $1 \leq i \leq n$ are affine coordinates on $U_0 = D_+(X_0)$, and P is the origin. Let $f = F(1, x_1, \ldots, x_n)$, and let $I \subset k[x_1, \ldots, x_n]$ be the ideal of $X \cap U_0$. Then f is still linear homogeneous, since H passes through the origin P. If $\mathbf{m} =< x_1, \ldots, x_n >$ is the maximal ideal of P, then $I \subset \mathbf{m}$; if $\overline{\mathbf{m}}$ is the maximal ideal of P in $A(X \cap U_0)$, then

$$\overline{\mathbf{m}}/\overline{\mathbf{m}}^2 \cong \mathbf{m}/(I + \mathbf{m}^2.)$$

Let $W = \ker(S_1 \to \overline{\mathbf{m}}/\overline{\mathbf{m}}^2$ be the vector space of linear homogeneous polynomials 'vanishing to order ≥ 2 along X near P'. Then $T_{P,(X\cap U_0)} = U_0 \cap \mathbf{T}_{P,X}$ is the linear subvariety of $U_0 = \mathbf{A}^n_k$ defined by

$$\{Q \in \mathbf{A}^n_k \mid G(Q) = 0 \; \forall \, G \in W\}.$$

Since this linear subvariety is *not* contained in H, we must have $f \notin W$. Hence the image $\overline{f}$ of $f \in \overline{\mathbf{m}}$ does not lie in $\overline{\mathbf{m}}^2$, *i.e.*, $\overline{f} \in \overline{\mathbf{m}}$ is part of a regular system of parameters. In particular, the ideal $\overline{f}\mathcal{O}_{P,X}$ is prime, and

$$\mathcal{O}_{P,X\cap H} = \mathcal{O}_{P,X}/\sqrt{\overline{f}\mathcal{O}_{P,X}} = \mathcal{O}_{P,X}/\overline{f}\mathcal{O}_{P,X}$$

is a regular local ring of dimension $d-1$. □

The above analysis shows that if $\mathbf{T}_{P,X} \subset H$, then $\overline{f} \in \overline{\mathbf{m}}^2$, so the quotient local ring is *not* regular, unless $X \subset H$ (since $\mathcal{O}_{P,X}$ is regular, hence an integral domain, the quotient local ring is not regular unless $\overline{f} = 0$).

The second version of the dual variety involves the *Grassmann variety* parametrizing d-dimensional linear subvarieties of $\mathbf{P}^n_k$, or equivalently, $d+1$-dimensional vector subspaces of k^{n+1}. We construct the Grassmannian in a slighly more intrinsic manner, as follows.

Let V be a vector space of dimension m. We construct a projective variety $\mathbf{G}(r,m)$ parametrizing r-dimensional subspace of V. Let

$$Z = \{v_1 \wedge \cdots \wedge v_r \mid v_1, \ldots, v_r \text{ are linearly independent}\} \subset \wedge^r V - \{0\}$$

be the set of non-zero decomposable tensors in $\wedge^r V$. The set of r-dimensional subspaces of V is naturally in bijection with the image of Z in $\mathbf{P}(\wedge^r V)$; the image of a subspace W is defined to be the class of $v_1 \wedge \cdots \wedge v_r$, where $v_1, \ldots, v_r$ is a basis for W. Any other basis is of the form $Av_1, \ldots, Av_r$ with $A \in \mathrm{GL}\,(W)$, the general linear group of linear automorphisms of W, and $Av_1 \wedge \cdots \wedge Av_r = \det(A) v_1 \wedge \cdots \wedge v_r$, from the definition of determinants; hence the map from the collection of subspaces to $\mathbf{P}(\wedge^r V)$ is well defined. Clearly the map is surjective onto the image of Z. To show that it is also injective, we need to show that if

$$v_1 \wedge \cdots \wedge v_r = v'_1 \wedge \cdots \wedge v'_r,$$

then $\{v_1, \ldots, v_r\}$ and $\{v'_1, \ldots, v'_r\}$ span the same r-dimensional subspace of V. But one verifies easily that if $v_1, \ldots, v_r$ are linearly independent, then the kernel of the linear map

$$\psi : V \to \wedge^{r+1} V, \ \ \psi(v) = v \wedge (v_1 \wedge \cdots \wedge v_r)$$

is precisely the linear span of $\{v_1, \ldots, v_r\}$ (to verify the claim, note that $\{v, v_1, \ldots, v_r\}$ is linearly independent precisely when $\psi(v) \neq 0$).

So it remains to show that $Z/\sim$, the image of Z in $\mathbf{P}(\wedge^r V)$, is a projective variety. Note that Z is conical, *i.e.*, if $w \in Z$, then $\lambda w \in Z$ for all $\lambda \in k^*$. Let

$$(Z/\sim) = Y \subset \mathbf{P}(\wedge^r V).$$

Now coordinates on $\mathbf{P}(\wedge^r V) \cong \mathbf{P}_k^{\binom{m}{r}-1}$ are given by elements of $(\wedge^r V)^* = \wedge^r V^*$. Choose a basis $e_1, \ldots, e_m$ of V, and dual basis $\ell_1, \ldots, \ell_m$ of V^*. Then $\ell_{i_1} \wedge \cdots \wedge \ell_{i_r}$, with $i_1 < \cdots < i_r$, form a basis for $\wedge^r V^*$. The set

$$U_{i_1, \cdots, i_r} = \{[\omega] \in \mathbf{P}(\wedge^r V) \mid \ell_{i_1} \wedge \cdots \wedge \ell_{i_r}(\omega) \neq 0\}$$

is the an affine space $\cong \mathbf{A}_k^{\binom{m}{r}-1}$, and‘these give the standard open covering of the projective space $\mathbf{P}(\wedge^r V)$. We will show that $Y_0 = Y \cap U_{1,\cdots,r}$ is a (closed) affine subvariety in $U_{1,\cdots,r}$; a similar argument proves that $Y \cap U_{i_1,\cdots,i_r}$ is an affine variety in $U_{i_1,\ldots,i_r}$ for all $i_1 < \ldots < i_r$. This implies $Y \subset \mathbf{P}(\wedge^r V)$ is Zariski closed.

Let $w_i = \sum_{j=1}^m w_{ij}e_j$ for $1 \leq j \leq r$ be a basis for a subspace W corresponding to a point of Y_0. Then the matrix

$$T = [w_{ij}]_{1\leq i,j\leq r}$$

is non-singular, since

$$\ell_1 \wedge \cdots \wedge \ell_r(v_1 \wedge \cdots \wedge v_r) = \det T.$$

Hence W has a *unique* basis v_i, $1 \leq i \leq r$, such that for this basis, the matrix T becomes the identity, *i.e.*, we have

$$v_i = e_i + \sum_{j=1}^{m-r} a_{ij}e_{r+j} \;\forall\, 1 \leq i \leq r$$

(equivalently, write $V = V_1 \oplus V_2$ where V_1 is spanned by $e_1, \ldots, e_r$, and V_2 is spanned by $e_{r+1}, \ldots, e_m$; then the projection onto V_1 with kernel V_2 maps W isomorphically onto V_1, and v_i is the preimage of e_i under this isomorphism).

Conversely, for any $r \times (m-r)$ matrix $[a_{ij}]$, the vectors $v_i = e_i + \sum_{j=1}^{m-r} a_{ij}e_{j+r}$ span an r-dimensional space W, with $[W] \in Y_0$, and the v_i then give the distinguished basis for W. Thus as a set, $Y_0 \cong k^{r(m-r)}$. Our aim is to show that $Y_0 \cong \mathbf{A}_k^{r(m-r)}$ suitably embedded as a closed subvariety of $U_{1,2,\ldots,r} \cong \mathbf{A}_k^{\binom{m}{r}-1}$.

Considering $Y_0 \subset U_0$, the coordinate functions on U_0 are $\psi_{i_1,\ldots,i_r}$ defined by

$$\psi_{i_1,\ldots,i_r}(W) = \ell_{i_1\wedge\cdots\wedge i_r}(v_1 \wedge \cdots \wedge v_r),$$

where $v_1, \ldots, v_r$ is the distinguished basis for W, and

$$i_1 < \cdots < i_r, \;\; \{i_1, \ldots, i_r\} \neq \{1, 2, \ldots, r\}.$$

Note that $\ell_1\wedge\cdots\wedge\ell_r(v_1\wedge\cdots\wedge v_r) = 1$, so the remaining $\ell_{i_1\wedge\cdots\wedge i_r}$ do determine affine coordinates.

Now $\psi_{i_1,\ldots,i_r}$ is the determinant of the $(r \times r)$-submatrix consisting of the $i_1^{\text{th}}, \ldots$ and i_r^{th} columns of the $(r \times m)$-matrix $[I_r, A]$, where I_r is the

$r \times r$ identity matrix, and $A = [a_{ij}]$. In particular, it is a polynomial with integer coefficients in the a_{ij}. On the other hand, for any $1 \leq i \leq r$ and $1 \leq j \leq m - r$, we compute that

$$\psi_{1,2,\ldots,i-1,i+1,\ldots,r,r+j}([W]) = (-1)^{r-i} a_{ij},$$

since the left side is the determinant of a matrix which is the identity I_r with its i^{th} column removed, and with the j^{th} column of A added on as the new r^{th} column.

Thus if we write

$$U_0 = \mathbf{A}_k^{\binom{m}{r}-1} = \mathbf{A}_k^{r(m-r)} \times \mathbf{A}_k^{\binom{m}{r}-1-r(m-r)},$$

where the first factor corresponds to the coordinate functions

$$(-1)^{r-i} \psi_{1,2,\ldots,i-1,i+1,\ldots,r,r+j},$$

and the second factor corresponds to the remaining $\psi_{i_1,\ldots,i_r}$, then Y_0 maps isomorphically onto $\mathbf{A}_k^{r(m-r)}$ under the first projection, and we identify Y_0 with the graph of a morphism

$$\mathbf{A}_k^{r(m-r)} \times \mathbf{A}_k^{\binom{m}{r}-1-r(m-r)}.$$

In particular $Y_0 \cong \mathbf{A}_k^{r(m-r)}$ and Y_0 is a Zariski closed subvariety of U_0.

This completes the proof that $Y = \mathbf{G}_k(r, m)$ is a projective variety, and further, shows that it has a covering by Zariski open subvarieties isomorphic to $\mathbf{A}_k^{r(m-r)}$. In particular $\mathbf{G}_k(r, m)$ is a non-singular irreducible projective variety of dimension $r(m - r)$.

Now again, one proves (using the above local coordinates, for example) that the *incidence variety*

$$I = \{(P, [W]) \in \mathbf{P}_k^n \times \mathbf{G}_k(d + 1, n + 1) \mid P \in W\}$$

is an irreducible, non-singular projective variety in $\mathbf{P}_k^n \times \mathbf{G}_k(d + 1, n + 1)$. This implies that

$$J = \{(P, [W]) \in X \times \mathbf{G}_k(d + 1, n + 1) \mid \mathbf{T}_{P,X} = W\}$$

is Zariski closed in $X \times \mathbf{G}_k(d+1, n+1)$, and gives the graph of a morphism $D_X : X \to \mathbf{G}_k(d + 1, n + 1)$, which we call the *dual morphism.*

We end this section with another application of the tangent variety, to a result on embeddings. An embedding of X into Y is an isomorphism of X with a closed subvariety of Y.

Theorem 8 (Whitney embedding theorem)
(i) Let X be a non-singular affine variety of dimension d. Then there is an embedding of X into $\mathbf{A}_k^{2d+1}$.
(ii) Let X be a non-singular projective variety of dimension d. Then there is an embedding of X into $\mathbf{P}_k^{2d+1}$.

Proof:
We give the proof in the affine case. The proof in the projective case is similar, and left to the reader.

We first note that the tangent variety is functorial for morphisms, *i.e.*, if $f : X \to Y$ is a morphism of affine varieties, there is a natural morphism $df : T_X \to T_Y$ giving rise to a commutative diagram

$$\begin{array}{ccc} T_X & \stackrel{df}{\to} & T_y \\ \downarrow & & \downarrow \\ X & \stackrel{f}{\to} & Y \end{array}$$

where the vertical arrows are the natural projections. Here df is defined as follows: if $P \in X$, the homomorphism $f^* : \mathcal{O}_{f(P),Y} \to \mathcal{O}_{P,X}$ yields a linear map

$$\mathbf{m}_{f(P)}/\mathbf{m}_{f(P)}3^2 \to \mathbf{m}_P/\mathbf{m}_P^2,$$

hence a map on dual spaces in the reverse direction. We now write this out explicitly in coordinates.

If $X \subset \mathbf{A}_k^n$, $Y \subset \mathbf{A}_k^m$ and $f = (f_1, \ldots, f_m)$, where $f_i \in k[x_1, \ldots, x_n]$, then one can check that $df : T_X \to T_Y \subset \mathbf{A}_k^m \times \mathbf{A}_k^m$ is the map

$$(f_1(x_1, \ldots, f_m(x_1, \ldots, x_n), \widetilde{f_1}, \widetilde{f_2}, \ldots, \widetilde{f_m}),$$

where for $1 \leq j \leq m$ we have

$$\widetilde{f_j}(x_1, \ldots, x_n, y_1, \ldots, y_n) = f_j(x_1, \ldots, x_n) + \sum_{i=1}^{n} (y_i - x_i) \frac{\partial f_j}{\partial x_i}(x_1, \ldots, x_n).$$

Here $T_X \subset \mathbf{A}_k^n \times \mathbf{A}_k^n$, and $y_1, \ldots, y_n$ are the coordinates on the second factor of $\mathbf{A}_k^n$. The definition of $\widetilde{f_j}$ is motivated by the formula (essentially the chain rule for differentiation)

$$\sum_{k=1}^{m} \frac{\partial h}{\partial z_k}(\widetilde{f_k} - f_k) = \sum_{i=1}^{n} \frac{\partial f^* h}{\partial x_i}(y_i - x_i),$$

valid for any polynomial h, which implies via the Jacobian criterion that $df(T_X) \subset T_Y$.

We make use of the following lemma.

Lemma 2 *Let $f : X \to Y$ be a morphism between affine varieties. Suppose*

(i) $A(X)$ is a finite $A(Y)$-module

(ii) f is injective and dominant (i.e., has Zariski dense image)

(iii) for each point $P \in X$, the tangent mapping $df : T_{P,X} \to T_{f(P),Y}$ is injective.

Then f is an isomorphism.

Proof:

Since f is dominant, f^* identifies $A(Y)$ with a subring of $A(X)$. We first claim that $f : X \to Y$ is in fact surjective, hence bijective. This is because $A(X)$ is a finite module over the subring $A(Y)$; hence for any point $Q \in Y$ with maximal ideal $\mathbf{m}$ of $A(Y)$, the ring $A(X) \otimes_{A(Y)} A(Y)_{\mathbf{m}}$ is a non-zero (since it contains $A(Y)_{\mathbf{m}}$), finite, $A(Y)_{\mathbf{m}}$-module. Hence $A(X)/\mathbf{m}A(X) \neq 0$, by Nakayama's lemma. Hence there is a maximal ideal $\mathbf{n}$ of $A(X)$ containing $\mathbf{m}A(X)$, which corresponds to a point P of X with $g(P) = Q$.

We have noted that $A(X) \otimes_{A(Y)} A(Y)_{\mathbf{m}}$ is a finite $A(Y)_{\mathbf{m}}$-module, for each maximal ideal $\mathbf{m}$ of $A(Y)$. Since f is bijective, there is a unique maximal ideal $\mathbf{n}$ of $A(X)$ lying over $\mathbf{m}$, and so $A(X) \otimes_{A(Y)} A(Y)_{\mathbf{m}} = A(X)_{\mathbf{n}}$. Let $P \in X$ correspond to $\mathbf{n}$, so that $f(P) \in Y$ corresponds to $\mathbf{m}$.

Since $df : T_{P,X} \to T_{f(P),Y}$ is injective, $\mathbf{m}/\mathbf{m}^2 \to \mathbf{n}/\mathbf{n}^2$ is surjective. Hence from Nakayama's lemma, $\mathbf{m}A(X)_{\mathbf{n}} = \mathbf{n}A(X)_{\mathbf{n}}$, and so $A(X)/\mathbf{m}A(X) = k$. Again from Nakayama's lemma, we get that $1 \in A(X)_{\mathbf{n}}$ generates $A(X)_{\mathbf{n}}$ as an $A(Y)_{\mathbf{m}}$ module, *i.e.*,

$$A(Y)_{\mathbf{m}} \to A(X)_{\mathbf{n}} = A(X) \otimes_{A(Y)} A(Y)_{\mathbf{m}}$$

is surjective. Since this is true for every maximal ideal $\mathbf{m}$ of $A(Y)$, we see that $A(Y) \to A(X)$ is surjective, *i.e.*, $f : X \to Y$ is a closed subvariety. Since f is bijective, f is an isomorphism, from the Nullstellensatz. □

Now to prove the theorem, start with some embedding of $f : X \hookrightarrow \mathbf{A}^n_k$. If $n \leq 2d + 1$ there is nothing to prove. So suppose $n > 2d + 1$. Let $\pi : \mathbf{A}^n_k \to \mathbf{A}^{2d+1}_k$ be the projection onto the first $2d + 1$ coordinates. We will show that there is a non-empty Zariski open set $U \subset \mathrm{GL}_n(k)$ such that for $\sigma \in U$, the composite $\pi \circ \sigma \circ f : X \to \mathbf{A}_k^{2d+1}$ is an embedding. Equivalently, if $x_1, \ldots, x_n$ are the given coordinate functions on X, and $y_1, \ldots, y_{2d+1}$ are $2d+1$ 'general' (homogeneous) k-linear combinations of the x_j, then $y_1, \ldots, y_{2d+1}$ give an embedding of X in $\mathbf{A}_k^{2d+1}$.

First, from the proof given for Noether normalization, there is a non-empty open set $U_1 \subset \mathrm{GL}_n(k)$ such that for $\sigma \in U_1$, the composite

$$X \xrightarrow{f} \mathbf{A}_k^n \xrightarrow{\pi' \circ \sigma} \mathbf{A}_k^d$$

makes $A(X)$ a finite module over $A(\mathbf{A}_k^d)$, a polynomial subring; here π' is the projection onto the first d coordinates. Since $\pi' \circ \sigma$ factors through $\pi \circ \sigma$, we see that there are homomorphisms of k-algebras

$$A(\mathbf{A}_k^d) \hookrightarrow A(\mathbf{A}_k^{2d+1}) \xrightarrow{(\pi \circ \sigma)^*} A(\mathbf{A}_k^n) \xrightarrow{f^*} A(X),$$

we see that $A(X)$ is also a finite $(\pi \circ \sigma \circ f)^* A(\mathbf{A}_k^{2d+1})$-module. In more concrete terms, if $A(X)$ is a finite module over its k-subalgebra generated by $y_1, \ldots, y_d$, then it is also finite over the larger subalgebra generated by $y_1, \ldots, y_{2d+1}$.

Next, consider the set Γ of all $\sigma \in \mathrm{M}_n(k)$, the set of $n \times n$ matrices, such that $\pi \circ \sigma(f(P)) = \pi \circ \sigma(f(Q))$, for some $P \neq Q$ in X. We want to prove this set is not Zariski dense in $\mathrm{M}_n(k) = \mathbf{A}_k^{n^2}$; if U_2 is the complement of its Zariski closure, then for $\sigma \in U_2$, the map $\pi \circ \sigma$ restricts to an injective map $f : X \to \mathbf{A}_k^{2d+1}$.

Let $\Delta_X \subset X \times X$ be the diagonal; then $X \times X - \Delta_X$ is a quasi-projective variety of dimension $2d$. Note that for any $R \in \mathbf{A}_k^n$, the morphism

$$L_R : \mathrm{M}_n(k) = \mathbf{A}_k^{n^2} \to \mathbf{A}_k^{2d+1} = k^{2d+1},$$

$$L_R(\sigma) = \pi \circ \sigma(R),$$

is a linear transformation $k^{n^2} \to k^{2d+1}$, which is surjective if $R \neq 0$, where $0 \in k^n = \mathbf{A}_k^n$ is the origin. Hence if $R \neq 0$, then $\ker L_R$ is a linear subvariety of $\mathbf{A}_k^{n^2}$ of dimension $n^2 - 2d - 1$. Consider the subvariety

$$I = \{((P,Q), \sigma) \in (X \times X - \Delta_X) \times \mathrm{M}_n(k) \mid L_{f(P)-f(Q)}(\sigma) = 0\},$$

where the equation on the right side is between vectors in k^{2d+1}. Clearly the image of I in $\mathrm{M}_n(k)$ is Γ.

