Riemann Surfaces

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Dedicated to Prof. K. R. Parthasarathy on turning 60.

This expository article on Riemann surfaces is an outgrowth of a Winter School on Riemann Surfaces held in December, 1995, organised by the Centre of Advanced Study in Mathematics of the University of Bombay, and sponsored by the University Grants Commission and the National Board for Higher Mathematics. It is a survey of some recent developments in the subject, which could not be included in the Winter School because of lack of time. We have however tried to make the exposition reasonably self-contained by recalling most of the basic definitions and concepts, as well as some key examples.

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1 Introduction, and some examples

Riemann surfaces are (Hausdorff) topological spaces on which one can do complex analysis of one complex variable. These generalizations of open subsets of the complex plane $\mathbb{C}$ occur in a natural way in a large number of areas of mathematics (including complex analysis in $\mathbb{C}$); their study is an active area of current research, though the subject is almost two centuries old.

We recall that a *Riemann surface* is a Hausdorff topological space X, together with a rule $\mathcal{O}_X$ which assigns to each open $U \subset X$ a set $\mathcal{O}_X(U)$ of functions $f : U \to \mathbb{C}$ (which will be called *holomorphic functions* on U) such that:

(i) (Sheaf property) if $\{U_i\}_{i\in I}$ are open subsets of X, and $U = \cup_{i\in I} U_i$, then a function $f : U \to \mathbb{C}$ lies in $\mathcal{O}_X(U) \Leftrightarrow f\,|_{U_i}: U_i \to \mathbb{C}$ lies in $\mathcal{O}_X(U_i)$ for all $i \in I$;

(ii) for each $x \in X$, there exists an open neighbourhood U of x, and a homeomorphism $f : U \to V$ onto an open subset $V \subset \mathbb{C}$, such that for any open $W \subset V$, a function $g : W \to \mathbb{C}$ is holomorphic on W (in the usual sense) $\Leftrightarrow g \circ f \in \mathcal{O}_X(g^{-1}(W))$.

The above definition is easily seen to be equivalent to the more usual one, according to which one is given a family of pairs (U_i, f_i), $i \in I$ (called *coordinate charts*), each consisting of an open set U_i and a homeomorphism f_i of U_i onto an open subset of $\mathbb{C}$ such that

(i)′ $\cup_i U_i = X$;

(ii)′ $f_j \circ f_i^{-1} : f_i(U_i \cap U_j) \to f_j(U_i \cap U_j) \subset \mathbb{C}$ is holomorphic whenever $U_i \cap U_j$ is nonempty.

A *holomorphic map* $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ between Riemann surfaces is a continuous map $f : X \to Y$ such that for all open subsets $V \subset Y$, and $g \in \mathcal{O}_Y(V)$, we have that $g \circ f \in \mathcal{O}_X(f^{-1}(V))$; f is said to be *biholomorphic* if f is a homeomorphism of X onto Y, and f and f^{-1} are holomorphic. However, it is easy to show that a holomorphic homeomorphism of one Riemann surface onto another is a biholomorphic map.

Clearly, $\mathbb{C}$ and its open subsets are Riemann surfaces in a natural way; if $U, V \subset \mathbb{C}$ are open sets, then a holomorphic map $f : U \to V$ is just a holomorphic function on U with $f(U) \subset V$. More generally, an open subset of a Riemann surface is a Riemann surface in a similar way.

Most of the properties of holomorphic functions on open subsets in $\mathbb{C}$ carry over to holomorphic maps between Riemann surfaces. For example, if $f, g : X \to Y$ are holomorphic maps and X is connected, then either $f \equiv g$, or $\{x \in X \mid f(x) = g(x)\}$ is a discrete[1] subset of X. Similarly, the open mapping theorem carries over in the obvious way. The theory of Riemann surfaces is concerned with the study of holomorphic maps between Riemann surfaces.

1.1 Examples of Riemann surfaces

Example 1.1: *The extended complex plane* $\overline{\mathbb{C}}$ *(or* $\mathbb{P}^1_{\mathbb{C}}$*).*

This is the first example of a "new" Riemann surface, *i.e.*, one which is not (biholomorphic to) an open subset of $\mathbb{C}$. As a topological space, $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ is just the one-point compactification of $\mathbb{C}$; in particular, $\overline{\mathbb{C}}$ is compact, and in fact homeomorphic to the two-sphere S^2. The Riemann surface structure can be defined, e.g., by two coordinate charts $(\mathbb{C}, Id)$ and $(\overline{\mathbb{C}} - \{0\}, w)$, where Id denotes the identity function, and

$$w(x) = \begin{cases} 0 & \text{if } x = \infty \in \overline{\mathbb{C}} \\ \dfrac{1}{x} & \text{if } x \in \mathbb{C} - \{0\}. \end{cases}$$

A *meromorphic function* on a Riemann surface X can be defined to be a holomorphic map $f : X \to \overline{\mathbb{C}}$ such that $f \not\equiv \infty$ on any connected component of X. When X is an open subset of $\mathbb{C}$, this reduces to the usual definition;

[1] A closed subset all of whose points are isolated.

further, when $X = \overline{\mathbb{C}}$, one can show that meromorphic functions are precisely the rational functions. The poles of a rational function f are the points in $f^{-1}(\infty)$.

The Riemann surface $\overline{\mathbb{C}}$ can be naturally identified with the 1-dimensional complex projective space $\mathbb{P}^1_{\mathbb{C}}$ (see Example 1.5).

Example 1.2: *The complex tori.*

These are compact (connected) Riemann surfaces homeomorphic to the two-torus $S^1 \times S^1$, and arise as follows. Choose $\omega_1, \omega_2 \in \mathbb{C}$ which are linearly independent over $\mathbb{R}$ (e.g., $\omega_1 = 1$, $\omega_2 \in \mathbb{C} \setminus \mathbb{R}$). Let $\Gamma = \Gamma(\omega_1, \omega_2)$ denote the additive subgroup of $\mathbb{C}$ generated by ω_1 and ω_2. Consider the quotient space E_Γ defined by the equivalence relation $z_1 \sim z_2 \Leftrightarrow z_1 - z_2 \in \Gamma$. Since the map

$$\mathbb{R}^2 \to \mathbb{C},$$

$$(s,t) \mapsto s\omega_1 + t\omega_2$$

is an $\mathbb{R}$-linear isomorphism carrying $\mathbb{Z} \oplus \mathbb{Z} \subset \mathbb{R}^2$ isomorphically onto Γ, the quotient space E_Γ is homeomorphic to

$$\mathbb{R}^2/\mathbb{Z} \oplus \mathbb{Z} = \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z} = S^1 \times S^1;$$

in particular it is a compact Hausdorff topological group. The Riemann surface structure on E_Γ is the one which makes the quotient map $\pi : \mathbb{C} \to E_\Gamma$ locally biholomorphic; thus if $U \subset E_\Gamma$ is an open set, then $f : U \to \mathbb{C}$ is holomorphic $\Leftrightarrow f \circ \pi : \pi^{-1}(U) \to \mathbb{C}$ is holomorphic. And meromorphic functions on E_Γ are precisely the doubly periodic meromorphic functions on $\mathbb{C}$ admitting ω_1 and ω_2 as periods.

Remark 1.1 Let X be a Riemann surface, and G be a subgroup of the group $\mathrm{Aut}\,(X)$ of biholomorphic maps $X \to X$. Suppose that any $p, q \in X$ have neighbourhoods U_p, U_q (respectively) such that $\{g \in G \mid g(U_p) \cap U_q \neq \emptyset\}$ is finite; G is then said to be a *discrete* subgroup of $\mathrm{Aut}\,(X)$. In this case, it can be shown that the orbit space $Y = X/G$ (whose points are G-orbits $\{gx \mid g \in G\}$), with the quotient topology induced by the natural quotient map $\pi : X \to Y$, is a Hausdorff space, and has a unique Riemann surface structure making π a holomorphic map. This result is easy to prove if G is *fixed-point free* (*i.e.*, if $g \in G$ is not the identity, then $gx \neq x$ for all $x \in X$). If X is simply connected, and G is fixed-point free, then G is isomorphic to the fundamental group $\pi_1(Y)$ of Y, and X is the universal covering space of Y. The complex tori are examples of this situation, where $X = \mathbb{C}$, and $G = \Gamma$, acting by translations.

The complex tori were the first (non-trivial) Riemann surfaces studied, and their complex analytic theory is completely understood. For example, if $E_{\Gamma'}$ corresponds to $\Gamma' = \Gamma(\omega'_1, \omega'_2)$, such that $\tau = \omega_1/\omega_2$, $\tau' = \omega'_1/\omega'_2$ have positive imaginary part, then E_Γ is biholomorphic to $E_{\Gamma'} \Leftrightarrow$ there exists

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z})$$

such that

$$\tau' = \frac{a\tau + b}{c\tau + d}. \tag{1.1}$$

Thus, if $\mathbb{H}$ is the upper half-plane in $\mathbb{C}$, then isomorphism classes of tori are in bijection with points of the quotient space $\mathbb{H}/\mathrm{SL}_2(\mathbb{Z})$ for the action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{H}$ given by the formula (1.1). More interestingly, every Riemann surface homeomorphic to $S^1 \times S^1$ is biholomorphic to some E_Γ.

Example 1.3: *Riemann surfaces of algebraic functions.*

These were the first "abstract" Riemann surfaces constructed by Riemann. The construction begins with an easy observation: if X is a Hausdorff space, and $\pi : X \to Y$ is a local homeomorphism[2] and if Y is a Riemann surface, then X becomes a Riemann surface by requiring that π be locally biholomorphic.

In the case at hand, X will be the set of all convergent power series $\sum_{n=0}^{\infty} a_n(z-b)^n$ which are obtained by analytic continuation (along some path in $\mathbb{C}$) of a fixed convergent power series $\sum_{n=0}^{\infty} \alpha_n(z-\beta)^n$; Y will be the complex plane $\mathbb{C}$, and $\pi : X \to Y$ will be defined by

$$\pi(\sum_{n=0}^{\infty} a_n(z-b)^n) = b.$$

(Thus, the series $\sum_{n=0}^{\infty} a_n(z-b)^n$ is obtained by analytic continuation along some path in $\mathbb{C}$ from β to b.) Note that there is a natural *Hausdorff* topology on X making π a local homeomorphism, and thus X becomes a connected Riemann surface. Defining $f : X \to \mathbb{C}$ by

$$f(\sum_{n=0}^{\infty} a_n(z-b)^n) = a_0,$$

we obtain a holomorphic function on X; this single-valued function on X represents the "multivalued" holomorphic function consisting of all analytic continuations of the given power series $\sum_{n=0}^{\infty} \alpha_n(z-\beta)^n$.

Now consider an *irreducible* polynomial $P(z,w) \in \mathbb{C}[z,w]$, the ring of polynomials in 2 variables (e.g., $P(z,w) = w^2 - z^2(1-z)$). We start from some $w_0(z) = \sum_{n=0}^{\infty} \alpha_n(z-\beta)^n$ such that $P(z, w_0(z)) = 0$ identically in a neighbourhood of $z = \beta$ (in our example, $w_0(z) = z\sqrt{(1-z)}$ and $\beta = 0$). Now perform the above construction on $w_0(z)$ to obtain a Riemann surface X. Because $P(z,w)$ is irreducible, one can prove that X will consist of *all* convergent power series $w(z) = \sum_{n=0}^{\infty} a_n(z-b)^n$ such that $P(z, w(z))$ is identically zero (near $z = b$). It is not hard to see that there is a finite set $F \subset \mathbb{C}$ such that for all $b \notin F$, the polynomial $P(b,w) \in \mathbb{C}[w]$ has $n = \deg_w P(z,w)$ distinct roots (here $\deg_w$ denotes the degree in the variable w), and $\pi^{-1}(b) \subset X$ consists of n

[2] *i.e.*, each $x \in X$ has an open neighbourhood U_x such that $\pi : U_x \to \pi(U_x)$ is a homeomorphism.

distinct points. (In our example, $F = \{0, 1\}$ and $n = 2$; note that $\pi^{-1}(0)$ also consists of 2 points, even though $0 \in F$!).

Now using some elementary algebraic topology, it is possible to adjoin finitely many points to X over each $b \in F$, and over $\infty \in \overline{\mathbb{C}}$, and obtain a compact Riemann surface $\overline{X}_P$, together with a holomorphic extension $\overline{\pi} : \overline{X}_P \to \overline{\mathbb{C}}$ of $\pi : X \to \mathbb{C} - F$. (At most n points are added over each point of $F \cup \{\infty\}$.) There will also be a single-valued meromorphic function $\overline{w}$ on $\overline{X}_P$ extending the obvious holomorphic function w on X, and the meromorphic function $P(\overline{\pi}, \overline{w})$ is identically 0 on $\overline{X}_P$. This is Riemann's construction of the compact Riemann surface associated to the irreducible polynomial P, or the "algebraic function" $w_0(z)$.

In our example $P(z, w) = w^2 - z^2(1 - z)$, the Riemann surface $\overline{X}_P$ will have only 1 point each over 1 and ∞, and is in fact biholomorphic to $\overline{\mathbb{C}}$. However, if we take $P(z, w) = w^2 - C(z)$ where $C(z)$ is a cubic polynomial with 3 distinct roots, then $\overline{X}_P$ is (biholomorphic to) a complex torus, and *all* complex tori arise in this way.

Example 1.4: *The method of Gauss.*

It was observed by Gauss that an $\mathbb{R}$-vector space V of dimension 2 can be made a $\mathbb{C}$-vector space by specifying any $\mathbb{R}$-linear map $J : V \to V$ with $J^2 = -1_V$ (where 1_V is the identity map); we have only to define $i \cdot v = J(v)$ for all $v \in V$. When $V = \mathbb{R}^2$, such a J is uniquely defined by specifying $J(1, 0) = (a, b)$ arbitrarily, so long as $b \neq 0$ (then $J^2 = -1_V$ will force $J(0, 1) = (\frac{-1-a^2}{b}, -a)$).

Now suppose $\Omega \subset \mathbb{R}^2$ is a connected open subset, and a, b are C^1 functions on Ω with $b > 0$. Then Ω can be made a Riemann surface as follows: for any $U \subset \Omega$, we declare a C^1 function $f = u + iv : U \to \mathbb{C}$ to be holomorphic iff, for each $x \in U$, the $\mathbb{R}$-linear map $df_x : \mathbb{R}^2 \to \mathbb{R}^2$ (defined by the Jacobian matrix at x of $(u, v) : U \to \mathbb{R}^2$) is $\mathbb{C}$-linear, where the image $\mathbb{R}^2$ has the usual $\mathbb{C}$-structure, while the source $\mathbb{R}^2$ has the $\mathbb{C}$-structure given by $J_x(1, 0) = (a(x), b(x))$. (It is not trivial to prove that this prescription makes Ω into a Riemann surface; the usual Riemann surface structure on Ω corresponds to $a \equiv 0$, $b \equiv 1$.) A beautiful theorem of Koebe asserts that Ω, with this new structure, is always biholomorphic to some open subset of the plane $\mathbb{C}$ (with its standard structure).

