K-theory of Quadrics*

V. Srinivas
(after Swan, Kapranov and Panin)

Let F be a field of any characteristic , and let V be an n-dimensional vector space over F. Let $q \in S^2(V^*)$ be a nondegenerate quadratic form on V. Let $\mathbf{P}(V^*) = \operatorname{Proj} S(V^*)$ be the projective space of lines in V. Then

$$H^0(\mathbf{P}(V^*), \mathcal{O}_{\mathbf{P}}(2)) \cong S^2(V^*) \ni q;$$

let Q be the zero scheme of q. Then Q is a smooth quadric hypersurface in $\mathbf{P}(V^*)$. Let

$$Cl(q) = (\bigoplus_{m \geq 0} V^{\otimes m}) / < v \otimes v - q(v)1 >$$

be the Clifford algebra of q, and let

$$Cl(q) = Cl_0(q) \oplus Cl_1(q)$$

where $Cl_0(q)$ is the even Clifford algebra (the image of $\bigoplus_m V^{\otimes 2m}$ in $Cl(q)$).

Theorem -1 *There is a natural isomorphism*

$$K_i(Q) \cong K_i(F)^{\oplus n-2} \oplus K_i(Cl_0(q)).$$

The main sources are [Sw], [Be], [Ka] and [Pa]. We remark that if S is a $\mathbf{Z}$-scheme, $\mathcal{E}$ a locally free sheaf on S with a nondegenerate quadratic form, then an analogous theorem is valid for the associated family of quadrics $Q \to S$, and can be proved by an easy extension of the proof given here. We leave the formulation of the precise statement, and the necessary modifications of the proof, to the interested reader.

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1 The main theorem

If X is a scheme, $\mathcal{P}(X)$ the exact category of locally free sheaves of finite rank on X, then

$$K_i(X) = K_i(\mathcal{P}(X)) = \pi_{i+1}(BQ\mathcal{P}(X), 0)$$

is Quillen's K-group. If $\mathcal{A}$ is a locally free $\mathcal{O}_X$-module of finite rank which is a sheaf of Azumaya algebras over its centre, let $\mathcal{P}(X, \mathcal{A})$ (respectively $\mathcal{P}(X, \mathcal{A}^{op})$) denote the exact category of locally free $\mathcal{O}_X$-modules which are left (respectively right) $\mathcal{A}$-modules. Let $K_i(X, \mathcal{A})$ (respectively $K_i(X, \mathcal{A}^{op})$) denote the i^{th} K-group of $\mathcal{P}(X, \mathcal{A})$ (respectively $\mathcal{P}(X, \mathcal{A}^{op})$).

Some facts from K-theory:

1. If $f : X \to Y$ is a projective and flat morphism, then there is a direct image map $f_* : K_i(X) \to K_i(Y)$. If $\mathcal{A}$ is a sheaf of algebras, as above, on Y, then there are direct image maps $f_* : K_i(X, f^*\mathcal{A}) \to K_i(Y, \mathcal{A})$, $f_* : K_i(X, f^*\mathcal{A}^{op}) \to K_i(Y, \mathcal{A}^{op})$.

2. If $F : \mathcal{C}' \times \mathcal{C}'' \to \mathcal{C}$ is a biexact functor (i.e. $F(A, --), F(--, B)$ are exact functors for any $A \in \mathcal{C}', B \in \mathcal{C}''$), then we have pairings

$$K_0(\mathcal{C}') \otimes K_i(\mathcal{C}'') \to K_i(\mathcal{C}), K_i(\mathcal{C}') \otimes K_0(\mathcal{C}'') \to K_i(\mathcal{C}).$$

In particular, we have pairings:

(i) $K_0(X) \otimes K_i(X) \to K_i(X)$
(ii) $K_i(X) \otimes K_0(X) \to K_i(X)$
(iii) $K_0(X) \otimes K_i(X, \mathcal{A}) \to K_i(X, \mathcal{A})$
(iv) $K_i(X) \otimes K_0(X, \mathcal{A}) \to K_i(X, \mathcal{A})$
(v) $K_0(X, \mathcal{A}^{op}) \otimes K_i(X, \mathcal{A}) \to K_i(X)$
(vi) $K_i(X, \mathcal{A}^{op}) \otimes K_0(X, \mathcal{A}) \to K_i(X)$

Analogues of (iii), (iv) are valid for $\mathcal{A}^{op}$. The pairings (v), (vii) are induced by the biexact functor

$$\mathcal{P}(X, \mathcal{A}^{op}) \times \mathcal{P}(X, \mathcal{A}) \to \mathcal{P}(X), \mathcal{E} \times \mathcal{F} \mapsto \mathcal{E} \otimes_{\mathcal{A}} \mathcal{F},$$

and are denoted by the symbol $\otimes_{\mathcal{A}}$ (the remaining pairings are all denoted $\cdot$). Commutative and associative laws hold whenever they make sense e.g. the two pairings

$$K_0(X) \otimes K_i(X, \mathcal{A}^{op}) \otimes K_0(X, \mathcal{A}) \to K_i(X)$$

are equal.

3. <u>Projection formulas:</u>
If $f : X \to Y$ is a projective, flat morphism, then

$$f_*(a \cdot f^* b) = f_*(a) \cdot b$$

in the following situations:

(i) if $a \in K_0(X), b \in K_i(Y)$, then the formula holds in $K_i(Y)$

(ii) if $a \in K_i(X), b \in K_0(Y)$, then the formula holds in $K_i(Y)$

(iii) if $\mathcal{A}$ is a sheaf of algebras on Y as above, and $a \in K_0(X), b \in K_i(Y, \mathcal{A})$, then the formula holds in $K_i(Y, \mathcal{A})$

(iv) if $\mathcal{A}$ is a sheaf of algebras on Y as above, and $a \in K_0(X, f^*\mathcal{A}), b \in K_i(Y)$, then the formula holds in $K_i(Y, \mathcal{A})$

(v) if $\mathcal{A}$ is a sheaf of algebras on Y as above, and $a \in K_i(X), b \in K_0(Y, \mathcal{A})$, then the formula holds in $K_i(Y, \mathcal{A})$

(vi) if $\mathcal{A}$ is a sheaf of algebras on Y as above, and $a \in K_i(X, f^*\mathcal{A}), b \in K_0(Y)$, then the formula holds in $K_i(Y, \mathcal{A})$.