Each fibre of $I \to X \times X - \Delta_X$ is an affine space of dimension $n^2 - 2d - 1$, so that

$$\dim I = \dim(X \times X - \Delta) + n^2 - 2d - 1 = n^2 - 1.$$

Hence the Zariski closure of the image $\Gamma \subset \mathrm{M}_n(k)$ of I has dimension $\leq n^2 - 1$, *i.e.*, is a proper subvariety of $\mathrm{M}_n(k) = \mathbf{A}_k^{n^2}$.

Finally, we claim that there is a non-empty Zariski open set $U_3 \subset \mathrm{M}_n(k) = \mathbf{A}_k^{n^2}$ such that for $\sigma \in U_3$, and each $P \in X$, the map

$$d(\pi \circ \sigma) : T_{f(P),f(X)} \to T_{\pi\circ\sigma\circ f(P),\mathbf{A}^{2d+1}}, \tag{5}$$

induced by $\pi \circ \sigma$, is injective.

Since $\pi \circ \sigma$ is linear, and $T_{f(X)} \subset f(X) \times \mathbf{A}_k^n$, we have

$$T_{f(P),X} \subset \{f(P)\} \times \mathbf{A}_k^n, \;\; T_{\pi\circ\sigma\circ f(P),\mathbf{A}^{2d+1}} \subset \{\pi \circ \sigma \circ f(P)\} \times \mathbf{A}_k^{2d+1},$$

and the map $d(\pi \circ \sigma)$ is just

$$d(\pi \circ \sigma)(f(P), Q) = (\pi \circ \sigma \circ f(P), \pi \circ \sigma(Q)),$$

that is, it is the restriction of a linear transformation to an affine linear subspace with origin $f(P)$. Hence if it is non-injective, there must exist $Q \neq f(P)$ with $Q \in T_{f(P),f(X)}$ such that $\pi \circ \sigma(Q) = \pi \circ \sigma \circ f(P)$.

For each fixed $(f(P), Q) \in T_{f(X)} \subset \mathbf{A}_k^n \times \mathbf{A}_k^n$, the morphism

$$D_{P,Q} : \mathbf{M}_n(k) = \mathbf{A}_k^d \to \mathbf{A}_k^{2d+1} = k^{2d+1},$$

$$D_{P,Q}(\sigma) = \pi \circ \sigma(Q - f(P)),$$

is a linear transformation, which is surjective if $Q \neq f(P)$ (*i.e.*, if the 'tangent vector' $Q - f(P)$ is non-zero). In fact $D_{P,Q}$ is just the natural map

$$\sigma \mapsto (\pi \circ \sigma)(Q - f(P)).$$

To prove the injectivity of the map in (5), consider the subvariety

$$J = \{(P, Q), \sigma) \in (T_{f(X)} - \Delta_{f(X)}) \times \mathrm{M}_n(k) \mid D_{P,Q}(\sigma) = 0\}.$$

We claim that the image of J in $\mathrm{M}_n(k)$ is not Zariski dense. Assuming this, let $U_3 \subset \mathrm{M}_n(k)$ be a non-empty Zariski open set in the complement of the image of J; then for $\sigma \in U_3$, the map (5) is non-zero on $Q - f(P)$ for all $Q \in T_{f(P),f(X)}$, for each $P \in X$ and each $Q \in T_{f(P),f(X)}$ with $Q \neq f(P)$. By the linearity of (5), this means (5) is injective.

We now compute the dimension of J. The fibre of J over $(f(P), Q)$ (with $Q \in T_{f(P),f(X)} - \{f(P)\}$) is an affine space of dimension $n^2 - 2d - 1$, so that $\dim J = n^2 - 2d - 1 + \dim T_{f(X)} = n^2 - 1 < \dim \mathrm{M}_n(k)$, as before (this is where the non-singularity of X is used; we then have $\dim T_{f(X)} = 2d$). Hence the projection $J \to \mathrm{M}_n(k)$ is not dominant.

Now take $U = U_1 \cap U_2 \cap U_3$. By construction, for any $\sigma \in U$, the map $\pi \circ \sigma \circ f = g : X \to \mathbf{A}_k^{2d+1}$ has the properties that (i) g^* makes $A(X)$ a finite $A(\mathbf{A}_k^{2d+1})$-module (ii) g is injective (iii) for each $P \in X$, the tangent map $T_{f(P),f(X)} \to T_{g(P),\mathbf{A}^{2d+1}}$ is injective.

Hence if we set $Y = \overline{g(X)}$, the Zariski closure of $g(X)$ in $\mathbf{A}_k^{2d+1}$, then Lemma 2 applied to $g : X \to Y$ shows that g is an isomorphism. $\square$

4 The Hodge Decomposition

4.1 Type decomposition of differential forms and Dolbeault cohomology

If X is a complex manifold of dimension n, then for each point $x \in X$ we can find an open neighbourhood U of x in X and local holomorphic coordinates $z_1, \ldots, z_n$ on U, identifying U with a polydisc in $\mathbf{C}^n$. If we write $z_j = x_j + \imath y_j$, then $dx_1, dy_1, \ldots, dx_n, dy_n$ give a real basis for the real cotangent space $T_y^* X$ to X at each point $y \in U$, and hence also a $\mathbf{C}$-basis for $T_y^* X_{\mathbf{C}} = T_y^* X \otimes_{\mathbf{R}} \mathbf{C}$. Then $dz_j = dx_j + \imath dy_j$, $d\overline{z}_j = dx_j - \imath dy_j$, $1 \leq j \leq n$, give another $\mathbf{C}$-basis for this space, for each y. Define the subspaces

$$T_y^{1,0} = \sum_{j=1}^{n} \mathbf{C} dz_j, \quad T_y^{0,1} = \sum_{j=1}^{n} \mathbf{C} d\overline{z}_j.$$

Similarly, the complexified tangent space $T_y X_{\mathbf{C}} = T_y X \otimes_{\mathbf{R}} \mathbf{C}$ has a basis given by

$$\frac{\partial}{\partial z_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} - \imath \frac{\partial}{\partial y_j}\right), \quad \frac{\partial}{\partial \overline{z_j}} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} + \imath \frac{\partial}{\partial y_j}\right), \quad 1 \leq j \leq n.$$

This basis is dual to the basis dz_j, $d\overline{z}_j$, $1 \leq j \leq n$. With this notation, the exterior derivative has the formula

$$df = \sum_{j=1}^{n} \frac{\partial f}{\partial x_j} dx_j + \frac{\partial f}{\partial y_j} dy_j = \sum_{j=1}^{n} \frac{\partial f}{\partial z_j} dz_j + \frac{\partial f}{\partial \overline{z}_j} d\overline{z}_j.$$

The Cauchy-Riemann equations, which describe the necessary and sufficient conditions for a smooth function f to be a holomorphic function, now reduce to

$$\frac{\partial f}{\partial \bar{z}_j} = 0, \;\; 1 \le j \le n.$$

If we choose another set of holomorphic coordinates $w_1, \ldots, w_n$ on an open set $V \subset X$, so that $w_j = w_j(z_1, \ldots, z_n)$ are holomorphic functions on $U \cap V$, then the Cauchy-Riemann equations imply that for all $y \in U \cap V$, the subspaces

$$T_y^{1,0} \subset T_y X_{\mathbf{C}}, \;\; T_y^{0,1} \subset T_y X_{\mathbf{C}}$$

defined by the two sets of coordinates z_j and w_j are the same, since dw_j is a linear combination of only the dz_k, and $d\overline{w}_j$ is similarly a linear combination of only the $d\bar{z}_k$. Thus these subspaces are *independent of the choice of local cordinates*, and define C^∞ complex sub-bundles of the complexified cotangent bundle

$$T_X^{1,0} \subset T^*_{X,\mathbf{C}}, \;\; T_X^{0,1} \subset T^*_{X,\mathbf{C}}.$$

Further, the matrix entries of the transition matrices for the vector bundle $T_X^{1,0}$ are the holomorphic functions $\dfrac{\partial w_j}{\partial z_k}$, so that $T_X^{0,1}$ is in fact a holomorphic vector bundle of rank n, the *holomorphic cotangent bundle*. The transition matrix entries for $T_X^{0,1}$ are the conjugate functions $\dfrac{\partial \overline{w}_j}{\partial \bar{z}_k} = \overline{\left(\dfrac{\partial w_j}{\partial z_k}\right)}$, so that $T_X^{0,1}$ is the complex conjugate vector bundle to $T_X^{1,0}$.

Similarly, we can decompose any C^∞ 1-form ω uniquely as a sum

$$\omega = \omega^{1,0} + \omega^{0,1},$$

and correspondingly write the exterior derivative operator

$$d = \partial + \overline{\partial},$$

where for any C^∞ function f,

$$\partial(f) = (df)^{1,0}, \overline{\partial}(f) = (df)^{0,1}.$$

Now the Cauchy-Riemann equations reduce further to

$$\overline{\partial}(f) = 0.$$

The decomposition of $T^*_{X,\mathbf{C}}$ into a direct sum of two complex sub-bundles induces a decompositon on the bundles of k-forms, for all $k \geq 0$,

$$\bigwedge^k T^*_{X,\mathbf{C}} = \bigoplus_{\substack{p,q \geq 0 \\ p+q=k}} T^{p,q}_X,$$

where

$$T^{p,q}_X = \bigwedge^p T^{1,0}_X \otimes \bigwedge^q T^{0,1}_X$$

is the bundle wih local basis

$$dz_{j_1} \wedge \cdots \wedge dz_{j_p} \wedge d\overline{z}_{k_1} \wedge \cdots \wedge d\overline{z}_{k_q}, \;\; 1 \leq j_1 < \cdots < j_p \leq n, \;\; 1 \leq k_1 < \cdots < k_q \leq n.$$

Thus any smooth k-form ω has a unique decomposition into smooth forms

$$\omega = \sum_{p=0}^{k} \omega^{p,k-p};$$

we say ω *has type* (p,q) if $\omega = \omega^{p,q}$ in this decomposition.

The decomposition into types is compatible with the exterior product of forms, in the sense that if ω is of type (p,q), and η is of type (r,s), then $\omega \wedge \eta$ is of type $(p+r, q+s)$. Similarly, the exterior derivative operator d on k-forms can be decomposed as

$$d = \partial + \overline{\partial},$$

where for ω of type (p,q),

$$\partial\omega = (d\omega)^{p+1,q}, \;\; \overline{\partial}\omega = (d\omega)^{p,q+1}.$$

Finally, the Cauchy-Riemann equations imply that for any smooth k-form ω of type $(k,0)$, ω is holomorphic $\Leftrightarrow \overline{\partial}\omega = 0$, and in this case, $d\omega = \partial\omega$.

The condition that $d \circ d = d^2 = 0$ implies the following identities:

$$\partial^2 = \overline{\partial}^2 = \partial\overline{\partial} + \overline{\partial}\partial = 0.$$

In particular, we can define the *Dolbeault cohomology groups* (in fact $\mathbf{C}$-vector spaces) by

$$H^{p,q}_{\overline{\partial}}(X) = \frac{\{\text{smooth } (p,q)\text{-forms } \omega \text{ with } \overline{\partial}\omega = 0\}}{\{\overline{\partial}\eta \text{ for smooth } (p,q-1)\text{-forms } \eta\}}.$$

We may compare this definition with that of the *de Rham cohomology groups*

$$H^k_{DR}(X, \mathbf{C}) = \frac{\{\text{smooth } k\text{-forms } \omega \text{ with } d\omega = 0\}}{\{d\eta \text{ for smooth } k-1\text{-forms } \eta\}}.$$

A further analogy between de Rham and Dolbeault cohomology is given by the following lemma, which is the analogue of the Poincaré lemma. Later, we will use it to derive a sheaf-theoretic interpretation of Dolbeault cohomology.

Lemma 3 ($\overline{\partial}$-Poincaré lemma) *If $\Delta = \Delta(s) = \{| z_j | < s,\ 1 \leq j \leq n\}$ is a polydisk in $\mathbf{C}^n$, then $H^{p,q}_{\overline{\partial}}(\Delta) = 0$ for all $q > 0$, and $H^{p,0}_{\overline{\partial}}(\Delta)$ is the space of holomorphic p-forms on Δ.*

Proof: That $H^{p,0}_{\overline{\partial}}(\Delta)$ is the space of holomorphic p-forms, is an immediate consequence of the Cauchy-Riemann equations. So we may assume $q > 0$.

Let $z_1, \ldots, z_n$ be the holomorphic coordinates on $\Delta(s)$. If ω is any smooth (p, q)-form, then we may uniquely write

$$\omega = \sum_I dz_I \wedge \omega_I,$$

where I runs over all (ordered) subsets of $\{1, \ldots, n\}$ of cardinality p, $dz_I = \wedge_{i \in I} dz_i$, and ω_I is a smooth $(0, q)$-form. Then

$$\overline{\partial}\omega = \sum_I dz_I \wedge \overline{\partial}\omega_I = 0 \ \Leftrightarrow\ \overline{\partial}\omega_I = 0 \text{ for all } I.$$

Hence we reduce at once to the case when $p = 0$. Now we proceed by induction on $q \geq 1$.

Let ω be a smooth $(0, q)$-form on $\Delta(s)$. We first show that for any $r < s$, there is a smooth $(0, q-1)$-form η with $\overline{\partial}\eta = \omega$ on $\Delta(r)$. We work by induction on k, such that ω is a linear combination (with smooth function coefficients) of wedge products of $d\overline{z}_1, \ldots, d\overline{z}_k$. The case $k = 1$ is trivial, so assume $k > 1$. Write

$$\omega = d\overline{z}_k \wedge \omega_1 + \omega_2,$$

where ω_1 is a smooth $q-1$-form which is a linear combination of wedges of $d\overline{z}_1, \ldots, d\overline{z}_{k-1}$, and ω_2 is a smooth q-form which is also a linear combination of wedges of the same differentials. Then $\overline{\partial}\omega = d\overline{z}_k \wedge \overline{\partial}\omega_1 + \overline{\partial}\omega_2 = 0$ implies that if we write

$$\omega_1 = \sum_{k \notin I} \omega_I d\overline{z}_I,$$

then

$$\frac{\partial \omega_I}{\partial \overline{z}_j} = 0 \ \ \forall\, j > k.$$

This means ω_I is *holomorphic in* z_j for all $j > k$. Now we solve the differential equation in 1 variable z_k

$$\frac{\partial \eta_I}{\partial \overline{z}_k} = \omega_I$$

by the integral formula

$$\eta_I(z) = \frac{1}{2\pi\imath} \int_{|\ w_k \leq s\ |} \omega_I(z_1, \ldots, z_{k-1}, w_k, z_{k+1}, \ldots, z_n) \frac{dw_k \wedge d\overline{w}_k}{w_k - z_k}.$$

The integral converges absolutely, since $s < r$, and gives a well-defined smooth function on open polydisk $\Delta(s)$, such that (by differentiating under the integral sign) η_I *is holomorphic in* z_j *for all* $j > k$; further, by a standard Stokes theorem argument in 1 (complex) variable w_k, we see also that

$$\frac{\partial \eta_I}{\partial \overline{z}_k} = \omega_I$$

holds. Hence if we set $\eta = \sum_I \eta_I d\overline{z}_I$, then $\omega - \overline{\partial}\eta$ is a $(0, q)$-form which involves only the differentials $d\overline{z}_1, \ldots, d\overline{z}_{k-1}$.

To finish the proof, let r_m be an increasing sequence of positive real numbers converging to r. Since we can η_m on $\Delta(r_m)$ with $\overline{\partial}\eta_m = \omega$, we can find such η_m on $\Delta(r)$ such that $\overline{\partial}(\eta_m) = \omega$ holds on $\Delta(r_m)$ (first choose η_m arbitrarily; then replace each η_m by $\varphi_m \eta_{m+1}$, for a suitable bump function φ_m which is 1 in a neighbourhood of $\Delta(r_m)$ and is supported within $\Delta(r_{m+1})$).

We wish to modify our sequence η_m to a new sequence $\widetilde{\eta}_m$ which converges uniformly on compact sets. We now work by induction on q. Suppose $q \geq 2$. Take $\widetilde{\eta}_i = \eta_i$ for $i \leq 2$. Now $\overline{\partial}(\eta_{m+1} - \widetilde{\eta}_m) = 0$ on $\Delta(r_m)$, so that by induction, we can find a smooth $(0, q-2)$-form β on $\Delta(r)$ such that $\overline{\partial}\beta = \eta_{m+1} - \widetilde{\eta}_m$ on $\Delta(r_{m-1})$. Take $\widetilde{\eta}_{m+1} = \eta_{m+1} + \overline{\partial}\beta$; then $\overline{\partial}\widetilde{\eta}_{m+1} = \omega$ on $\Delta(r_{m+1})$, and $\widetilde{\eta}_{m+1} = \widetilde{\eta}_m$ on $\Delta(r_{m-1})$. Hence $\{\widetilde{\eta}_m\}$ converges uniformly on compact subsets to a smooth $(0, q-1)$-form η with $\overline{\partial}\eta = \omega$ on $\Delta(r)$.

If $q = 1$, then η_m is a sequence of smooth functions. Modify it to a sequence $\widetilde{\eta}_m$ as follows: $\widetilde{\eta}_i = \eta_i$ for $i \leq 2$; if $\widetilde{\eta}_m$ is already determined, then $\overline{\partial}(\eta_{m+1} - \widetilde{\eta}_m) = 0$ on $\Delta(r_m)$, *i.e.*, $\eta_{m+1} - \widetilde{\eta}_m$ is a holomorphic function on

$\Delta(r_m)$. Hence it is represented by a convergent power series in $\Delta(r_m)$. Let β be a polynomial obtained by truncating this power series so that

$$\sup_{\Delta(r_{m-1})} |\, \eta_{m+1} - \widetilde{\eta}_m - \beta \,| < 2^{-m}.$$

Set

$$\widetilde{\eta}_{m+1} = \eta_{m+1} - \beta.$$

Then $\overline{\partial}\widetilde{\eta}_{m+1} = \omega$ on $\Delta(r_{m+1})$, $\widetilde{\eta}_{m+1} - \widetilde{\eta}_m$ is holomorphic on $\Delta(r_m)$ and uniformly bounded by 2^{-m} on $\Delta(r_{m-1})$. Hence $\lim_m \widetilde{\eta} = \eta$ is smooth, and satisfies $\overline{\partial}\eta = \omega$ on $\Delta(r)$. □

4.2 Harmonic forms and the Hodge theorem

Since $T_X^{0,1}$ is the complex conjugate bundle to the holomorphic cotangent bundle $T_X^{1,0}$, a smooth section h of $T_X^{1,0} \otimes T_X^{0,1}$ is identified with a smoothly varying family of $\mathbf{R}$-bilinear forms on the holomorphic tangent spaces, $h_x : T_x \otimes_{\mathbf{R}} T_x X \to \mathbf{C}$, which are $\mathbf{C}$-linear in the first argument and conjugate linear in the second argument. If in addition h_x is positive definite Hermitian, we call h a *smooth Hermitian metric* on X. Thus we can locally write $h = \sum_{j,k} h_{jk} dz_j \otimes d\overline{z}_k$; then h defines a smooth Hermitian metric if the matrix $[h_{jk}(x)]$ is positive definite Hermitian for each x.

The real and imaginary parts of a positive definite Hermitian form on a complex vector space V respectively yield a positive definite inner product, and a skew-symmetric form, on the underlying real vector space of V. Hence, $Re(h)$, the real part of h, gives a *Riemannian metric* on X, while the imaginary part of h yields a real 2-form on X. The imaginary part of h is expressible as $\sum_{j,k} h_{jk} dz_j \wedge d\overline{z}_k$ for a positive definite Hermitian matrix h_{jk}.