This method of Gauss of introducing a complex structure, and its generalizations, are of great importance in the study of Riemann surfaces and their higher dimensional analogues.

Example 1.5: *Smooth projective curves.*

Recall that the n-dimensional complex projective space $\mathbb{P}^n = \mathbb{P}^n_{\mathbb{C}}$ is the set of all one-dimensional $\mathbb{C}$-vector subspaces of $\mathbb{C}^{n+1}$; $\mathbb{P}^n$ becomes a compact Hausdorff space by regarding it as the quotient space of $\mathbb{C}^{n+1} - \{0\}$ by the equivalence relation $u \sim v \Leftrightarrow u = \lambda v$ for some $\lambda \in \mathbb{C} \setminus \{0\}$. Clearly $\mathbb{P}^0$ is a point; also, $\mathbb{C}^n$ is identified with an open subset of $\mathbb{P}^n$, as the image of $\mathbb{C}^n \times \{1\} \subset \mathbb{C}^{n+1} \setminus \{0\}$. The complement of $\mathbb{C}^n$ in $\mathbb{P}^n$ is easily seen to be $\mathbb{P}^{n-1}$;

in particular, $\mathbb{P}^1$ is a compact space obtained by adjoining 1 point to $\mathbb{C}$, and is hence identified with $\overline{\mathbb{C}}$.

Suppose now that $H \in \mathbb{C}[x, y, z]$ is a *homogeneous* polynomial such that $\dfrac{\partial H}{\partial x}, \dfrac{\partial H}{\partial y}, \dfrac{\partial H}{\partial z}$ have no common zero in $\mathbb{C}^3 - \{0\}$. Then the set C of (equivalence classes of) zeroes of H defines a closed subset of $\mathbb{P}^2$, which becomes a compact, connected Riemann surface in a natural way: if $(x_0, y_0, z_0) \in C$, and (say) $\frac{\partial H}{\partial x}(x_0, y_0, z_0) \neq 0$, then it can be verified that either y/z or z/y (or both) defines a local coordinate near (x_0, y_0, z_0). The Riemann surface C is exactly what is called in Algebraic Geometry a "smooth projective plane curve".

More generally, suppose $C \subset \mathbb{P}^n$ $(n \geq 2)$ is the set of (equivalence classes of) common zeroes of finitely many homogeneous polynomials $H_i \in \mathbb{C}[z_0, \ldots, z_n]$, and the Jacobian matrix $\left(\frac{\partial H_i}{\partial z_j}\right)$ has rank $n-1$ at all points of C. Then C again acquires the structure of a Riemann surface in a similar way; of course C will be compact (as $\mathbb{P}^n$ is so), but C need not be connected in general.

For example, for each $\lambda \in \mathbb{C} - \{0, 1\}$, the cubic polynomial (in 3 variables) $f_\lambda = zy^2 - x(x-z)(x-\lambda z)$ defines a compact Riemann surface $C_\lambda \subset \mathbb{P}^2$; this Riemann surface is biholomorphic to a complex torus, and every complex torus is biholomorphic to a C_λ.

1.2 Topology of a compact Riemann surface

A compact Riemann surface is, by virtue of the Riemann surface structure, a compact, *oriented* C^∞ differential manifold of dimension 2. As such, it has a C^∞ triangulation. As explained in the book [Mas], any such triangulation can be "simplified", to obtain a topological (or C^∞) classification of connected compact Riemann surfaces: any such is homeomorphic (resp. diffeomorphic) to a "g-holed torus" (or a "sphere with g handles") for a unique integer $g \geq 0$. For $g = 0$, this is a 2-sphere S^2; for $g \geq 1$, this can be realized as an identification space of a $4g$-sided polygon in a standard way. We call g the (topological) *genus* of X.

Some consequences of this are the following. First, the (singular) homology and cohomology of a (connected) compact Riemann surface has the following description:

$$H_i(X, \mathbb{Z}) \cong H^{2-i}(X, \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } i = 0, 2, \\ \mathbb{Z}^{\oplus 2g} & \text{if } i = 1, \\ 0 & \text{otherwise.} \end{cases}$$

This implies analogous results for homology and cohomology with coefficients in an arbitrary abelian group. The isomorphism between homology and cohomology is a particular case of the *Poincaré duality theorem.* There is a skew-symmetric pairing (the *cup-product*, or *intersection pairing*)

$$H^1(X, \mathbb{Z}) \otimes H^1(X, \mathbb{Z}) \to H^2(X, \mathbb{Z}) \cong \mathbb{Z};$$

it is non-degenerate (over $\mathbb{Z}$) by Poincaré duality, *i.e.*, induces an isomorphism $H^1(X,\mathbb{Z}) \cong \mathrm{Hom}\,(H^1(X,\mathbb{Z}),\mathbb{Z})$. In particular, the matrix determined by this pairing (with respect to any choice of $\mathbb{Z}$-basis for $H^1(X,\mathbb{Z})$) has determinant 1.

Finally, one has a presentation for the fundamental group $\pi_1(X,x)$ (with respect to any choice of the base point $x \in X$),

$$\pi_1(X,x) \cong F(a_1,\ldots,a_g,b_1,\ldots,b_g)/ < a_1b_1a_1^{-1}b_1^{-1}\cdots a_gb_ga_g^{-1}b_g^{-1} >,$$

as the quotient of the free group on the $2g$ generators a_i, b_j modulo the normal subgroup generated by the product of the commutators. This presentation follows from the description as a $4g$-sided polygon with identifications, using the van Kampen theorem (see [Mas]).

2 Divisors and the Riemann-Roch theorem

Let X be a Riemann surface. A *divisor* on X is a function $D : X \to \mathbb{Z}$ such that the set

$$\mathrm{supp}\,(D) = \{P \in X \mid D(P) \neq 0\},$$

called the *support* of D, is discrete. The set of divisors $\mathrm{Div}\,(X)$ forms an abelian group under pointwise addition, which is a subgroup of $\prod_{P\in X}\mathbb{Z}$. If $D(P) \geq 0$ for all $P \in X$, we call D an *effective* divisor, and write $D \geq 0$.

In particular, if X is compact, then any divisor on X has finite support, so $\mathrm{Div}\,(X)$ is the free abelian group on the points of X. In this case we may write a divisor as $D = \sum_i n_iP_i$, for points $P_i \in X$ and integers n_i; this notation means that $D(P_i) = n_i$, and $D(P) = 0$ if P is distinct from any of the P_i.

Let $f : X \to \mathbb{C}\cup\{\infty\} = \mathbb{P}^1_{\mathbb{C}}$ be an invertible meromorphic function on X (*i.e.*, $1/f$ is also meromorphic). For any $P \in X$, there is a coordinate neighbourhood (U, z) of P, with $z(P) = 0$. Then $f\mid_U = z^rg$ for a unique integer r, and function $g : U \to \mathbb{C}\cup\{\infty\}$ which is holomorphic and non-zero at P. The integer r is easily seen to be independent of the choice of (U, z); we define the *order* of f at P to be $\mathrm{ord}_P(f) = r$. Now $D : X \to \mathbb{Z}$, $D(P) = \mathrm{ord}_P(f)$, defines a divisor on X, which we denote by $\mathrm{div}\,(f)$. Such a divisor is called a *principal divisor*. Note that if f, g are two such functions on X, then

$$\mathrm{div}\,(fg) = \mathrm{div}\,(f) + \mathrm{div}\,(g).$$

Similarly, if ω is a meromorphic 1-form on X which is non-zero on any component of X, then on any coordinate neighbourhood (U, z) we may write $\omega = f(z)dz$, where f is meromorphic and invertible. We can then define $\mathrm{ord}_P(\omega) = \mathrm{ord}_P(f)$, for any $P \in U$. One verifies easily that $\mathrm{ord}_P(\omega)$ depends only on P and ω, and not on the coordinate (U, z). Thus we obtain a divisor $\mathrm{div}\,(\omega) = \sum_P \mathrm{ord}_P(\omega)P$.

Sometimes we will also use the notation $\mathrm{ord}_P(D)$, where D is a divisor, to mean the integer $D(P)$.

From now onwards, we will assume X is a *compact, connected* Riemann surface. For any divisor D on X, define

(i) an integer $\deg(D) = \sum_{P \in X} D(P)$ (this makes sense since the sum has only finitely many non-zero terms)

(ii) a vector space

$$L(D) = \{f \text{ meromorphic function on } X \mid \ f = 0, \text{ or } D + \operatorname{div}(f) \geq 0\}.$$

For example, if $D = \sum_i n_i P_i$ with $n_i > 0$ for all i (*i.e.*, D is effective), then $L(D)$ consists of functions which are holomorphic on $X \setminus \operatorname{supp}(D)$, which have a pole of order $\leq n_i$ at P_i for each i. So we can describe $L(D)$ as the space of meromorphic functions with "poles bounded by D". When D is effective, $L(D)$ always contains at least the constant functions, *i.e.*, $\dim L(D) \geq 1$.

An important problem in understanding function theory on X is what we nowadays call the *Riemann-Roch problem*: to determine (or efficiently estimate) $\dim L(D)$ for any divisor X on a compact Riemann surface X.

If D is effective of degree n, then one has an easy *upper bound* $\dim L(D) \leq n + 1$, obtained by considering principal parts of Laurent expansions in local coordinates around each of the P_i.

For any divisor D and non-zero meromorphic function f, we have an isomorphism of $\mathbb{C}$-vector spaces

$$L(D + \operatorname{div}(f)) \xrightarrow{\cong} L(D),$$

given by $g \mapsto fg$, since

$$(D + \operatorname{div}(f)) + \operatorname{div}(g) \geq 0 \quad \Leftrightarrow \quad D + \operatorname{div}(fg) \geq 0.$$

This implies that

$$\dim L(D) = \dim L(D + \operatorname{div}(f)).$$

This motivates the notion of *linear equivalence* of divisors: D is linearly equivalent to E if there exists a non-zero meromorphic function f such that $E = D + \operatorname{div}(f)$; thus the Riemann-Roch problems for D and E are equivalent. Define the *divisor class group* $\operatorname{Cl}(X)$ of a compact Riemann surface X to be the quotient

$$\operatorname{Cl}(X) = \frac{\operatorname{Div}(X)}{P(X)},$$

where $P(X)$ is the subgroup of $\operatorname{Div}(X)$ consisting of principal divisors (divisors of non-zero meromorphic functions). Thus divisors D, E have the same image in $\operatorname{Cl}(X)$ precisely when they are linearly equivalent.

We can now state the following important result.

Theorem 2.1 *(i) For any non-zero meromorphic function f on X, we have* $\deg(\mathrm{div}\,(f)) = 0$. *Hence there is a well defined degree map*

$$\deg : \mathrm{Cl}\,(X) \to \mathbb{Z}.$$

(ii) For any divisor D on X, $L(D)$ is a finite dimensional vector space. If $\deg(D) < 0$, *we have $L(D) = 0$.*

(iii) (Riemann-Roch) *Let X be a compact Riemann surface of genus g. Then there exists a divisor K on X, called a* canonical divisor*, with* $\deg(K) = 2g - 2$, *such that for any divisor D on X, we have*

$$\dim L(D) - \dim L(K - D) = \deg(D) + 1 - g.$$

Any two such canonical divisors K, K' are linearly equivalent. If ω is any non-zero meromorphic 1-form on X, then we may take $K = \mathrm{div}\,(\omega)$.

(iv) (Residue theorem) *For any meromorphic 1-form ω on X, the sum of the residues of ω vanishes.*

Here, in the residue theorem, the residue of a 1-form ω at P may be defined to be the integral of $\frac{1}{2\pi i}\omega$ over a small (positively oriented) contour around P. From (i) of the Theorem, we may define the group

$$\mathrm{Cl}^{\,0}(X) = \ker(\deg : \mathrm{Cl}\,(X) \to \mathbb{Z}).$$

We begin by listing a few simple corollaries of the above results.

Corollary 2.2 *Let X be a compact Riemann surface. Then there exists a non-constant meromorphic function on X.*

Proof: Let $P \in X$ be a point, and $D = (g+1)P$. Then

$$\dim L(D) \geq \deg(D) + 1 - g = 2.$$

Hence $L(D)$ contains a non-constant meromorphic function, which is holomorphic on $X - \{P\}$, and has a pole of order at most $g+1$ at P. □

Corollary 2.3 *Let X be a compact, simply connected Riemann surface. Then X is biholomorphic to the Riemann sphere $\mathbb{P}^1_{\mathbb{C}}$.*

Proof: Since X is simply connected, we have $H^1(X, \mathbb{C}) = 0$, and $g = 0$. Hence for any point P on X, we have $\dim L(P) \geq 2$. Let f be a non-constant function in $L(P)$. Then f must have a pole at P, which is of order 1, and be holomorphic elsewhere. This map f defines a holomorphic mapping $X \to \mathbb{P}^1_{\mathbb{C}}$, whose fibre over ∞ consists of 1 point P, which is not a point of ramification. Hence f is a biholomorphic map. □

Corollary 2.4 *Let $\Omega(X)$ be the vector space of holomorphic 1-forms on X. Then $\dim \Omega(X) = g$. We have $\deg K_X = 2g - 2$, where K_X is the divisor of any non-zero holomorphic (or meromorphic) 1-form on X.*

Proof: The Riemann-Roch theorem for $D = 0$ gives $\dim L(K_X) = g$. If ω is a meromorphic 1-form on X, then taking $K_X = \mathrm{div}\,(\omega)$, we see at once from the definitions that there is an isomorphism $L(K_X) \to \Omega(X)$, given by $f \mapsto f\omega$.