Analogous formulae hold with $\mathcal{A}^{op}$. Finally, we have

$$f_*(a \otimes_{f^*\mathcal{A}} f^* b) = f_* a \otimes_{\mathcal{A}} b$$

in $K_i(Y)$, whenever

(i) $a \in K_0(X, f^*\mathcal{A}^{op}), b \in K_i(Y, \mathcal{A})$, or

(ii) $a \in K_i(X, f^*\mathcal{A}^{op}), b \in K_0(Y, \mathcal{A})$.

4. <u>Fibre product formulas:</u>
If we have a fibre product diagram of schemes

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ f' \downarrow & & \downarrow f \\ Y' & \xrightarrow{g} & Y \end{array}$$

where f, f' are projective and flat, then

(i) $f'_* \circ g'^* = g^* \circ f_* : K_i(X) \to K_i(Y')$

(ii) if $\mathcal{A}$ is a sheafof algebras on Y as above, then $f'_* \circ g'^* = g^* \circ f_* : K_i(X, f^*\mathcal{A}) \to K_i(Y', g^*\mathcal{A})$.

We now state the key proposition, which is proved in the next section, and is the point of departure from Swan's original proof.

Proposition 0 *Let Q be a quadric as in the theorem. There exist locally free sheaves (which are functorial in Q) $\mathcal{E}_1, \mathcal{E}_2, \cdots \mathcal{E}_{n-3}$, and $\mathcal{U} \in \mathcal{P}(Q, Cl_0(q)^{op}), \mathcal{V} \in \mathcal{P}(Q, Cl_0(q))$, which are faithful $Cl_0(q)$ modules, such that there is a resolution on $Q \times Q$*

$$0 \to \mathcal{U} \Box_{Cl_0(q)} \mathcal{V} \to \mathcal{O}_Q(-n+3) \Box \mathcal{E}_{n-3} \to \cdots \\ \to \mathcal{O}_Q(-1) \Box \mathcal{E}_1 \to \mathcal{O}_{Q \times Q} \to \mathcal{O}_\Delta \to 0$$

where $\Delta \subset Q \times Q$ is the diagonal.

Here $\mathcal{F} \Box \mathcal{G} = p_1^* \mathcal{F} \otimes_{\mathcal{O}_{Q \times Q}} p_2^* \mathcal{G}$; and $\mathcal{F} \Box_{Cl_0(q)} \mathcal{G} = p_1^* \mathcal{F} \otimes_{(Cl_0(q) \otimes_F \mathcal{O}_{Q \times Q})} p_2^* \mathcal{G}$, if $\mathcal{F} \in \mathcal{P}(Q, Cl_0(q)^{op}), \mathcal{G} \in \mathcal{P}(Q, Cl_0(q))$.

We prove the theorem in the following more precise form. Let

$$\Phi_j : \mathcal{P}(F) \to \mathcal{P}(Q), 0 \leq j \leq n-3; \Phi_{n-2} : \mathcal{P}(Cl_0(q)) \to \mathcal{P}(Q)$$

be defined by

$$\Phi_j(W) = \mathcal{O}_Q(-j) \otimes_F W, 0 \leq j \leq n-3, \text{ for } W \in \mathcal{P}(F),$$

$$\Phi_{n-2}(W) = \mathcal{U} \otimes_{Cl_0(q)} W \text{ for } W \in \mathcal{P}(Cl_0(q)).$$

Let φ_j be the map on K_i induced by Φ_j.

Theorem 1 *The map*

$$\varphi = \sum_{j=0}^{n-2} \varphi_j : K_i(F)^{\oplus n-2} \oplus K_i(Cl_0(q)) \to K_i(Q)$$

is an isomorphism.

Let

$$a_j = [\mathcal{O}_Q(-j)], b_j = (-1)^j [\mathcal{E}_j] \in K_0(Q) \ (\mathcal{E}_0 = \mathcal{O}_Q),$$
$$a_{n-2} = [\mathcal{U}] \in K_0(Q, Cl_0(q)^{op}),$$
$$b_{n-2} = [\mathcal{V}] \in K_0(Q, Cl_0(q)),$$

where we abuse notation, and write $Cl_0(q)$ also for $Cl_0(q) \otimes_F \mathcal{O}_Q$. Then

$$\varphi_j(x) = a_j \cdot f^* x, \text{if } 0 \leq j \leq n-3,$$

$$\varphi_{n-2}(x) = a_{n-2} \otimes_{Cl_0(q)} f^*x,$$

where $f : Q \to \operatorname{Spec} F$ is the structure morphism. Hence

$$\varphi(x_0, x_1, \cdots, x_{n-2}) = \sum_{j=0}^{n-3} a_j \cdot f^*x_j + a_{n-2} \otimes_{Cl_0(q)} f^*x_{n-2}$$

Surjectivity of φ: For $z \in K_0(Q \times Q)$, let $z_* \in \operatorname{End}(K_i(Q))$ be defined by

$$z_*(x) = p_{1*}(z \cdot p_2^*x).$$

If $h : Q \to Q$ is any morphism, and $\Gamma \subset Q \times Q$ is the *transpose* of its graph, $[\Gamma] = [\mathcal{O}_\Gamma] \in K_0(Q \times Q)$, then one verifies easily that $[\Gamma]_*$ is the endomorphism of $K_i(Q)$ induced by h. In particular, $[\Delta]_*$ is the identity endomorphism of $K_i(Q)$.