A *coframe* for the metric h on an open set $U \subset X$ is defined to be an n-tuple of smooth forms $\varphi_1, \ldots, \varphi_n$ of type $(1,0)$ on U such that $h = \sum_{i=1}^n \varphi_i \otimes \overline{\varphi}_i$ on U, *i.e.*, which correspond to the choice of an orthonormal basis with respect to $h(x)$ of the holomorphic tangent space $T_x X$, at each point $x \in U$. Such coframes always exist locally, by the applying the Gram-Schmidt process to $dz_1, \ldots, dz_n$. Then the imaginary part of h is locally given by the 2-form

$$\omega = \frac{1}{2} \sum_{j=1}^{n} \varphi_j \wedge \overline{\varphi}_j,$$

from which we see that ω is a $(1,1)$-form, with $\overline{\omega} = \omega$ (*i.e.*, ω is a *real* 2-form). In this case, a *volume form* for X (with respect to the Riemannian metric determined by the real part of h) is given by

$$\Phi = \frac{\omega^n}{n!} = \frac{(-1)^{n(n-1)/2}(\imath)^n}{2^n}\varphi_1 \wedge \cdots \wedge \varphi_n \wedge \overline{\varphi}_1 \wedge \cdots \wedge \overline{\varphi}_n.$$

This is an easy computation, using the fact that if $\varphi_j = \alpha_j + \imath\beta_j$, then the associated Riemannian metric is $\sum_{j,k} \alpha_j \otimes \alpha_k + \beta_j \otimes \beta_k$, and the volume element corresponding to this is

$$\alpha_1 \wedge \beta_1 \wedge \alpha_2 \wedge \beta_2 \wedge \cdots \wedge \alpha_n \wedge \beta_n.$$

The metric h induces Hermitian metrics on all the tensor bundles $T_X^{p,q}$, where $\{\varphi_I \wedge \overline{\varphi}_J\}$, for all multi-indices I, J with cardinality p, q respectively, form an orthogonal set of elements each of length given by $\| \varphi_I \wedge \overline{\varphi}_J \|^2 = 2^{p+q}$ (note that $\| dz \|^2 = 2$ on $\mathbf{C}$). We can then define the *Hodge star operator* on (p,q)-forms as a bundle map

$$* : T_X^{p,q} \to T_X^{n-p,n-q},$$

where

$$*\varphi_I \wedge \overline{\varphi}_J = 2^{p+2-n}\varepsilon_{IJ}\varphi_{\widetilde{I}} \wedge \overline{\varphi_{\widetilde{J}}},$$

where $\widetilde{I} = \{1, \ldots, n\} - I$, $\widetilde{J} = \{1, \ldots, n\} - J$ are the complementary sets of indices, and ε_{IJ} is the sign of the permutation

$$(1, 2, \ldots, n, 1', 2', \ldots, n') \mapsto (i_1, \ldots, i_p, j'_1, \ldots, j'_q, \widetilde{i}{-1}, \ldots, \widetilde{i}_{n-p}, \widetilde{j}'_1, \ldots, \widetilde{j}'_{n-q}).$$

Then one verifies that $**\,\omega = (-1)^{p+q}\omega$ on (p,q)-forms ω. The important property of $*$ is that the inner product on (p,q) forms is given by

$$h(\omega, \eta)\Phi = \omega \wedge *\eta.$$

Thus if $A^{p,q}(X)$ is the space of global smooth forms on X of type (p,q), and X is a *compact* complex manifold, then we have a positive definite Hermitian inner product on $A^{p,q}$ defined by

$$(\omega, \eta) = \int_X h(\omega, \eta)\Phi = \int_X \omega \wedge *\eta.$$

This makes $A^{p,q}(X)$ into a pre-Hilbert space. We compute that of $\psi \in A^{p,q-1}(X)$, then

$$(\overline{\partial}\psi, \eta) = \int_X \overline{\partial}\psi \wedge *\eta$$

$$= (-1)^{p+q} \int_X \psi \wedge \overline{\partial} * \eta + \int_X \overline{\partial}(\psi \wedge *\eta)$$
$$= (-1)^{p+q} \int_X \psi \wedge \overline{\partial} * \eta + \int_X d(\psi \wedge *\eta),$$

since $\overline{\partial} = d$ on forms of type $(n, n-1)$. But $\int_X d(\psi \wedge *\eta) = 0$ by Stokes' theorem. Hence we deduce that

$$(\overline{\partial}\psi, \eta) = -\int_X \psi \wedge *(*\overline{\partial} * \eta).$$

Hence $\overline{\partial}$ has an *adjoint* $\overline{\partial}^* = - * \overline{\partial} *$ with respect to the Hermitian inner product on $A^{p,q}(X)$.

Lemma 4 *A ∂-closed (p,q)-form ψ has minimal norm in its cohomology class in $H^{p,q}_{\overline{\partial}}(X)$ $\Leftrightarrow \overline{\partial}^*\psi = 0$.*

Proof: If $\overline{\partial}^*\psi = 0$ then for any $(p, q-1)$-form η with $\overline{\partial}\eta \neq 0$, we have

$$\begin{aligned}
\| \psi + \overline{\partial}\eta \|^2 = \quad & (\psi + \overline{\partial}\eta, \psi + \overline{\partial}\eta) \\
= \quad & \| \psi \|^2 + \| \overline{\partial}\eta \|^2 + 2\,\mathrm{Re}\,(\psi, \overline{\partial}\eta) \\
= \quad & \| \psi \|^2 + \| \overline{\partial}\eta \|^2 + 2\,\mathrm{Re}\,(\overline{\partial}^*\psi, \eta) \\
= \quad & \| \psi \|^2 + \| \overline{\partial}\eta \|^2 \\
> \quad & \| \psi \|^2
\end{aligned}$$

Hence ψ has minimal norm. Conversely, if ψ has minimal norm, then for any $\eta \in A^{p,q-1}(X)$,

$$\frac{\partial}{\partial t} \| \psi + t\overline{\partial}\eta \|^2 |_{t=0} = 0.$$

This gives $2\,\mathrm{Re}\,(\psi, \overline{\partial}\eta) = 0$. Applying the same argument to $\imath\eta$ gives also $2\,\mathrm{Im}\,(\psi, \overline{\partial}\eta) = 0$. Hence

$$(\overline{\partial}^*\psi, \eta) = (\psi, \overline{\partial}\eta) = 0.$$

□

Notice that from the lemma, a Dolbeault cohomology class contains a *unique* form of minimal norm, if one exists.

Definition: A (p,q) form $\omega \in A^{p,q}(X)$ is called *$\overline{\partial}$-harmonic* if $\overline{\partial}\omega = \overline{\partial}^*\omega = 0$.

Equivalently, ω is $\overline{\partial}$-harmonic $\Leftrightarrow \Delta_{\overline{\partial}}\omega = (\overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial})\omega = 0$. Indeed, if $\Delta_{\overline{\partial}}\omega = 0$, then

$$0 = (\Delta_{\overline{\partial}}(\omega), \omega) = (\overline{\partial}\overline{\partial}^*\omega, \omega) + (\overline{\partial}^*\overline{\partial}\omega, \omega) = (\| \overline{\partial}\omega \|^2 + \| \overline{\partial}^*\omega \|^2$$

which implies that ω is $\overline{\partial}$-harmonic; the converse clearly holds. Notice that the operator $\Delta_{\overline{\partial}}$ is *self-adjoint*.

One of the first major results of Hodge theory is the following. The proof is beyond the scope of these lectures.

Theorem 9 ($\overline{\partial}$-Hodge Theorem) *Let $\mathcal{H}^{p,q}(X)$ denote the space of harmonic forms of type (p, q) on X. Then for each p, q we have:*

(i) $\mathcal{H}^{p,q}(X)$ is finite dimensional

(ii) the orthogonal projection $\mathcal{H} : A^{p,q}(X) \to \mathcal{H}^{p,q}(X)$ is well-defined, and there exists a unique operator (the Green's operator) $G = G^{p,q} : A^{p,q}(X) \to A^{p,q}(X)$ *with kernel $\mathcal{H}^{p,q}$, such that G commutes with both $\overline{\partial}$ and $\overline{\partial}^*$, and*

$$\mathcal{H} + \Delta_{\overline{\partial}} \circ G = \text{ identity}$$

on $A^{p,q}(X)$.

Thus there is an orthogonal direct sum decomposition for each p, q

$$A^{p,q}(X) = \mathcal{H}^{p,q}(X) + \overline{\partial}A^{p,q-1}(X) + \partial^* A^{p,q+1}(X).$$

Corollary 3 *There is a natural identification of $\mathcal{H}^{p,q}(X)$ with $H^{p,q}_{\overline{\partial}}(X)$.*

Proof: If ω is $\overline{\partial}$-closed, then

$$\begin{aligned} \omega &= \mathcal{H}(\omega) + \overline{\partial}(\overline{\partial}^* G\omega) + \overline{\partial}^*\overline{\partial}G\omega \\ &= \mathcal{H}(\omega) + \overline{\partial}(\overline{\partial}^* G\omega) + \overline{\partial}^* G\overline{\partial}\omega \\ &= \mathcal{H}(\omega) + \overline{\partial}(\overline{\partial}^* G\omega). \end{aligned}$$

Hence ω and $\mathcal{H}(\omega)$ represent the same element of $H^{p,q}_{\overline{\partial}}(X)$, *i.e.*, $\mathcal{H}^{p,q}(X) \to H^{p,q}_{\overline{\partial}}(X)$ is surjective. If $\omega \in \mathcal{H}^{p,q}(X)$ is $\overline{\partial}$-exact, *i.e.*, $\omega = \overline{\partial}\eta$, then

$$\| \omega \|^2 = (\omega, \overline{\partial}\eta) = (\overline{\partial}^*\omega, \eta) = 0,$$

that is, $\omega = 0$. $\square$

In a similar vein, one has the Hodge theorem for the exterior derivative operator d. Let $A^k(X)$ be the vector space of smooth ($\mathbf{C}$-valued) k-forms on X. Then the Hodge star operator gives a map $*: A^k(X) \to A^{2n-k}(X)$, and hence we obtain a Hermitian inner product on $A^k(X)$ by

$$(\omega, \eta) = \int_X \omega \wedge *\eta,$$

as before. This is in fact the Hermitian extension of a positive definite inner prodcut on the $\mathbf{R}$-subspace of *real forms* in $A^k(X)$. This is because the Hodge $*$-operator is in fact a real operator.

Now the exterior derivative operator d has an adjoint $d^* = - * d *$, and we can define the corresponding *Laplace-Beltrami operator* $\Delta_d = dd^* + d^*d$. A smooth k-form ω is *d-harmonic* (or just *harmonic*) if $d\omega = d^*\omega = 0$, or equivalently, if $\Delta_d \omega = 0$.

Theorem 10 ('Real' Hodge theorem) *Let $\mathcal{H}^k(X) \subset A^k(X)$ denote the space of harmonic forms of degree k on X. Then for each k, we have:*

(i) $\mathcal{H}^k(X)$ is finite dimensional

(ii) the orthogonal projection $\mathcal{H} : A^k(X) \to \mathcal{H}^k(X)$ is well-defined, and there exists a unique operator (the Green's operator) $G = G^k : A^k(X) \to A^k(X)$ *with kernel $\mathcal{H}^k$, such that G commutes with both d and d^*, and*

$$\mathcal{H} + \Delta_d \circ G = \textit{ identity}$$

on $A^k(X)$.

Thus there is an orthogonal direct sum decomposition for each k

$$A^{p,q}(X) = \mathcal{H}^{p,q}(X) + d(A^{k-1}(X)) + d^*(A^{k+1}(X)).$$

Finally, d, d^*, $\mathcal{H}$, G are all real operators*, i.e., commute with complex conjugation on forms.*

Corollary 4 *There is a natural identification of $\mathcal{H}^k(X)$ with the de Rham cohomology $H^k_{DR}(X)$, compatible with complex conjugation.*

4.3 The Kähler condition

Recall that a Hermitian metric h has an associated $(1,1)$ form ω, given by the imaginary part of h. We say h is a *Kähler metric* if $d\omega = 0$. Two basic examples of Káhler metrics on a complex manifold are (i) the standard Hermitian metric $\sum_{j=1}^n dz_j \otimes d\overline{z}_j$ on $\mathbf{C}^n$, and (ii) the *Fubini-Study metric* on $\mathbf{P}^n_{\mathbf{C}}$ —- if $X_0, \ldots, X_n$ are standard homogenous coordinates on $\mathbf{P}^n_{\mathbf{C}}$, and U is the open set with $X_0 \neq 0$, let $z_j = X_j/X_0$ be the standard affine coordinates on U; then

$$\omega = \frac{1}{2\pi}\left[\frac{\sum_j dz_j \wedge d\overline{z}_j}{1+\sum_j z_j\overline{z}_j} - \frac{(\sum_j \overline{z}_j dz_j) \wedge (\sum_j w_j d\overline{w}_j)}{(1+\sum_j z_j\overline{z}_j)^2}\right]$$

is a $(1,1)$ form on U, which extends to a $(1,1)$ form on $\mathbf{P}^n_{\mathbf{C}}$ which is invariant under the group $\mathrm{PGL}_n(\mathbf{C})$ of projective linear transformations, and is the imaginary part of a unique, $\mathrm{PGL}_n(\mathbf{C})$-invariant Hermitian metric on $\mathbf{P}^n_{\mathbf{C}}$. Since one computes readily that $d\omega = 0$, we see that the Fubini-Study metric is Kähler. Further, one can compute that $H^2_{DR}(\mathbf{P}^n_{\mathbf{C}}) = \mathbf{C}$ for any $n \geq 1$, and the class of ω is a generator; under the isomorphism $H^2_{DR}(\mathbf{P}^n_{\mathbf{C}}) \cong H^2(\mathbf{P}^n_{\mathbf{C}}, \mathbf{C}) = H^2(\mathbf{P}^n_{\mathbf{C}}, \mathbf{Z}) \otimes \mathbf{C}$ given by *de Rham's theorem*, in fact ω is a generator of $H^2(\mathbf{P}^n_{\mathbf{C}}, \mathbf{Z}) = \mathbf{Z}$.

Lemma 5 *Let $X \subset Y$ be a complex submanifold of a complex manifold Y, and let h be a Hermitian metric on Y. Then the restriction $h\,|_X$ of h to X is a Hermitian metric on X, such that if $i : X \to Y$ is the inclusion, then the corresponding $(1,1)$ forms are related by*

$$i^*\omega_h = \omega_{h|_X}.$$

In particular, if h is a Kähler metric on Y, then $h\,|_X$ is a Kähler metric on X.

Proof: Let $\dim X = m$, $\dim Y = n$. For each $x \in X$, $T_xX \subset T_xY$, and we have a corresponding restriction map $T^*_xY \twoheadrightarrow T^*_xX$. By Gram Schmid, we can find a coframe $\varphi_1, \ldots, \varphi_n$ for Y near x, such that φ_j, $m+1 \leq j \leq n$ lie in the kernel of the restriction map to X. Then $i^*\varphi_j$, $1 \leq j \leq m$ are a coframe for $h\,|_X$, and clearly

$$i^*\omega_h = i^*(\sum_{j=1}^n \varphi_j \wedge \overline{\varphi}_j) = \sum_{j=1}^m i^*\varphi_j \wedge i^*\overline{\varphi}_j = \omega_{h|_X}.$$

□

Thus, any non-singular complex projective algebraic variety X has a Kähler Hermitian metric. If h is a Kähler metric on X, the associated $(1,1)$-form ω is called the *Kähler form* on X. We call such a pair (X,h) or (X,ω) a *Kähler manifold*; however, we will often abuse notation and refer to X itself as a Kähler manifold, meaning that there is a choice of Käher metric implicit in the discussion.

Lemma 6 *Let X be a compact Kähler manifold.*

(i) The even de Rham cohomology groups $H^{2k}_{DR}(X)$ are all non-zero, $0 \leq k \leq n = \dim X$.

(ii) The space of holomorphic k-forms injects into $H^k_{DR}(X)$, i.e., all non-zero holomorphic k-forms are d-closed, and non d-exact.

(iii) If $V \subset X$ is any closed analytic subvariety, of codimension p, then the corresponding fundamental cohomology class $\eta_V \in H^{2p}_{DR}(X)$ *is non-zero.*

Proof: (i) Since ω is d-closed, ω^k represents an element of $H^{2k}_{DR}(X)$, for each $1 \leq k \leq n = \dim X$. If $\omega^k = d\eta$, then $\omega^n = d(\eta \wedge \omega^{k-n})$ is exact as well, and so $\int_X \omega^n = 0$. This contradicts that ω^n is a non-zero positive multiple of the volume form of X.

(ii) If η is a holomorphic k-form, $k \leq n = \dim X$, and we have a local expression
$$\eta = \sum_I \eta_I \varphi_I$$
for a local coframe, then we compute that
$$\eta \wedge \overline{\eta} = \sum_{I,J} \eta_I \overline{\eta_J} \varphi_I \wedge \overline{\varphi}_J;$$
since
$$\omega^{n-k} = C_k \sum_{\#K=n-k} \varphi_K \wedge \overline{\varphi}_K$$
for a certain non-zero constant C_k, we then have
$$\eta \wedge \overline{\eta} \wedge \omega^{n-k} = (\frac{C_k}{(n-k)!} \sum_I \mid \eta_I \mid^2) \cdot \Phi.$$

Hence

$$\int_X \eta \wedge \overline{\eta} \wedge \omega^{n-k} \neq 0$$

if $\eta \neq 0$. On the other hand, if $\eta = d\mu$ is exact, then $\eta \wedge \overline{\eta} \wedge \omega^{n-k} = d(\mu \wedge \overline{\eta} \wedge \omega^{n-k})$ is exact as well, and hence has vanishing integral over X; hence if η is non-zero holomorphic, then η is non-exact. This also forces η to be closed, else $d\eta = \partial\eta$ would be a non-zero, exact holomorphic $k+1$-form.

(iii) We first recall the definition of the fundamental class η_V in the de Rham cohomology $H^{2p}_{DR}(X)$. From the de Rham theorem and Poincaré duality, the pairing $H^{2p}_{DR}(X) \otimes H^{2n-2p}_{DR}(X) \to \mathbf{C}$, $\omega \otimes \eta \mapsto \int_X \omega \wedge \eta$, is a perfect pairing of finite dimensional $\mathbf{C}$-vector spaces. Now V determines a functional

$$H^{2n-2p}_{DR}(X) \to \mathbf{C},$$

$$\omega \mapsto \int_V *\omega \mid_{V^*},$$

where $V^* \subset V$ is the dense open subset of non-singular points (*i.e.*, the set of points of V where V is a complex submanifold of X). One first needs to prove that this is well-defined, *i.e.*, that the integral is finite for any closed form ω, and vanishes for exact forms; the first property can be proved by locally representing V as a branched covering of a polydisk in $\mathbf{C}^m$ ($m = \dim V$), and the second follows from a version of Stokes' theorem (see the book [GH], pages 32-33 for details). Now one appeals to *Wirtinger's theorem*, which states that

$$\int_{V^*} \omega^{n-p} \mid_{V^*} = (n-p)!\,\mathrm{Volume}\,(V^*),$$

where the volume is measured using the volume form of the Riemannian metric induced on V^* from that on X; this is of course finite, as a particular case of our earlier remarks, and it is evidently positive. Hence the functional $\eta_V \in H^{2p}_{DR}(X)$ is non-zero. (Incidentally, Wirtinger's theorem is a consequence of lemma 5, since $h \mid_{V^*}$ is the induced Kähler metric from X, with corresponding Kähler form $\omega \mid_{V^*}$; now the $n-p$-th power of this is proportional to the volume form on V^*.) □

Remark: The condition (iii) of the above lemma has been used by Hironaka to construct examples of non-singular 'abstract' algebraic varieties over $\mathbf{C}$ which are compact, but not projective. See [H], Appendix. His argument is actually valid over any algebraically closed field, but then uses the intersection theory of algebraic cycle classes in place of de Rham cohomology.

Let X be a compact Kähler manifold with Kähler form ω. Let $L : A^{p,q}(X) \to A^{p+1,q+1}(X)$ denote the *Lefschetz operator* defined by

$$L(\eta) = \omega \wedge \eta,$$

and let $\Lambda = L^* : A^{p+1,q+1}(X) \to A^{p,q}(X)$ be its adjoint. Define new operators

$$d^c = \frac{1}{4\pi}(\overline{\partial} - \partial), \quad \Delta_\partial = \overline{\Delta_{\overline{\partial}}} = \partial\partial^* + \partial^*\partial.$$

One computes at once that

$$dd^c + d^c d = 0$$

and

$$dd^c = \frac{1}{2\pi}\partial\overline{\partial}.$$

Further, taking complex conjugates, the analogue of the Hodge theorem (Theorem 9) is valid for Δ_∂ as well.

