23 Feb. 91.

Dear Pati,

I decided to type up the reply to your question, so that you have a proper record of it. I worked it out with Paramu's help. As you see, it is a bit intricate.

Let $(X,0) \hookrightarrow (\mathbf{C}^N,0)$ be the germ of an isolated 3-dim. singularity.

Proposition 1 *Let X be a representative of the germ $(X,0)$. Then after a linear change of coordinates, and shrinking the representative X if necessary, the following is true:*
let

$$D_i = X - \{0\} \cap \{z_i = 0\}, \quad 1 \le i \le n,$$

and let

$$E_{ijk} = \text{ zero locus (with its possibly non-reduced structure) of } \omega_{ijk} = d\,z_i \wedge d\,z_j \wedge d\,z_k \text{ on } X - \{0\}, \text{ for any } 1 \le i < j < k \le N.$$

Then

(i) D_i *is smooth for all* $1 \le i \le N$

(ii) E_{ijk} *is smooth for all* $1 \le i < j < k \le N$

(iii) D_i, D_j *meet transversally for all* $i \ne j$

(iv) E_{ijk}, $E_{i'j'k'}$ *meet transversally for all* $(i,j,k) \ne (i',j',k')$

(v) E_{ijk} *and* D_l *meet transversally for all* (i,j,k) *and all* l.

Here,

(i) in (iv) above, some of the coordinates of (i,j,k), (i',j',k') are allowed to coincide, and

(ii) in (v) above, the cases $l = i,j,k$ are allowed.

We need to use the following transversality lemma, which is (I guess) folklore (but I don't know a reference). Its a slight generalisation of Kleiman's Bertini theorem, as stated in Hartshorne's book, and has a similar proof (there is also an article of Kleiman *The transversality of the general translate* which has related arguments).

Lemma 1 *Let Y be a complex manifold, and let $\mathcal{E}_1, \cdots, \mathcal{E}_k$ be locally free sheaves of ranks $s_1, \cdots, s_k$ respectively. Let V_i be an n_i-dimensional space of global sections of $\mathcal{E}_i$ which generates $\mathcal{E}_i$, for each $1 \le i \le k$. Let r_i be a positive integer, with $r_i \le n_i$, and set $m_i = \min\{r_i, s_i\}$; let $\mathbf{G}_i$ be the Grassmannian of*

r_i-dimensional subspaces of V_i, and set $\mathbf{G} = \prod_{i=1}^k \mathbf{G}_i$. For $t = (t_1, \cdots, t_k) \in \mathbf{G}$, if $W_i(t) \subset V_i$ are the corresponding subspaces, let

$$Y_t(\mathcal{E}_1, \cdots, \mathcal{E}_k) = \{y \in Y \mid \text{ for each } 1 \le i \le k,\ W_i(t) \otimes \mathcal{O}_Y \to \mathcal{E}_i \text{ is not surjective}\},$$

with its natural (possibly non-reduced) structure as an analytic space, defined locally in Y by the vanishing of determinants.

Then there is a dense subset $U \subset \mathbf{G}$, whose complement is a countable union of locally closed analytic subsets of $\mathbf{G}$ of smaller dimension, such that for $t \in U$,

(i) $Y_t(\mathcal{E}_1, \cdots, \mathcal{E}_k)$ is empty, or has codimension $\sum_{i=1}^k (r_i - m_i + 1)(s_i - m_i + 1)$ in Y

(ii) the singular locus of $Y_t(\mathcal{E}_1, \cdots, \mathcal{E}_k)$ is empty, or has codimension $\sum_{i=1}^k (r_i - m_i + 2)(s_i - m_i + 2)$ in Y

(iii) the singular locus of the singular locus of $Y_t(\mathcal{E}_1, \cdots, \mathcal{E}_k)$ is empty, or has codimension $\sum_{i=1}^k (r_i - m_i + 3)(s_i - m_i + 3)$ in Y

(iv) $Y_t(\mathcal{E})$ is Cohen-Macaulay (i.e., its local rings are Cohen-Macaulay).