Now the Riemann-Roch theorem for $D = K_X$ gives $\deg K_X = 2g - 2$, since $\dim L(D) = g$, and $\dim L(K_X - D) = L(0) = 1$. □

We now give a sheaf theoretic interpretation of $L(D)$ (see the Appendix A for a brief discussion of sheaf theory). We will associate to each divisor D on a Riemann surface X a sheaf $\mathcal{O}_X(D)$. For any open subset U of X, define[3]

$$\mathcal{O}_X(D)(U) = \{f \text{ meromorphic on } U \mid \mathrm{ord}_P(f) + D(P) \geq 0 \ \ \forall\, P \in U\}.$$

Here, if f vanishes in a neighbourhood of P (e.g. if $f = 0$) then we take $\mathrm{ord}_P(f) = \infty$, so the condition $\mathrm{ord}_P(f) + D(P) \geq 0$ is taken to be true for any D. Now one easily verifies that $U \mapsto \mathcal{O}_X(D)(U)$ defines a sheaf of $\mathbb{C}$-vector spaces on X. By definition, the space of global sections $\mathcal{O}_X(D)(X)$ is just $L(D)$. The sheaf $\mathcal{O}_X(D)$ is in fact a sheaf of modules over the sheaf $\mathcal{O}_X$ of holomorphic functions on X. The space of global sections of $\mathcal{O}_X(D)$ is the zeroeth sheaf cohomology group $H^0(X, \mathcal{O}_X(D))$.

Theorem 2.5 (Serre duality) *Let D be a divisor on X. Then $H^1(X, \mathcal{O}_X(D))$ is a finite dimensional vector space, and there is a natural isomorphism of the dual vector space $H^1(X, \mathcal{O}_X(D))^\vee$ with $L(K_X - D)$.*

One can further show that $H^i(X, \mathcal{O}_X(D))$ vanishes for $i \geq 2$.

For expository reasons, we have stated the Riemann-Roch and Serre duality theorems as above. In fact, a convenient proof of the Riemann-Roch theorem goes in several steps: first show that $H^i(X, \mathcal{O}_X(D))$ is a finite dimensional vector space for all i, which vanishes for $i \geq 2$; then prove the Riemann-Roch theorem in the 'Euler characteristic' form

$$\chi(\mathcal{O}_X(D)) = \dim H^0(X, \mathcal{O}_X(D)) - \dim H^1(X, \mathcal{O}_X(D)) = \deg(D) + \chi(\mathcal{O}_X).$$

One proof uses Hilbert space methods, combined with properties of normal families of analytic functions (Montel's theorem in complex analysis). Now *define* the genus g of X to be the dimension of $H^1(X, \mathcal{O}_X)$. Then the version of Riemann-Roch stated in the Theorem follows from the Serre duality theorem, which is proved independently (for example, one can prove Serre duality using the Euler characteristic form of Riemann-Roch, together with properties of residues of differentials; see [Fo]).

[3] This slightly tricky definition of $\mathcal{O}_X(D)(U)$ is because U need not be connected.

Finally, one needs to identify the above modified notion of genus (the so-called *arithmetic genus*) with the *topological genus*. Again, one way to do this involves sheaf cohomology: one has the holomorphic de Rham complex

$$0 \to \mathbb{C}_X \to \mathcal{O}_X \xrightarrow{d} \Omega^1_X \to 0, \tag{2.2}$$

where $\mathbb{C}_X$ is the constant sheaf $\mathbb{C}$ on X, and $\mathcal{O}_X, \Omega^1_X$ respectively denote the sheaves of holomorphic functions and holomorphic differentials on X. Now from the corresponding exact cohomology sequence, one extracts a short exact sequence of finite dimensional $\mathbb{C}$-vector spaces

$$0 \to H^0(X, \Omega^1_X) \to H^1(X, \mathbb{C}) \to H^1(X, \mathcal{O}_X) \to 0. \tag{2.3}$$

From topology, the middle term has dimension $2g_{top}$, where g_{top} is the topological genus; from Serre duality, the extreme terms are dual to each other, hence have the same dimension g. This implies that $g = g_{top}$.

The cohomological point of view also yields another interesting result. The Mittag-Leffler theorem in complex analysis states that one can find a holomorphic function $f(z)$ on $\mathbb{C} - S$, where S is any discrete subset of $\mathbb{C}$, such that the Laurent expansion of f at each point of S has a prescribed principal part. From the proof of corollary 2.3, we see that on a compact Riemann surface of genus ≥ 1, it is impossible to find a function holomorphic outside a single point P, and with a simple pole at P. Hence we cannot have a strict analogue of the Mittag-Leffler theorem on a compact Riemann surface. However, we have the following result.

Theorem 2.6 (Mittag-Leffler theorem) *Let X be a compact Riemann surface of genus g, $S = \{P_1, \ldots, P_r\}$ points of X, and (U_i, z_i) a local coordinate at P_i for each i. Assume given convergent power series*[4] *without constant terms*

$$f_i(t) = \sum_{n=1}^{\infty} a_{in} t^n, \quad 1 \leq i \leq r.$$

Let $P \in X - S$ be an arbitrary point. Then there is a holomorphic function f on $X - (\{P\} \cup S)$ such that

(i) f has a pole of order $\leq 2g - 1$ at P

(ii) for each $1 \leq i \leq r$, the Laurent expansion of $f\,|_{U_i}$ with respect to the chosen coordinate z_i is given by

$$f(z_i) = g_i(z_i) + f_i(z_i^{-1})$$

where g_i is holomorphic in a neighbourhood of P_i.

[4] With infinite radius of convergence.

We do not prove this result here, but note that using the machinery of sheaf cohomology, it can be easily deduced from the vanishing of the sheaf cohomology group $H^1(X, \mathcal{O}_X((2g-1)P))$, which in turn follows from Serre duality.

Another topic which we touch on briefly is the *Riemann-Hurwitz formula*. If $f : Y \to X$ is a non-constant holomorphic map between compact Riemann surfaces, then one can show that the following properties hold (see [Fo], Chapter 1).

(i) There is an integer $d > 0$, called the *degree* of f, and a finite subset $S \subset X$ such that for any $P \notin S$, we have that $f^{-1}(P)$ consists of d points, while for $P \in S$, $f^{-1}(P)$ consists of $< d$ points (here S could be empty). The map $f : X - f^{-1}(S) \to Y - S$ is a *covering space* of degree d, in the topological sense.

(ii) There is a finite set $T \subset Y$, consisting of *ramification points*, with $f(T) = S$, such that for each $Q \in T$, $P = f(Q)$, we can find coordinate patches (U, z) on X at P and (V, w) on Y at Q such that $f(V) \subset U$, and $z \circ f = w^{e_Q} : V \to \mathbb{C}$, for some integer $e_Q > 1$ (*i.e.*, locally f is the map $z = w^{e_Q}$). This integer is called the *ramification index* of f at Q, and is independent of the choice of such local coordinates.

(iii) For any $Q \in Y - T$, if $P = f(Q)$, we can choose coordinate patches (U, z) on X at P, and (V, w) on Y at Q, so that $w = z \circ f : V \to \mathbb{C}$ (*i.e.*, locally, f is the map $z = w$). In this case we may define $e_Q = 1$.

(iv) For any $P \in X$, we have

$$\sum_{Q \in f^{-1}(P)} e_Q = d.$$

If $f : Y \to X$ is as above, and D is any divisor of X, define $f^*D(Q) = e_Q D(f(Q))$ (where we identify divisors with certain $\mathbb{Z}$-valued functions). Thus if

$$D = \sum n_i P_i,$$

then

$$f^*D = \sum n_i (\sum_{Q \in f^{-1}(P_i)} e_Q Q).$$

From property (iv) above, note that

$$\deg f^*(D) = d(\deg D) = (\deg f)(\deg D).$$

Also, $f^*(\mathrm{div}\,(g)) = \mathrm{div}\,(g \circ f)$, so that there are induced maps $f^* : \mathrm{Cl}\,(X) \to \mathrm{Cl}\,(Y)$ and $f^* : \mathrm{Cl}^0(X) \to \mathrm{Cl}^0(Y)$.

The formula $\deg K_X = 2g - 2$, combined with the interpretation of K_X as the divisor of a non-zero meromorphic differential, has the following important consequence.

Theorem 2.7 (Riemann-Hurwitz formula) *Let $f : Y \to X$ be a non-constant holomorphic map between compact Riemann surfaces. Let $g(X) =$ genus of X, $g(Y) =$ genus of Y. Let $P_1, \ldots, P_r$ be the points of Y where f is ramified, and let $e_i =$ ramification index of f at P_i. Then*

$$K_Y = f^*(K_X) + \sum_{i=1}^{r}(e_i - 1)P_i$$

holds in $\mathrm{Cl}\,(Y)$, *and*

$$2g(Y) - 2 = (\deg f)(2g(X) - 2) + \sum_{i=1}^{r}(e_i - 1).$$

Proof: Let ω be a non-zero meromorphic differential on X. Then $f^*\omega$ is a non-zero meromorphic differential on Y. If $\mathrm{div}\,(\omega) = D$, then we claim that

$$\mathrm{div}\,(f^*\omega) = f^*(D) + \sum_{Q \in Y}(e_Q - 1)Q,$$

where the right side is meaningful since $e_Q = 1$ for all but a finite number of Q. This will imply the first formula; on computing degrees on both sides, the second formula follows.

To prove the claim, suppose $Q \in Y$, $P = f(Q)$. Let (U, z) be a coordinate on X at P, and (V, w) a coordinate on Y at Q such that $f(V) \subset U$, and $z \circ f = w^{e_Q}$. Then $f^*dz = e_Q w^{e_Q - 1}dw$. If $\mathrm{ord}_P(\omega) = r$, then $\omega\,|_U = z^r h(z)\,dz$, where $h(z)$ is holomorphic and non-zero at $z(P)$. Then

$$f^*\omega\,|_V = (w^{e_Q})^r h(w^{e_Q}) e_Q w^{e_Q - 1}dw.$$

Hence

$$\mathrm{ord}_Q(f^*\omega) = re_Q + e_Q - 1 = e_Q \mathrm{ord}_P(\omega) + e_Q - 1 = \mathrm{ord}_Q(f^*\mathrm{div}\,\omega) + e_Q - 1.$$

Since Q was arbitrary, the claim is proved. □

Remark 2.8 The numerical formula relating genera and ramification indices can also be proved by topological arguments.

We mention two consequences of the Riemann-Hurwitz formula, which the reader may try to prove as an exercise. First, if $f : Y \to X$ is a non-constant map between compact Riemann surfaces, then $g(Y) \geq g(X)$, with equality possible only if $g(Y) \leq 1$, or f is an isomorphism. Combined with Theorem 2.10, this implies *Lüroth's theorem* in algebra: any subfield K of a pure transcendental extension $\mathbb{C}(t)$ of $\mathbb{C}$, with $\mathbb{C} \subsetneq K$, is again a pure transcendental extension of $\mathbb{C}$. A second application of the theorem is as follows: granting that for any Riemann surface X of genus $g \geq 2$, the group $\mathrm{Aut}\,(X)$ of holomorphic automorphisms is finite, then in fact $\mathrm{Aut}\,(X)$ has cardinality $\leq 84(g - 1)$ (the proof is by applying the Riemann-Hurwitz formula to the quotient map $X \to X/\mathrm{Aut}\,(X)$; see [Ha], IV, Ex. 2.5).

As a final application of the Riemann-Roch theorem, we state the following result.

Theorem 2.9 *Let X be a compact Riemann surface. Then there is a biholomorphic map from X onto a non-singular, projective algebraic curve.*

We give an idea of the proof when X has genus 1. Fix a point $O \in X$ (an origin). From the Riemann-Roch theorem, we see that $\dim L(D) = \deg D$ for any divisor D of degree > 0 (since $\deg K_X = 0$, and so $L(K_X - D) = 0$).

In particular, $L(O)$ is 1-dimensional, *i.e.*, consists of the constant functions, and $L(2O)$ is 2-dimensional. Let $x \in L(2O)$ be non-constant. Then x has a double pole at O, and no other poles. Hence x defines a holomorphic mapping $x : X \to \mathbb{P}^1_{\mathbb{C}}$ of degree 2. Each fibre of this mapping consists of either 2 distinct unramified points, or 1 point with ramification index 2. From the Riemann-Hurwitz formula, we deduce that there are 4 ramification points.

Now $L(3O)$ is 3-dimensional; let $y \in L(3O) - L(2O)$. Then y has a triple pole at O, and no other poles. Notice that

(i) $1, x, y, x^2, xy$ have poles at O of orders $0, 2, 3, 4, 5$ respectively, and hence are linearly independent elements of $L(5O)$

(ii) $1, x, x^2, x^3, y, xy, y^2$ are 7 elements in $L(6O)$, which is 6-dimensional

(iii) y^2, x^3 are in $L(6O) - L(5O)$.

Hence there is a non-trivial linear relation

$$py^2 + qxy = ax^3 + bx^2 + cx + d$$

with $a, b, c, d, p, q \in \mathbb{C}$ and $ap \neq 0$. We may replace y by $p^{1/2}y + rx$, and then x by $a^{1/3}x + s$, for suitable $r, s \in \mathbb{C}$ (for any choice of the square and cube roots), to get new functions $x \in L(2O)$, $y \in L(3O)$ as above for which the relation takes the simplified form

$$y^2 = x^3 + ux + v$$

for some complex numbers u, v. This equation defines an *algebraic plane curve* $A \subset \mathbb{C}^2$.

We claim the mapping $(x, y) : X - \{O\} \to A$ is injective. Indeed, if $x(P) = x(Q) = t \in \mathbb{C}$ with $P \neq Q$, then $x - t \in L(2O)$ has zeroes at P, Q. These must then be simple zeroes, and $\mathrm{div}\,(x - t) = P + Q - 2O$. If we also have $y(P) = y(Q) = t'$, then $\mathrm{div}\,(y - t') = P + Q + R - 3O$ for some (unique) point $R \in X - \{O\}$, since $y \in L(3O) - L(2O)$. Then

$$\mathrm{div}\,(\frac{y - t'}{x - t}) = R - O,$$

which (as in the proof of corollary 2.3) would imply that $X \cong \mathbb{P}^1_{\mathbb{C}}$, the isomorphism being given by the meromorphic function

$$f = \frac{y - t'}{x - t}.$$

Hence this is impossible, and so $y(P) \neq y(Q)$. In particular, $t^3 + ut + v \neq 0$.

The mapping $x : X \to \mathbb{P}^1_{\mathbb{C}}$ is ramified at 4 points, one of which is clearly O (since x has a double pole there). At any unramified point P, we saw that $t = x(P)$ is not a root of $x^3 + ux + v = 0$. Hence the roots of this cubic equation are the only possible values of x at a point of ramification in $X - \{O\}$. Hence this cubic must have 3 distinct roots, and they are precisely the remaining points of ramification of x. Since y has zeroes at each of these points, and $y \in L(3O)$, these 3 points must be precisely the zeroes of y, and must all be *simple* zeroes.

Since the map $x : A \to \mathbb{C}$ also has 2 fibres over points $t \in \mathbb{C}$ where $t^3+ut+v \neq 0$, and singleton fibres at the roots of the cubic, we see that $X - \{O\} \to A$ is in fact *bijective*.