Lemma 7 (Kähler identitites) *The following formulas hold.*

(i) (Basic Identity) $[L, d^*] = 4\pi d^c$ *and* $[\Lambda, d] = -4\pi d^{c*}$.

(ii) $[\Lambda, \overline{\partial}] = -\imath\partial^*$ *and* $[\Lambda, \partial] = \imath\overline{\partial}^*$.

(iii) $[L, d] = [\Lambda, d^*] = 0$.

(iv) $[L, \Delta_d] = [\Lambda, \Delta_d] = 0$, *and* $\partial\overline{\partial}^* + \overline{\partial}^*\partial = \partial^*\overline{\partial} + \overline{\partial}\partial^* = 0$.

(v) $\Delta_\partial = \Delta_{\overline{\partial}} = \dfrac{1}{2}\Delta_d$.

Proof: The two forms of the Basic Identity, as well as the two formulas in (ii) are (respectively) equivalent to each other, by adjointess. Further, by decomposition into types, (ii) is equivalent to (i). These identities are proved by first proving analogous identities for the standard metric on $\mathbf{C}^n$, where the operators are now regarded as acting on compactly supported smooth forms on $\mathbf{C}^n$ (so that we still have Hermitian inner products on spaces of forms, and adjoints make sense). This involves elementary, but tedious, computations (see pages 111-114 in [GH]). The general case is reduced to this by showing that for any $x \in X$, the Käler metric has a unitary coframe $\varphi 1, \ldots, \varphi_n$ near x such that $d\varphi_j(x) = 0$ for all j. Now one argues that, if one tries to carry out the same computations as in the Euclidean case, say for the identity $[\Lambda, \overline{\partial}] = -\imath\partial^*$, we get all the same terms as before, as well as additional terms which contain factors of first derivatives $\overline{\partial}\varphi_j$ (on $\mathbf{C}^n$, we would get $\overline{\partial}(z_j) = 0$). But at x, all these additional terms will vanish, so the desired identity will hold at x.

Since ω is closed, $d(\omega \wedge \eta) = \omega \wedge d\eta$, and so $[L, d] = 0$; adjointness gives $[\Lambda, d^*] = 0$. We now compute that

$$[L, \Delta_d] = Ldd^* + Ld^*d - dd^*L - d^*dL$$

$$= dLd^* + Ld^*d - dd^*L - d^*Ld = d(Ld^* - d^*L) + (Ld^* - d^*L)d = 4\pi(dd^c + d^c d) = 0.$$

By adjointness, $[\Lambda, \Delta_d] = 0$ (since Δ_d is self-adjoint). Next, since $\overline{\partial}^* = -\imath[\Lambda, \partial]$, we get that

$$\imath(\partial\overline{\partial}^* + \overline{\partial}^*\partial) = \partial[\Lambda, \partial] + [\Lambda, \partial]\partial$$

$$= \partial\Lambda\partial - \partial^2\Lambda + \Lambda\partial^2 - \partial\Lambda\partial = 0.$$

Now adjointness gives $\partial^*\overline{\partial} + \overline{\partial}\partial^* = 0$.

Hence

$$\Delta_d = (\partial + \overline{\partial})(\partial^* + \overline{\partial}^*) + (\partial^* + \overline{\partial}^*)(\partial + \overline{\partial}) = \Delta_\partial + \Delta_{\overline{\partial}}.$$

It remains to show that $\Delta_{\overline{\partial}} = \Delta_\partial$. For this we use

$$-\imath\Delta_\partial = \partial[\Lambda, \overline{\partial}] + [\Lambda, \overline{\partial}]\partial$$

$$= \partial\Lambda\overline{\partial} - \partial\overline{\partial}\Lambda + \Lambda\overline{\partial}\partial - \overline{\partial}\Lambda\partial,$$

and so

$$\imath\Delta_{\overline{\partial}} = \overline{\partial}[\Lambda, \partial] + \Lambda, \partial]\overline{\partial}$$

$$= \overline{\partial}\Lambda\partial - \overline{\partial}\partial\Lambda + \Lambda\partial\overline{\partial} - \partial\Lambda\overline{\partial}$$

$$= \imath\Delta_\partial,$$

since $\partial\overline{\partial} + \overline{\partial}\partial = 0$. □

As a consequence, we obtain that *$\overline{\partial}$-harmonic forms are d-harmonic.* We also get that Δ_d *preserves the type decomposition*; in particular, if $\omega = \sum_{p,q} \omega^{p,q}$, then ω is harmonic $\Leftrightarrow$ each $\omega^{p,q}$ is harmonic. Finally, we obtain the decomposition

$$\mathcal{H}^k(X) = \bigoplus_{p \geq 0} \mathcal{H}^{p,k-p}_{\overline{\partial}},$$

which can be viewed as a decomposition on cohomology groups

$$H^k_{DR}(X) = \bigoplus_{p\geq 0} H^{p,k-p}_{\overline{\partial}}(X).$$

Identifying $H^k_{DR}(X)$ with the singular cohomology $H^k(X, \mathbf{C})$, we obtain the *Hodge Decomposition*

$$H^k(X, \mathbf{C}) = \bigoplus_{p\geq 0} H^{p,k-p}(X),$$

where $H^{p,k-p}(X)$ is the image of $\mathcal{H}^{p,k-p}_{\overline{\partial}}$ under the above series of identifications; we then have also

$$\overline{H^{p,k-p}(X)} = H^{k-p,p}(X) \ \text{(Hodge symmetry)}$$

Since $H^{p,0}_{\overline{\partial}}(X) = \mathcal{H}^{p,0}_{\overline{\partial}}$ is naturally identified with the space of holomorphic p-forms, we get that holomorphic forms on X are harmonic for any Kähler metric on X. Another consequence is that *the odd Betti numbers of a compact Kähler manifold X are even*, since

$$\dim_{\mathbf{C}} H^{2k+1}(X, \mathbf{C}) = \sum_{p=0}^{k} \dim_{\mathbf{C}} H^{p,2k+1-p}(X) + \dim_{\mathbf{C}} H^{2k+1-p,p}(X) = 2\sum_{p=0}^{k} \dim H^{p,2k+1-p}(X).$$

4.4 The Hard Lefschetz Theorem

Theorem 11 (Hard Lefschetz Theorem) *Let X be compact Kähler of dimension n. Then the map*

$$L^k : H^{n-k}(X, \mathbf{C}) \to H^{n+k}(X, \mathbf{C})$$

is an isomorphism, for each $k \leq n$. If we define the primitive cohomology *by*

$$P^{n-k}(X, \mathbf{C}) = \ker\left(L^{k+1} H^{n-k}(X, \mathbf{C}) \to H^{n+k+2}(X, \mathbf{C})\right),$$

then we have the Lefschetz decomposition

$$H^m(X, \mathbf{C}) = \bigoplus_{k\geq 0} L^k P^{m-2k}(X, \mathbf{C}).$$

Note that the Lefschetz decomposition is compatible with the Hodge decomposition, in the sense that if we define

$$P^{r,s}(X) = P^{r+s}(X, \mathbf{C}) \cap H^{r,s}(X),$$

then

$$P^m(X, \mathbf{C}) = \bigoplus_{k \geq 0} P^{k,m-k}(X).$$

If the class of ω in $H^2_{DR}(X) \cong H^2(X, \mathbf{Z}) \otimes \mathbf{C}$ corresponds to a rational cohomology class under the above de Rham isomorphsim, then the Hard Lefschetz Theorem, the definition of primitive cohomology and the Lefschetz Decomposition are valid for cohomology with rational coefficients as well. However this is not the case with integral cohomology, in genenral. If the Kähler class is induced from a projective embedding $X \hookrightarrow \mathbf{P}^N_{\mathbf{C}}$, then the Kähler class is indeed an integral, hence rational cohomology class. In this case, L has the following geometric/topological interpretation: it is the cup product with the fundamental cohomology class of a *hyperplane intersection* $H \cap X$, where $H \cong \mathbf{P}^{N1}_{\mathbf{C}}$ is a projective linear hyperplane intersecting X transversally. This is because the Kähler class of the Fubini-Study metric on $\mathbf{P}^N_{\mathbf{C}}$ is the fundamental cohomology class of any hyperplane.

We may use the Hard Lefschetz Theorem to state the *Hodge-Riemann bilinear relations.* Define a bilinear form on $H^{n-k}(X, \mathbf{C})$ ($k \geq 0$) by the formula

$$Q(\eta, \psi) = \int_X \eta \wedge \psi \wedge \omega^k.$$

Since ω is a real form, Q in fact defines a real valued bilinear form on $H^{n-k}(X, \mathbf{R})$. Also $H^{p,q}$ and $H^{p',q'}$ are orthogonal with respect to Q unless $(p, q) = (q', p')$.

Now the bilinear relations assert that for any $\xi \in P^{r,s}(X)$,

$$\imath^{r-s}(-1)^{(n-r-s)(n-r-s-1)/2} Q(\xi, \overline{\xi}) > 0.$$

In particular, if $r + s$ is even,

$$\imath^{r-s}(-1)^{(n-r-s)(n-r-s-1)/2} Q$$

defines a *positive definite quadratic form* on

$$(P^{r,s}(X) \oplus P^{s,r}(X)) \cap H^{r+s}(X, \mathbf{R}).$$

If $r+s$ is odd, then Q defines a *non-degenerate alternating form* on $P^{r+s}(X)$. Since $Q(L^k\eta, L^k\psi)Q(\eta, \psi)$ for any primitive classes η, ψ we have by the Lefschetz decomposition that Q*is non-degenerate on* $H^{n-k}(X, \mathbf{C})$.

5 Topology of Varieties

In this lecture we discuss several aspects of the topological structure of algebraic varieties. We begin with an outline of the proof of the Lefschetz Hyperplane Theorem via Morse Theory, following Andreotti and Frankel (as exposed in [Mi]).

5.1 Review of Morse Theory

We first review, without proofs, some of the basis facts from Morse Theory. We begin by recalling the basic definitions.

Let M be a smooth manifold, and $f : M \to \mathbf{R}$ a smooth function. A point $x \in M$ is called a *critical point* for f if $df(x) = 0$; equivalently, if $x_1, \ldots, x_n$ are local coordinates on M near x, then $\frac{\partial f}{\partial x_i}(x) = 0$ for all $1 \leq i \leq n$. We say that a critical point x is *non-degenerate* if, in any such system of local coordinates, the matrix of second partial derivatives

$$H = \left[\frac{\partial^2 f}{\partial x_i \partial x_j}(x) \right]$$

is non-singular. One checks easily that the definition of non-degeneracy of a critical point is independent of the choice of local coordinates. Finally, a *critical value* of f is the image of a critical point of f.

The matrix of second partial derivatives (the *Hessian*)

$$H = \left[\frac{\partial^2 f}{\partial x_i \partial x_j}(x) \right]$$

is symmetric, and defines a quadratic form

$$q(y_1, \ldots, y_n) = \sum_{i,j} \frac{\partial^2 f}{\partial x_i \partial x_j}(x) y_i y_j.$$

The equivalence class of this form, *i.e.*, the *nullity* and *index*, are independent of the choice of local coordinates; here recall that the nullity is the number of 0-eigenvalues of H, and the index is the number of *negative eigenvalues* of H. One has the following lemma.

Lemma 8 *Let $f : M \to \mathbf{R}$ be a smooth function on a smooth n-manifold, and $x \in M$ a non-degenerate critical point. Then there is a system of local* *coordinates $x_1, \ldots, x_n$ on M in a neighbourhood U of x such that on U, we have*

$$f(x_1, \ldots, x_n) = f(x) + x_1^2 + \cdots + x_{n-k}^2 - x_{n-k+1}^2 - \cdots - x_n^2.$$

The integer k depends only on f, and is the index of the critical point.

In particular, any non-degenerate critical point is isolated.

Finally, a *Morse function* on a manifold M is a smooth function f such that

(i) the sets $M_{f,c} = f^{-1}((-\infty, c])$ are compact, for all $c \in \mathbf{R}$

(ii) the critical points of f are all non-degenerate.

Theorem 12 (Main Theorem of Morse Theory) *Let $f : M \to \mathbf{R}$ be a Morse function on a smooth manifold M. Then M has the homotopy type of a CW complex, whose k-cells are in bijection with the critical points of f of index k, for each $k \geq 0$.*

The proofs of the above lemma and theorem can be found in §2-3 of Milnor's book [Mi] on Morse Theory.

5.2 The distance squared as a Morse function

Let $M \subset \mathbf{R}^m$ be a closed smooth submanifold (by the Whitney Embedding Theorem, every smooth n manifold M can be realized as a closed smooth submanifold of $\mathbf{R}^{2n+1}$). Let $p \in \mathbf{R}^m$ be a point, and let $f_p : M \to \mathbf{R}$ be the function given by the square of the Euclidean distance from p,

$$f_p(x) = \| x - p \|^2 = < x, x > + < p, p > -2 < x, p > . \qquad (6)$$

Here $< \, , \, >$ denotes the Euclidean inner product. Following the treatment of Milnor's book [Mi], §6, we discuss when this is a Morse function, and show how to compute the index of f at a critical point of f in terms of other data.

Let N denote the normal bundle of M. It may be regarded as the set

$$N = \{(x, v) \in M \times R^m \mid \ v \text{ is normal to } M \text{ at } x\}.$$

Consider the map

$$\varphi : N \to \mathbf{R}^m, \quad (x, v) \mapsto x + v.$$

Then φ is a smooth map between smooth m-dimensional manifolds, whose Jacobian is non-singular at points $(x, 0)$ (the zero-section of N). Recall that a critical point of φ is a point where the Jacobian (with respect to any system of local coordinates) is singular; a critical value is the image of a critical point. By Sard's Theorem, we know that *the set of critical values of φ has measure 0* in $\mathbf{R}^m$ (see [Mi2], Chapter 2, for a proof of Sard's Theorem).

We call a point $y \in \mathbf{R}^m$ a *focal point of multiplicity* μ of (M, x) if $y = \varphi(x, v)$, where (x, v) is a critical point of φ, and

$$\mu = \dim_{\mathbf{R}} \ker(d_{x,v}\varphi : T_{x,v}N \to T_y\mathbf{R}^m).$$

Note that if y is a focal point of (M, x), then $v = y - x$ is normal to M at x. We say $y \in \mathbf{R}^m$ is a focal point of M if it is a focal point of (M, x) for some x; equivalently, y is a critical value for φ.

The interest in this concept for us is seen in the following lemma.

Proposition 13 *(i) $x \in M$ is a critical point for f_p $\Leftrightarrow$ $v = p - x$ is normal to M at x*

(ii) $x \in M$ is a degenerate critical point for $f_p : M \to \mathbf{R}$ if and only if p is a focal point of (M, x)

(iii) If x is a non-degenerate critical point for f_p, then the index of f_p at x equals the number of focal points for (M, x), counted with multiplicity, which lie on the line segment joining x and p. This index is always $\leq n = \dim M$.

In particular, for almost all $p \in \mathbf{R}^m$, f_p is a Morse function on M.

Proof: This is a local computation. Suppose $x \in M$. Let $u_1, \ldots, u_n$ be local coordinates on M on a neighbourhood U of x, and let

$$w_1 = (w_{11}, \ldots, w_{1m}), \ldots, w_{m-n} = (w_{m-n\,1}, \ldots, w_{m-n\,m})$$

be local vector ($\mathbf{R}^m$) valued functions on U whose values at any $y \in U$ give an orthonormal basis of the normal space N_yM to M at y. If $\pi : N \to M$ is the projection, then we have coordinates $u_1, \ldots, u_n, t_1, \ldots, t_{m-n}$ on $\pi^{-1}(U)$, where

$$(u_1, \ldots, u_n, t_1, \ldots, t_{m-n}) \mapsto (x(u), v),$$

$$x(u) = (x_1(u_1, \ldots, u_n), \ldots, x_m(u_1, \ldots, u_n)), \quad v = \sum_j t_j w_j(u_1, \ldots, u_n)).$$

In terms of these corrdinates, φ is given by

$$\varphi(u_1, \dots, u_n, t_1, \dots, t_{m-n}) = x + \sum_{j=1}^{m-n} t_j w_j.$$

This has partial derivatives

$$\frac{\partial \varphi}{\partial u_i} = \frac{\partial x}{\partial u_i} + \sum_j t_j \frac{\partial w_j}{\partial u_i},$$

$$\frac{\partial \varphi}{\partial t_j} = w_j.$$

Now form the matrix of inner products of these m partial derivative vectors with the m linearly independent vectors

$$\frac{\partial x}{\partial u_i} = (\frac{\partial x_1}{\partial u_i}, \cdots, \frac{\partial x_m}{\partial u_i}), \ \ 1 \leq i \leq n,$$

$$w_1, \dots, w_{m-n}.$$

This is equivalent to multiplying the Jacobian matrix of φ by a non-singular matrix (and hence preserves the rank). We obtain the $m \times m$ matrix

$$\begin{bmatrix} J & K \\ 0_{m-n \times m} & I_{m-n} \end{bmatrix}$$

where

$$J = \left[< \frac{\partial x}{\partial u_i}, \frac{\partial x}{\partial u_k} > + \sum_j t_j < \frac{\partial w_j}{\partial u_i}, \frac{\partial x}{\partial u_k} > \right]_{1 \leq i \leq n, 1 \leq k \leq n},$$

$$K = \left[\sum_j t_j < \frac{\partial w_j}{\delta u_i}, w_l > \right]_{1 \leq i \leq n, 1 \leq l \leq m-n},$$

$0_{r \times s}$ is an $r \times s$ matrix of zeroes, and I_r is an identity matrix of size r. This follows by a simple computation using the formulas

$$< w_j, \frac{\partial x}{\partial u_i} >= 0 \ \ \forall \, 1 \leq i \leq n, 1 \leq j \leq m-n$$

$$< w_j, w_l >= \delta_{jl} \ \ \forall \, 1 \leq j, l \leq m-n$$

(the first set of formulas express that each w_j is normal to M, and the second, that the w_j give an orthonormal basis for the nromal space at each point.)

Thus, $(u_1, \ldots, u_n, t_1, \ldots, t_{m-n})$ is a critical point precisely when the matrix

$$J = \left[< \frac{\partial x}{\partial u_i}, \frac{\partial x}{\partial u_k} > + \sum_j t_j < \frac{\partial w_j}{\partial u_i}, \frac{\partial x}{\partial u_k} > \right]_{1 \le i \le n, 1 \le k \le n}$$

is singular. Now using

$$0 = \frac{\partial}{\partial u_i} < w_j, \frac{\partial x}{\partial u_k} > = < \frac{\partial w_j}{\partial u_i}, \frac{\partial x}{\partial u_k} > + < w_j, \frac{\partial^2 x}{\partial u_i \partial u_k} >,$$

we may rewrite J as

$$[g_{ik} - < v, \ell_{ik} >],$$

where

$$g_{ik} = < \frac{\partial x}{\partial u_i}, \frac{\partial x}{\partial u_k} >, \ \ \ell_{ik} = \frac{\partial^2 x}{\partial u_i \partial u_k}, \ v = \sum_j t_j w_j;$$

here v is a nomal vector to M at $x = x(u_1, \ldots, u_n)$.

Hence we have proved:

Lemma 9 *For $v \in \mathbf{R}^m$, the vector $x + v$ is a focal point for (M, x) with multiplicity $\mu \Leftrightarrow$ the matrix*

$$J = [g_{ik} - < v, \ell_{ik} >]$$

is singular, with nullity (i.e., dimension of kernel) equal to μ.

We now want to relate this to critical points for the distance function f_p. We have from equation (6) that

$$\frac{\partial f_p}{\partial u_i} = 2 < \frac{\partial x}{\partial u_i}, x - p >,$$

so that x is critical for $f_p \Leftrightarrow v = p - x$ is normal to M at x; in this case, the Hessian at x is

$$\left[\frac{\partial^2 f_p}{\partial u_i \partial u_k} \right] = [2(g_{ik} - < v, \ell_{ik} >)] = 2J.$$

Hence from the lemma, we conclude that x is a degenerate critical point for $f_p \Leftrightarrow p$ is a focal point for (M, x).