From the proposition,

$$[\Delta] = \sum_{j=0}^{n-3} a_j \Box b_j + a_{n-2} \Box_{Cl_0(q)} b_{n-2}$$

in $K_0(Q \times Q)$, where $x \Box y$ denotes $p_1^*x \cdot p_2^*y$ (and $\Box_{Cl_0(q)}$ has a similar meaning). Hence, if $x \in K_i(Q)$, then

$$x = [\Delta]_*(x) = p_{1*}([\Delta] \cdot p_2^*x)$$
$$= \sum_{j=0}^{n-3} p_{1*}(a_j \Box b_j \cdot p_2^*x) + p_{1*}(a_{n-2} \Box_{Cl_0(q)} b_{n-2} \cdot p_2^*x)$$
$$= \sum_{j=0}^{n-3} a_j \cdot p_{1*} \circ p_2^*(b_j \cdot x) + a_{n-2} \otimes_{Cl_0(q)} p_{1*} \circ p_2^*(b_{n-2} \cdot x)$$

by the projection formulae. Now

$$\begin{array}{ccc} Q \times Q & \xrightarrow{p_2} & Q \\ p_1 \downarrow & & \downarrow f \\ Q & \xrightarrow{f} & \operatorname{Spec} F \end{array}$$

is a fibre product diagram; hence $p_{1*} \circ p_2^* = f^* \circ f_*$. We thus get

$$x = \sum_{j=0}^{n-3} a_j \cdot f^* \circ f_*(b_j \cdot x) + a_{n-2} \otimes_{Cl_0(q)} f^* \circ f_*(b_{n-2} \cdot x)$$
$$= \sum_{j=0}^{n-2} \varphi_j(f_*(b_j \cdot x))$$

which is contained in image φ.

Injectivity of φ

Let $\mathcal{P}(Q,f) \subset \mathcal{P}(Q)$ be the full subcategory of locally free sheaves $\mathcal{E}$ satisfying $H^m(Q,\mathcal{E}(k)) = 0$ for all $m > 0, k \geq 0$. Note that $\mathcal{O}_Q(-j) \in \mathcal{P}(Q,f)$ for $j \leq n-3$; hence for any F-vector space W, we have $\mathcal{O}_Q(-j) \otimes_F W \in \mathcal{P}(Q,f)$.

Lemma -2 *For any $W \in \mathcal{P}(Cl_0(q))$, we have $\mathcal{U} \otimes_{Cl_0(q)} W \in \mathcal{P}(Q,f)$, and $H^0(Q,\mathcal{U} \otimes_{Cl_0(q)} W \otimes \mathcal{O}_Q(k)) = 0$ for $k \geq 0$.*

Proof: Since $Cl_0(q)$ is a finite dimensional semisimple algebra over F, any W as above is projective; hence it suffices to prove the lemma for $W = Cl_0(q)$. It suffices to prove the lemma after a base change to Spec $\overline{F(Q)} =$ Spec K, the geometric generic point of Q. If V is a faithful $Cl_0(q)$-module, then $Cl_0(q)$ is itself a direct summand of the module $V^{\oplus N}$ for some $N > 0$. Hence it suffices to prove the lemma for $W = V$, where V is a given faithful module. Let V be the stalk of $\mathcal{V}$ at the geometric generic point. The resolution of the diagonal given by the Proposition yields an exact sheaf sequence on Q_K

$$0 \to \mathcal{U}_K \otimes_{Cl_0(q)_K} V \to \mathcal{O}_{Q_K}(-n+3) \otimes_K E_{n-3} \to \cdots$$
$$\to \mathcal{O}_{Q_K}(-1) \otimes_K E_1 \to \mathcal{O}_{Q_K} \to \mathcal{O}_\eta \to 0$$

where η is the closed point of Q_K determined by the geometric generic point, and E_j is the stalk of $\mathcal{E}_j$ at the geometric generic point. If $\mathcal{E} = \mathcal{U}_K \otimes_{Cl_0(q)_K} V$, then by splitting the above exact sequence into short exact sequences, we obtain

$$\begin{aligned} H^m(\mathcal{E}(k)) &\cong H^{m-n+2}(\mathcal{O}_\eta) = 0, 0 \leq m < n-2, k \geq 0; \\ H^{n-2}(\mathcal{E}(k)) &\cong \text{coker}\,(H^0(\mathcal{O}_{Q_K}(k)) \to H^0(\mathcal{O}_\eta)) = 0 \end{aligned}$$

□

Let $\Psi_j : \mathcal{P}(Q,f) \to \mathcal{P}(F)$ be the exact functor given by $\Psi_j(\mathcal{F}) = H^0(Q,\mathcal{F}(j))$, for $0 \leq j \leq n-3$. Let $\psi_j : K_i(Q) \to K_i(F)$ be the induced map. Then we see that

$$\psi_j \circ \varphi_j = \text{identity}\ ,\ 0 \leq j \leq n-3$$

$$\psi_k \circ \varphi_j = 0, j > k.$$

Thus to prove that φ is injective, we are reduced to proving that φ_{n-2} is injective.

We now distinguish between the cases of even and odd rank.

Case 1 (n odd):

In this case, $Cl_0(q)$ is a central simple algebra over F. Let X be the associated Severi-Brauer scheme. Then $H^i(X, \mathcal{O}_X) = 0$ for $i > 0$, and $H^0(X, \mathcal{O}_X) = F$. Hence if $\pi : X \to \operatorname{Spec} F$ is the structure map, then $\pi_* : K_0(X) \to K_0(F)$ satisfies $\pi_*(1) = 1$. Hence, from the projection formula, $\pi_* \circ \pi^* =$ identity in $\operatorname{End}(K_i(F))$ and in $\operatorname{End}(K_i(Cl_0(q)))$. Thus π^* is injective.

Let $\pi' : X \times Q \to Q$, $f' : X \times Q \to X$ be the projections. We have a fibre product diagram

$$\begin{array}{ccc} X \times Q \times Q & \stackrel{p_2'}{\to} & X \times Q \\ p_1' \downarrow & & \downarrow f' \\ X \times Q & \stackrel{f'}{\to} & X \end{array}$$

Let $\mathcal{E}_j' = \pi'^* \mathcal{E}_j$. Since X is the Severi-Brauer variety associated to $Cl_0(q)$, there is a locally free sheaf $\mathcal{E}$ on X such that $\mathcal{O}_X \otimes_F Cl_0(q) \cong \mathcal{E}nd(\mathcal{E})$. Let

$$\begin{aligned} \mathcal{U}' &= \pi'^* \mathcal{U} \otimes_{\mathcal{E}nd(\mathcal{E})} \mathcal{E}, \\ \mathcal{V}' &= \mathcal{E}^* \otimes_{\mathcal{E}nd(\mathcal{E})} \pi'^* \mathcal{V}. \end{aligned}$$