If we differentiate x, y with respect to a local parameter at $P \in X$, then either (i) P is not a ramification point of $x : X - \{O\} \to \mathbb{C}$, in which case $x'(P) \neq 0$, or else (ii) P is a ramification point, in which case y has a simple zero at P, so that $y'(P) \neq 0$. Thus the map $(x, y) : X - \{O\} \to \mathbb{C}^2$ is a *holomorphic immersion*.

It is now easy to see that A is in fact a non-compact Riemann surface in $\mathbb{C}^2$, such that $X - \{O\} \to A$ is biholomorphic. Thus, we can say that the open Riemann surface $X - \{O\}$ 'is' the algebraic curve A!

We can extend the above analysis to include the point O as well, by considering the projective algebraic plane curve associated to A, which is defined by the homogeneous polynomial equation

$$y^2 z = x^3 + uxz^2 + vz^3.$$

One can show that this defines a smooth projective plane curve $\overline{A}$ (see Example 1.5), which is a 1-point compactification of A. Now one can show that $X \cong \overline{A}$ is biholomorphic.

The proof of the algebraicity theorem in general is along similar lines, but is more elaborate.

There is a refinement of this result, as follows. If $X \subset \mathbb{P}^n_{\mathbb{C}}$ is a smooth projective curve, then one can further prove that any meromorphic function on X is the restriction to X of a rational function in the coordinate functions of $\mathbb{P}^n_{\mathbb{C}}$. This implies that the field $\mathbb{C}(X)$ of meromorphic functions on any compact Riemann surface X is a so-called *algebraic function field in 1 variable* over $\mathbb{C}$, *i.e.*, a finitely generated extension field which has transcendence degree 1. If $f : X \to Y$ is a non-constant holomrophic map between Riemann surfaces, then there is an inclusion of fields $f^* : \mathbb{C}(Y) \to \mathbb{C}(X)$, given by $h \mapsto h \circ f$.

Conversely, one has the following.

Theorem 2.10 *(i) Let K be an algebraic function field of 1 variable over $\mathbb{C}$. Then there exists a compact Riemann surface X with $\mathbb{C}(X) \cong K$ (as $\mathbb{C}$-algebras).*

(ii) If X, Y are two compact Riemann surfaces, and $\varphi : \mathbb{C}(X) \to \mathbb{C}(Y)$ is a homomorphism of $\mathbb{C}$-algebras, then there is a unique holomorphic map $f : Y \to X$ (which is non-constant) such that $\varphi = f^*$.

Thus the notions of a compact Riemann surface, and an algebraic function field in 1 variable, are essentially equivalent. It can be shown that both of these are equivalent to the notion of a smooth projective algebraic curve over $\mathbb{C}$ (*i.e.*, any holomorphic mapping between smooth, projective curves is a morphism of algebraic varieties).

3 The Jacobian variety

General references for the material in this section are the lecture notes [Mu], [Mu2] and the book [ACGH].

Let X be a compact (connected) Riemann surface of genus $g \geq 1$ (this topic is uninteresting if $g = 0$, *i.e.*, for the Riemann surface $\mathbb{P}^1$). From the exact sequence (2.3), one has a surjection

$$\psi : H^1(X, \mathbb{C}) \to H^1(X, \mathcal{O}_X),$$

and hence a natural $\mathbb{R}$-linear map

$$\varphi : H^1(X, \mathbb{R}) \to H^1(X, \mathcal{O}_X).$$

We claim that φ is injective; equivalently, from (2.3),

$$H^0(X, \Omega^1_X) \cap H^1(X, \mathbb{R}) = 0$$

within $H^1(X, \mathbb{C})$. To see this, by de Rham's theorem, we must show that if ω is a holomorphic 1-form whose imaginary part, regarded as a C^∞ 1-form, is exact, then in fact $\omega = 0$. But if $\mathrm{Im}\,(\omega) = df$, then one checks easily that f is a harmonic function on X; now the maximum modulus principle implies that f is constant, which easily implies that $\omega = 0$.

In particular, since $H^1(X, \mathbb{Z}) \subset H^1(X, \mathbb{R})$ is a lattice (a discrete subgroup with compact quotient), we have that

$$\mathrm{image}(H^1(X, \mathbb{Z}) \to H^1(X, \mathcal{O}_X))$$

is a lattice of rank $2g$ in the g-dimensional complex vector space $H^1(X, \mathcal{O}_X)$. Hence the quotient

$$J(X) = \frac{H^1(X, \mathcal{O}_X)}{H^1(X, \mathbb{Z})}$$

is a g-dimensional complex torus.

Theorem 3.1 *$J(X)$ is a projective algebraic variety, such that the addition map $J(X) \times J(X) \to J(X)$ is a morphism of varieties.*

The idea of the proof is as follows. Identifying $H^1(X, \mathbb{R})$ with $H^1(X, \mathcal{O}_X)$, one computes that the intersection pairing on $H^1(X, \mathbb{R})$ corresponds to the imaginary part of a positive definite Hermitian form on $H^1(X, \mathcal{O}_X)$. This form is used to construct *theta functions* (see [Mu1], or [Mu2], Chapter 2) with respect to the lattice

$$\mathrm{image}(H^1(X, \mathbb{Z}) \to H^1(X, \mathcal{O}_X)).$$

Ratios of such theta functions yield meromorphic functions on $J(X)$, which are used to construct a projective embedding of $J(X)$. Now a general result implies that the group operation on $J(X)$, which is clearly holomorphic, is in fact algebraic.

We call $J(X)$ the *Jacobian variety* of X; it is an example of an *Abelian variety* (a projective algebraic variety with an algebraic group operation). We have $\dim J(X) = g$.

Let A be a complex torus, *i.e.*, we can write $A = V/\Lambda$, where V is a complex vector space, and $\Lambda \subset V$ is a lattice, that is, a discrete subgroup of maximal rank. One may take V to be the *Lie algebra* of the complex Lie group A; then Λ is the kernel of the exponential mapping $\mathrm{Lie}\,(A) \to A$, and is identifed with the fundamental group of A.

It is a theorem, essentially due to Riemann, that $A = V/\Lambda$ is an algebraic variety, which is then automatically a projective algebraic manifold, precisely when there exists a positive definite Hermitian form H on the vector space V, such that the alternating bilinear form $\mathrm{Im}\,(H)$ is integer valued on Λ. The choice of such a Hermitian form H (or of its imaginary part, which suffices to determine H) is called a *polarization* on the torus A. The proof that a polarized torus (A, H) is algebraic is along similar lines to that for $J(X)$; using H, one can construct suitable theta functions to projectively embed A. On the other hand, if A is already known to be projective algebraic, then a (Kähler) 2-form on projective space (a certain non-zero closed 2-form invariant under the projective linear group) can be used to construct a polarization H on A.

There is a numerical invariant associated to a polarization H on a torus $A = V/\Lambda$. The imaginary part of H yields an alternating, integer valued bilinear form on Λ; let M be the matrix for this alternating form in some basis for Λ (which is free abelian of rank $2\dim V = 2g$, say). Then M is a $2g \times 2g$ alternating matrix with integer entries, and non-zero determinant; the determinant is independent of the choice of basis. Then $\det(M)$ is the square of an integer, from the theory of the Pfaffian (see [La3]); the positive square root of $\det(M)$ is called the *degree* of the polarization H. If $\det(M) = 1$, we call H a *principal polarization*.

As we remarked earlier, for the torus $J(X)$, we have $V = H^1(X, \mathcal{O}_X)$, $\Lambda = H^1(X, \mathbb{Z})$, and $\mathrm{Im}\,(H)$ is the intersection form, which is of course integer valued (one can also describe the Hermitian form H explicitly, using the intersection form and the complex vector space structure, but we do not need this here). Since by Poincaré duality, the intersection pairing

$$H^1(X,\mathbb{Z}) \otimes H^1(X,\mathbb{Z}) \to H^2(X,\mathbb{Z}) = \mathbb{Z}$$

is non-degenerate, with determinant 1, we see that the polarization on $J(X)$ has degree 1, *i.e.*, $J(X)$ is principally polarized.

Theorem 3.2 *There is a natural identification of abelian groups* $\mathrm{Cl}^0(X) \cong J(X)$.

A sketch of a proof is as follows. Let $\mathcal{O}_X^*$ denote the sheaf of invertible holomorphic functions. One has the following exact sequence of sheaves

$$0 \to \mathbb{Z} \stackrel{1\mapsto 2\pi i}{\longrightarrow} \mathcal{O}_X \stackrel{f\mapsto e^f}{\longrightarrow} \mathcal{O}_X^* \to 0,$$

the *exponential sequence*, with associated long exact sequence of cohomology groups

$$\cdots H^0(X,\mathcal{O}_X) \to H^0(X,\mathcal{O}_X^*) \to H^1(X,\mathbb{Z}) \to H^1(X,\mathcal{O}_X) \to H^1(X,\mathcal{O}_X^*) \to H^2(X,\mathbb{Z}) \cdots$$

Here, $H^0(X,\mathcal{O}_X) \to H^0(X,\mathcal{O}_X^*)$ is the exponential map $\mathbb{C} \to \mathbb{C}^*$, which is surjective. Next, $H^1(X,\mathcal{O}_X^*)$ is identified with the group of isomorphism classes of line bundles (or locally free sheaves of rank 1), and hence in turn with the divisor class group $\mathrm{Cl}\,(X)$; the isomorphism associates to a divisor D the class of the sheaf $\mathcal{O}_X(D)$. Finally, the boundary map $H^1(X,\mathcal{O}_X^*) \to H^2(X,\mathbb{Z}) = \mathbb{Z}$ can be identified with the degree map $\mathrm{Cl}\,(X) \to \mathbb{Z}$. Thus we have an identification of $J(X)$ with the group $\mathrm{Cl}^0(X)$ of divisors of degree 0 modulo linear equivalence.

Another way to understand the isomorphism in Theorem 3.2 is as follows. By Serre duality, we may view $H^1(X,\mathcal{O}_X)$ as the dual space $H^0(X,\Omega^1_X)^\vee$. There is also an isomorphism (Poincaré duality) $H_1(X,\mathbb{Z}) \cong H^1(X,\mathbb{Z})$. Thus we must have a natural map $H_1(X,\mathbb{Z}) \to H^0(X,\Omega^1_X)^\vee$, that is, a bilinear pairing

$$H_1(X,\mathbb{Z}) \times H^0(X,\Omega^1_X) \to \mathbb{C}.$$

One can show that this is just the natural pairing

$$[\gamma] \times \omega \mapsto \int_\gamma \omega$$

given by integrating holomorphic 1-forms on homology classes (since holomorphic 1-forms are closed, this is well-defined, by Stokes theorem). Thus we have another description

$$J(X) = \frac{H^0(X,\Omega^1_X)^\vee}{H_1(X,\mathbb{Z})}.$$

Now we can define a map $AJ_X : \mathrm{Cl}^0(X) \to J(X)$, called the *Abel-Jacobi map*, as follows: if $D = \sum_i([x_i] - [y_i])$ is a divisor of degree 0, associate to it the linear functional on holomorphic 1-forms given by

$$\omega \mapsto \sum_i \int_{y_i}^{x_i} \omega.$$

This depends on the choice of paths joining y_i to x_i (and on the particular way of decomposing D); however, if we make different choices, one verifies that the difference between the two functionals is obtained by integration over some closed loop, *i.e.*, is the functional associated to an element of $H_1(X, \mathbb{Z})$. Now a more precise form of Theorem 3.2 states that the map AJ_X is an isomorphism of abelian groups. In particular, one has *Abel's Theorem*, which characterizes principal divisors on X as those divisors D such that, for a suitable choice of paths as above, the corresponding functional can be made to vanish identically.

In particular, let $x_0 \in X$ be a chosen base point. Then we have a set-theoretic map $X \to \mathrm{Cl}^0(X)$ given by $x \mapsto [x] - [x_0]$. Composing with the Abel-Jacobi map, we obtain a map

$$f : X \to J(X),$$

such that $f(x_0) = 0 \in J(X)$. It is easy to see that f is in fact holomorphic; from the description of the map AJ_X, this can be reduced to the fact that in any open disk in $\mathbb{C}$, the function $h(z) = \int_{z_0}^{z} g(w)dw$ is holomorphic, for any holomorphic function g on the disk, and any chosen base point z_0. Further, the Riemann-Roch theorem can be used to prove that f is an *embedding*. In particular, when $g = 1$, f is an *isomorphism* of X with $J(X)$, proving that any compact Riemann surface of genus 1 is biholomorphic to a complex torus.

Now since X and $J(X)$ are both projective algebraic, general theorems imply that f is an algebraic morphism. In particular, one can restrict rational functions on $J(X)$ to $f(X)$ to obtain rational functions on X; in particular, one can use theta functions to give a concrete description of rational functions on X (see [Mu2], Chapter 2).

It turns out that if X is defined (as an algebraic subvariety of some projective space) by equations with coefficients in a subfield k of $\mathbb{C}$, then so is $J(X)$; if x_0 also has coordinates in k, then f is described by polynomials with coefficients in k. This is important for number-theoretical applications, which we touch on later in this article.

We now mention 2 other important topics in connection with the Jacobian variety. A natural question is:

> to what extent does its Jacobian variety $J(X)$ determine a compact Riemann surface X?

There are examples (see [H], [HN], [Lg]) of non-isomorphic Riemann surfaces X_1, X_2 such that $J(X_1) \cong J(X_2)$ as complex tori (by a result of Narasimhan and Nori [NN], however, a given complex torus can be the Jacobian of at most a *finite* number of distinct Riemann surfaces). So $J(X)$ alone does not determine X. However, the *Torelli theorem* states that the pair $(J(X), H)$ consisting of $J(X)$ with its natural principal polarization, does determine X — in other words, if $(J(X_1), H_1) \cong (J(X_2), H_2)$, then $X_1 \cong X_2$.

A more precise statement is as follows. Suppose there is an isomorphism of the cohomology groups $f : H^1(X_1, \mathbb{Z}) \to H^1(X_2, \mathbb{Z})$ such that

(i) f is compatible with the intersection products on both groups

(ii) the induced $\mathbb{R}$-linear map $H^1(X_1, \mathbb{R}) \to H^1(X_2, \mathbb{R})$ is in fact $\mathbb{C}$-linear, where we use the ($\mathbb{R}$-linear) isomorphisms $H^1(X_j, \mathbb{R}) \to H^1(X_j, \mathcal{O}_{X_j})$, $j = 1, 2$ to regard the two real vector spaces as complex vector spaces.