Now suppose p is not a focal point for M, so that f_p is a Morse function. As above, x is critical for f_p precisely when $v = p - x$ is normal to M at x. The index of f_p at x is then given by the number of negative eigenvalues, counted with multiplicity, of

$$J = [g_{ik} - < v, \ell_{ik} >] .$$

We may always make a linear change of coordinates (in the u_i) so that g_{ik} is the identity matrix; then the index of f_p at x is the number of eigenvalues of $[< v, \ell_{ik} >]$ which are > 1. If t is such an eigenvalue, then clearly $x + t^{-1}v$ is a focal point for (M, x), lying on the line segment joining x to p. This implies that index of f_p at x equals the number of focal points for (M, x), counted with multiplicity, which lie on the line segment joining x and p. This finishes the proof of Proposition 9. □

Remark: Note that since $\varphi : N \to \mathbf{R}^m$ is always a diffeomorphism on a neighbourhood of the 0-section $M \times \{0\} \subset N$, $x \in M$ is never a focal point for (M, x).

Corollary 5 *Any smooth manifold of dimension n has the homotopy type of a CW complex of dimension $\leq n$.*

5.3 The Lefschetz Hyperplane Theorem

Theorem 14 (Weak Lefschetz, first form) *Let $M \subset \mathbf{C}^N$ be a closed complex submanifold of dimension n (for example, a nonsingular affine variety of dimension n). Then M has the homotopy type of a CW complex of dimension $\leq n$.*

Proof: We make use of 2 lemmas.

Lemma 10 *Let $Q(z_1, \ldots, z_n) = \sum a_{ij} z_i z_j$ be a quadratic form in n complex variables, and let*

$$\widetilde{Q}(x_1, \ldots, x_n, y_1, \ldots, y_n) = \text{ Real part of } Q(x_1 + \imath y_1, \ldots, x_n + \imath y_n).$$

If e is an eigenvalue of $\widetilde{Q}$ of multiplicity μ, then $-e$ is also an eigenvalue of multiplicity μ.

Proof: Since $Q(\imath z_1, \ldots, \imath z_n) = -Q(z_1, \ldots, z_n)$, the quadratic forms $\widetilde{Q}$ and $-\widetilde{Q}$ are related by an orthogonal transformation, and hence have the same eigenvalues, with multiplicites. □

Lemma 11 *If $x \in M$, and $x + v \in \mathbf{C}^N = \mathbf{R}^{2N}$ is a focal point for (M, x) with multiplicity μ, then $x - v$ is also a focal point for M with the same multiplicity.*

Proof: For $v, w \in \mathbf{C}^N = \mathbf{R}^{2n}$, we can consider the Euclidean innder product $< v, w >$ as the real part of a Hermitian inner product,

$$<< v, w >>= \sum_{i=1}^{N} v_i \overline{w}_i.$$

Choose anaytic coordinates $z_1, \ldots, z_n$ on M near x so that $z_i(x) = 0$ for all i. Let v be a vector normal to M at x. Let $w_1, \ldots, w_N$ be the coordinate functions on M, so that $w_i = w_i(z_1, \ldots, z_n)$ are holomorphic functions in a neighbourhood of the origin. Then $<< w, v >>$ is a holomorphic function of $z_1, \ldots, z_n$ near the origin, and so has a power series expansion

$$<< w, v >>= \sum_{i=1}^{N} w_i(z_1, \ldots, z_n)\overline{v_i} = \text{constant} + Q(z_1, \ldots, z_n) + \text{higher degree terms},$$

where Q is a homogeneous quadratic polynomial in the z_j (there is no linear term since v is normal to M at x). Hence if we set $z_i = x_i + \imath y_i$, then $< w, v >$ has a real power series expansion

$$< w, v >= \text{constant} + \widetilde{Q}(x_1, \ldots, x_n, y_1, \ldots, y_n) + \text{ higher terms},$$

where the notation $\widetilde{Q}$ is as in lemma 10. Since the eigenvalues of $\widetilde{Q}$ occur in opposite pairs, the focal points of (M, x) along the normal line $x + \mathbf{R}v$ occur in pairs $x \pm tv$. □

Now we complete the proof of theorem 14, as follows. Choose a point $p \in \mathbf{C}^N = \mathbf{R}^{2N}$ such that f_p is a Morse function for M. If x is any critical point for f_p, then the index of f_p at x is the number of focal points, counted with multiplicity, lying on the line segment joining x to p (where $p - x$ is normal to M at x). There are at most $2n$ such focal points on the line $x + \mathbf{R}(p - x)$, and if $x + t(p - x)$ is a focal point with multiplicity μ, so is $x - t(p - x)$. Hence at most n of these, counted with multiplicity, can have the form $x + t(p - x)$ with $0 < t < 1$. Thus the index of any critical point of f_p is always $\leq n = \dim_{\mathbf{C}} M$. In particular, Theorem 12 imples that M has the homotopy type of a CW complex of dimension $\leq n$. □

Corollary 6 (Lefschetz hyperplane theorem) *Let $X \subset \mathbf{P}^N_{\mathbf{C}}$ be a projective variety of dimension n, and $H \subset \mathbf{P}^N_{\mathbf{C}}$ a hyperplane containing the singular locus of X, i.e., such that $X - X \cap H$ is a non-singular affine variety of pure dimension n. Then $H_i(X \cap H, \mathbf{Z}) \to H_i(X, \mathbf{Z})$ is an isomorphism for $i < n-1$, and is surjective for $i = n-1$.*

Proof: From the long exact sequence of homology groups of the pair $(X, X \cap H)$, the above result is equivalent to the vanishing of $H_i(X, X \cap H; \mathbf{Z})$ for $i \leq n-1$. By Lefschetz duality, $H_i(X, X \cap H; \mathbf{Z}) \cong H^{2n-i}(X - X \cap H; \mathbf{Z})$, and since $X - X \cap \mathbf{Z}$ is non-singular affine of dimension n, its cohomology groups $H^j(X - X \cap H; \mathbf{Z})$ vanish for $j > n$. □

A slight refinement of the above argument yields the following stronger conclusion.

Theorem 15 (Lefschetz) *Let $X \subset \mathbf{P}^N_{\mathbf{C}}$ and H be as above. Then*

$$\pi_i(X, X \cap H) = 0 \text{ for } i < n.$$

Proof: (Sketch) Let p be a point in $\mathbf{C}^N = \mathbf{P}^N_{\mathbf{C}} - H$ such that f_p is a Morse function for $X - X \cap H$. Consider the function $f : X \to \mathbf{R}$,

$$f(x) = \begin{cases} 0 & \text{if } x \in X \cap H \\ \frac{1}{f_p(x)} & \text{otherwise} \end{cases}$$

This is again sort of a Morse function on $X - X \cap H$, such that each critical point now has index $\geq n$. Hence, for any $\varepsilon > 0$, a variant of Theorem 12 implies that if $X_\varepsilon = f^{-1}([0, \varepsilon])$, then (X, X_ε) has the homotopy type a relative CW complex with finitely many cells, each of dimension $\geq k$. Hence $\pi_i(X, X_\varepsilon) = 0$ for all $i < k$. Now one argues that X_ε has the same homotopy type as $X \cap H$, for small enough ε, for example since there exists a triangulation of X such that $X \cap H$ is a subcomplex. □

5.4 Example: hypersurfaces and complete intersections

Let $X \subset \mathbf{P}^{n+1}_{\mathbf{C}}$ be a hypersurface of degree d. Then we may regard X as a hyperplane section of $\mathbf{P}^{n+1}_{\mathbf{C}}$ embedded in some projective space $\mathbf{P}^N_{\mathbf{C}}$ via the d-tuple embedding. From the Lefshctz Hyperplane theorems for homology and homotopy (Corollary 6 and Theorem 15), it follows that

$$\pi_i(X) \to \pi_i(\mathbf{P}^{n+1}_{\mathbf{C}}), \quad H_i(X, \mathbf{Z}) \to H_i(\mathbf{P}^{n+1}_{\mathbf{C}}, \mathbf{Z})$$

are isomorphisms for $i < n$ and are surjective for $i = n$. This gives that if $n \geq 2$, then X is *simply connected*. Further,

$$H_i(X, \mathbf{Z}) = \begin{cases} 0 & \text{if } i < n \text{ is odd} \\ \mathbf{Z} & \text{if } i < n \text{ is even} \\ \mathbf{Z} \oplus (?) & \text{if } i = n \text{ is even.} \end{cases}$$

Suppose further that X is non-singular. Then Poincaré duality implies that $H_i(X, \mathbf{Z}) \cong H^{2n-i}(X, \mathbf{Z})$ for all i. Since also $H^i(X, \mathbf{Z}) \cong \mathrm{Hom}\,(H_i(X, \mathbf{Z}), \mathbf{Z})$ for $i < n$ (by the Universal Coefficient Theorem in topology, since the homology in degrees $< n$ is torsion-free), we get that

$$H^i(X, \mathbf{Z}) = \begin{cases} 0 & \text{if } i \neq n \text{ is odd} \\ \mathbf{Z} & \text{if } i \neq n \text{ is even} \end{cases}$$

Further,

$$H^n(X, \mathbf{Q}) = \begin{cases} \mathbf{Q} \oplus PH^n(X, \mathbf{Z}) & \text{if } i = n \text{ is even} \\ PH^n(X, \mathbf{Q}) & \text{if } i = n \text{ is odd.} \end{cases}$$

Here $PH^n(X, \mathbf{Q}) = \ker(H^n(X, \mathbf{Q}) \to H^{n+2}(X, \mathbf{Q}))$ is the primitive middle-dimensional cohomology (in the sense of the Lefschetz decomposition, give by the Hard Lefschetz theorem).

Now we may proceed further as follows. Recall that a *complete intersection* of dimension n in $\mathbf{P}^N_{\mathbf{C}}$ is a subvariety X, such that for some homogeneous polynomials $F_1, \ldots, F_{N-n}$, the homogeneous ideal $I(X)$ is generated by $F_1, \ldots, F_{N-n}$. Let X be a non-singular complete intersection of dimension n. Bertini's Theorem then implies that there exist $N - n$ non-singular hypersurfaces $X_1, \ldots, X_{N-n}$ such that $X = X_1 \cap \cdots \cap X_{N-n}$, and all of the intersections $X_1 \cap \cdots \cap X_i$, $2 \leq i \leq N-n$ are transverse. Then $Y_i = X_1 \cap \cdots \cap X_i$ is non-singular, for each i, and Y_{i+1} is a non-singular hypersurface section of Y_i. Hence by the Weak Lefschetz theorems for homology and homotopy, we see by induction that

$$\pi_j(Y_i) \to \pi_j(\mathbf{P}^N_{\mathbf{C}}), \;\; H_j(Y_i, \mathbf{Z}) \to H_j(Y_i, \mathbf{Z})$$

are isomorphisms for $j < \dim Y_i = N - i$ and are surjective for $j = N - i$. Since $Y_{N-n} = X$, we conclude the following (using also Poincaré duality).

Corollary 7 *For any smooth projective complete intersection X of dimension n, we have*

$$H_i(X, \mathbf{Z}) = \begin{cases} 0 & \textit{if } i \neq n \textit{ is odd} \\ \mathbf{Z} & \textit{if } i \neq n \textit{ is even} \\ \mathbf{Z} \oplus (?) & \textit{if } i = n \textit{ is even.} \end{cases}$$

$$H^i(X,\mathbf{Z}) = \begin{cases} 0 & \textit{if } i \neq n \textit{ is odd} \\ \mathbf{Z} & \textit{if } i \neq n \textit{ is even} \end{cases}$$

$$H^n(X,\mathbf{Q}) = \begin{cases} \mathbf{Q} \oplus PH^n(X,\mathbf{Z}) & \textit{if } i = n \textit{ is even} \\ PH^n(X,\mathbf{Q}) & \textit{if } i = n \textit{ is odd.} \end{cases}$$

If X is a smooth surface in $\mathbf{P}^3_{\mathbf{C}}$, then by the above, the only 'unknown' cohomology group is $H^2(X,\mathbf{Z})$. Since X is simply connected, the Universal Coefficient theorem implies that this cohomology group is torsion-free, hence is free abelian of finite rank, equal to the second Betti number b_2. We now sketch an argument to compute b_2 of a surface in $\mathbf{P}^3_{\mathbf{C}}$. This depends on the following result, which is interesting in its own right. Recall that the *topological Euler characteristic* of a space X is defined to be

$$\chi_{top}(X) = \sum_{i\geq 0}(-1)^i b_i(X),$$

where $b_i(X) = \dim_{\mathbf{Q}} H_i(X,\mathbf{Q})$ is the i-th Betti number of X; for $\chi_{top}(X)$ to be well-defined, we must have that $b_i < \infty$ for all i, and is non-zero for only finitely many i. Since any projective algebraic variety over $\mathbf{C}$ is known to have a finite triangulation, its topological Euler characteristic is well-defined.

Proposition 16 *Let $f : X \to C$ be a morphism from a projective variety X to a smooth projective curve C, over the complex number field $\mathbf{C}$. Assume that there exists a finite set $S = \{x_1,\ldots,x_n\} \subset C$ such that $f^{-1}(C-S) \to C-S$ is a smooth morphism (i.e., has non-zero differential everywhere). Let F be any fiber of f over a point of C_S, and let $F_i = f^{-1}(x_i)$ be the "singular fibers" of f. Then the topological Euler characteristic of X satisfies the formula*

$$\chi_{top}(X) = \chi_{top}(C)\chi_{top}(F) + \sum_{i=1}^{n}(\chi_{top}(F_i) - \chi_{top}(F)).$$

Proof: Since $f : f^{-1}(C-S) \to C-S$ is a smooth and proper morphism (since f is proper), it is in fact a C^∞ locally trivial fiber bundle, all of whose fibers are diffeomorphic to F.

Lemma 12 *Let $\pi : E \to B$ be a locally trivial fiber bundle with fiber F, where B, F are finite CW complexes. Then $\chi_{top}(E)$ is defined, and*

$$\chi_{top}(E) = \chi_{top}(B)\chi_{top}(F).$$

Proof: We can find a finite open covering of B by subsets U_i, $1 \leq i \leq m$, such that $f^{-1}(U_i) \cong U_i \times F$ for all i, and $V_i = U_1 \cup \cdots \cup U_i$ and $U_i \cap V_{i-1}$ have well-defined topological Euler characteristics, for all $i \geq 2$. Now $\pi^{-1}(U_i) \cong U_i \times F$, so by the Kunneth formula,

$$\chi_{top}(\pi^{-1}(A)) = \chi_{top}(A)\chi_{top}(F)$$

for any subset $A \subset U_i$ for which $\chi_{top}(A)$ is well-defined. Then from the Mayer Vietoris sequences (homology is with $\mathbf{Q}$-coefficients)

$$\cdots H_i(U_i \cap V_{i-1}) \to H_i(U_i) \oplus H_i(V_{i-1}) \to H_i(V_i) \to H_{i-1}(V_{i-1} \cap U_i) \to \cdots$$

and

$$\cdots H_i(\pi^{-1}(U_i \cap V_{i-1})) \to H_i(\pi^{-1}(U_i)) \oplus H_i(\pi^{-1}(V_{i-1})) \to \\ H_i(\pi^{-1}(V_i)) \to H_{i-1}(\pi^{-1}(V_{i-1} \cap U_i)) \to \cdots$$

we conclude that

$$\chi_{top}(V_i) = \chi_{top}(U_i) + \chi_{top}(V_{i-1}) - \chi_{top}(U_i \cap V_{i-1}),$$

$$\chi_{top}(\pi^{-1}(V_i)) = \chi_{top}(\pi^{-1}(U_i)) + \chi_{top}(\pi^{-1}(V_{i-1})) - \chi_{top}(\pi^{-1}(U_i \cap V_{i-1})),$$

and hence by induction on i, that

$$\chi_{top}(\pi^{-1}(V_i)) = \chi_{top}(V_i)\chi_{top}(F),$$

for all i. The conclusion of the lemma is the case $i = m$. $\square$

In our context, choose a small neighbourhood W of S in C, such that W is a disjoint union of open disks W_i centred at each x_i. Let V be a smaller neighbourhood which is a union of concentric disks. Then from the lemma,

$$\chi_{top}(f^{-1}(C - V)) = \chi_{top}(C - V)\chi_{top}(F).$$

Further, if $S_i \subset W_i - V_i$ is a small circle in W_i around V_i, then $S_i \hookrightarrow W_i - V_i$ is a homotopy equivalence; hence also $f^{-1}(S_i) \hookrightarrow f^{-1}(W_i - V_i)$ is a homotopy equivalences, since $f^{-1}(W_i - V_i) \to W_i - V_i$ is a locally trivial fiber bundle (with fiber F). By the lemma, $\chi_{top}(f^{-1}(S_i)) = \chi_{top}(S_i)\chi_{top}(F) = 0$, since S_i is a circle, and hence has vanishing Euler characteristic. Hence

$$\chi_{top}(W_i - V_i) = \chi_{top}(f^{-1}(W_i - V_i)) = 0$$

for all i. Finally, one can show that $F_i \hookrightarrow f^{-1}(W_i)$ is a homotopy equivalence, provided W_i are sufficiently small disks around x_i; this follows, for example from the fact that X and C have triangulations, such that $S \subset C$, $f^{-1}(S) \subset X$ are subcomplexes. Hence

$$\chi_{top}(f^{-1}(W_i)) = \chi_{top}(F_i) = \chi_{top}(W_i)\chi_{top}(F) + \chi_{top}(F_i) - \chi_{top}(F),$$

since $\chi_{top}(W_i) = 1$ (as W_i is contractible). Again arguing as in the proof of the lemma, using the Mayer-Vietoris exact sequences for the open cover $\{W, C - V\}$ of C, and the induced covering of X, we deduce the formula claimed in the proposition. □

Now to apply this to study surfaces in $\mathbf{P}^3_{\mathbf{C}}$, we use the technique of *Lefschetz pencils*. Let $L \subset \mathbf{P}^3_{\mathbf{C}}$ be a line, and consider all hyperplanes $H \subset \mathbf{P}^3_{\mathbf{C}}$ such that $L \subset H$. There is a line $\widehat{L}$ in the dual projective space $\widehat{\mathbf{P}}^3_{\mathbf{C}}$ parametrizing these hyperplanes H. Now let $F(x, y, z, w) = 0$ be the defining equation for the surface $X \subset \mathbf{P}^3_{\mathbf{C}}$ of degree d. The tangent plane to X at $P = (a : b : c : d)$ is

$$x\frac{\partial F}{\partial x}(a, b, c, d) + y\frac{\partial F}{\partial y}(a, b, c, d) + z\frac{\partial F}{\partial z}(a, b, c, d) + w\frac{\partial F}{\partial w}(a, b, c, d) = 0.$$

This corresponds to the point in $\widehat{\mathbf{P}}^3_{\mathbf{C}}$ with coordinates

$$(\frac{\partial F}{\partial x}(a, b, c, d) : \frac{\partial F}{\partial y}(a, b, c, d) : \frac{\partial F}{\partial z}(a, b, c, d) : \frac{\partial F}{\partial w}(a, b, c, d)).$$

Consider the morphism $D : \mathbf{P}^3_{\mathbf{C}} \to \widehat{\mathbf{P}}^3_{\mathbf{C}}$ given by

$$D(a : b : c : d) = (\frac{\partial F}{\partial x}(a, b, c, d) : \frac{\partial F}{\partial y}(a, b, c, d) : \frac{\partial F}{\partial z}(a, b, c, d) : \frac{\partial F}{\partial w}(a, b, c, d)).$$

Then $X \to D(X)$ is dual morphism of X. One knows that this is in fact *birational* (for a given X, this can of course be checked explicitly), and if $\widehat{X} = D(X)$, then one has the "double duality theorem" $\widehat{(\widehat{X})} = X$ (the dual of a singular hypersurface is defined to be the closure of the dual of its non-singular locus). Further, local calculations show that D is an isomorphism near $x \in X \iff$ the tangent hyperplane $\mathbf{P}T_x \cap X$ is a curve with an ordinary double point at x (a plane curve singularity with a local analytic equation $z_1 z_2 = 0$). Of course $\mathbf{P}T_x \cap X$ is a curve in $\mathbf{P}T_x X \cong \mathbf{P}^2_{\mathbf{C}}$ of degree d.

In particular, one may choose the line L so that the dual line $\widehat{L}$ intersects $D(X) = \widehat{X}$ only at smooth points of $\widehat{X}$, and the intersection is transverse.

This will imply that if H corresponds to a point of $\widehat{L} \cap \widehat{X}$, then $H = \mathbf{P}T_x X$ for a unique point $x \in X$ (namely $x = D^{-1}(\widehat{H})$), and $H \cap X$ is a plane curve of degree d with one ordinary double point. We can also assume that L meets X transversally; this will force the finite set $D^{-1}(\widehat{X} \cap \widehat{L}) \cap X$ to be disjoint from L (if the tangent hyperplane to X at a point $x \in L$ contains L, then $L \cap X$ is a hyperplane section through x of the singular curve $X \cap \mathbf{P}T_x X$, which will force x to be a singular point of $L \cap X$).