Proof: Let $\mathbf{G}'_i$ be the Grassmannian of s_i dimensional quotients of V_i, and let $f_i : Y \to \mathbf{G}'_i$ be the classifying map of $\mathcal{E}_i$. Let

$$\Gamma_Y = \{(t, y) \in \mathbf{G} \times Y \mid y \in Y_t(\mathcal{E}_1, \cdots, \mathcal{E}_k)\}.$$

If $\mathbf{G}' = \prod_{i=1}^k \mathbf{G}'_i$, and $\mathcal{Q}_i$ is the universal quotient bundle of rank s_i on the Grassmannian $\mathbf{G}'_i$, then there is a 'universal' such subvariety

$$\Gamma = \{(t, y) \in \mathbf{G} \times \mathbf{G}' \mid y \in \mathbf{G}'_t(\mathcal{Q}_1, \cdots, \mathcal{Q}_k)\},$$

which is Γ_Y for the case $Y = \mathbf{G}'$, $\mathcal{E}_i = \mathcal{Q}_i$.

The projections $\Gamma \to G$, $\Gamma \to \mathbf{G}'$ are known to be locally trivial fibre bundles (for the Zariski topology), whose fibres are irreducible and Cohen-Macaulay; further,
(a) Γ has codimension $\sum_{i=1}^k (r_i - m_i + 1)(s_i - m_i + 1)$ in $\mathbf{G} \times \mathbf{G}'$;
(b) the singular locus Γ_{sing} of Γ has codimension $\sum_{i=1}^k (n_i - m_i + 2)(s_i - m_i + 2)$ in $\mathbf{G} \times \mathbf{G}'$ – it is the locus of pairs (t, y) where for each i, the map $W_i(t) \to \mathcal{Q}_i \otimes \mathbf{C}(y_i)$ has rank $\le m_i - 2$;
(c) the singular locus of Γ_{sing} has codimension $\sum_{i=1}^k (r_i - m_i + 3)(s_i - m_i + 3)$ in $\mathbf{G} \times \mathbf{G}'$ – it is the locus of pairs (t, y) such that for each i the rank of $W_i(t) \to \mathcal{Q}_i \otimes \mathbf{C}(y_i)$ is $\le m_i - 3$.
These assertions are easily (??!) reduced to the case $k = 1$, since

$$\mathbf{G}'_t(\mathcal{Q}_1, \cdots, \mathcal{Q}_k) = \prod_{i=1}^k (\mathbf{G}'_i)_{t_i}(\mathcal{Q}_i)$$

(for the case $k = 1$, see, for example, Fulton's *Intersection Theory*, Ch. 14, especially Thm. 14.3). Hence the same estimates hold for the codimensions in $\mathbf{G}$ of any fibre of $\Gamma \to \mathbf{G}'$, and for that of its singular locus, etc.

Now $\Gamma_Y = \Gamma \times_{\mathbf{G}'} Y \to Y$ is also a locally trivial fibre bundle over the complex manifold Y, so the same estimates hold for the codimension in $\mathbf{G} \times Y$ of Γ_Y, that of its singular locus, etc. The first three assertions of the lemma now follow from Sard's theorem applied to the projection to $\mathbf{G}$. Since Γ, X are Cohen-Macaulay, so is $\mathbf{G} \times Y$; since $\mathbf{G} \times \mathbf{G}'$ is smooth, the fibre product $\Gamma_Y \subset \mathbf{G} \times X$ is locally defined by the vanishimg of a regular sequence, and so is Cohen-Macaulay. Hence any fibre $Y_y(\mathcal{E})$ of $\Gamma_Y \to \mathbf{G}$, which has the 'expected' codimension in X, is Cohen-Macaulay (again, since $\mathbf{G}$ is smooth, such a fibre is defined by the vanishing of a regular sequence). □

I don't think the estimate for the dimension of the 'singular locus of the singular locus', or the Cohen-Macaulay property, are needed in the proof of the above proposition, but it may be useful in other contexts. One has a similar statement for the higher iterated singular loci.