Let g be the composite map $X \times Q \times Q \to X$. Then we have a natural (Morita) isomorphism

$$\begin{aligned} \pi'^* \mathcal{U} \square_{\mathcal{E}nd(\mathcal{E})} \pi'^* \mathcal{V} &\cong p_1'^*(\pi'^* \mathcal{U}) \otimes_{g^* \mathcal{E}nd(\mathcal{E})} p_2'^*(\pi'^* \mathcal{V}) \\ &\cong p_1'^*(\mathcal{U}') \otimes_{\mathcal{O}_{X \times Q \times Q}} p_2'^*(\mathcal{V}') \\ &= \mathcal{U}' \square \mathcal{V}'. \end{aligned}$$

There is an equivalence of categories $\Theta : \mathcal{P}(X, \mathcal{E}nd(\mathcal{E})) \simeq \mathcal{P}(X)$, given by $\mathcal{F} \mapsto \mathcal{E}^* \otimes_{\mathcal{E}nd(\mathcal{E})} \mathcal{F}$; further, we have an isomorphism

$$\pi'^* \mathcal{U} \otimes_{\mathcal{E}nd(\mathcal{E})} \mathcal{F} \cong \mathcal{U}' \otimes_{\mathcal{O}_{X \times Q}} \Theta(\mathcal{F}).$$

Let $\Delta' = X \times \Delta \subset X \times Q \times Q$. The resolution for the diagonal on $Q \times Q$ pulls back under π' to a resolution for Δ'

$$\begin{aligned} 0 \to \mathcal{U}' \square \mathcal{V}' \to \mathcal{O}_{X \times Q}(-n+3) \square \mathcal{E}_{n-3}' \to \cdots \\ \to \mathcal{O}_{X \times Q}(-1) \square \mathcal{E}_1' \to \mathcal{O}_{X \times Q \times Q} \to \mathcal{O}_{\Delta'} \to 0. \end{aligned}$$

Let $a'_j = [\mathcal{O}_{X\times Q}(-j)], b'_j = (-1)^j[\mathcal{E}'_j]$ in $K_0(X \times Q)$ for $0 \leq j \leq n-3$ (where $b'_0 = 1$), and let $a'_{n-2} = [\mathcal{U}'], b'_{n-2} = (-1)^{n-2}[\mathcal{V}'] \in K_0(X \times Q)$. Let $\varphi'_j : K_i(X) \to K_i(X \times Q)$ be given by $x \mapsto a'_j \cdot f'^*x$, and let φ' : $K_i(X)^{\oplus n-1} \to K_i(X \times Q)$ be their sum. Since $\pi'^* \circ \varphi_j = \pi^* \circ \varphi'_j$ (this is clear for $j < n-2$, and follows for $j = n-2$ from the Morita equivalence discussed above), to prove the injectivity of φ, it suffices to prove that of φ'.

Let $c_{ij} = f'_*(a'_i \cdot b'_j) \in K_0(X)$. Suppose that for some $x_j \in K_i(X)$, we have

$$\sum_j a'_j \cdot f'^*(x_j) = 0.$$

Then we compute that

$$\begin{aligned} 0 &= f'_*(b_k \cdot (\sum_j a'_j \cdot f'^* x_j)) \\ &= \sum_j c_{jk} \cdot x_j. \end{aligned}$$

Hence it suffices to show that the determinant of the matrix $[c_{ij}]$ is a unit in $K_0(X)$. Let L be the algebraic closure of the function field $F(X)$. There is a surjective homomorphism of rings $K_0(X) \to K_0(L) = \mathbf{Z}$ whose kernel is a nilpotent ideal. So it suffices to prove that the determinant of $[c_{ij}]$ maps to a unit in $K_0(L)$.

Let $\widetilde{a}_j, \widetilde{b}_j$ be the images of a_j, b_j respectively under the natural map $K_0(Q) \to K_0(Q_L)$, where Q_L is the quadric obtained from Q by extending the ground field F to L. We again obtain a resolution for the diagonal of Q_L. If $\widetilde{f} : Q_L \to \operatorname{Spec} L$ is the structure map, then by the calculation we made earlier, we have a formula for any $x \in K_i(Q_L)$,

$$x = \sum_j \widetilde{a}_j \cdot \widetilde{f}^* \circ \widetilde{f}_*(\widetilde{b}_j \cdot x).$$

In particular, the $\widetilde{a}_j$ generate $K_0(Q_L)$ as a $K_0(L) = \mathbf{Z}$-module. However, $K_0(Q_L) \cong \mathbf{Z}^{\oplus n-1}$ from the standard decomposition of a quadric over an algebraically closed field into Schubert cells. Hence the $\widetilde{a}_i$ are a $\mathbf{Z}$-basis for $K_0(Q_L)$.The above equation, with $x = \widetilde{a}_j$, now yields $\widetilde{f}^*\widetilde{c}_{ij} = \delta_{ij}$, the Kronecker delta. Since $H^0(Q_L, \mathcal{O}_{Q_L}) = L$, and $H^m(Q_L, \mathcal{O}_{Q_L}) = 0$ for $m > 0$, we have $\widetilde{f}_* \circ \widetilde{f}^* =$ identity. Hence $\widetilde{c}_{ij} = \delta_{ij}$.

Case 2 (n even)

In this case, $Cl_0(q)$ is a central simple algebra over a reduced commutative F-algebra A which has dimension 2 over F.

Suppose $A \cong F \times F$ as an algebra. Then $Cl_0(Q) \cong B \times B$ for some central simple F-algebra B. Now an argument as in Case 1, with the Severi-Brauer variety Y associated to B in place of X, yields the injectivity of φ. Indeed, the role of X could be played by any smooth projective variety Y satisfying $H^0(Y, \mathcal{O}_Y) = F, H^i(Y, \mathcal{O}_Y) = 0$ for $i > 0$, and such that on Y, the pullback of $Cl_0(q)$ is a direct product of two endomorphism algebras of locally free sheaves.

Suppose A is a quadratic extension field of F. If Q_A is the quadric obtained from Q by base change, then $Cl_0(q_A) \cong Cl_0(q) \times Cl_0(q)$ as A- algebras. In particular, the maps $K_i(Cl_0(q)) \to K_i(Cl_0(q_A))$ are injective. Since the theorem is valid for q_A, we deduce that the map $\varphi_{n-2} : K_i(Cl_0(q)) \to K_i(Q)$ is injective. But we had reduced the proof of injectivity of φ to this.