Let

$$\widetilde{X} = \{(x,t) \in X \times \widehat{L} \mid x \in H_t\}.$$

One sees by a simple local calculation that $\widetilde{X} \to X$ is the blow up of X at the set of d points $L \cap X$. Hence one sees easily that

$$\chi_{top}(\widetilde{X}) = d + \chi_{top}(X).$$

On the other hand, $f : \widetilde{X} \to \widehat{L} \cong \mathbf{P}^1_{\mathbf{C}}$ has singular fibres over the points $\widehat{L} \cap \widehat{X}$. Now $D^{-1}(\widehat{L})$ is a subvariety of $\mathbf{P}^3_{\mathbf{C}}$ defined by 2 homogeneous polynomials of degree $d-1$ (linear combinations of partial derivatives of $F(x,y,z,w)$). Hence by Bezout's Theorem for $\mathbf{P}^3_{\mathbf{C}}$, $\widehat{L} \cap \widehat{X} \cong D^{-1}(\widehat{L}) \cap X$ consists of $d(d-1)^2$ points. Finally, the general fiber F of f is a non-singular plane curve of degree d, which has genus $\dfrac{(d-1)(d-2)}{2}$, hence has Euler characteristic $d^2 - 3d$; any singular fiber F_i is a plane curve of degree d with 1 ordinary double point, so that $\chi_{top}(F_i) - \chi_{top}(F) = 1$. Hence we see that

$$\begin{aligned} \chi_{top}(\widetilde{X}) &= \chi_{top}(\mathbf{P}^1_{\mathbf{C}})(d^2 - 3d) + d(d-1)^2 \\ &= d(d-1)^2 - 2(d^2 - 3d) = d^3 - 4d^2 + 7d. \end{aligned}$$

Hence

$$\chi_{top}(X) = d^3 - 4d^2 + 6d.$$

But the Betti numbers of X satisfy $b_0 = b_4 = 1$, and $b_1 = b_3 = 0$. Hence

$$b_2 = \chi_{top}(X) - 2 = d^3 - 4d^2 + 6d - 2.$$

For example, this formula gives the following.

(i) If $d = 1$, then $b_2 = 1$, which is consistent with the fact that X is a plane.

(ii) If $d = 2$, then $b_2 = 2$, which is consistent with the fact that $X \cong \mathbf{P}^1_{\mathbf{C}} \times \mathbf{P}^1_{\mathbf{C}}$.

(iii) If $d = 3$, then $b_2 = 7$; in fact it is known that a general cubic surface in $\mathbf{P}^3_{\mathbf{C}}$ is the blow up of $\mathbf{P}^2$ at 6 distinct points, so our formula is consistent with this.

(iv) If $d = 4$, then $b_2 = 22$. In fact, a quartic surface in $\mathbf{P}^3_{\mathbf{C}}$ is an example of a *K3 surface*; one knows that these types of surfaces have $b_2 = 22$.

5.5 Barth theorems

We saw above that by the Lefschetz hyperplane theorem, the homology and homotopy groups of a projective complete intersection of dimension n agree with those of projective space, in degrees $< n$. A line of argument, originating with ideas of Barth, implies that similar conclusions can be obtained about the homology and homotopy of an arbitrary non-singular projective variety $X \subset \mathbf{P}^N_{\mathbf{C}}$ of dimension n, provided the *codimension* $N - n$ is "small" compared to the dimension.

A new approach was found to such results by Fulton and Hansen, via the following "connectedness theorem".

Theorem 17 Connectedness Theorem *Let X be a projective variety of dimension n, $f : X \to \mathbf{P}^m_{\mathbf{C}} \times \mathbf{P}^m_{\mathbf{C}}$ a morphism with finite fibers. Let $\Delta \subset \mathbf{P}^m_{\mathbf{C}} \times \mathbf{P}^m_{\mathbf{C}}$ be the diagonal. Then*

(i) if $n \geq m$, then $f^{-1}(\Delta)$ is non-empty

(ii) if $n > m$, then $f^{-1}(\Delta)$ is connected

(iii) if $n > m$ and X is locally analytically irreducible, then $\pi_1(f^{-1}(\Delta)) \to \pi_1(X)$ is surjective

(iv) if X is a local complete intersection at each point not in $f^{-1}(\Delta)$, then

$$\pi_i(X, f^{-1}(\Delta)) \cong \pi_i(\mathbf{P}^m_{\mathbf{C}} \times \mathbf{P}^m_{\mathbf{C}}, \Delta)$$

for all $i \leq n - m$.

For a proof, see [Fu1] and the (extensive) bibliography given there. Note that

$$\pi_i(\mathbf{P}^m_{\mathbf{C}} \times \mathbf{P}^m_{\mathbf{C}}, \Delta) = \begin{cases} \mathbf{Z} & \text{if } i = 2 \\ 0 & \text{otherwise, for } i \leq 2m \end{cases}$$

As a corollary, we get the following, proved originally for non-singular V by Barth (for homology) and Larsen (for homotopy).

Corollary 8 (Barth-Larsen theorem) *In $V \subset \mathbf{P}^m_{\mathbf{C}}$ is an n-dimensional local complete intersection (for example, a nonsingular variety), then*

$$\pi_i(\mathbf{P}^m_{\mathbf{C}}, V) = H_i(\mathbf{P}^m_{\mathbf{C}}, V; \mathbf{Z}) = 0 \text{ for } i \leq 2n - m + 1.$$

To prove the corollary, one applies the Connectedness Theorem to the inclusion of $X = V \times V$ into $\mathbf{P}^m_{\mathbf{C}} \times \mathbf{P}^m_{\mathbf{C}}$, to try to conclude that

$$\pi_i(V) = \pi_i(V \times V \cap \Delta) \twoheadrightarrow \pi_i(V \times V) = \pi_i(V) \times \pi_i(V).$$

But if the diagonal homomorphism $A \to A \times A$ of an abelian group is surjective, then clearly $A = 0$. This argument is not quite correct, since $\pi_2(\mathbf{P}^m_{\mathbf{C}} \times \mathbf{P}^m_{\mathbf{C}}, \Delta) = \mathbf{Z}$ is non-zero, but the argument can be modified to take care of this problem.

Similarly, one can prove the following.

Corollary 9 *Let $Y \subset \mathbf{P}^m_{\mathbf{C}}$ be a local complete intersection of codimension d, and $h : V \to \mathbf{P}^m_{\mathbf{C}}$ a morphism with finite fibres, where* $\dim V = n$, *and V is also a local complete intersection. Then*

$$\pi_i(V, h^{-1}(Y)) \to \pi_i(\mathbf{P}^m_{\mathbf{C}}, Y)$$

is an isomorphism for $i \leq n - d$, and is surjective for $i = n - d + 1$.

The proof proceeds by applying the Connectedness Theorem to $V \times Y \to \mathbf{P}^m_{\mathbf{C}} \times \mathbf{P}^m_{\mathbf{C}}$. Taking $Y = V$, we get the earlier corollary, while taking YL, a projective linear subspace, we obtain a version of the Lefschetz hyperplane theorem. æ

6 Sheaves

Let X be a topological space. The open sets in X are partially ordered by inclusion, hence may be regarded as a category[3] $\mathcal{T}_X$, whose objects are the open sets in X, and a unique morphism $U \to V$ if $U \subset V$.

A *presheaf of sets* on X is a functor $\mathcal{T}_X^{op} \to \mathbf{Set}$, where **Set** is the category of sets.

Thus if $\mathcal{F}$ is a presheaf on X, then for each open set $U \subset X$, we are given a set $\mathcal{F}(U)$, and for any smaller open subset $V \subset U$, a *restriction*

[3] Category theory and homological algebra are briefly reviewed in an appendix to this section.

map $\rho_{UV} : \mathcal{F}(U) \to \mathcal{F}(V)$, such that $\rho_{VW} \circ \rho_{UV} = \rho_{UW}$ for $W \subset V \subset U$. Elements of $\mathcal{F}(U)$ are called *sections* of $\mathcal{F}$ over U; if $U = X$, they are called *global sections*. We sometimes also use the notation $\Gamma(U, \mathcal{F})$ instead of $\mathcal{F}(U)$.

Morphisms of presheaves are just natural transformations of functors. A presheaf $\mathcal{F}'$ is called a *sub-presheaf* of $\mathcal{F}$ if $\mathcal{F}'(U) \subset \mathcal{F}(U)$ for each U, and the restriction maps for $\mathcal{F}'$ are obtained by restricting those for $\mathcal{F}$. If $f : \mathcal{F} \to \mathcal{G}$ is a morphism of presheaves, then $U \mapsto (\text{image}\mathcal{F}(U))$ is a presheaf, which is a sub-presheaf of $\mathcal{G}$.

For example, let A be a set. For any topological space X, let $\mathcal{F}(U) = A$ for all open sets U, and let ρ_{UV} be the identity for all $V \subset U$. Then $\mathcal{F}$ is a presheaf on X called the *constant presheaf* associated to A.

A presheaf $\mathcal{F}$ is called a *sheaf* if for any open set U of X and any open cover $\{U_\alpha\}_{\alpha \in \mathcal{A}}$ of U, the following conditions hold.

(i) For any sections $s, t \in \mathcal{F}(U)$, if $\rho_{UU_\alpha}(s) = \rho_{UU_\alpha}(t)$ for all $\alpha \in \mathcal{A}$, then $s = t$.

(ii) Let $U_{\alpha\beta} = U_\alpha \cap U_\beta$ for any $\alpha, \beta \in \mathcal{A}$; then for any family of sections $s_\alpha \in \mathcal{F}(U_\alpha)$, $\alpha \in \mathcal{A}$, such that $\rho_{U_\alpha U_{\alpha\beta}}(s_\alpha) = \rho_{U_\beta U_{\alpha\beta}}(s_\beta)$ for all $\alpha, \beta \in \mathcal{A}$, there exists a (necessarily unique, by (i)) $s \in \mathcal{F}(U)$ such that $\rho_{UU_\alpha}(s) = s_\alpha$ for all $\alpha \in \mathcal{A}$.

(iii) If $U = \phi$ is empty, then $\mathcal{F}(U)$ is a 1-point set (*i.e.*, a *final object* in the category **Set**).

Morphisms of sheaves are defined to be morphisms of the underlying presheaves. Thus we can make sense of subsheaves of a sheaf. However, if $f : \mathcal{F} \to \mathcal{G}$ is a morphism of sheaves, the image presheaf is *not* a sheaf in general.

Some basic examples of sheaves for this course are as follows.

(i) The *structure sheaf* $\mathcal{O}_X$ of an algebraic variety X (with its Zariski topology) over a given field k; this is the sheaf given by $\mathcal{O}_X(U) = \mathcal{O}(U)$, the ring of regular functions on U.

(ii) The *structure sheaf* $\mathcal{A}_M$ of a smooth manifold M; here $\mathcal{A}_M(U)$ is the **C**-algebra of complex valued smooth (*i.e.*, C^∞) functions on U; we may similarly consider real valued functions.

(iii) The *structure sheaf* $\mathcal{O}_X$ of a complex manifold X; this is the sheaf $\mathcal{O}_X(U) = \mathcal{H}(U)$, the **C**-algebra of holomorphic functions on U.

(iv) The sheaf of smooth differential forms $\mathcal{A}^n_M$ on a smooth manifold M; here $\mathcal{A}^n_M(U)$ is the **C**-vector space of smooth n-forms on U.

(v) The sheaf Ω^n_X of holomorphic n forms on a complex manifold X.

(vi) The sheaf $\Omega^n_{X/k}$ of algebraic n-forms on a non-singular variety over the field k.

The *stalk* $\mathcal{F}_x$ of a presheaf $\mathcal{F}$ at $x \in X$ is defined as

$$\mathcal{F}_x = \varinjlim_{U \ni x} \mathcal{F}(U).$$

Define

$$\mathcal{G}(\mathcal{F})(U) = \prod_{x \in U} \mathcal{F}_x.$$

(If $U = \phi$, define $\mathcal{G}(\mathcal{F})(U)$ to be a final object in **Set**.) Then $\mathcal{G}(\mathcal{F})$ is a sheaf, such that all the restriction maps ρ_{UV} are surjective; a sheaf with this property is called *flasque* (or *flabby*). For each open set U, there is a natural map $\mathcal{F}(U) \to \prod_{x \in U} \mathcal{F}_x$, giving a morphism of presheaves $\mathcal{F} \to \mathcal{G}(\mathcal{F})$. If $\mathcal{F}$ is a sheaf, this is injective, giving an isomorphism of $\mathcal{F}$ with its image. In general, the image of $\mathcal{F}$ is a sub-presheaf. Let $a(\mathcal{F})$ be the intersection of all the subsheaves of $\mathcal{G}(\mathcal{F})$ which contain the image of $\mathcal{F}$ (since $\mathcal{G}(\mathcal{F})$ is one such, the family of subsheaves is non-empty, and clearly any intersection of subsheaves is a subsheaf). If $f : \mathcal{F} \to \mathcal{F}'$ is a morphism of presheaves, there is an induced morphism of sheaves $\mathcal{G}(\mathcal{F}) \to \mathcal{G}(\mathcal{F}')$ compatible with f, and hence a morphism $a(\mathcal{F}) \to a(\mathcal{F}')$. In particular, if $\mathcal{F}'$ is a sheaf, so that $\mathcal{F}' \to a(\mathcal{F}')$ is an isomorphism, we see that f factors uniquely through $\mathcal{F} \to a(\mathcal{F})$. Thus a is a functor from presheaves to sheaves on X, which is left adjoint to the inclusion functor from sheaves to presheaves. We call $a(\mathcal{F})$ the *sheaf associated to the presheaf* $\mathcal{F}$.

A presheaf of abelian groups (or rings, or modules over a ring ...) on X is a functor from $\mathcal{T}_X^{op}$ to the category **Ab** of abelian groups (or rings, or modules over a ring, ...). It is a sheaf if the analogues of the conditions (i), (ii), (iii) above are satisfied. If $\mathcal{F}$ is a presheaf of abelian groups, $\mathcal{G}(\mathcal{F})$, $a(\mathcal{F})$ are sheaves of abelian groups; a similar claim holds for sheaves of rings, modules, etc. In particular, for any abelian group A, we have the *constant sheaf* A_X associated to A, which is the sheaf associated to the constant presheaf determined by A (discussed earlier). If A is a ring, A_X is a sheaf of rings. An important example is the sheaf $\mathbf{Z}_X$ of rings determined by the ring **Z** of integers.

More generally, we may consider sheaves with values in any category with arbitrary products and finite inverse limits, and which has a final object, since the sheaf conditions may be rephrased using only these notions.

Let $\mathcal{F}$ be a sheaf of abelian groups on X, and $s \in \mathcal{F}(U)$. Then the *support* of s is the set $|\, s \,| = \{x \in U \mid s_x \neq 0\}$, where s_x is the image of s in the stalk $\mathcal{F}_x$. One sees easily that $|\, s \,| \subset U$ is closed. We define the support of $\mathcal{F}$ to be the union of the supports of its sections, which is the set $|\, \mathcal{F} \,| = \{x \in X \mid \mathcal{F}_x \neq 0\}$. This need not be closed in general. However, we will see later that this is the case for *coherent* sheaves of $\mathcal{O}_X$-modules on a 'reasonable' scheme X.

Let $\mathcal{O}_X$ be a presheaf of rings on a topological space X. A *presheaf of $\mathcal{O}_X$-modules* is a presheaf $\mathcal{F}$ of abelian groups together with an $\mathcal{O}_X(U)$-module structure on each abelian group $\mathcal{F}(U)$, such that if $V \subset U$, then $\rho_{UV} : \mathcal{F}(U) \to \mathcal{F}(V)$ is $\mathcal{O}_X(U)$-linear, where $\mathcal{F}(V)$ is regarded as an $\mathcal{O}_X(U)$-module via the ring homomorphism $\rho_{UV} : \mathcal{O}_X(U) \to \mathcal{O}_X(V)$ and the given $\mathcal{O}_X(V)$-module structure. If $\mathcal{O}_X$ is a sheaf of rings, a *sheaf of $\mathcal{O}_X$-modules* is a sheaf of abelian groups which has the structure of a presheaf of $\mathcal{O}_X$-modules. A sheaf of $\mathbf{Z}_X$-modules is just a sheaf of abelian groups. In another direction, if $X = \{x\}$, then all presheaves which satisfy the sheaf condition (iii) are in fact sheaves; a sheaf of rings $\mathcal{O}_X$ is identified with a ring R (the stalk of $\mathcal{O}_X$ at x), and the category of sheaves of $\mathcal{O}_X$-modules is identified with the category of R-modules.

Convention: whenever we consider *sheaves* of $\mathcal{O}_X$-modules, *we will assume that $\mathcal{O}_X$ is a sheaf of rings.*

The category of presheaves of $\mathcal{O}_X$-modules on a topological space X forms an abelian category in a natural way. The category of sheaves of $\mathcal{O}_X$-modules is a full additive subcategory, which is also an abelian category; for any morphism $f : \mathcal{F} \to \mathcal{F}'$, the sheaf kernel of f is the presheaf kernel, but the sheaf cokernel is defined to be $a(\operatorname{coker}_p(f))$ where 'coker_p' denotes the presheaf cokernel. In particular, one sees that a sequence

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$

of sheaves of $\mathcal{O}_X$-modules is *exact* iff

$$0 \to \mathcal{G}(\mathcal{F}') \to \mathcal{G}(\mathcal{F}) \to \mathcal{G}(\mathcal{F}'') \to 0$$

is exact as a sequence of presheaves; this is equivalent to the exactness of

$$0 \to \mathcal{F}'_x \to \mathcal{F}_x \to \mathcal{F}''_x \to 0$$

for each $x \in X$.

The category of presheaves of $\mathcal{O}_X$-modules has direct sums, and direct and inverse limits over directed sets. A finite (presheaf) direct sum of sheaves of $\mathcal{O}_X$-modules is a sheaf. The inverse limit presheaf of a directed family of sheaves of $\mathcal{O}_X$-modules is in fact a sheaf, but the direct limit in the category of sheaves of $\mathcal{O}_X$-modules is the sheaf associated to the presheaf direct limit. However there is one case where the presheaf and sheaf direct limits coincide: when the topological space X is *Noetherian*, *i.e.*, satisfies the *descending chain condition for closed subsets*, that any strictly descending chain of closed subsets of X is finite.

We mention some other basic operations on presheaves and sheaves. If $f : X \to Y$ is a continuous map, and $\mathcal{F}$ is a presheaf on X, then we can define a presheaf $f_*\mathcal{F}$ on Y by $f_*\mathcal{F}(U) = \mathcal{F}(f^{-1}(U))$. We call $f_*\mathcal{F}$ the *direct image* of $\mathcal{F}$. If $\mathcal{F}$ is a sheaf, so is $f_*\mathcal{F}$. If $\mathcal{O}_X$ is a sheaf of rings on X, then $f_*\mathcal{O}_X$ is a sheaf of rings, and for any $\mathcal{O}_X$-module $\mathcal{F}$, the direct image $f_*\mathcal{F}$ is an $f_*\mathcal{O}_X$-module in a natural way. The direct image functor is *left exact*.

The direct image functor f_* from presheaves (or sheaves) of abelian groups on X to those on Y has a left adjoint f^{-1}, called the *inverse image* functor. On presheaves, it is defined (on objects) by

$$(f^{-1}\mathcal{F})(U) = \varinjlim_{V \supset f(U)} \mathcal{F}(V).$$

This clearly defines a presheaf on X, and the adjointness property

$$\mathrm{Hom}\,(f^{-1}\mathcal{F}', \mathcal{F}) \cong \mathrm{Hom}\,(\mathcal{F}', f_*\mathcal{F})$$

is easily verified. The sheaf inverse image is the sheaf associated to the presheaf inverse image; the adjointness property follows from the adjointness at the level of presheaves, and the adjointness of the 'associated sheaf' functor a. If $f(x) = y$, then for any presheaf $\mathcal{F}$ on Y, we have an identification of stalks $f^{-1}(\mathcal{F})_x \cong \mathcal{F}_y$. In particular, f^{-1} is an exact functor. If $\mathcal{O}_Y$ is a sheaf of rings, then so is $f^{-1}\mathcal{O}_Y$, and f^{-1} takes $\mathcal{O}_Y$-modules into $f^{-1}\mathcal{O}_Y$-modules, and converts $\mathcal{O}_Y$-linear maps into $f^{-1}\mathcal{O}_Y$-linear ones.