If Y is an irreducible analytic space, call a subset $U \subset Y$ *big* if it is the complement of a countable union of locally closed analytic subsets of smaller dimension. Then

(i) any countable intersection of big subsets is big

(ii) if $f : Y' \to Y$ is a morphism of irreducible analytic spaces, $U \subset Y$ a big subset, then $f^{-1}(U) \subset Y'$ is big

(iii) if $f : Y' \to Y$ is a morphism of irreducible analytic spaces with open, dense image and irreducible fibres, $U' \subset Y'$ a subset whose complement is the countable union of locally closed analytic subsets, such that for some big subset $U \subset Y$, $f^{-1}(y) \cap U' \subset f^{-1}(y)$ is big for all $y \in U$, then $U' \subset Y'$ is big (the hypotheses here can probably be relaxed). This is because

Let $G = GL_N(\mathbf{C})$, which we regard as parametrizing possible bases in $\mathbf{C}^N$. By the above remarks, it suffices to show that each of the smoothness/transversality conditions is (individually) valid for all bases of $\mathbf{C}^N$ in some big subset of G.

We now observe the following. Note that when we assert below that an variety is smooth, the variety always comes equipped with a natural structure as a possibly non-reduced analytic space, and we are asserting that this analytic space is a complex manifold.

1. For a given index i, there is a big subset of G on which the i^{th} hyperplane section of $X - \{0\}$ is a smooth surface.

   We apply lemma 1 with $Y = X - \{0\}$, $k = 1$, $\mathcal{E} = \mathcal{O}_{X-\{0\}}$, $V = \mathbf{C}^N$ and $r = 1$. Here elements of V are considered as homogeneous linear functions on $X - \{0\}$. This gives a big subset $U \subset \mathbf{P}^{N-1}$, the projective space of lines in $\mathbf{C}^N$, parametrizing smooth hyperplane sections. The choice of the index i yields a surjective mapping $\varphi_i : G \to \mathbf{P}^{N-1}$, giving it the structure of a homogeneous space; the desired big subset of G is $\varphi^{-1}(U)$.

2. For given $i \neq j$, there is a big subset of G on which the intersection of the i^{th} and j^{th} hyperplane sections is transversal.

   We apply lemma 1 with $Y = X - \{0\}$, $k = 1$, $\mathcal{E} = \mathcal{O}_{X-\{0\}}{}^{\oplus 2}$, $V = \mathbf{C}^N \oplus \mathbf{C}^N$ and $r = 1$. Elements of V are ordered pairs of homogeneous linear functions. This yields a big subset U' of $\mathbf{P}^{2N-1} = \mathbf{P}(\mathbf{C}^N \oplus \mathbf{C}^N)$. The choice of indices $i \neq j$ yields a map $\varphi_{ij} : G \to \mathbf{P}^{2N-1}$ whose image is a Zariski open subset, and $\varphi_{ij}^{-1}(U')$ is the desired big subset of G.

3. For a given triplet (i, j, k), there is a big subset of G such that the corresponding E_{ijk} is a smooth surface.

   We apply lemma 1 with $Y = X - \{0\}$, $k = 1$, $\mathcal{E} = \Omega^1_{X-\{0\}}$, $r = 3$. Elements of V yield differentials of homogeneous linear functions on $X - \{0\}$. The Grassmannian of 3 dim. subspaces of $\mathbf{C}^N$ is a homogeneous space for G, and the choice of indices yields a surjection $\varphi_{ijk} : G \to \mathbf{G}(3, \mathbf{C}^N)$; the inverse image under this of the big set of lemma 1 is the desired subset of G.

4. For a given triplet (i, j, k) and a given index $l \notin \{i, j, k\}$, there is a big open subset of G such that the corresponding intersection $E_{ijk} \cap D_l$ is transversal.