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2 The resolution of the diagonal

Let $B_i = H^0(Q, \mathcal{O}_Q(i))$, so that $B = \oplus_m B_m$ is the homogeneous coordinate ring of Q. Let $B_+ \subset B$ be the ideal generated by elements of positive degree. Let

$$\begin{aligned} T(V) &= \oplus_m V^{\otimes m} \\ A &= T(V)[t]/ < v \otimes v - q(v)t > \end{aligned}$$

where t is an indeterminate which commutes with $T(V)$. Then A is graded with $\deg t = 2$. Clearly,

$$\begin{aligned} A/tA &\cong \wedge(V), \text{ (the exterior algebra on } V), \\ A/(t-1)A &\cong Cl(q). \end{aligned}$$

Let $A_i \subset A$ be the subset of homogeneous elements of degree i. If

$$(v_1, v_2) = q(v_1 + v_2) - q(v_1) - q(v_2)$$

defines the bilinear form (,) associated to q, the relation

$$v_1 v_2 + v_2 v_1 = (v_1, v_2)t$$

holds in A for any v_1, v_2 in $V = A_1$

Since $A_1 = V$, $B_1 = V^*$, we have pairings

$$B_i \otimes V^* \to B_{i+1}$$
$$V \otimes A_i \to A_{i+1}$$
$$A_i \otimes V \to A_{i+1}$$

where the second and third pairings correspond to left and right multiplication by elements of A_1, respectively. In particular, if $\varphi \in \mathrm{End}\,(V) = V^* \otimes V$, then we have two maps

$$\begin{array}{rll} l(\varphi): & A_i^* \otimes_F B_j & \to A_{i-1} \otimes B_{j+1}, \\ r(\varphi): & A_i^* \otimes_F B_j & \to A_{i-1} \otimes B_{j+1} \end{array}$$

corresponding to left and right multiplication by elements of V. Clearly the diagram

$$\begin{array}{ccc} A_i^* \otimes_F B_j & \stackrel{l(\varphi)}{\to} & A_{i-1}^* \otimes_F B_{j+1} \\ r(\varphi)\downarrow & & \downarrow r(\varphi) \\ A_{i-1}^* \otimes_F B_{j+1} & \stackrel{l(\varphi)}{\to} & A_{i-2}^* \otimes_F B_{j+2} \end{array}$$

Lemma -4 *If* $1 \in \mathrm{End}\,(V)$ *denotes the identity endomorphism, then*

$$l(1) \circ l(1) = r(1) \circ r(1) = 0$$

in

$$\mathrm{Hom}\,(A_i^* \otimes_F B_j, A_{i-2}^* \otimes B_{j+2}).$$

Proof: Let $v_1, v_2, \cdots, v_n$ be a basis for V,and let $v_1^*, v_2^*, \cdots, v_n^*$ be the dual basis; then $1 = \sum_i v_i^* \otimes v_i$. If $a \otimes b \in A_i^* \otimes_F B_j$, then

$$l(1)(a \otimes b) = \sum_i (v_i \cdot a) \otimes (v_i^* b)$$

and so

$$\begin{aligned} l(1) \circ l(1)(a \otimes b) &= \sum_{i,j=1}^n ((v_j v_i) \cdot a) \otimes (v_i^* v_j^* b) \\ &= \sum_{i=1}^n ((v_i^2) \cdot a) \otimes (v_i^{*2} b) + \sum_{i=1}^n \sum_{j<i} ((v_i v_j + v_j v_i) \cdot a) \otimes (v_i^* v_j^* b) \\ &= \sum_{i=1}^n (q(v_i) t \cdot a) \otimes (v_i^{*2} b) + \sum_{i=1}^n \sum_{j<i} ((v_i, v_j) t \cdot a) \otimes (v_i^* v_j^* b) \\ &= t \cdot a \otimes q(v_1^*, \cdots, v_n^*) b \\ &= 0 \end{aligned}$$

where $q(v_1^*, \cdots, v_n^*) \in S^2(V^*)$ maps to 0 in B. An analogous argument works for $r(1)$. □

Thus we obtain two complexes of graded B modules

$$\cdots \stackrel{l(1)}{\to} A_i^* \otimes_F B(-i) \stackrel{l(1)}{\to} A_{i-1}^* \otimes_F B(-i+1) \stackrel{l(1)}{\to} \cdots$$
$$\stackrel{l(1)}{\to} A_1^* \otimes_F B(-1) \stackrel{l(1)}{\to} B \to 0$$
$$\cdots \stackrel{r(1)}{\to} A_i^* \otimes_F B(-i) \stackrel{r(1)}{\to} A_{i-1}^* \otimes_F B(-i+1) \stackrel{r(1)}{\to} \cdots$$
$$\stackrel{r(1)}{\to} A_1^* \otimes_F B(-1) \stackrel{r(1)}{\to} B \to 0$$

(where for a graded module $M = \oplus M_m$, $M(i)$ is M with the new grading $M(i)_m = M_{m+i}$). Clearly both complexes have $H_0 = B/B_+ = F$.

Lemma -3 *The above complexes give graded resolutions of B/B_+ as a graded B module.*

Proof: We give the proof for the complex with $l(1)$; the argument for $r(1)$ is very similar.

For any graded left A-module $M = \oplus M_m$, the action of $A_1 = V$ allows us to define maps

$$l(\varphi) : M_i^* \otimes B(-i) \to M_{i-1}^* \otimes B(-i+1)$$

as in the case $M = A$, and we have $l(1) \circ l(1) = 0$. Let $M^* \otimes B$ denote the resulting complex of graded B modules. Clearly $M \mapsto M^* \otimes B$ is an exact contravariant functor from graded left A modules to complexes of graded B modules.

We have an exact sequence of graded left A modules

$$0 \to A(-2) \to A \to A/tA \to 0.$$

This gives an exact sequence of complexes

$$0 \to (A/tA)^* \otimes B \to A^* \otimes B \to A^* \otimes B[-2] \to 0$$

(where the $[-2]$ denotes the shift operator on complexes).Thus we have an exact homology sequence

$$\cdots H_i((A/tA)^* \otimes B) \to$$
$$H_i(A^* \otimes B) \to H_{i-2}(A^* \otimes B) \stackrel{\partial}{\to} H_{i-1}((A/tA)^* \otimes B) \to \cdots \qquad (\#)$$

Now (A/tA) is the exterior algebra of V, so that for any $v \in V$, the map $\wedge^i(V)^* \to \wedge^{i-1}(V)^*$ dual to left multiplication by v is contraction on the left with v. Hence $(A/tA)^* \otimes B$ is identified with the Koszul complex $K_.(B)$ for $V^* = B_1$ over B.