In particular, if $j : U \hookrightarrow X$ is the inclusion of an open subset, we have $(j^{-1}\mathcal{F})(V) = \mathcal{F}(V)$ for any open set $V \subset U$. We also denote $j^{-1}\mathcal{F}$ by $\mathcal{F}\mid_U$. The functor j^{-1} from sheaves of abelian groups on X to those on U has a left adjoint $j_!$, called *extension by 0*, where $j_!\mathcal{F}$ is the sheaf associated to the presheaf

$$V \mapsto \begin{cases} \mathcal{F}(V) & \text{if } V \subset U \\ 0 & \text{if } V \not\subset U \end{cases}$$

The sheaf $j_!\mathcal{F}$ is characterized by the properties that $j^{-1}j_!\mathcal{F} \cong \mathcal{F}$ and $(j_!\mathcal{F})_x = 0$ for $x \in X - U$. Note that there is a natural inclusion $j_!(\mathcal{F} \mid_U) \to \mathcal{F}$ for any sheaf $\mathcal{F}$ os abelain groups.

If $i : Z \hookrightarrow X$ is the inclusion of a closed subset, let $\mathcal{F} \mid_Z = i^{-1}\mathcal{F}$. The functor i_* gives an equivalence of categories between sheaves of abelian groups on Z and the full subcategory of sheaves of abelian groups $\mathcal{F}$ on X with $\mathcal{F} \mid_{X-Z} = 0$.

If $\mathcal{O}_X$ is a sheaf of rings on X, let $\mathcal{O}_U = \mathcal{O}_X \mid_U$. If $\mathcal{F}, \mathcal{G}$ are presheaves of $\mathcal{O}_X$-modules, define a presheaf of abelian groups $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ by the assignment $U \mapsto \mathrm{Hom}_{\mathcal{O}_U}(\mathcal{F} \mid_U, \mathcal{G} \mid_U)$. If $\mathcal{F}$, $\mathcal{G}$ are sheaves, so is $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$. If $\mathcal{O}_X$ is a sheaf of commutative rings, $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ is a sheaf of $\mathcal{O}_X$-modules in a natural way. In particular, if $\mathcal{O}_X$ is commutative, we have a notion of *dual*; the dual $\mathcal{F}^*$ of a sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules is $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)$.

If $\mathcal{O}_X$ is a sheaf of rings, not necessarily commutative, let $\mathcal{O}_X^{\mathrm{op}}$ be the corresponding sheaf of opposite rings, so that an $\mathcal{O}_X^{\mathrm{op}}$-module is a right $\mathcal{O}_X$-module. For any $\mathcal{O}_X^{\mathrm{op}}$-module $\mathcal{F}$, and any sheaf $\mathcal{H}$ of abelian groups, the sheaf $\mathcal{H}om_{\mathbf{Z}_X}(\mathcal{F}, \mathcal{H})$ is an $\mathcal{O}_X$-module in a natural way, via the action $(s \cdot \varphi)(f) = \varphi(s \cdot f)$ for sections $s \in \mathcal{O}_X^{\mathrm{op}}(U) = \mathcal{O}_X(U)$, $f \in \mathcal{F}(U)$ and $\varphi \in \mathcal{H}om_{\mathbf{Z}_X}(\mathcal{F}, \mathcal{H})(U)$.

Let $\mathcal{O}_X$ be a sheaf of rings, $\mathcal{F}$ an $\mathcal{O}_X^{\mathrm{op}}$-module. The functor $\mathcal{H} \mapsto \mathcal{H}om_{\mathbf{Z}_X}(\mathcal{F}, \mathcal{H})$ (from the category of $\mathbf{Z}_X$-modules to that of $\mathcal{O}_X$-modules) has a left adjoint $\mathcal{G} \mapsto \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$. Thus, by definition, there are natural isomorphisms

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{G}, \mathcal{H}om_{\mathbf{Z}_X}(\mathcal{F}, \mathcal{H})) \cong \mathrm{Hom}_{\mathbf{Z}_X}(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}, \mathcal{H}),$$

which characterizes $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ in terms of the usual universal property for bilinear maps of sheaves $\mathcal{F} \times \mathcal{G} \to \mathcal{H}$. One checks that the sheaf associated to the presheaf $U \mapsto \mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U)$ satisfies this universal property, so that this defines the sheaf $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$. When $\mathcal{O}_X$ is commutative, if $\mathcal{H}$ is also an $\mathcal{O}_X$-module, then we have a commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{G}, \mathcal{H}om_{\mathbf{Z}_X}(\mathcal{F}, \mathcal{H})) & \xrightarrow{\cong} & \mathrm{Hom}_{\mathbf{Z}_X}(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}, \mathcal{H}) \\ \uparrow & & \uparrow \\ \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{G}, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{H})) & \xrightarrow{\cong} & \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}, \mathcal{H}) \end{array}$$

where the vertical arrows are each induced by the natural inclusion of the abelian group of $\mathcal{O}_X$-linear maps into that of $\mathbf{Z}_X$-linear ones.

In a similar fashion, one may define symmetric powers, exterior powers, etc. when $\mathcal{O}_X$ is commutative.

Convention: from now onwards, we will assume $\mathcal{O}_X$ is a sheaf of *commutative* rings, unless explicitly mentioned otherwise. Some statments made below may have generalizations to the non-commutative case; we leave these to the interested reader.

One way to define a sheaf on a space X is through patching: let $\{U_i\}_{i\in I}$ be an open cover of X, and let $\mathcal{F}_i$ be a sheaf on U_i, for each i, such that
(i) for any pair of distinct indices i, j there is an isomorphism

$$\varphi_{ij} : \mathcal{F}_i \mid_{U_i\cap U_j} \xrightarrow{\cong} \mathcal{F}_j \mid_{U_i\cap U_j}$$

(ii) $\varphi_{ji} = \varphi_{ij}^{-1}$
(iii) for any 3 distinct indices i, j, k we have $\varphi_{jk} \circ \varphi_{ij} = \varphi_{ik}$ on $U_i \cap U_j \cap U_k$.
Then there is a sheaf $\mathcal{F}$, such that there are isomorphisms $\varphi_i : \mathcal{F} \mid_{U_i} \to \mathcal{F}_i$ compatible with the φ_{ij}; further such an $\mathcal{F}$ is unique upto unique isomorphism compatible with the φ_{ij}. One way to construct $\mathcal{F}$ is to define a presheaf $\mathcal{F}_0$ as follows: let $\mathcal{U}$ be the collection of open subsets of X which are contained in some U_i, and choose a function $f : \mathcal{U} \to I$ such that $V \subset U_{f(V)}$ for all $V \in \mathcal{U}$. Define $\mathcal{F}_0(V) = 0$ for $V \notin \mathcal{U}$, and $\mathcal{F}_0(V) = \mathcal{F}_{f(V)}(V)$ for $V \in \mathcal{U}$. Using the isomorphisms φ_{ij} we see that there are natural restriction maps making $\mathcal{F}_0$ a presheaf, together with given isomorphisms $\mathcal{F}_0 \mid_{U_i} \cong \mathcal{F}_i$. Then $\mathcal{F} = a(\mathcal{F}_0)$ is the desired sheaf obtained by patching the $\mathcal{F}_i$ using the isomorphisms φ_{ij}. We leave it to the reader to check the uniqueness assertion.

An $\mathcal{O}_X$-module is *free* of rank n if it is isomorphic to $\mathcal{O}_X^{\oplus n}$. An $\mathcal{O}_X$-module $\mathcal{F}$ is called *locally free* (of finite rank) if each $x \in X$ has an open neighbourhood U such that $\mathcal{F} \mid_U$ is a free $\mathcal{O}_U = \mathcal{O}_X \mid_U$-module of finite rank. A locally free $\mathcal{O}_X$-module of rank 1 is called an *invertible* $\mathcal{O}_X$-module.

Locally free modules have several good properties. For example, if $\mathcal{E}$ is locally free, then the functors $\mathcal{F} \mapsto \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{F}$, $\mathcal{F} \mapsto \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{F})$ are exact. We also have isomorphisms of functors (in $\mathcal{F}$) $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{F}) \cong \mathcal{E}^* \otimes_{\mathcal{O}_X} \mathcal{F}$, and $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{E}, \mathcal{F}) \cong (cE^* \otimes_{\mathcal{O}_X} \mathcal{F})(X)$. The natural map $\mathcal{E} \to (\mathcal{E}^*)^*$ from $\mathcal{E}$ to its double dual is an isomorphism. For any locally free $\mathcal{O}_X$-module $\mathcal{E}$, there is a natural $\mathcal{O}_X$-linear surjection $\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{E}^* \to \mathcal{O}_X$, which is an isomorphism if $\mathcal{E}$ is invertible. However, note that in general, locally free $\mathcal{O}_X$-modules are *not* projective objects in the category of $\mathcal{O}_X$-modules.

Recall that an object I of an abelian category $\mathcal{A}$ is *injective* if the functor $X \mapsto \mathrm{Hom}_{\mathcal{A}}(X, I)$ is exact. For any (possibly non-commutative) sheaf of rings $\mathcal{O}_X$, the abelian category of sheaves of $\mathcal{O}_X$-modules *has enough injectives*, *i.e.*, for any sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules, there is a monomorphism $\mathcal{F} \to \mathcal{I}$, where $\mathcal{I}$ is an injective $\mathcal{O}_X$-module. To prove this, one notes that if $\mathcal{J}$ is an injective sheaf of abelian groups, then $\mathcal{I} = \mathcal{H}om_{\mathbf{Z}_X}(\mathcal{O}_X^{\rm op}, \mathcal{J})$, which is naturally an $\mathcal{O}_X$-module, is in fact injective; this follows from the natural isomorphism

$$\operatorname{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{H}om_{\mathbf{Z}_X}(\mathcal{O}_X^{\rm op}, \mathcal{J})) \cong \operatorname{Hom}_{\mathbf{Z}_X}(\mathcal{O}_X^{\rm op} \otimes_{\mathcal{O}_X} \mathcal{F}, \mathcal{J}) = \operatorname{Hom}_{\mathbf{Z}}(\mathcal{F}, \mathcal{J})$$

for any sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules. This reduces us to proving the result when $\mathcal{O}_X = \mathbf{Z}_X$. One sees easily that if $\{I_x\}_{x\in X}$ is a family of injective (= divisible) abelian groups indexed by points of X, and $\mathcal{I}(U) = \prod_{x\in U} I_x$, then $\mathcal{I}$ is an injective sheaf of abelian groups. Now for any sheaf $\mathcal{F}$ of abelian groups, if we choose inclusions $\mathcal{F}_x \hookrightarrow I_x$ into injective abelian groups, then we obtain an injection of sheaves $\mathcal{G}(\mathcal{F}) \hookrightarrow \mathcal{I}$, where $\mathcal{I}$ is defined by the chosen family $\{I_x\}_{x\in X}$; composing with the natural injection $\mathcal{F} \hookrightarrow \mathcal{G}(\mathcal{F})$ (since $\mathcal{F}$ is a sheaf, the natural map is an inclusion), we are done.

Thus any sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules has an *injective resolution*

$$0 \to \mathcal{F} \to \mathcal{I}_0 \to \mathcal{I}_1 \to \cdots \to \mathcal{I}_n \to \cdots$$

in the category of $\mathcal{O}_X$-modules, and this is unique upto chain homotopy (by standard arguments using the universal property of an injective object). Hence for any left exact functor F from the category of sheaves of $\mathcal{O}_X$-modules to an abelian category, we may define its *derived functors* R^iF by

$$R^iF(\mathcal{F}) = i^{\rm th} \text{ cohomology object of the complex } F(I_\bullet).$$

If $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is an exact sequence of sheaves, we have functorial boundary maps $R^iF(\mathcal{F}'') \to R^{i+1}F(\mathcal{F}')$ giving a long exact sequence of derived functors (where we identify R^0F with F)

$$\begin{aligned} 0 \to F(\mathcal{F}') \to F(\mathcal{F}) \to F(\mathcal{F}'') \to R^1F(\mathcal{F}') \to \cdots \to \\ R^iF(\mathcal{F}') \to R^iF(\mathcal{F}) \to R^iF(\mathcal{F}'') \to R^{i+1}(\mathcal{F}') \to \cdots \end{aligned}$$

Any natural transformation between left exact funtors induces a unique natural transformation between their derived functors, compatible with boundary maps in the respective long exact sequences.

Important examples of left exact functors on sheaves and their derived functors are as follows.

(i) Let $f : X \to Y$ be a continuous map, $\mathcal{O}_X$ a sheaf of (possibly non-commutative) rings on X. Then f_* is a left exact functor from $\mathcal{O}_X$-modules to $f_*\mathcal{O}_X$-modules, whose derived functors R^if_* are called the *higher direct image* functors of the map f. In particular, if $Y = \{y\}$ is a point, then $f_*\mathcal{O}_X$ is identifed with the ring $R = \mathcal{O}_X(X)$, and $f_*\mathcal{F}$ is identified with the R-module $\mathcal{F}(X)$ of global sections. The sheaves $R^i f_*\mathcal{F}$ yield R-modules $H^i(X, \mathcal{F})$ called the *cohomology groups* (really, cohomology R-modules) of $\mathcal{F}$.

(ii) Let $\mathcal{G}$ be an $\mathcal{O}_X$-module. Then

$$\mathcal{F} \mapsto \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{G}, \mathcal{F}), \quad \mathcal{F} \mapsto \mathcal{H}om_{\mathcal{O}_X}(\mathcal{G}, \mathcal{F})$$

are left exact functors. Their i^{th} derived functors are denoted by $\operatorname{Ext}^i_{\mathcal{O}_X}(\mathcal{G}, \mathcal{F})$ and $\mathcal{E}xt^i_{\mathcal{O}_X}(\mathcal{G}, \mathcal{F})$, respectively.

We have a natural isomorphism $\operatorname{Hom}_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{F}) \cong \mathcal{F}(X) = H^0(X, \mathcal{F})$. Hence there are natural isomorphisms $\operatorname{Ext}^i_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{F}) \cong H^i(X, \mathcal{F})$. Note that if $\mathcal{O}_X$ is commutative, and $\mathcal{E}$ is a locally free $\mathcal{O}_X$-module, then

$$\mathcal{E}xt^i_{\mathcal{O}_X}(\mathcal{E}, \mathcal{F}) = 0 \text{ for all } i > 0,$$

and there are natural isomorphisms

$$\operatorname{Ext}^i_{\mathcal{O}_X}(\mathcal{E}, \mathcal{F}) \cong H^i(X, \mathcal{E}^* \otimes_{\mathcal{O}_X} \mathcal{F}).$$

Derived functors may also be computed using *acyclic resolutions*, *i.e.*, if $0 \to \mathcal{F} \to \mathcal{F}_\bullet$ is a resolution, and F a left exact functor with $R^iF(\mathcal{F}_j) = 0$ for all $i > 0$, $j \geq 0$, then the i^{th} cohomology object of the complex $F(\mathcal{F}_\bullet)$ is naturally isomorphic to $R^iF(\mathcal{F})$.

We claim that flasque sheaves of abelian groups are acyclic for f_* for any map $f : X \to Y$. Indeed, one shows that the following statements hold (see [H], II, Ex. 1.16).

(i) Injective sheaves of abelian groups are flasque. Indeed, if $j : U \hookrightarrow X$ is an open set, then the map $\operatorname{Hom}_{\mathbf{Z}_X}(\mathbf{Z}_X, \mathcal{I}) \to \operatorname{Hom}_{\mathbf{Z}_X}(j_!\mathbf{Z}_U, \mathcal{I})$, induced by the inclusion of sheaves $j_!\mathbf{Z}_U \to \mathbf{Z}_X$, is *surjective* for any injective sheaf $\mathcal{I}$, *i.e.*, $\rho_{X,U} : \mathcal{I}(X) \to \mathcal{I}(U)$ is surjective. Similarly, working with $\mathcal{O}_X$ and $j_!\mathcal{O}_U$, we see that injective $\mathcal{O}_X$-modules are flasque for any (possibly non-commutative) sheaf of rings $\mathcal{O}_X$.

(ii) If $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is exact with $\mathcal{F}'$ flasque, then $\mathcal{F}(U) \to \mathcal{F}''(U)$ is surjective for each open $U \subset X$.

(iii) If $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is exact with $\mathcal{F}', \mathcal{F}$ flasque, then $\mathcal{F}''$ is flasque.

From (i) and (iii), the quotient of an injective sheaf by a flasque subsheaf is flasque. From (ii), given a short exact sequence $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ with $\mathcal{F}'$ flasque, we get that for any continuous map $f : X \to Y$, the direct image sequence $0 \to f_*\mathcal{F}' \to f_*\mathcal{F} \to f_*\mathcal{F}'' \to 0$ is exact. Hence if $\mathcal{F}$ is a flasque sheaf of $\mathcal{O}_X$-modules, $0 \to \mathcal{F} \to \mathcal{I}_\bullet$ is an injective resolution by sheaves of $\mathcal{O}_X$-modules, then $0 \to f_*\mathcal{F} \to f_*\mathcal{I}_\bullet$ is exact for any continuous map $f : X \to Y$. Hence $R^i f_*\mathcal{F} = 0$ for all $i > 0$.

Since injective $\mathcal{O}_X$-modules are flasque, we see that the cohomology (or higher direct images) of an $\mathcal{O}_X$-module $\mathcal{F}$, computed with resolutions by injective $\mathcal{O}_X$-modules, equals the cohomology (or higher direct images) of the underlying sheaf of abelian groups $\mathcal{F}$. Another application is the following: if $f : X \to Y$ is the inclusion of a closed subset, then f_* is an exact functor from sheaves of abelian groups on X to those on Y, which sends flasque sheaves on X to flasque sheaves on Y. Hence $R^i f_*\mathcal{F} = 0$ for all $i > 0$, and there are natural isomorphisms $H^i(X, \mathcal{F}) \cong H^i(Y, f_*\mathcal{F})$ for all $i \geq 0$ (an injective resolution of $\mathcal{F}$ on X yields a flasque resolution of $f_*\mathcal{F}$ with the same complex of global sections).

Another important class of acyclic sheaves are *fine* sheaves on a paracompact space. For a proof, see [Sw] or [W]. Recall from general topology that (i) any metric space is paracompact (ii) a second countable T_3-space[4] is paracompact. In particular, a simplicial complex with countably many cells is paracompact. Thus all the spaces usually encountered in algebraic geometry over **C** (with their Euclidean topology, rather than Zariski topology) are paracompact.

A sheaf $\mathcal{F}$ is *fine* if for any open set $U \subset X$, and any locally finite[5] open cover $\mathcal{U} = \{U_\alpha\}$ of U, there exist endomorphisms $t_\alpha : \mathcal{F} \mid_U \to \mathcal{F} \mid_U$ such that $\operatorname{supp} t_\alpha \subset U_\alpha$ for all α. The t_α act like partitions of unity, allowing us to patch up locally defined sections of the sheaf. One can prove the acyclicity of fine sheaves by showing that if $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ is exact, and $\mathcal{F}$ is fine, then for any open set $U \subset X$, the map $\mathcal{G}(U) \to \mathcal{H}(U)$ is surjective. Since $\mathcal{F} \mid_U$ is also fine, we reduce to considering the case when $U = X$; now given a section $s \in \mathcal{H}(X)$, and local liftings $s_i \in \mathcal{G}(U_i)$ of s, one uses the partitions of unity to modify these lifts s_i by sections $t_i \in \mathcal{F}(U_i)$, so that $s_i + t_i$ patch together to give a global lift of s — we leave the details of this argument to the reader. Now we apply this lemma to an exact sequence where $\mathcal{F}$

[4]A T_3 space is a Hausdorff space in which any point and a disjoint closed set can be separated by open neighbourhoods.

[5]Recall that a family of subsets of X is locally finite if each point of X has a neighbourhood which intersects only a finite number of sets in the family.

is fine and $\mathcal{G}$ is flasque (say, injective), to conclude that $\mathcal{H}$ is flasque; this implies that $\mathcal{F}$ is acyclic, from the long exact sequence of derived functors associated to the short exact sequence of sheaves $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$.