   We apply lemma 1 with $Y = X - \{0\}$, $k = 2$, $\mathcal{E}_1 = \mathcal{O}_{X-\{0\}}$, $\mathcal{E}_2 = \Omega^1_{X-\{0\}}$, $V_1 \cong V_2 \cong \mathbf{C}^N$, $r_1 = 1$, $r_2 = 3$, where V_1 is the space of homogeneous linear functions, and V_2 the space of differentials of homogeneous linear functions. Now use the fact that the choice of indices yields a map $\psi_{l,ijk} : G \to \mathbf{P}^{N-1} \times \mathbf{G}(3, \mathbf{C}^N)$; since $l \notin \{i, j, k\}$, its image is a Zariski open subset.

5. For a given pair of triplets (i, j, k) and (i', j', k') *with no common indices*, there is a big subset of G on which the intersection $E_{ijk} \cap E_{i'j'k'}$ is transversal.

   We apply lemma 1 with $Y = X - \{0\}$, $k = 2$, $\mathcal{E}_1 \cong \mathcal{E}_2 \cong \Omega^1_{X-\{0\}}$, $V_1 \cong V_2 \cong \mathbf{C}^N$ (considered as differentials of homogeneous linear functions), $r_1 = r_2 = 3$. The choice of indices yields a map $\psi_{ijk,i'j'k'} : G \to \mathbf{G}(3, \mathbf{C}^N) \times \mathbf{G}(3, \mathbf{C}^N)$ whose image is a Zariski open subset, since all 6 indices are distinct.

6. Let i, j, k be distinct indices, and choose $l \in \{i, j, k\}$; then there is a big subset of G for which $D_l \cap E_{ijk}$ is transversal.

   The choice of indices yields a map $\psi_{l,ijk} : G \to \mathbf{P}^{N-1} \times \mathbf{G}(3, \mathbf{C}^N)$ whose image is the closed subvariety $\mathbf{F}(1,3)$ consisting of pairs of subspaces (L, W) of $\mathbf{C}^N$ with $\dim L = 1$, $\dim W = 3$ and $L \subset W$ (this is a flag variety). We use the criterion (iii) above, applied to the projection $p : \mathbf{F}(1,3) \to \mathbf{G}(3, \mathbf{C}^N)$ (which is surjective with irreducible fibres), in order to produce a big subset of $\mathbf{F}(1,3)$ parametrizing smooth intersections.

First, the subset $U' \subset \mathbf{F}(1,3)$ parametrizing such transversal intersections is the complement of a countable union of locally closed analytic subsets, since it is the image of an analytic set under an analytic map (the projection to $\mathbf{F}(1,3)$ from the 'universal' such intersection, which is a closed analytic subset of $\mathbf{F}(1,3) \times X - \{0\}$). So we are reduced to showing that for some big set $U \subset \mathbf{G}(3, \mathbf{C}^N)$, and any fixed $W \in U$, there is a big subset of the fibre $p^{-1}(W) \cong \mathbf{P}^2$, such that the points corresponding to smooth intersections form a big subset. There are big subsets U_1, U_2 of $\mathbf{G}(3, \mathbf{C}^N)$ such that (a) for $W \in U_1$, the subvariety of $X - \{0\}$ on which W fails to generate $\Omega^1_{X-\{0\}}$ is a smooth surface S_W, and (b) for $W \in U_2$, W generates $\mathcal{O}_{X-\{0\}}$ (we may have to shrink the representative of the germ to do this); equivalently, if z_1, z_2, z_3 is a basis for W, then $\{0\}$ is the locus of common zeroes of the z_i. Now take $U = U_1 \cap U_2$. For $W \in U$, $S_W \subset X - \{0\}$ is a smooth complex surface; apply lemma 1 with $Y = S_W$, $k = 1$ $\mathcal{E} = \mathcal{O}_Y$, $r = 1$ and $V = W$; this yields a big subset of $\mathbf{P}^2 = p^{-1}(W)$ parametrising smooth hyperplane sections of S_W by hyperplanes defined by the vanishing of elements of W.

7. Let (i, j, k), (i', j', k') be two sets of indices with exactly two indices in common; then there is a big subset of G on which $E_{ijk} \cap E_{i'j'k'}$ is transversal except at a finite set.