We have an exact sequence of complexes

$$0 \to K_.(S(V^*))(-2) \xrightarrow{q} K_.(S(V^*)) \to K_.(B) \to 0$$

which gives an exact homology sequence

$$\to H_i(K_.(S(V^*))) \xrightarrow{q} H_i(K_.(S(V^*))) \to H_i(K_.(B)) \to H_{i-1}(K_.(S(V^*))) \to$$

But $K_.(S(V^*))$ is exact except at $H_0 = F$, which is in degree 0; since deg $q = 2$, multiplication by q is 0 on $H_0(K_.(S(V^*)))$, and we deduce that $H_i(K_.(B)) = 0$ for $i \neq 0, 1$, and $H_i(K_.(B)) \cong F$ for $i = 0, 1$.

From the exact sequence (#) we now see that

$$H_i(A^* \otimes B) \cong H_{i+2}(A^* \otimes B),\ i \geq 1$$

and there is an exact sequence

$$0 \to H_2(A^* \otimes B) \to H_0(A^* \otimes B) \xrightarrow{\partial} H_1((A/tA)^* \otimes B) \to H_1(A^* \otimes B) \to 0.$$

If we show that $\partial : F \to F$ is nonzero, then $H_i(A^* \otimes B) = 0$ for $i = 1, 2$ and hence for all $i > 0$. We do this by computing the map ∂.

Let $v_1, v_2, \cdots, v_n$ be a basis for V, and let $v_1^*, \cdots, v_n^*$ be the dual basis. Note that a generator for $H_1((A/tA)^* \otimes B)$ is given by a nontrivial relation in B, with coefficients in $B_1 = V^*$, between the v_i^*, which is induced by the quadratic form q. Explicitly, we may choose the class of

$$\sum_{i=1}^{n} q(v_i) v_i^* \otimes v_i^* + \sum_{i<j} (v_i, v_j) v_i^* \otimes v_j^*$$

(which lies in $(A/tA)_1^* \otimes_F B_1$).

We have

$$A_2 = \sum_{i>j} F v_i v_j + Ft,$$

so that if $a \in A_2^*$ is defined by

$$\begin{aligned} a(v_i v_j) &= 0, \text{ if } i > j, \\ a(t) &= 1, \end{aligned}$$

then a maps to a generator $\alpha \in H_0(A^* \otimes B)$ under the map of complexes $A^* \otimes B \to A^* \otimes B[-2]$ dual to multiplication by t. Then $\partial\alpha$ is represented by the class in $H_1((A/tA)^* \otimes B)$ of $l(1)(a \otimes 1)$. We have

$$l(1)(a \otimes 1) = \sum_{i=1}^{n} (v_i \cdot a) \otimes v_i^*,$$

where $v_i \cdot a \in A_1^* = V^*$ satisfies

$$v_i \cdot a(v_j) = \begin{cases} 0 & \text{if } i > j \\ q(v_i) & \text{if } i = j \\ (v_i, v_j) & \text{if } i < j \end{cases}$$

Hence

$$l(1)(a \otimes 1) = \sum_{i=1}^{n} q(v_i) v_i^* \otimes v_i^* + \sum_{i<j} (v_i, v_j) v_i^* \otimes v_j^*,$$

which is precisely the generator for $H_1((A/tA)^* \otimes B)$ considered above. □

From the theorem of Serre about the correspondence between coherent sheaves and graded modules, the lemma yields exact sequences of locally free sheaves on Q

$$\cdots \overset{l(1)}{\to} A_i^* \otimes_F \mathcal{O}_Q(-i) \overset{l(1)}{\to} \cdots \overset{l(1)}{\to} A_1^* \otimes_F \mathcal{O}_Q(-1) \overset{l(1)}{\to} \mathcal{O}_Q \to 0$$
$$\cdots \overset{r(1)}{\to} A_i^* \otimes_F \mathcal{O}_Q(-i) \overset{r(1)}{\to} \cdots \overset{r(1)}{\to} A_1^* \otimes_F \mathcal{O}_Q(-1) \overset{r(1)}{\to} \mathcal{O}_Q \to 0.$$

Let

$$\mathcal{E}_i = \{\ker r(1) : A_i^* \otimes_F \mathcal{O}_Q(-i) \to A_{i-1}^* \otimes_F \mathcal{O}_Q(-i+1)\} \otimes_{\mathcal{O}_Q} \mathcal{O}_Q(i)$$

Then we have a resolution

$$0 \to \mathcal{E}_i \to A_i^* \otimes \mathcal{O}_Q \overset{r(1)}{\to} A_{i-1}^* \otimes \mathcal{O}_Q(1) \overset{r(1)}{\to} \cdots$$
$$\overset{r(1)}{\to} A_1^* \otimes \mathcal{O}_Q(i-1) \overset{r(1)}{\to} \mathcal{O}_Q(i) \to 0.$$

In particular, $\mathcal{E}_i$ is locally free on Q, and $\mathcal{E}_0 = \mathcal{O}_Q$. If $v \in V = A_1$, then the dual of left multiplication by v on A yields a map of complexes

$$\begin{array}{ccccccccccc} 0 \to & A_i^* \otimes \mathcal{O}_Q & \overset{r(1)}{\to} & A_{i-1}^* \otimes \mathcal{O}_Q(1) & \overset{r(1)}{\to} \cdots & \overset{r(1)}{\to} & A_1^* \otimes \mathcal{O}_Q(i-1) & \overset{r(1)}{\to} & \mathcal{O}_Q(i) & \to 0 \\ & v\cdot\downarrow & & v\cdot\downarrow & & & v\cdot\downarrow & & v\cdot\downarrow & \\ 0 \to & A_{i-1}^* \otimes \mathcal{O}_Q & \overset{r(1)}{\to} & A_{i-2}^* \otimes \mathcal{O}_Q(1) & \overset{r(1)}{\to} \cdots & \overset{r(1)}{\to} & \mathcal{O}_Q(i-1) & \to & 0 & \end{array}$$