(We note here that our definition of a fine sheaf differs slightly from that in the literature ([Sw], for example); it is usually only assumed that $\mathcal{F}$ has a partition of unity associated to an open cover of X (rather than of an arbitrary open U). But this extra condition is always satisfied in practice. This modification was suggested to me by R. R. Simha.)

If X is a topological space, $\mathcal{O}_X$ a fine sheaf of (commutative) rings, then any sheaf of $\mathcal{O}_X$-modules is fine. Examples of fine sheaves of rings are (i) the sheaf of continuous functions on a paracompact space (ii) the sheaf of C^∞ functions on a smooth manifold (*i.e.*, a C^∞ differential manifold). For any locally finite open cover $\{U_\alpha\}$, the endomorphisms t_α then become functions, which can be even chosen to have values in the interval $[0,1] \subset \mathbf{R}$, and are called a *partition of unity* subordinate to the covering $\{U_\alpha\}$.

Fine sheaves are used to prove that on a paracompact, locally contractible space X, the sheaf cohomology groups $H^n(X, A_X)$ with coefficients in a constant sheaf A_X, associated to an abelian group A, are naturally isomorphic to the singular cohomology groups $H^n(X, A)$ defined in algebraic topology.

Thus, for example, on a smooth manifold, the constant sheaf $\mathbf{C}$ has a resolution by the de Rham complex of sheaves of C^∞ differential forms (with complex values)

$$0 \to \mathbf{C}_X \to \mathcal{A}_X \to \mathcal{A}^1_X \to \mathcal{A}^2_X \to \cdots,$$

where $\mathcal{A}^j_X$ is the sheaf of smooth j-forms, and the maps in the complex are given by exterior differentiation of forms. We deduce that

$$H^n(X, \mathbf{C}_X) \cong \frac{\text{closed smooth } n\text{-forms}}{\text{exact smooth } n\text{-forms}}.$$

On the other hand, one can construct another fine resolution of $\mathbf{C}$

$$0 \to \mathbf{C}_X \to \mathcal{S}^0_X \to \mathcal{S}^1_X \to \mathcal{S}^2_X \to \cdots$$

where $\mathcal{S}^j_X$ is the sheaf of singular complex valued cochains on X (this is the sheaf associated to the presheaf of singular cochains). One shows that $\mathcal{S}^j_X$ is fine, and the complex $\Gamma(X, \mathcal{S}^\bullet)$ is the complex of singular cochains modulo the subcomplex of locally trivial cochains (cochains which vanish on simplices of sufficiently small support). By a subdivision argument, one shows the complex of locally trivial cochains is acyclic, so that $H^n(X, \mathbf{C}_X)$ is identified with singular cohomology. We obtain deRham's theorem, identifying the quotient of the closed forms modulo the subspace of exact ones with the singular cohomology of X. For a detailed proof, see [Sw] or [W]. A similar theorem is valid for cohomology with real coefficients, and real valued differential forms.

One important tool in computing sheaf cohomology is Leray's theorem, which relates the cohomology groups defined above to Čech cohomology. We first recall the definition of Čech cohomology (in a simple context sufficient for our needs). Let $\mathcal{U} = \{U_i\}_{i \in I}$ be an open covering of a topological space X, where we fix a well ordering of the index set I, and let $\mathcal{F}$ be a sheaf of abelian groups on X. Define groups

$$\check{C}^p(\mathcal{U}, \mathcal{F}) = \prod_{i_0 < i_1 < \cdots < i_p} \mathcal{F}(U_{i_0} \cap \cdots \cap U_{i_p})$$

and maps $\delta^p : \check{C}^p(\mathcal{U}, \mathcal{F}) \to \check{C}^{p+1}(\mathcal{U}, \mathcal{F})$ by

$$(\delta^p \alpha)_{i_0, \ldots, i_{p+1}} = \sum_{j=0}^{p+1} (-1)^j \alpha_{i_0, \ldots, \widehat{i_j}, \ldots, i_{p+1}} |_{U_{i_0} \cap \cdots \cap U_{i_{p+1}}},$$

where $\widehat{i_j}$ means that the index i_j is omitted. Then $(\check{C}^{\bullet}(\mathcal{U}, \mathcal{F}), \delta^{\bullet})$ is a complex, called the *Čech complex* of $\mathcal{F}$ with respect to $\mathcal{U}$, whose cohomology groups are called the *Čech cohomology* groups of $\mathcal{F}$ with respect to $\mathcal{U}$, and are denoted by $\check{H}^i(\mathcal{U}, \mathcal{F})$.

There is a natural map $\check{H}^i(\mathcal{U}, \mathcal{F}) \to H^i(X, \mathcal{F})$ for each i (see [H], III, (4.4)). *Leray's theorem* asserts that if $H^j(U_{i_0} \cap \cdots \cap U_{i_p}, \mathcal{F}) = 0$ for all finite intersections of open sets in the covering, and for all $j > 0$, then these natural maps are isomorphisms (see [H], III, Ex. 4.11).

A Review of categories and homological algebra

We will assume some acquaintence with the notions of *categories* and *functors.*

Thus, in a category $\mathcal{C}$, one is given a collection Ob $\mathcal{C}$ of *objects.* For any pair of objects A, B we are given a set $\mathrm{Hom}_{\mathcal{C}}(A, B)$ of *morphisms* (or *arrows*) between them, with an associative composition law, such that each object A has an identity arrow 1_A which is a left and right identity for composition of morphisms.

If $\mathcal{C}, \mathcal{D}$ are categories, a *(covariant) functor* $F : \mathcal{C} \to \mathcal{D}$ between them associates to each object A of $\mathcal{C}$ an object $F(A)$ of $\mathcal{D}$, and for any pair A, B of objects of $\mathcal{C}$, a map of sets

$$\mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Hom}_{\mathcal{C}}(F(A), F(B)),$$

$$f \mapsto F(f),$$

such that $F(1_A) = 1_{F(A)}$, and $F(f \circ g) = F(f) \circ F(g)$.

If $\mathcal{C}$ is any category, define its *opposite category* $\mathcal{C}^{\mathrm{op}}$ to have the same objects, with

$$\mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(A, B) := \mathrm{Hom}_{\mathcal{C}}(B, A),$$

and with composition of arrows reversed from $\mathcal{C}$ (*i.e.*, $f^{\mathrm{op}} \circ g^{\mathrm{op}} = (g \circ f)^{\mathrm{op}}$). A *contravariant functor* $\mathcal{C} \to \mathcal{D}$ is a functor $\mathcal{C}^{\mathrm{op}} \to \mathcal{D}$.

If $\mathcal{C}$, $\mathcal{D}$ are categories, their *product* $\mathcal{C} \times \mathcal{D}$ has $\mathrm{Ob}\,\mathcal{C} \times \mathcal{D} = \mathrm{Ob}\,\mathcal{C} \times \mathrm{Ob}\,\mathcal{D}$, and

$$\mathrm{Hom}_{\mathcal{C}\times\mathcal{D}}((A, B), (C, D)) = \mathrm{Hom}_{\mathcal{C}}(A, C) \times \mathrm{Hom}_{\mathcal{D}}(B, D),$$

with composition rule $(f, g) \circ (h, k) = (f \circ h, g \circ k)$.

If F, G are functors from $\mathcal{C}$ to $\mathcal{D}$, then a *natural transformation* $\eta : F \to G$ is a rule which associates to each object A of $\mathcal{C}$ an element $\eta(A) \in \mathrm{Hom}_{\mathcal{D}}(F(A), G(A))$, such that for any morphism $f : A \to B$ in $\mathcal{C}$, the diagram below commutes:

$$\begin{array}{ccc} F(A) & \stackrel{\eta(A)}{\longrightarrow} & G(A) \\ F(f)\downarrow & & \downarrow G(f) \\ F(B) & \stackrel{\eta(B)}{\longrightarrow} & G(B) \end{array}$$

A *natural isomorphism* between functors is a natural transformation η such that for any object A, the morphism $\eta(A)$ is an isomorphism. A functor $F : \mathcal{C} \to \mathcal{D}$ is called an *equivalence of categories* if there is a functor $G : \mathcal{D} \to \mathcal{C}$ such that the two composities $F \circ G$ and $G \circ F$ are each naturally equivalent to the respective identity functors.

Let **Set** denote the category of sets. A pair of functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ are said to be an *adjoint pair* (in which case we say F is *left adjoint* to G, and G is *right adjoint* to F) if there is a natural isomorphism of functors $\mathcal{D}^{\mathrm{op}} \times \mathcal{C} \to \mathbf{Set}$,

$$\mathrm{Hom}_{\mathcal{C}}(F(A), B) \stackrel{\cong}{\longrightarrow} \mathrm{Hom}_{\mathcal{D}}(A, G(B)).$$

For example, if N is an abelian group, and $\mathcal{C} = \mathcal{D} = \mathcal{A}b$, the category of abelian groups, then $F(A) = N \otimes_{\mathbf{Z}} A$ and $G(B) = \mathrm{Hom}_{\mathbf{Z}}(N, B)$ are an adjoint pair; this is equivalent to the universal property of tensor products. Given a functor, the construction of an adjoint for it amounts to solving a class of universal mapping problems: for example, given G, for any A we want in particular to define $F(A)$ such that there is an arrow $A \to GF(A)$, corresponding under the natural isomorphism to the identity arrow of A; this is universal among arrows from A to objects $G(C)$, in the obvious sense.

An *initial object* of $\mathcal{C}$ is an object O such that for any object A, there is a unique morphism $O \to A$. A *final object* of $\mathcal{C}$ is an object E such that there is a unique morphism $A \to E$ for each object A. For example, the empty set is the initial object for **Set**, and a one-point set is a final object for **Set**.

An *additive category* is a category $\mathcal{C}$ such that (i) $\mathrm{Hom}_{\mathcal{C}}(A, B)$ has the structure of an abelian group, such that composition of morphisms is bilinear (ii) there is an object 0 which is both an initial and a final object (iii) for a pair of objects A, B there is a direct sum $A \oplus B$, which is also a direct product, *i.e.*, there are morphisms $i_1 : A \to A \oplus B$, $i_2 : B \to A \oplus B$, $j_1 : A \oplus B \to A$, $j_2 : A \oplus B \to B$ such that $j_1 \circ i_1 = 1_A$, $j_2 \circ i_2 = 1_B$, $j_1 \circ i_2 = 0$, $j_2 \circ i_1 = 0$, and $j_1 \circ i_1 + j_2 \circ i_2 = 1_{A \oplus B}$.

An *additive functor* between additive categories is a functor which preserves 0 objects and direct sums, such that the maps on Hom sets are homomorphisms of abelian groups.

If $\mathcal{A}$ is additive, an arrow $k : C \to A$ is called a *kernel* of $f : A \to B$ if for any D, the sequence of abelian groups (with maps induced by composition with k and f respectively)

$$0 \to \mathrm{Hom}_{\mathcal{A}}(D, C) \to \mathrm{Hom}_{\mathcal{A}}(D, A) \to \mathrm{Hom}_{\mathcal{A}}(D, B)$$

is exact. The definiton of a *cokernel* of f is dual: it is an arrow $B \to C'$ such that for any D, the sequence of abelian groups

$$0 \to \mathrm{Hom}_{\mathcal{A}}(C', D) \to \mathrm{Hom}_{\mathcal{A}}(B, D) \to \mathrm{Hom}_{\mathcal{A}}(A, D)$$

is exact. A kernel or cokernel, if it exists, is unique upto unique isomorphism; we let $\ker f$ (or $\ker(f)$) denote the kernel of f, and $\mathrm{coker}\, f$ (or $\mathrm{coker}\,(f)$) the cokernel.

Suppose every arrow in $\mathcal{A}$ has a kernel and cokernel; then for any arrow f, we have a unique arrow

$$\mathrm{coker}\,(\ker(f)) \to \ker(\mathrm{coker}\,(f)).$$

An *abelian category* is an additive category $\mathcal{A}$ such that every arrow has a kernel and cokernel, and the above arrow $\mathrm{coker}\,(\ker(f)) \to \ker(\mathrm{coker}\,(f))$ is an isomorphism, for any morphism f. This last condition means f cann be factored into the composition of a surjection (coker $= 0$), an isomorphism and an injection (ker $= 0$). Equivalently, an arrow with 0 kernel and cokernel is an isomorphism. There are additive categories $\mathcal{C}$ which are not abelian, where every arrow has a kernel and cokernel (for example, the category of Banach spaces and linear continuous maps, or the category of vector spaces with a given filtration, and linear maps preserving the filtrations).

Abelian categories provide the natural context for doing homological algebra; the usual basic lemmas (snake lemma, five lemma) which hold for the category of abelian groups also hold in any abelian category; in particular, a short exact sequence of complexes in an abelian category gives rise to a long excat sequence of (co)homology objects. The category of (co)chain complexes in an abelian category $\mathcal{A}$ is again abelian, where the kernel and cokernel of a morphism of complexes is defined term by term. One has a notion of (co)chain homotopy between two morphisms of complexes, and homotopic morphisms induce the same morphism on (co)homomolgy objects. From now onwards, we work only with cochain complexes and cohomology objects; the theory of chain complexes and homology is dual (*i.e.*, is obtained by working in the opposite category). For example, if

$$0 \to C_0 \to C_1 \to C_2 \to \cdots, 0 \to D_0 \to D_1 \to D_2 \cdots$$

are complexes, and $f_\bullet, g_\bullet : C_\bullet \to D_\bullet$ are 2 morphisms of complexes, a homotopy between them is a sequence of arrows $k_n : C_n \to D_{n-1}$ (with $k_0 = 0$), such that

$$\partial_D \circ k_n + (-1)^n k_{n+1} \circ \partial_C = f_n - g_n$$

for all n.

Let $C^\bullet[m]$ be the shifted complex with $(C^\bullet[m])^n = C^{m+n}$, and differential $(-1)^{mn}\partial_C$ on $C^{m+n} = (C^\bullet[m])^n$. Then $H^n(C^\bullet[m]) = H^{n+m}(C^\bullet)$. For any morphism $f^\bullet : C^\bullet \to D^\bullet$ between complexes, we can define its *mapping cone* by

$$C(f^\bullet)^n = C^{n+1} \oplus D^n$$

with differential $C(f^\bullet)^n \to C(f^\bullet)^{n+1}$ represented by the matrix

$$\begin{bmatrix} (-1)^{n+1}\partial_C & f^{n+1} \\ 0 & \partial_D \end{bmatrix}$$

Then there is a short exact sequence of complexes

$$0 \to D^{\bullet} \to C(f^{\bullet}) \to C^{\bullet}[1] \to 0,$$

such that the boundary morphism in the long exact sequence of cohomology objects

$$H^{n-1}(C^{\bullet}[1]) \to H^n(D^{\bullet})$$

is identified with $f^n : H^n(C^{\bullet}) \to H^n(D^{\bullet})$. In particular, $f^{\bullet}$ induces an isomorphism on all cohomology objects $\Leftrightarrow C(f^{\bullet})$ is exact.

An *injective object* of an abelian category $\mathcal{A}$ is an object I such that the functor $\mathrm{Hom}_{\mathcal{A}}(-, I)$ is exact. Equivalently, for any injective map $i : A \to B$, any arrow $f : A \to I$ factors though i (*i.e.*, f extends to B). A *projective object* is an injective object in the oposite abelian category.

We have the following lemma.

Lemma 13 *Let*

$$0 \to A \to C_0 \to C_1 \to \cdots$$

be a resolution, and let

$$0 \to B \to I_0 \to I_1 \to \ dots$$

be a complex, with I_j injective for all j. Then the natural map from the group of homotopy classes of maps of complexes $C_{\bullet} \to I_{\bullet}$ to $\mathrm{Hom}\,(A, B)$ *is an isomorphism of abelian groups (i.e., , any $f : A \to B$ lifts to a map of complexes $C_{\bullet} \to I_{\bullet}$, which is unique upto homotopy).*

This is easily proved by starting with a map f, and using the defining property of injective objects to construct the map of complexes inductively. Similarly given two such lifts, a homotopy between them may be constructed inductively.

We say that an abelian category *has enough injectives* if every object is a subobject of an injective object. Now suppose $F : \mathcal{A} \to \mathcal{B}$ is a left exact (additive) functor between abelian categories $\mathcal{A}$ and $\mathcal{B}$, and $\mathcal{A}$ has enough injectives. We define the *right derived functors* $R^nF(A)$ by

$$R^nF(A) = H^n(F(I_{\bullet}))$$

where

$$0 \to A \to I_0 \to I_1 \to I_2 \to \cdots$$

is an injective resolution of A. From the lemma, any two injective resolutions are homotopy equivalent, so have the same cohomology; hence the derived functors are well defined. Clearly $R^0F(A) \cong F(A)$, *i.e.*, R^0 is isomorphic to the given functor F.

If $0 \to A \to B \to C \to 0$ is an exact sequence in $\mathcal{A}$, then we can extend it to a compatible short exact sequence of injective resolutions as follows: first choose injective resolutions $0 \to A \to I_0 \to I_1 \to \cdots$ and $0 \to C \to J_0 \to J_1 \to \cdots$. Then we can inductively construct an injective map $B \to I_0 \oplus J_0$ and differentials $I_n \oplus J_n \to I_{n+1} \oplus J_{n+1}$ such that

(i) $0 \to B \to (I_0 \oplus J_0) \to (I_1 \oplus J_1) \to \cdots$ is a resolution, which we denote by $0 \to B \to K_\bullet$, and

(ii) the split exact sequences $0 \to I_n \to I_n \oplus J_n \to J_n \to 0$ fit together into an exact sequence of complexes $0 \to I_\bullet \to K_\bullet \to J_\bullet \to 0$.

Thus for any left exact functor $F : \mathcal{A} \to \mathcal{B}$, we obtain a short exact sequence of complexes

$$0 \to F(I_\bullet \to F(K_\bullet) \to F(J_\bullet) \to 0,$$

which gives rise to a long exact sequence of derived functors

$$0 \to F(A) \to F(B) \to F(C) \to R^1F(A) \to R^1F(B) \to \\ R^1F(C) \to R^2F(A) \to R^2F(B) \to \cdots$$

The notion of derived functors of objects can be generalized to 'derived functors of complexes', in the following sense. A map of complexes is called a *quasi-isomorphism* if it induces isomorphisms on cohomology objects, or equivalently, if its mapping cone is exact. Suppose $\mathcal{A}$ has enough injectives. Given a complex $0 \to C_0 \to C_1 \to C_2 \to \cdots$ in $\mathcal{A}$, there is a morphism of complexes $C_\bullet \to I_\bullet$ where $0 \to I_0 \to I_1 \to \cdots$ is a complex of injectives, such that $C^\bullet \to I^\bullet$ is a quasi-isomorphism, and any two such complexes of injectives are homotopy equivalent via unique (upto homotopy) cochain maps. If $C_\bullet$ is a single object A in degree 0, $I_\bullet$ is just an injective resolution of A. Define the *hyper-derived functors* of $C^\bullet$ to be

$$\mathbf{R}^nF(C^\bullet) = H^n(I^\bullet).$$

The definition is immediately extended to all complexes bounded below by shifting, using the formula

$$\mathbf{R}^nF(C^\bullet[m]) = \mathbf{R}^{n+m}F(C^\bullet).$$

Again, one shows that

(i) a short exact sequence of complexes (which are bounded below) induces a long exact sequence of hyper-derived functors

(ii) a quasi-isomorphism between (bounded below) complexes induces isomorphisms on hypercohomology

(iii) there are two functorial convergent spectral sequences

$$_{I}E_1^{p,q} = R^qF(C^p) \Rightarrow \mathbf{R}^{p+q}F(C^\bullet)$$

and

$$_{II}E_2^{p,q} = R^pF(H^q(C^{-d}) \Rightarrow \mathbf{R}^{p+q}F(C^\bullet)$$

In particular, a morphism of complexes $C^\bullet \to D^\bullet$ inducing isomorphisms $R^nF(C^m) \to R^nF(D^m)$ for all m, n induces isomorphisms on hyperderived functors $\mathbf{R}^nF(C^\bullet) \to \mathbf{R}^nF(D^\bullet)$.

Hyperderived functors (like *hypercohomology groups* of a complex of sheaves) arise naturally in algebraic geometry. For example, with the last remark above as a starting point, Grothendieck has given a purely algebraic definition of 'de Rham cohomology groups' for an algebraic variety over k, which are k-vector spaces, and for non-singular varieties over $\mathbf{C}$ agree with the usual de Rham cohomology (and hence with singular cohomology).

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