Then f is induced by a unique biholomorphic map $\widetilde{f} : X_2 \to X_1$. Of course an isomorphism $(J(X_1), H_1) \cong (J(X_2), H_2)$ does yield such an f. For proofs of the Torelli theorem, see the books [Mu], [ACGH] and references given there.

We give a third formulation of the Torelli theorem which is perhaps the most concrete. If X is a compact Riemann surface of genus g, then one can find a basis for $H_1(X, \mathbb{Z})$ consisting of the classes of loops $a_1, \ldots, a_g, b_1, \ldots, b_g$ such that if $a_1^*, \ldots, b_g^*$ is the corresponding (dual) basis for $H^1(X, \mathbb{Z})$, then the intersection pairing has the simple form

$$< a_j^*, a_k^* >=< b_j^*, b_k^* >= 0,$$

$$< a_j^*, b_k^* >= \begin{cases} 0 & \text{if } j \neq k \\ 1 & \text{if } j = k \end{cases}$$

Such a basis for $H_1(X, \mathbb{Z})$ is called a *symplectic basis*, and is well determined up to the action of an element of the (integral) symplectic group $\mathrm{Sp}\,(2g, \mathbb{Z})$. One way to obtain such a basis is to express X as a quotient of a $4g$ sided polygon in the standard way; the images of oriented edges of the polygon are loops giving the desired basis.

Now we use the description $J(X) = H^0(X, \Omega^1_X)^\vee/(\mathrm{image} H_1(X, \mathbb{Z}))$. We know that $H_1(X, \mathbb{Z}) \to H^0(X, \Omega^1_X)^\vee$ is injective; further, the positive definite Hermitian form on $H^0(X, \Omega^1_X)^\vee$ is real valued, hence symmetric and positive definite, on the real g-dimensional subspace $\sum_j \mathbb{R}a_j$ (as the imaginary part, which corresponds to the intersection form, vanishes). Hence $\sum_j \mathbb{R}a_j$ contains an orthonormal set of cardinality g, *i.e.*, a $\mathbb{C}$-basis for $H^0(X, \Omega^1_X)^\vee$. Thus, $a_1, \ldots, a_g$ is a $\mathbb{C}$-basis for $H^0(X, \Omega^1_X)^\vee$. Hence we can find a (dual) basis for $H^0(X, \Omega^1_X)$ consisting of forms $\omega_1, \ldots, \omega_g$ such that

$$\int_{a_j} \omega_k = \begin{cases} 1 & \text{if } j = k, \\ 0 & \text{otherwise.} \end{cases}$$

Then we have a complex $g \times g$ matrix

$$\Omega = \left[\int_{b_j} \omega_k \right]_{1 \le j,k \le g} .$$

This is called a *Riemann matrix* for the curve X; it depends on the choice of symplectic basis. Now one can work out the conditions that the intersection form is the imaginary part of a positive definite Hermitian form; these are the so-called *Riemann bilinear relations*:

(i) Ω is a symmetric matrix, and

(ii) $\mathrm{Im}\,(\Omega)$, the imaginary part of the (symmetric) matrix Ω, defines a positive definite inner-product on $\mathbb{R}^g$.

If one changes the symplectic basis for $H_1(X, \mathbb{Z})$ by a $2g \times 2g$ matrix in the symplectic group $\mathrm{Sp}\,(2g, \mathbb{Z})$, written in $(g \times g)$-block form

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix},$$

then one computes that Ω is replaced by $(A\Omega + B)(C\Omega + D)^{-1}$. So we may finally restate the Torelli theorem as follows: given compact Riemann surfaces X and Y, such that their respective Riemann matrices Ω, Ω' are related as above by an element of $\mathrm{Sp}\,(2g, \mathbb{Z})$, then X and Y are biholomorphic.

This also brings us naturally to the second important topic alluded to earlier. This is the *Schottky problem*, which we first state loosely as follows:

> can one describe (or characterize) the set of all Riemann matrices of Riemann surfaces?

More precisely, one observes that in the discussion of Riemann matrices and Riemann's bilinear relations, one only needs that the imaginary part of the Hermitian form yields an integral, alternating form which has determinant 1; then standard linear algebra implies that one can find a symplectic basis for the lattice relative to this alternating form, and hence define an analogue of the Riemann matrix, which satisfies the Riemann bilinear relations. Conversely, given a $g \times g$ matrix Ω of complex numbers satisfying the Riemann relations, then $A = \mathbb{C}^g/\Lambda$, where $\Lambda = \mathbb{Z}^g + \Omega\mathbb{Z}^g$ does define a complex torus. Further, one checks that under the identification

$$\mathbb{Z}^g \times \mathbb{Z}^g \xrightarrow{\cong} \Lambda, \ (m, n) \mapsto \Omega m + n,$$

the alternating form

$$< (m_1, n_1), (m_2, n_2) >= m_1 \cdot n_2 - m_2 \cdot n_1$$

(where $\cdot$ is the dot product of g-vectors) is transported to a form on Λ which is the imaginary part of a positive definite Hermitian form on $\mathbb{C}^g$. This Hermitian form determines a principal polarization on the torus A.

Let $\mathbb{H}_g$ denote the set of all $g \times g$ complex matrices Ω satisfying the above two Riemann bilinear relations; it is an open subset of the $\mathbb{C}$-vector space of symmetric matrices. The integral symplectic group acts on $\mathbb{H}_g$ (on the left) by the formula

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \cdot \Omega = (A\Omega + B)(C\Omega + D)^{-1}.$$

From the discussion above, the quotient space

$$\mathcal{A}_g = \mathbb{H}_g / \mathrm{Sp}\,(2g, \mathbb{Z})$$

parametrizes isomorphism classes of principally polarized abelian varieties. The space $\mathcal{A}_g$, which by construction is (almost) a complex manifold[5] , can in fact be shown to be a (Zariski) open subset of a projective algebraic variety $\overline{\mathcal{A}_g}$; the algebraic structure is natural, in the following sense. Assume given a surjective morphism $f : X \to Y$ of algebraic varieties, with a section $\sigma : Y \to X$, such that the fibres $(f^{-1}(y), \sigma(y))$ form an algebraic family of principally polarized abelian varieties of dimension g (this notion can be made precise) with $\sigma(y) \in f^{-1}(y)$ as the origin. Then we have a set-theoretic map $Y \to \mathcal{A}_g$, given by

$$y \mapsto \text{isomorphism class of } f^{-1}(y).$$

The algebraic structure on $\mathcal{A}_g$ is the unique one such that for any such family f, the induced map $Y \to \mathcal{A}_g$ is a morphism of algebraic varieties. We express this by saying that $\mathcal{A}_g$ is a (coarse) moduli space for principally polarized abelian varieties of dimension g (the notions of a *moduli problem*, and *moduli space*, are discussed in some more detail later in this article).

Let $\mathcal{M}_g$ be the set of isomorphism classes of compact Riemann surfaces of genus g. Then the Torelli theorem implies that there is an injective map $\mathcal{M}_g \to \mathcal{A}_g$. One can prove that the image is a (Zariski) open subset of a subvariety of $\mathcal{A}_g$, such that the boundary points of $\mathcal{M}_g$ correspond to 'generalized Jacobians' of certain 'Riemann surfaces with singularities'.

A more precise version of the Schottky problem is the following: give equations (in terms of suitable coordinates on $\mathcal{A}_g$) describing the closure of the image of $\mathcal{M}_g$. There are certain natural equations satisfied by the closure of $\mathcal{M}_g$ in $\mathcal{A}_g$, which amount to the condition that the corresponding theta functions satisfy certain differential equations; the *Novikov Conjecture* asserts that the closure of $\mathcal{M}_g$ is precisely the set of solutions of these equations. The differential equations themselves are motivated by the so-called 'KdV equation' (or 'soliton equation') which arises in many other contexts in mathematics and physics (like fluid dynamics, and string theory, to name two).

[5] $\mathcal{A}_g$ has mild singularities, called quotient singularities; for example, it is a rational homology manifold.

The Schottky problem was solved fairly recently [AD]; shortly after that, the Novikov Conjecture was proved [S]. We do not attempt any discussion of these papers, and refer the interested reader to the original sources, as well as to the articles [Gu], [A].

4 Non-compact Riemann surfaces

Let $f = \sum_{n=0}^{\infty} a_n(z-b)^n$ be a convergent power series. Consider the set $C(f)$ of all convergent power series $\sum_{n=0}^{\infty} c_n(z-b)^n$ obtained by analytically continuing f along loops (closed paths) at b. Then, acording to a famous theorem of Poincaré and Volterra, the set $C(f)$ is at most countable.

Now, we know that there is a connected Riemann surface X_f and a holomorphic map $\pi : X_f \to \mathbb{C}$ such that $\pi^{-1}(b) = C_f$; hence C_f is a discrete subset of X_f. Thus, if we know that X_f is a countable union of compact subsets, then the countability of $C(f)$ would follow. And we have, in fact,

Theorem 4.1 (Rado) *Every connected Riemann surface is a countable union of compact subsets.*

One method of proving this theorem, which is useful in many contexts, has to do with harmonic functions and the Dirichlet problem, which we now briely explain. A function $h : X \to \mathbb{R}$ on a Riemann surface X is *harmonic* if it is locally the real part of a holomorphic function. Given an open subset Ω of X, the *Dirichlet problem* for Ω consists in finding a continuous function $h : \overline{\Omega} \to \mathbb{R}$ which is harmonic in Ω, and coincides with a given continuous function b on the boundary $\partial\Omega := \overline{\Omega} \setminus \Omega$. A method of Perron (Ahlfors [Ahl], Forster [Fo]) shows that the Dirichlet problem can be solved for Ω if, for example, $\partial\Omega$ is a finite disjoint union of Jordan curves[6] on X.

To prove Rado's theorem, we take $\Omega = X \setminus (K_1 \cup K_2)$, where K_1, K_2 are disjoint closed disks in X contained in a coordinate disk in X, and use Perron's method to obtain a non-constant harmonic function on Ω. It is then easy to show that Ω, and hence X, is a countable union of compact sets (see Forster [Fo]).

Using solvability of the Dirichlet problem, one can also prove fairly easily the following basic results.

Theorem 4.2 *Let Ω be a connected open subset of a Riemann surface X. Suppose that (i) $\overline{\Omega}$ is compact, and (ii) $\partial\Omega$ is the disjoint union of (differentiable) Jordan curves. Then there exists a* compact *Riemann surface Y which contains an open subset Ω' biholomorphic to Ω, and such that $Y - \Omega'$ is a finite disjoint union of compact sets homeomorphic to the closed unit disk in $\mathbb{C}$.*

[6] Homeomorphic images of the circle S^1.

Theorem 4.3 (Riemann Mapping Theorem, *or* Uniformization Theorem) *Let X be a connected, simply connected Riemann surface. Then*

(i) if X is compact, X is biholomorphic to $\overline{\mathbb{C}} = \mathbb{P}^1_{\mathbb{C}}$;

(ii) if X is non-compact, X is biholomorphic to either $\mathbb{C}$, or the unit disk in $\mathbb{C}$.

Both these theorems can be deduced rather easily from the following result, which can be proved using solvability of the Dirichlet problem (see Ahlfors [Ahl]).

Theorem 4.4 *A Riemann surface which is homeomorphic to an annulus $\{R_1 < \mid z \mid < R_2\}$ is biholomorphic to a unique annulus $\{1 < \mid z \mid < R \leq \infty\}$, or to $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$.*

By the Riemann-Roch theorem 2.1, we have detailed information about meromorphic functions on compact Riemann surfaces. Theorem 4.2 can be used to solve the analogues of the Weierstrass and Mittag-Leffler problems on relatively compact open subsets of any non-compact Riemann surface X. To do the same on the whole of X, one needs (as in the case of domains in $\mathbb{C}$) a *Runge approximation theorem.*

Theorem 4.5 (Behnke-Stein) *Any connected non-compact Riemann surface X can be written as an increasing union of relatively compact open subsets Ω_n such that, for each n, the pair (Ω_{n+1}, Ω_n) has the* Runge property*:*
for any holomorphic function f on Ω_n, any compact set $K \subset \Omega_n$ and any $\varepsilon > 0$, there exists a holomorphic function g on Ω_{n+1} such that $\mid g(z) - f(z) \mid < \varepsilon$ for all $z \in K$.

This implies that the theorems of Weierstrass and Mittag-Leffler are valid on *any* non-compact Riemann surface. In particular, on any non-compact Riemann surface, there exists a holomorphic function which takes arbitrarily prescribed values at points of any given discrete set.

The Uniformization Theorem is perhaps the single most useful result in the theory of Riemann surfaces. To explain some of its applications, we recall that, by topology, every connected Riemann surface admits a *universal covering* $\widetilde{X}$, which is a (Hausdorff) topological space X equipped with a map $\pi : \widetilde{X} \to X$ such that (i) $\widetilde{X}$ is connected and simply connected (ii) π is a local homeomorphism (iii) the fundamental group $\pi_1(X)$ of X acts discretely as a fixed-point free group of homeomorphisms of X onto itself, and the orbits of the action are precisely the fibres $\pi^{-1}(x)$, $x \in X$, of π. As π is a local homeomorphism, there is a unique Riemann surface structure on $\widetilde{X}$ which makes π locally biholomorphic, and it is then clear that $\pi_1(X)$ acts by holomorphic automorphisms on $\widetilde{X}$.

Now the Uniformization Theorem says that $\widetilde{X}$ is (biholomorphic to) either (i) $\mathbb{P}^1_{\mathbb{C}} = \overline{\mathbb{C}}$, (ii) $\mathbb{C}$ or (iii) the unit disk Δ (or equivalently, the upper half plane

$\mathbb{H}$) in $\mathbb{C}$. But every $g \in \mathrm{Aut}\,(\mathbb{P}^1) = \mathrm{PGL}_2(\mathbb{C})$ has a fixed point, and the fixed-point free automorphisms of $\mathbb{C}$ are precisely the translations $z \mapsto z + a$. It follows rather easily that

(i) $\mathbb{P}^1$ is the only Riemann surface X with universal cover $\widetilde{X} = \mathbb{P}^1$;

(ii) $\mathbb{C}$, $\mathbb{C}^* = \mathbb{C} - \{0\}$ and the complex tori $X = \mathbb{C}/\Gamma$ are the only Riemann surfaces X with $\widetilde{X} = \mathbb{C}$.

All other Riemann surfaces X arise as $X = \mathbb{H}/\Gamma$, where Γ is a discrete, fixed-point free subgroup of $\mathrm{Aut}\,(\mathbb{H})$ isomorphic to $\pi_1(X)$. Thus the Uniformization Theorem enables us, in principle, to list "all" (connected) Riemann surfaces in a fairly concrete way.