Hence we have a pairing

$$V \otimes_F \mathcal{E}_i \to \mathcal{E}_{i-1}.$$

Since $V^* = H^0(Q, \mathcal{O}_Q(1))$, we have a pairing

$$V^* \otimes_F \mathcal{O}_Q(-i) \to \mathcal{O}_Q(-i+1).$$

Hence, the identity endomorphism of V induces a map on $Q \times Q$

$$\mathcal{O}_Q(-i)\Box\mathcal{E}_i \stackrel{l(1)}{\to} \mathcal{O}_Q(-i+1)\Box\mathcal{E}_{i-1}.$$

Lemma -2

$$\cdots \stackrel{l(1)}{\to} \mathcal{O}_Q(-i)\Box\mathcal{E}_i \stackrel{l(1)}{\to} \mathcal{O}_Q(-i+1)\Box\mathcal{E}_{i-1} \cdots$$
$$\stackrel{l(1)}{\to} \mathcal{O}_Q(-1)\Box\mathcal{E}_1 \stackrel{l(1)}{\to} \mathcal{O}_{Q\times Q} \to \mathcal{O}_\Delta \to 0$$

is a resolution of the diagonal on $Q \times Q$.

Proof: Let $B^{(2)} = \oplus_m B_m \otimes_F B_m$. Then $B^{(2)}$ is the homogeneous coordinate ring of $Q \times Q$ in the Segre embedding. The homogeneous coordinate ring of Δ in this embedding is $B(\Delta) = \oplus_m B_{2m}$, and the embedding $\Delta \subset Q \times Q$ corresponds to the surjection $B^{(2)} \to \oplus_m B_{2m}$ induced by the maps $B_m \otimes_F B_m \to B_{2m}$ given by the ring structure of B.

Let $E_{(i)}$ be the graded B-module defined by

$$E_{(i)} = (\ker r(1) : A_i^* \otimes B(-i) \to A_{i-1}^* \otimes B(-i+1)) \otimes_B B(i)$$

so that $E_{(i)}$ is associated to the sheaf $\mathcal{E}_i$.

For any two graded B-modules M_1, M_2 let $M_1\Box M_2$ be the graded $B^{(2)}$ module given by

$$(M_1\Box M_2)_i = (M_1)_i \otimes_F (M_2)_i.$$

We have a double complex of graded $B^{(2)}$ modules $C^{\cdot\cdot}$

$$\begin{array}{cccccccccc}
0 & & & & & & & & & \\
\uparrow & & & & & & & & & \\
\stackrel{l(1)}{\to} B(-i)\Box B(i) \to & & 0 & & & & & & & \\
\vdots & & & & & & & & & \\
 & & \cdots & 0 & & & & & & \\
 & & & \uparrow & & & & & & \\
 & & \cdots & \stackrel{l(1)}{\to} B(-2)\Box B(2) & \to & 0 & & & & \\
r(1)\uparrow & & & r(1)\uparrow & & \uparrow & & & & \\
\stackrel{l(1)}{\to} A^*_{i-1} \otimes_F B(-i)\Box B(1) & \stackrel{l(1)}{\to} & \cdots & \stackrel{l(1)}{\to} A^*_1 \otimes_F B(-2)\Box B(1) & \stackrel{l(1)}{\to} & B(-1)\Box B(1) & \to & 0 & & \\
r(1)\uparrow & & & r(1)\uparrow & & r(1)\uparrow & & \uparrow & & \\
\stackrel{l(1)}{\to} A^*_i \otimes_F B(-i)\Box B & \stackrel{l(1)}{\to} & \cdots & \stackrel{l(1)}{\to} A^*_2 \otimes_F B(-2)\Box B & \stackrel{l(1)}{\to} & A^*_1 \otimes_F B(-1)\Box B & \stackrel{l(1)}{\to} & B^{(2)} & \to & 0
\end{array}$$

Here the $l(1)$'s are induced by the dual to left multiplication by A_1 on the A_j^*, and the B module structure on a $B^{(2)}$ module induced by $B \subset B^{(2)}$ by $b \mapsto b \otimes 1$. The maps $r(1)$ are induced by right multiplication by A_1, and the B module structure induced by $b \mapsto 1 \otimes b$. Since the sub double complex of elements which are homogeneous of degree m has only a finite number of nonzero terms, for any m, the two spectral sequences both converge to the cohomology of the total complex (we regard the double complex as concentrated in the second quadrant).

If $Tot(C^{\cdot\cdot})$ is the total complex, there is a surjection $f : H^0(Tot(C^{\cdot\cdot})) \to B(\Delta)$ induced by the maps $C^{-i,i} = B(-i)\Box B(i) \to B(\Delta)$, $b_1 \otimes b_2 \mapsto b_1 b_2$. To see that this induces a map on $H^0(Tot)$, it suffices to prove that the diagram

$$\begin{array}{ccc} B(-i-1)\Box B(i+1) & \to & B(\Delta) \\ r(1)\uparrow & & \uparrow \\ A_1^* \otimes_F B(-i-1)\Box B(i)] & \stackrel{l(1)}{\to} & B(-i)\Box B(i) \end{array}$$

is commutative. Identifying $A_1^* \cong V^* \cong B_1$, this follows from the associativity of multiplication in B. Clearly f is a surjection.

The columns of $C^{\cdot\cdot}$ are exact, except possibly at H^0, and we have

$$E_1^{-i,0} = H^0(C^{-i,\cdot}) = B(-i)\Box E_{(i)};$$

clearly the differentials d_1 are just the maps $l(1)$. In particular, $E_1^{0,0} = B^{(2)}$; the edge homomorphism $E_1^{0,0} \to H^0(Tot)$,when composed with f, induces a map $B^{(2)} \to B(\Delta)$, which is readily computed to be the natural surjection. Hence, the lemma follows from Serre's theorem, if we prove that $H^i(Tot) = 0$ for $i < 0$, and f is an isomorphism.