   Let $\mathbf{F}(2,3) \subset \mathbf{G}(2, \mathbf{C}^N) \times \mathbf{G}(3, \mathbf{C}^N)$ be the flag variety parametrizing flags of subspaces $W_1 \subset W_2 \subset \mathbf{C}^N$ with $\dim W_1 = 2$, $\dim W_2 = 3$, and let $p : \mathbf{F}(2,3) \to \mathbf{G}(2, \mathbf{C}^n)$ be the projection. Let $\mathbf{F} = \mathbf{F}(2,3) \times_{\mathbf{G}(2,\mathbf{C}^N)} \mathbf{F}(2,3)$ be the fibre product, paramatrizing pairs of such flags with the same 2 dim. subspace W_1. The choice of indices (i, j, k), (i', j', k') gives a morphism $\psi : G \to \mathbf{F}$ whose image is the complement of the diagonal, hence is Zariski open and dense. So it suffices to find a big subset of $\mathbf{F}$ corresponding to transversal intersections.

   Let $U' \subset \mathbf{F}$ be the subset of the complement of the diagonal such that for $(W_1, W_2, W_2') \in \mathbf{F}$ (with $\dim W_1 = 2$, $\dim W_2 = \dim W_2' = 3$ and $W_1 = W_2 \cap W_2'$), then the subvariety of $X - \{0\}$ where neither W_2 nor W_2' generate $\Omega^1_{X-\{0\}}$ is a reduced curve (this is the desired transversality condition). Clearly (?!) U' is the complement of a countable union of locally closed analytic subsets.

   Consider the projection $q : \mathbf{F} \to \mathbf{G}(3, \mathbf{C}^N)$ (induced by the first projection $\mathbf{F} \to \mathbf{F}(2,3)$, followed by the projection $\mathbf{F}(2,3) \to \mathbf{G}(2,3)$; thus

$q(W_1, W_2, W_2') = W_2$. For $W \in \mathbf{G}(3, \mathbf{C}^N)$, the fibre $q^{-1}(W)$ is an irreducible subvariety of the flag variety $\mathbf{F}(2,3)$. There is a big subset of $\mathbf{G}(3, \mathbf{C}^N)$ corresponding to subspaces W such that the subvariety $S_W \subset X - \{0\}$, where W does not generate $\Omega^1_{X-\{0\}}$, is a smooth surface. The fibre of $\mathbf{F}(2,3) \to \mathbf{G}(3, \mathbf{C}^N)$ over W is $\cong \mathbf{P}^2$, parametrizing two dimensional subspaces of W; there is a big subset parametrizing the $W_1 \subset W$ such that the subvariety of S_W, where $W_1 \otimes \mathcal{O}_{S_W} \to \Omega^1_{X-\{0\}} \otimes \mathcal{O}_{S_W}$ has rank < 2, is a finite set T_{W_1} (apply lemma 1 with $Y = S_W$, $\mathcal{E} = \Omega^1_{X-\{0\}} \otimes \mathcal{O}_{S_W}$, etc. to conclude this degeneracy locus is zero dimensional). The fibre of $p_1 : \mathbf{F} \to \mathbf{F}(2,3)$ over (W_1, W) is isomorphic to the projective space of lines in $\mathbf{C}^N/W_1$; on $S_W - T_{W_1}$,

$$\mathcal{E} = \Omega^1_{X-\{0\}} \otimes \mathcal{O}_{S_W}/(\text{im } W_1 \otimes \mathcal{O}_{S_W})$$

is an invertible sheaf, generated by the space $\mathbf{C}^N/W_1$ of global sections; now lemma 1 yields a big subset of $p_1^{-1}(W_1, W)$ on which the desired transversality condition holds.

8. Let (i,j,k), (i',j',k') be two sets of indices with exactly 1 index in common; then there is a big subset of G on which $E_{ijk} \cap E_{i'j'k'}$ is transversal. This is similar to the previous case.

As you can see, its probably best to check the arguments carefully before you quote them!

Bye,