Example 4.1: Consider the Riemann surface $X = \mathbb{C} \setminus \{0, 1\}$. It is standard that its fundamental group $\pi_1(X)$ is non-abelian (it is the free group on 2 generators). It follows that its universal covering $\widetilde{X}$ cannot be $\mathbb{C}$, since all fixed-point free subgroups of $\mathrm{Aut}\,(\mathbb{C})$ are abelian. Hence $\widetilde{X} = \mathbb{H}$. This statement implies the Picard Theorem, that any entire function $f : \mathbb{C} \to \mathbb{C} - \{0, 1\} = X$ is constant (indeed, since $\mathbb{C}$ is simply connected, topology implies that there is a continuous lifting $\widetilde{f} : \mathbb{C} \to \widetilde{X} = \mathbb{H}$ of f; now $\widetilde{f}$ is a bounded, entire function).

Example 4.2: The Uniformization Theorem yields an easy proof of the fact that every Riemann surface is *triangulable* (Rado's Theorem). It can also be used to get complete information on the topological structure of a compact Riemann surface (c.f. Nevanlinna [Ne], or Springer [Sp]), as described earlier in §1.2 of this article.

Classification problems regarding non-compact Riemann surfaces seem to be very hard; for example, it is not easy to decide whether a given simply connected Riemann surface is either $\mathbb{H}$ or $\mathbb{C}$. There has been a lot of work on special classes of non-compact Riemann surfaces and inclusion relations among these classes, e.g. the class O_H (respectively O_A) of Riemann surfaces admitting no non-constant bounded harmonic (respectively holomorphic) functions, etc. (see Ahlfors-Sario [AS]). But the area does not appear to be very active now.

5 Moduli of Riemann surfaces

In general, a classification problem for certain mathematical objects leads to a *moduli problem* if, intuitively speaking, the objects in question can vary continuously. To see this in a concrete example from complex analysis, consider the set $\mathcal{D}$ of biholomophic equivalence classes of bounded, doubly connected domains $D \subset \mathbb{C}$ (*i.e.*, domains D such that $\mathbb{C} \setminus D$ has precisely 2 connected components). It is well-known (see [Ahl]) that each such D is biholomorphic to a unique annulus[7]

$$A_r = \{z \in \mathbb{C} \mid 1 < |z| < r\},$$

where $r \in (0, \infty)$. Thus, the classification problem in this case leads to the topological space $(1, \infty)$, and it is natural to ask in what sense $\mathcal{D}$ "is" this 1-dimensional Hausdorff space (instead of merely the underlying set).

The only compact Riemann surface of genus 0 is $\mathbb{P}^1_{\mathbb{C}}$. However, if $g \geq 1$, there are different isomorphism classes of Riemann surfaces of genus g, depending continuously on complex parameters. For any $g \geq 1$, let $\mathcal{M}_g$ be the set of isomorphism classes of compact Riemann surfaces of genus g. We will see that it has a natural structure as a topological space, and in fact, as an algebraic variety; it is called the *moduli space* for Riemann surfaces of genus g.

First we consider the case when $g = 1$. We had seen (example 1.2) that the moduli space $\mathcal{M}_1$ can be identified (as a set) with the quotient space $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$, where $\mathbb{H} = \{\tau \in \mathbb{C} \mid \mathrm{Im}(\tau) > 0\}$ is the upper half-plane in $\mathbb{C}$, and $\mathrm{SL}_2(\mathbb{Z})$ acts on $\mathbb{H}$ through fractional linear transformations $\tau \mapsto \frac{a\tau+b}{c\tau+d}$.

We now give a different-looking description of $\mathcal{M}_1$. We had also seen that any compact Riemann surface X of genus 1 can be realized as a smooth projective plane curve, defined by a homogeneous cubic equation

$$y^2 z = f(x, z),$$

where $f(x, 1)$ is a cubic polynomial with distinct roots. The meromorphic function x/z on X yields a holomorphic map $X \to \mathbb{P}^1$ of degree 2 ramified at 4 points, namely ∞ and the 3 roots of $f(x, 1) = 0$. One can show that two curves X, X' of genus 1 are isomorphic (biholomorphic) precisely if there is an automorphism of $\mathbb{P}^1_{\mathbb{C}}$ mapping the 4 ramification points of $X \to \mathbb{P}^1$ to the corresponding points for X'. The group $\mathrm{Aut}(\mathbb{P}^1)$ is known to be $\mathrm{PGL}_2(\mathbb{C}) = \mathrm{GL}_2(\mathbb{C})/(\text{scalars})$, acting through linear fractional transformations. Thus, if $U \subset (\mathbb{P}^1)^4$ is the open subset consisting of 4-tuples of distinct points, then $G = S_4 \times \mathrm{Aut}(\mathbb{P}^1)$ acts on U, where the symmetric group S_4 permutes the coordinates, and $\mathrm{Aut}(\mathbb{P}^1)$ acts diagonally. The parameter space $\mathcal{M}_1$ for Riemann surfaces of genus 1 is thus identified with the quotient space U/G.

We may further normalize the cubic polynomial f to be of the form $f_\lambda(x, z) = x(x-z)(x-\lambda z)$, where $\lambda \neq 0, 1$. Let X_λ be the curve defined by $y^2 z = f_\lambda(x, z)$. This amounts to forming the quotient $U/\mathrm{Aut}(\mathbb{P}^1)$; the quotient is identified with $\mathbb{P}^1 - \{0, 1, \infty\}$, with coordinate λ. Then X_λ is biholomorphic to X_μ iff there is an element of $\mathrm{Aut}(\mathbb{P}^1)$ carrying $\{0, 1, \infty, \lambda\}$ onto $\{0, 1, \infty, \mu\}$. This amounts to μ lying in the orbit of λ under the permutation group S_4 (which acts through its quotient $\cong S_3$, as it turns out), and the parameter space U/G may be then identified with the quotient $(\mathbb{C} - \{0, 1\})/S_3$, which we may compute to be $\mathbb{C}$, with $\frac{(\lambda^2-\lambda+1)^3}{\lambda^2(1-\lambda)^2}$ as a coordinate.

[7] Classically, $\log(r)$ was called the "modulus" of D.

One can prove that the point in $\mathbb{C}$ corresponding to the isomorphism class of $X = \mathbb{C}/\mathbb{Z} + \mathbb{Z}\tau$ is just the classical j-invariant $j(\tau)$, if we make an appropriate choice of the coordinate function on $(\mathbb{C} - \{0,1\})/S_3$. Thus, one has an identification $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H} \cong \mathbb{C}$ as well, *i.e.*, the natural Riemann surface structures on the set $\mathcal{M}_1$ obtained by the two (rather different) constructions are the same! The proof is not entirely trivial, however; Picard's theorem (that an entire function omitting 2 values is constant) can be deduced as a consequence.

The "reason" why both constructions gave the same Riemann surface structure on $\mathcal{M}_1$ is that both give a *universal* parameter space for "families" of Riemann surfaces of genus 1, depending continuously (or holomorphically, or algebraically) on parameters. Then, as is usual with objects satisfying universal mapping properties, the universal property would force uniqueness.

We will later make the above notions of "families" more precise. The notion of an algebraic family is easiest to define, at least in some contexts: for example, an algebraic family of plane curves of degree d parametrized by a variety T is given by a collection of homogeneous polynomials $f_t(x, y, z)$ of degree d, whose coefficients are all algebraic (regular) functions on T with no common zero.

Thus, in our example above, the Riemann surfaces (or algebraic curves) X_λ form an algebraic family. Further, G can be viewed as an algebraic group, and U as an algebraic variety; the quotient map $U \to U/G \cong (\mathbb{C} - \{0,1\})/S_3 \cong \mathbb{C}$ is seen to be a morphism of algebraic varieties. Thus, the algebraic construction of $\mathcal{M}_1$ reduces to the problem of construction of a quotient of an algebraic variety modulo the action of an algebraic group; in this case, we have determined this quotient explicitly.

In the case of compact Riemann surfaces of genus $g \geq 2$, there is a construction of a parameter space for isomorphism classes, which again identifies it with a quotient of an algebraic variety modulo the action of an algebraic group of automorphisms. This is a purely algebraic construction, and is important (among other things) for number-theoretic reasons. There is also a purely analytic construction of this space via Teichmüller theory, which has been most successful in terms of obtaining explicit results about the topology and geometry of $\mathcal{M}_g$. We discuss these two constructions separately.

5.1 $\mathcal{M}_g$ via Teichmüller theory

Let G denote the fundamental group of a compact Riemann surface of genus g, and define Γ_g, the *Teichmüller modular group* or *mapping class group* (of genus g), to be the quotient group

$$\Gamma_g = \Gamma = \frac{\mathrm{Aut}\, G}{\mathrm{Int}\, G},$$

where $\mathrm{Int}\, G$ is the subgroup of inner automorphisms of G. The main result of Teichmüller theory is the following.

Theorem 5.1 *There exists a bounded domain* $\mathbb{T}_g \subset \mathbb{C}^{3g-3}$*, and an open set* $\Omega_g \subset \mathbb{T}_g \times \mathbb{P}^1$*, such that*

(i) $\mathbb{T}_g$ *is homeomorphic to* $\mathbb{R}^{6g-6}$*;*

(ii) Γ_g *acts as a discrete group of holomorphic automorphisms of* $\mathbb{T}_g$*;*

(iii) G *acts as a discrete, fixed-point free group of automorphisms of* Ω_g*, preserving the fibres of the projection* $\pi : \Omega_g \to \mathbb{T}_g$*, and acting on each fibre through linear fractional transformations;*

(iv) the induced mapping $\Omega_g/G \to \mathbb{T}_g$ *is continuous, and for each* $t \in \mathbb{T}_g$*, the quotient* $X_t = \pi^{-1}(t)/G$ *is a compact Riemann surface of genus* g*, such that the map* $\pi^{-1}(t) \to X_t$ *is holomorphic;*

(v) every compact Riemann surface is biholomorphic to X_t*, for some* $t \in \mathbb{T}_g$*, and* X_t*,* $X_{t'}$ *are biholomorphic precisely when* t, t' *are in the same* Γ_g*-orbit; in particular, there is a natural bijection* $\mathcal{M}_g \cong \mathbb{T}_g/\Gamma_g$*;*

(vi) for any $t \in \mathbb{T}_g$*, the isotropy group* $\Gamma_g(t) = \{g \in \Gamma_g \mid g \cdot t = t\}$ *is a finite group, naturally isomorphic to the group* $\mathrm{Aut}\,(X_t)$ *of holomorphic automorphisms of the compact Riemann surface* X_t*.*

Remark 5.2 Observe that if we take $\mathbb{T}_1$ to be $\mathbb{H}$ (the unit disk in its unbounded *avatar*), $\Omega_1 = \mathbb{H} \times \mathbb{C}$, $G = \mathbb{Z} \oplus \mathbb{Z}$, and $\Gamma_1 = \mathrm{SL}_2(\mathbb{Z})$, then the conclusions (ii)-(v) of the theorem hold in the case $g = 1$ as well; the conclusion (vi) needs to be modified to read: $\Gamma_1(t) = \mathrm{Aut}\,(X_t, 0)$, where $\mathrm{Aut}\,(X_t, 0)$ is the group of holomorphic *group* automorphisms (*i.e.*, preserving the group structure) of the torus X_t.

The above theorem was formulated by Teichmüller in the 30's, but was finally proved only in the 60's by Lipman Bers ([Be]). The space $\mathbb{T}_g$, called the *Teichmüller space* of genus g, was constructed abstractly as a metric space by Teichmüller, and came with a natural action of Γ_g. Teichmüller himself proved only that Γ_g is homeomorphic to $\mathbb{R}^{6g-6}$, and that the action of Γ_g has the properties (iv) and (v) of the theorem, in a suitable sense. It must be noted that Riemann himself had stated that the space of compact Riemann surfaces of genus $g \geq 2$ has complex dimension $3g - 3$, and had given some heuristic justification for this.

To understand in what sense the above theorem solves the "moduli problem", we must finally give the definition of a *holomorphic family* of compact Riemann surfaces. Suppose Z is a connected complex manifold of dimension n (for example, a connected open set in $\mathbb{C}^n$). Then a holomorphic family of compact Riemann surfaces of genus g parametrized by Z is a triple $(\mathcal{C}, Z, p : \mathcal{C} \to Z)$, where

(i) $\mathcal{C}$ is a connected complex manifold of dimension $n + 1$

(ii) p is a holomorphic, surjective map whose Jacobian matrix has maximal rank everywhere on $\mathcal{C}$

(iii) the fibres of $\mathcal{C}$ (which, by (i), (ii) and the holomorphic implicit function theorem, must necessarily be complex submanifolds of $\mathcal{C}$ of dimension 1) are compact Riemann surfaces of genus g.

For example, one may take $Z = \mathbb{C} - \{0, 1\}$, $\mathcal{C} = \{((x, y, z), \lambda) \in \mathbb{P}^1 \times Z \mid y^2 z - x(x - z)(x - \lambda z) = 0\}$, and $p : \mathcal{C} \to Z$ to be the natural projection map; here $g = 1$.

Given such a family $p : \mathcal{C} \to Z$, we have the obvious set-theoretic map

$$f : Z \to \mathcal{M}_g, \quad (z \in Z) \mapsto (\text{isomorphism class of } p^{-1}(z)).$$

It is natural to require that, for the "correct" topology on $\mathcal{M}_g$, this map f should be continuous; in fact, $\mathcal{M}_g$ should have the strongest[8] topology for which this holds. For example, the topology of $\mathcal{M}_g$ should be T_1 (*i.e.*, points are closed) iff, in all such families, the set

$$\{z \in Z \mid p^{-1}(z) \text{ is isomorphic to a fixed Riemann surface } X\}$$

is closed, for each compact Riemann surface X (of genus g).

Now *define* the topology on $\mathcal{M}_g$ as the strongest one for which all such "classifying maps" f are continuous. To get a non-trivial topology on $\mathcal{M}_g$, we must then construct plenty of families for which the classifying maps f are non-constant. In a sense, this is one of the major steps in tackling any moduli problem.

The Teichmüller-Bers theorem *constructs a holomorphic family of compact Riemann surfaces of genus g* over $Z = \mathbb{T}_g$, with $\mathcal{C} = \Omega_g/G$. Indeed, $\mathcal{C}$ is a complex manifold, since the action of G is discrete, fixed-point free and through holomorphic automorphisms of Ω_g; since $\Omega_g \to \mathcal{C}$ is locally biholomorphic, the Jacobian matrix condition holds for $\mathcal{C} \to Z$ (since it clearly holds for $\Omega_g \to Z$). The classifying map $\mathbb{T}_g \to \mathcal{M}_g$ is just the quotient map modulo the action of Γ_g, by (v) of the theorem. Thus, the correct topology on $\mathcal{M}_g$ (as defined above) can be no finer than the quotient topology, under the identification $T_g/\Gamma_g \cong \mathcal{M}_g$.