To do this, we consider the other spectral sequence for the double complex. The complex $C^{\cdot,j}$ is obtained (upto a shift of [-j]) from the exact sequence of graded B modules

$$\cdots \stackrel{l(1)}{\to} A_i^* \otimes_F B(-i) \stackrel{l(1)}{\to} A_{i-1}^* \otimes_F B(-i+1) \stackrel{l(1)}{\to} \cdots$$
$$\stackrel{l(1)}{\to} A_1^* \otimes_F B(-1) \stackrel{l(1)}{\to} B \to 0$$

by applying the functor $\otimes_B B(-j)\Box B(j)$ where the B module structure on the $\Box$ term is via $b \mapsto b \otimes 1$. In particular it is exact except at H^{-j}, and

$$H^{-j}(C^{\cdot,j}) = F \otimes_B B(-j)\Box B(j) = F \otimes B_{2j},$$

concentrated in degree j. Hence $E_1^{i,j} = 0$ except when $i = -j$, and the spectral sequence degenerates at E_1. Thus $H^i(Tot) = 0$ for $i \neq 0$, and

$H^0(Tot)_i \cong B_{2i}$. Since f is a surjection of graded $B^{(2)}$ modules, whose spaces of homogeneous elements of degree i are isomorphic finite dimensional vector spaces, f is an isomorphism. $\square$

We now prove the proposition. We truncate the above resolution of $\mathcal{O}_\Delta$ as follows:

$$0 \to \mathcal{F} \to \mathcal{O}_Q(-n+3)\square\mathcal{E}_{n-3} \stackrel{l(1)}{\to} \cdots \to \mathcal{O}_{Q\times Q} \to \mathcal{O}_\Delta \to 0.$$

It suffices to prove that $\mathcal{F} \cong \mathcal{U}\square_{Cl_0(q)}\mathcal{V}$ for suitable $\mathcal{U}$, $\mathcal{V}$.

Since $\mathcal{E}_i$ has a left resolution

$$\cdots \stackrel{r(1)}{\to} A^*_{i+2} \otimes_F \mathcal{O}_Q(-2) \stackrel{r(1)}{\to} A^*_{i+1} \otimes_F \mathcal{O}_Q(-1) \to \mathcal{E}_i \to 0$$

$\mathcal{F}$ has a resolution by the total complex of the double complex of sheaves on $Q \times Q$

$$\begin{array}{ccccc}
\cdots & \stackrel{l(1)}{\to} & A^*_n \otimes_F \mathcal{O}_Q(-n+1)\square\mathcal{O}_Q(-1) & \stackrel{l(1)}{\to} & A^*_{n-1} \otimes_F \mathcal{O}_Q(-n+2)\square\mathcal{O}_Q(-1) \\
 & & r(1)\uparrow & & r(1)\uparrow \\
\cdots & \stackrel{l(1)}{\to} & A^*_{n+1} \otimes_F \mathcal{O}_Q(-n+1)\square\mathcal{O}_Q(-2) & \stackrel{l(1)}{\to} & A^*_n \otimes_F \mathcal{O}_Q(-n+2)\square\mathcal{O}_Q(-2) \\
 & & r(1)\uparrow & & r(1)\uparrow \\
 & & \vdots & & \vdots
\end{array}$$

From the natural identification (of F-algebras)

$$A/(t-1)A \cong Cl(q)$$

we obtain isomorphisms

$$A_i \cong Cl_{\bar{i}}(q) \text{ for } i \geq n-1$$

where $\bar{i}$ denotes $i(\mathrm{mod}\, 2)$. These are clearly compatible with the isomorphisms $A_i \cong A_{i+2}$ for $i \geq n-1$ given by multiplication by t. We can then rewrite the above double complex as

$$\begin{array}{ccccc}
\cdots & \stackrel{l(1)}{\to} & Cl_{\overline{n}}(q)^* \otimes_F \mathcal{O}_Q(-n+1)\square\mathcal{O}_Q(-1) & \stackrel{l(1)}{\to} & Cl_{\overline{n-1}}(q)^* \otimes_F \mathcal{O}_Q(-n+2)\square\mathcal{O}_Q(-1) \\
 & & r(1)\uparrow & & r(1)\uparrow \\
\cdots & \stackrel{l(1)}{\to} & Cl_{\overline{n+1}}(q)^* \otimes_F \mathcal{O}_Q(-n+1)\square\mathcal{O}_Q(-2) & \stackrel{l(1)}{\to} & Cl_{\overline{n}}(q)^* \otimes_F \mathcal{O}_Q(-n+2)\square\mathcal{O}_Q(-2) \\
 & & r(1)\uparrow & & r(1)\uparrow \\
 & & \vdots & & \vdots
\end{array}$$

Now $l(1)$, $r(1)$ are regarded as maps induced by the dual of left (respectively right) multiplication in the Clifford algebra.

There are natural isomorphisms

$$Cl_{\bar{i}}(q) \otimes_{Cl_0(q)} Cl_{\bar{j}}(q) \cong Cl_{\overline{i+j}}(q)$$

for all i, j. In particular, the above double complex can be regarded as the double complex $(\alpha)\square_{Cl_0(q)}(\beta)$, where (α), (β) are the following complexes:

$$\cdots \stackrel{l(1)}{\to} Cl_{\bar{i}}(q)^* \otimes_F \mathcal{O}_Q(-i) \stackrel{l(1)}{\to} Cl_{\overline{i-1}}(q)^* \otimes_F \mathcal{O}_Q(-i+1) \stackrel{l(1)}{\to} \cdots$$
$$\stackrel{l(1)}{\to} Cl_{\overline{n-1}}(q)^* \otimes_F \mathcal{O}_Q(-n+1) \stackrel{l(1)}{\to} Cl_{\overline{n-2}}(q)^* \otimes_F \mathcal{O}_Q(-n+2) \to 0 \qquad \cdots(\alpha)$$

$$\cdots \stackrel{r(1)}{\to} Cl_{\bar{j}}(q)^* \otimes_F \mathcal{O}_Q(-j) \stackrel{r(1)}{\to} Cl_{\overline{j-1}}(q)^* \otimes_F \mathcal{O}_Q(-j+1) \stackrel{r(1)}{\to} \cdots$$
$$\stackrel{r(1)}{\to} Cl_{\bar{2}}(q)^* \otimes_F \mathcal{O}_Q(-2) \stackrel{r(1)}{\to} Cl_{\bar{1}}(q)^* \otimes_F \mathcal{O}_Q(-1) \to 0 \qquad \cdots(\beta)$$

Thus $\mathcal{F} \cong \mathcal{U}\square_{Cl_0(q)}\mathcal{V}$ where $\mathcal{U} \cong H^0((\alpha))$ and $\mathcal{V} \cong H^0((\beta))$. This proves the Proposition.

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