Now the Teichmüller family $\mathcal{C} \to \mathbb{T}_g$ has the following "semi-universal" property: if $Y \to Z$ is *any* holomorphic family of compact Riemann surfaces of genus g, then the (set-theoretic) classifying map $Z \to T_g/\Gamma_g$ lifts locally[9] to a *holomorphic* map $Z \to \mathbb{T}_g$. It follows that $Z \to T_g/\Gamma_g$ is continuous, *i.e.*, the quotient topology on $\mathcal{M}_g = T_g/\Gamma_g$ is the correct topology.

As a corollary, the we deduce that $\mathcal{M}_g$ is a connected, Hausdorff space. Because the action of Γ_g on $\mathbb{T}_g$ has fixed points, $\mathcal{M}_g$ is not a (complex) manifold,

[8] Finest, *i.e.*, with the most open sets.

[9] That is, each $x \in Z$ has an open coordinate neighbourhood U, such that the restricted map $U \to T_g/\Gamma_g$ lifts to a holomorphic map $U \to \mathbb{T}_g$.

but it does have a complex analytic structure: namely, that of a so-called *normal complex space*. It is possible to define the notion of a holomorphic family of compact Riemann surfaces $p : Y \to Z$ on a complex space Z. One can show that the classifying map $Z \to \mathcal{M}_g$ determined by any such family is a holomorphic map of complex spaces; we express this by saying that $\mathcal{M}_g$ is a *coarse moduli space* (for compact Riemann surfaces of genus g).

It would be ideal if there were a "universal" holomorphic family $u : \mathcal{U} \to \mathcal{M}_g$ such that an arbitrary family $Y \to Z$ is obtained from u via the pull-back under a holomorphic map $Z \to \mathcal{M}_g$. However, it can be proved that *there is no such universal family* $u : \mathcal{U} \to \mathcal{M}_g$. This explains the need to go to the Teichmüller space, which is a "ramified covering" of $\mathcal{M}_g$.

The above phenomenon often occurs in moduli problems. For a "sufficiently good" moduli problem, for which there is moduli space M, as well as a universal family $U \to M$ of the objects one is classifying, one says that M is a *fine* moduli space. At the other extreme, we mention here that, in some moduli problems (for example, that of holomorphic vector bundles on $\mathbb{P}^1$ of a fixed rank ≥ 2), the "natural" topology on the moduli space, as defined via families, is non-Hausdorff.

We mention some topological consequences of the construction of $\mathcal{M}_g$ via Teichmüller theory. Since $\mathcal{M}_g$ is the quotient of $\mathbb{T}_g$ by a group of automorphisms, whose stabilizers are all *finite* groups, one can prove that the cohomology groups $H^i(\mathcal{M}_g, \mathbb{Q})$ can be identified with the "group cohomology" $H^i(\Gamma_g, \mathbb{Q})$, which may (in principle) be computed purely in terms of the group Γ_g (as an abstract group).

J. Harer ([Harer]) used the Teichmüller construction to show that $H^2(\mathcal{M}_g, \mathbb{Z}) = \mathbb{Z}$; another result of his ([Harer2]) is a *stability theorem*: $H^k(\mathcal{M}_g, \mathbb{Z}) \cong H^k(\mathcal{M}_{g+1}, \mathbb{Z})$ if $g \geq 3k + 1$. He has also proved that $\mathcal{M}_g$ is homotopy equivalent to a simplicial complex of dimension $4g - 4$; in particular, $H^i(\mathcal{M}_g, \mathbb{Z}) = 0$ for $i > 4g - 4$. D. Johnson [Jo] has interesting results on the related *Torelli group*, namely the subgroup of Γ_g of automorphisms of $G = \pi_1$ consisting of automorphisms acting trivially on the abelianization $G^{ab} = G/[G, G]$ (here $[G, G]$ is the commutator subgroup). The geometric consequences of some of these results are still being understood.

5.2 $\mathcal{M}_g$ via invariant theory

As stated earlier, there is a different algebraic construction of $\mathcal{M}_g$, for $g \geq 2$, analogous to the construction of $\mathcal{M}_1$ using plane cubic curves.

One begins by considering the space of n-fold "pluricanonical forms" on a compact Riemann surface X of genus g, which is just the space $L(nK_X)$ for a canonical divisor K_X (elements of this space may be identified with holomorphic tensors which are locally expressible as $f(z)(dz)^{\otimes n}$, with evident transition formulas). If $n \geq 3$, the Riemann-Roch theorem implies that $\dim L(nK_X) = (2n - 1)(g - 1) = N + 1$, say. If $f_0, \ldots, f_N$ is a basis for this vector space, then

$x \mapsto (f_0(x), \ldots, f_N(x))$ gives a (holomorphic) mapping $V \to \mathbb{C}^{N+1} - \{0\}$, where V is the complement of the finite set of poles, and of the common zeroes, of the f_j in X. From the Riemann-Roch theorem, one can show that the composite $V \to \mathbb{P}^N$ extends to a biholomorphic map from X onto a non-singular curve $Y \subset \mathbb{P}^N$. This construction is essentially intrinsic to the Riemann surface X, except that it involves the choice of the basis $f_0, \ldots, f_N$.

A. Grothendieck has constructed a certain universal (algebraic) family $\mathcal{C} \to \mathcal{H}$, where $\mathcal{H}$ is a projective algebraic variety called the *Hilbert scheme*[10]. The fibres of $\mathcal{C} \to \mathcal{H}$ are precisely the subvarieties (or subschemes) of $\mathbb{P}^N$ of degree[11] $n(2g-2)$, with a sheaf $\mathcal{O}_X$ of algebraic (or holomorphic) functions with Euler characteristic $\chi(\mathcal{O}_X) = 1 - g$ (over $\mathbb{C}$, this is computed via sheaf cohomology as defined in the appendix; however, there is an equivalent algebraic definition using the Hilbert-Samuel polynomial which works in general). An open subset $U \subset \mathcal{H}$ (in the Zariski topology) will parametrize those fibres which are non-singular curves of genus g. As with the Teichmüller family, we see that (i) every compact Riemann surface is isomorphic to the fibre over some point of U, and (ii) the fibres over $x, y \in U$ are isomorphic iff both are obtained as embeddings of a fixed Riemann surface X, with (possibly) different choices of bases for the vector space $L(nK_X)$.

There is a natural action of the algebraic group $\mathrm{Aut}\,(\mathbb{P}^N) = \mathrm{PGL}_{N+1}(\mathbb{C})$ on $\mathcal{H}$, which (on points) is given by $(g, [Z]) \mapsto [g(Z)]$, where $Z \subset \mathbb{P}^N$ is a subvariety (or subscheme) corresponding to a point $[Z]$ of $\mathcal{H}$ (*i.e.*, Z has the appropriate degree and holomorphic Euler characteristic), $g \in \mathrm{Aut}\,(\mathbb{P}^N)$, and $g(Z)$ is the image of Z under translation by g (clearly $g(Z)$ again determines a point of $\mathcal{H}$). Also, the open set $U \subset \mathcal{H}$ parametrizing smooth projective curves is clearly stable under $\mathrm{Aut}\,(\mathbb{P}^N)$. Hence the quotient set $U/\mathrm{Aut}\,(\mathbb{P}^N)$ is naturally in bijection with $\mathcal{M}_g$.

Mumford's *geometric invariant theory* (see [GIT]) gives a way of constructing quotients of algebraic varieties modulo algebraic actions of linear algebraic groups, whenever these exist as varieties. This is done using the notion of *stable points* for such an action. One first considers a projective algebraic variety T, together with an algebraic action of a linear algebraic group (*i.e.*, an algebraic matrix group) G on T. Then, one must *choose* a possibly new embedding $T \subset \mathbb{P}^M$ such that G acts on $\mathbb{P}^M$ via a linear representation $G \to \mathrm{GL}_{M+1}(\mathbb{C})$, where $\mathrm{GL}_{M+1}(\mathbb{C}) \to \mathrm{Aut}\,(\mathbb{P}^M)$ is the obvious quotient map onto $\mathrm{PGL}_{M+1}(\mathbb{C})$. The choice of such an embedding gives a parameter in Mumford's theory; it is called a *linearization* of the action of G on T.

Now define a point $t \in T$ to be *semi-stable* for the given action (and linearization) if there exists a G-invariant hypersurface $Y_t \subset \mathbb{P}^M$ with $t \notin Y_t$; we say t is *stable* if in addition, Y_t can be chosen so that all G-orbits on the (affine)

[10] Actually, $\mathcal{H}$ is a scheme, which is a generalization of an algebraic variety, on which one is also allowed to have non-zero nilpotent functions; however, the subsets we will need to consider will be algebraic varieties in the usual sense.

[11] This means a "general" hyperplane intersects the subvariety in $n(2g-2)$ points.

open subvariety $T \setminus Y_t$ are closed. The main theorem of geometric invariant theory is the following.

Theorem 5.3 *Let G act on T, with a given linearization, corresponding to an embedding $T \subset \mathbb{P}^M$. Let T^{ss} be the set of semi-stable points of T, and T^s the subset of stable points. Then*

(i) T^s, T^{ss} are G-invariant Zariski open subsets of T

(ii) there is a projective variety $\overline{M}$, and a surjective algebraic morphism $\psi : T^{ss} \to \overline{M}$, with affine fibres, such that $\psi(t) = \psi(t')$ iff $\overline{Gt} \cap \overline{Gt'} \neq \emptyset$

(iii) ψ is an open map between algebraic varieties (i.e., the inverse image of a Zariski open subset of T^{ss} is Zariski open in $\overline{M}$)

(iv) $\psi^{-1}\psi(t)$ consists of a single G-orbit iff $t \in T^s$

(v) $M = \psi(T^s)$ is a Zariski open subset of $\overline{M}$

(vi) points of $\overline{M}$ are in bijection with closed orbits in T^{ss}.

Thus, a good quotient, namely M, exists for the action of G on T^s, and M comes equipped with a "good compactification" $\overline{M}$, which is a projective algebraic variety.

In our example of smooth curves, Mumford shows (see [GIT], Chapter 5) that there is a certain natural G-linearization of the action of G on U, where $G = \mathrm{Aut}\,(\mathbb{P}^N)$; if $\overline{U}$ is the closure of U in the corresponding projective space, then Mumford proves that $U \subset \overline{U}^s$. Hence the quotient U/G exists, as a Zariski open subset of $\overline{U}^s/\mathrm{Aut}\,(\mathbb{P}^N)$. In fact, since the whole theory is purely algebraic, it works in a similar fashion over arbitrary fields, and even over the ring of integers. Thus Mumford is able to construct a "moduli scheme" $\mathbb{M}_g$ over $\mathbb{Z}$, whose generic fibre is (as a $\mathbb{C}$-variety) just $\mathcal{M}_g$, and whose reduction modulo any prime p is the moduli space of smooth projective curves of genus g over fields of characteristic p.

Needless to say, the appropriate universal property of this quotient structure U/G implies that, considered as a (normal) complex space, it coincides with the space $\mathcal{M}_g$ constructed using Teichmüller theory.

In the above case, it turns out that $\overline{U}^s = \overline{U}^{ss}$, and the good compactification $\overline{\mathcal{M}}_g$ can be identified as the coarse moduli space for "stable curves of genus g"; the boundary points (*i.e.*, points of $\overline{\mathcal{M}}_g \setminus \mathcal{M}_g$) correspond to certain "Riemann surfaces with singularities" (the singularities are restricted in a certain explicitly described way). The term "stable curve" is consistent with the fact that such curves correspond to stable points. However, we will see later that in dealing with the moduli spaces of vector bundles, we do encounter semi-stable points which are not stable.

Another point of interest is related to the Jacobian variety and Torelli's theorem. We had discussed the analytic construction of $\mathcal{A}_g$, the parameter space of principally polarized complex tori (or abelian varieties) of dimension g. Mumford's geometric invariant theory again gives an algebraic construction, by starting with a suitable projective embedding, and using a suitable Hilbert scheme. Since a holomorphic (or algebraic) family of Riemann surfaces of genus g gives rise to a similar family of Jacobians, *i.e.*, a family of principally polarized complex tori, there is a natural map $\mathcal{M}_g \to \mathcal{A}_g$ which is a morphism of algebraic varieties (and hence a holomorphic mapping between complex spaces.) This "explains" the statement made earlier that the image of $\mathcal{M}_g$ in $\mathcal{A}_g$ is an algebraic variety.

The algebraic construction of $\mathcal{M}_g$ has been useful in number theory; for example, it featured critically in Faltings' proof of the Mordell Conjecture (we further discuss this topic later in this article). The present focus has been on natural subvarieties of $\mathcal{M}_g$, and intersection theory for these (which, in principle, amounts to counting the number of fibres with some distinguished property in any given family of Riemann surfaces of genus g, in terms of "intrinsic invariants" of the family); see [Mu3]. Recent impetus has been received from string theory in theoretical physics, leading to conjectures which have been studied by mathematicians like Kontsevich (see [Lo] for a recent exposition of these ideas).

We mention in passing that $\mathcal{M}_g$, though non-compact, is far from being an affine algebraic variety. On the one hand, this can be seen by the fact that there are no non-constant algebraic regular (or even holomorphic) functions on $\mathcal{M}_g$. Another evidence of this "non-affineness" is that one can construct non-constant families of compact Riemann surfaces of genus g with a compact parameter space; one construction of such families was exhibited by Kodaira, and a variant was also used in the proof of the Mordell conjecture (this was an idea of Parshin).

For a nice overview of this topic, with many references for further reading, see [Har].

6 Riemann surfaces and number theory

Due to our lack of expertise, and to limit the scope of the discussion, we mention here only a few of the connections of Riemann surface theory with number theory.

We first discuss the Diophantine properties of a Riemann surface (or algebraic curve) X, which is defined (as a subset of some projective space) by polynomial equations with coefficients in an algebraic number field F, *i.e.*, a subfield $F \subset \mathbb{C}$ which is a finitely generated algebraic extension field of $\mathbb{Q}$). We then say that X is *defined over* F. One wants a description of 2 types of point sets: (i) points on X with coordinates in F (called F*-rational points*), and (ii) points with integer coordinates on an affine curve $X - \{P_1, \ldots, P_r\}$, for suitable points P_j in X. An excellent survey of current ideas on these topics, with particular emphasis on connections with geometry, is the book of S. Lang (see

[La]).

We first discuss rational points. We begin with Riemann surfaces of genus 1, which are also called *elliptic curves* (see [Si] for more details). If X is an elliptic curve, with a chosen base point P, then as seen earlier, one can find meromorphic functions x, y which are holomorphic on $X - \{P\}$, which have poles of order 2 and 3 respectively at P, such that we have an algebraic relation

$$y^2 = f(x) = ax^3 + bx + c,$$

where $f(x)$ is a cubic polynomial with distinct roots. There is a group structure on the points of X, such that P is the identity element; this group operation $X \times X \to X$ is an algebraic morphism. One way to realize this group operation is through the bijection $X \to \mathrm{Cl}^0(X)$, $Q \mapsto [Q] - [P]$, composed with the group isomorphism $\mathrm{Cl}^0(X) \cong J(X)$; equivalently, the Abel-Jacobi mapping $AJ_X : X \to J(X)$ (associated to the chosen base point P) is an isomorphism of algebraic varieties.

Now suppose f has coefficients in an algebraic number field F. Then in fact the group operation on X can be described by polynomial functions with coefficients in F. Hence one can show that, for any field $K \subset \mathbb{C}$ containing F, the set $X(K)$ of points with coordinates in K is in fact a subgroup of X.

It is now natural to ask what the structure of the group $X(F)$ is. The *Mordell-Weil theorem* implies that $X(F)$ is a finitely generated abelian group. Thus, since it is a subgroup of the complex torus $X = X(\mathbb{C}) \cong \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z}$, we see that

$$X(F) \cong \mathbb{Z}^{\oplus r(X)} \oplus \mathbb{Z}/a\mathbb{Z} \oplus \mathbb{Z}/b\mathbb{Z},$$

for unique positive integers a, b with $a \mid b$, and a unique integer $r(X) \geq 0$.

A remarkable theorem of Mazur states that for $F = \mathbb{Q}$, there are only 15 possibilities for the torsion subgroup of $X(\mathbb{Q})$, *i.e.*, for the pair of integers a, b. An analogous result is conjectured for any algebraic number field F (the conjectured finite list of possible torsion subgroups would, presumably, depend on F).

For a given elliptic curve X, a result of Lutz and Nagell in fact gives a simple procedure to compute the torsion subgroup of $X(F)$, by finding an explicit upper bound on its order, in terms of the orders of the 'reductions modulo primes (of F)' of the algebraic curve X (see [Si] for details).

We explain further what is meant by 'reduction modulo a prime of F'. For simplicity, suppose $F = \mathbb{Q}$; then 'prime' refers to a prime number in the usual sense. Without loss of generailty, we may assume that the defining homogeneous cubic polynomial $F(x, y, z)$ of X in $\mathbb{P}^2$ has relatively prime, integer coefficients. Then it makes sense to consider the solutions of the congruence $F(x, y, z) \equiv 0 \pmod{p}$, for any prime number p; equivalently, we consider the corresponding cubic equation over the finite field $\mathbb{Z}/p\mathbb{Z}$. Let X_p be the corresponding curve; it is the 'reduction modulo p' of the curve X.

For all but a finite set of p, one sees that X_p is a non-singular projective curve of genus 1 over $\mathbb{Z}/p\mathbb{Z}$, which then has a group structure; now $X_p(\mathbb{Z}/p\mathbb{Z})$ (*i.e.*, the set of solutions of the congruence) will form a *finite* abelian group, whose order can be explicitly determined. We say X has *good reduction* at such a prime p. One can show that there is a well-defined homomorphism $X(\mathbb{Q}) \to X_p(\mathbb{Z}/p\mathbb{Z})$, which is injective on the torsion subgroup of order relatively prime to p. The Lutz-Nagell theorem is a stronger form of this assertion (which includes all primes p, not just those of good reduction). The constructions over an arbitrary algebraic number field are similar, involving congruences modulo prime ideals.

The most important remaining problem is to determine $r(X)$ (and further, to identify $r(X)$ points on X which generate $X(F)$ modulo torsion). There is no known formula for $r(X)$ in terms of 'simpler' invariants of X, nor is an effective procedure known to find a generating set of $X(F)$. However, one of the deep conjectures in the subject gives a conjectural method for determining $r(X)$. We now explain this further.

If p is a prime of good reduction for X, then define

$$a_p = p + 1 - (\text{cardinality of } X_p(\mathbb{Z}/p\mathbb{Z})),$$

and set

$$L_p(X, s) = \frac{1}{1 - a_p p^{-s} + p^{1-2s}},$$

where s is a complex parameter. There is a certain more technical definition of a_p for the remaining p, which we do not go into here (see [Si], Appendix C, § 16). Then one defines

$$L(X, s) = \prod_{p \text{ prime}} L_p(X, s),$$

which is seen to converge for $\operatorname{Re} s > 3/2$ (since one has that $\mid a_p \mid \leq 2\sqrt{p}$, by a result of Hasse, which is a special case of Weil's Riemann Hypothesis for curves over finite fields). It is conjectured that $L(X, s)$ has an analytic continuation to the whole complex plane, and satisfies a funtional equation relating its values at s and $2-s$, analogous to the functional equation of the Riemann Zeta function (a more precise form of the functional equation is given in [Si]). This conjecture has been recently proved by A. Wiles, for $F = \mathbb{Q}$, for a certain class of elliptic curves X (the so-called *semi-stable* curves), while proving Fermat's Last Theorem.

The *conjecture of Birch and Swinnerton-Dyer* states that $r(X)$ equals the order of vanishing of $L(X, s)$ at $s = 1$. A more precise version of the conjecture describes the coefficient of the leading term $(s-1)^{r(X)}$ of $L(X, s)$ at $s = 1$. The conjecture has been verified in some cases (see [Si] and references given there, particularly the works of Coates and Wiles, and of Gross and Zagier, as well as recent work of Kolyvagin [Ko]).

Now we consider the situation with Riemann surfaces of genus ≥ 2. Here, the main result is the *Mordell Conjecture*. It states that if X is of genus ≥ 2

and defined over a number field F, then $X(F)$ is always a *finite* set. This was proved by Faltings in 1983. Later, other proofs of the Mordell Conjecture were given, including a second proof by Faltings; perhaps the most accessible (though by no means easy) proof is that given in Bombieri [Bom]. As an application, we have that if X is the Riemann surface associated to a smooth projective plane curve of degree $d \geq 4$, defined by a homogeneous polynomial $f(x, y, z) = 0$ with coefficients in F, then $X(F)$ is finite, and so the equation $f(x, y, z) = 0$ has only finitely many solutions in F, upto multiples.

As mentioned earlier, if X is defined over F, then so is its Jacobian $J(X)$. The *Mordell-Weil Theorem* states that if A is an abelian variety defined over F, an algebraic number field, then $A(F)$ is a finitely generated abelian group. If $P \in X(F)$ is a base point, then we have the embedding $f : X \to J(X)$, $f(Q) = AJ_X([Q] - [P])$, given by the Abel-Jacobi map. Then $X(F)$ is the intersection of $f(X)$ with the finitely generated subgroup $J(X)(F) \subset J(X)$. In fact, Lang conjectured (and Faltings proved, giving his second proof of the Mordell conjecture) that in any abelian variety A, an algebraic curve $C \subset A$ has finite intersection with any finitely generated subgroup of A, unless C is an elliptic curve which is a translate of a subgroup variety of A.

We now discuss the topic of integer points. Let $X \subset \mathbb{P}^n$ be a projective non-singular curve defined over $\mathbb{Q}$ (for simplicity), and let $U = X - X \cap H$, where H is a hyperplane not containing X. Choosing homogeneous coordinates $x_0, \ldots, x_n$ on $\mathbb{P}^n$ so that H is given by $x_0 = 0$, we can identify $\mathbb{P}^n - H$ with $\mathbb{C}^n$, with inverse isomorphism given by $(t_1, \ldots, t_n) \mapsto (1 : t_0 : \cdots : t_n)$. Then U is a subset of $\mathbb{C}^n$ defined by the vanishing of a finite set of polynomials with integer coefficients

$$f_1(x_1, \ldots, x_n) = \cdots = f_r(x_1, \ldots, x_n) = 0.$$

It now makes sense to consider $U(\mathbb{Z})$, the set of integer solutions of this system of equations. More generally, one considers the set $U(S^{-1}\mathbb{Z})$ of 'S-integer points', where S is a finite set of primes, and $S^{-1}\mathbb{Z}$ is the set of rationals whose denominators are divisible only by primes from the set S. The sets $U(\mathbb{Z})$, $U(S^{-1}\mathbb{Z})$ depend on the choice of the embedding of U in $\mathbb{C}^n$.

The main result on integer points is a theorem of Siegel: in the above context, $U(S^{-1}\mathbb{Z})$ is *finite* for any finite set S, if either (i) X has genus > 0, or (ii) X has genus 0 (*i.e.*, $X = \mathbb{P}^1_{\mathbb{C}}$) and $X - U$ has cardinality ≥ 3. In case $U \cong \mathbb{C}$ or $U \cong \mathbb{C} - \{0\}$, there are obvious counterexamples given by lines and conics. Siegel's results are also valid if we replace $\mathbb{Z}$ by the ring of algebraic integers in an algebraic number field F.

A nice way to "understand" Mordell's conjecture (Faltings' theorem) and Siegel's result is the following. By the uniformization theorem (Theorem 4.3), for any (connected) Riemann surface U, the universal covering Riemann surface of U is either $\mathbb{P}^1_{\mathbb{C}}$, the complex plane $\mathbb{C}$, or the unit disk Δ in $\mathbb{C}$. If $U = X - T$ for a compact Riemann surface X and a finite set T, then (i) the universal cover is $\mathbb{P}^1_{\mathbb{C}}$ only if $U = X = \mathbb{P}^1_{\mathbb{C}}$ (ii) the universal cover is $\mathbb{C}$ when either X

has genus 0, and T has cardinality 1 or 2, or $X = U$ has genus 1. In case the universal cover is the disk, one may view U as 'hyperbolic' in the sense that U has a metric of constant negative curvature. Thus, the Mordell conjecture and Siegel's theorem state that 'hyperbolic' Riemann surfaces have finitely many rational/integral points. These are the prototypes of far-reaching conjectures relating 'hyperbolicity' of algebraic varieties to Diophantine properties; a precise articulation of this vision has been given by Vojta, explained in [Vo] (see also [La]).

There is one further topic related to Diophantine questions: the question of *effectivity*. What this means is the following: given a curve which is known (say, by the Mordell conjecture, or Siegel's theorem) to have finitely many rational/integral points; is there then a finite procedure which can be used to determine all the points, at least in principle? At present, this is an open problem, in general. However, work of Baker and others shows that the problem has a positive solution for some special types of equations; see Baker's book [Ba] for more details. The theorems of Faltings and Siegel are *not* effective, since (in a sense) they assume the existence of an infinity of (rational or integral) points and derive a contradiction; however, one can (in principle) use their proofs to find an upper bound for the *number* of such points.

In a rather different direction, we discuss some connections of Riemann surfaces with the theory of transcendental numbers. We begin by considering the open Riemann surface $\mathbb{C}^* = \mathbb{C} - \{0\}$, which we may regard as an algebraic curve, the hyperbola defined by $xy = 1$. Notice that this curve has a defining equation with rational coefficients, and has a regular, algebraic differential form defined over $\mathbb{Q}$,

$$\omega = y\,dx = \frac{dx}{x}.$$

We also have $H_1(\mathbb{C}^*, \mathbb{Z}) = \mathbb{Z}$, generated by a positively oriented loop γ winding around 0 once. There is then one basic period

$$\int_\gamma \omega = 2\pi i.$$

More generally, if α is a non-zero algebraic number, then for any choice of a path δ within $\mathbb{C}^*$ from 1 to α, we have

$$\int_\delta \omega = \log(\alpha),$$

where 'log' is a suitable branch of the logarithm (the case of a period arises when we take $\alpha = 1$). The classical results of Hermite and Lindemann on the transcendence of e and π, and more generally, the transcendence of any non-zero value of $\log(\alpha)$ for any non-zero algebraic number α, may thus be viewed as a transcendence result on integrals of rational differential forms.

More generally, the following can be proved (see [La2]). Let X be a compact Riemann surface regarded as an algebraic curve defined over an algebraic number field F. Let $a, b \in X(F)$ be two F-rational points, γ a path in X joining a to b. Let $\omega_1, \ldots, \omega_g$ be a basis for the holomorphic 1-forms on X given by forms which are rational over F (*i.e.*, are each expressible as fdg for some meromorphic functions f, g defined over F). Then at least 1 of the numbers

$$\int_\gamma \omega_1, \ldots, \int_\gamma \omega_g$$

is transcendental, unless they are all 0. In particular, for any non-zero homology class $[\gamma]$, there exists a period $\int_\gamma \omega$ which is transcendental, for some F-rational differential ω. The cases of e and π considered above are included by allowing the Riemann surface to have 'singularities', and/or be affine.

We mention 2 interesting examples of this result. Let a, b be rational numbers which are non-integral. Then

$$\beta(a, b) = \int_0^1 x^{a-1}(1-x)^{b-1}dx = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$$

is transcendental; this follows by considering the Riemann surface $y^n = x^{na}(1-x)^{nb}$, where n is the smallest common denominator for the fractions a, b. Another example is given by the *Chowla-Selberg formula*. Here, one considers the case of an elliptic curve X with complex multiplication, which is defined over a number field F; then as a Riemann surface, $X = \mathbb{C}/\Lambda$ where Λ is (essentially) the ring of algebraic integers in an imaginary quadratic field $\mathbb{Q}(\sqrt{-d})$, for some integer $d > 0$. This means that there is a holomorphic differential η whose lattice of periods is $\Lambda = \mathbb{Z} + \mathbb{Z}\tau$, for some τ in the upper half plane, which is also an algebraic integer in $\mathbb{Q}(\sqrt{-d})$. On the other hand, one can find a holomorphic differential ω which is also defined over F; since X has genus 1, we must have $\omega = C\eta$ for some complex constant C, and so ω has period lattice $C\mathbb{Z} + C\mathbb{Z}\tau$. The Chowla-Selberg formula is a formula for C, where we have suitably normalized the choices of ω, η; independent of the specific normalizations, C is determined upto multiplication by a non-zero element of F (see [Gr], and also the original paper [CS]). Now the theorem on transcendence of periods implies that the number C is transcendental. If $d = p$, a prime number, for example, then one has

$$C = \sqrt{\pi} \prod_{a=1}^{p-1} \Gamma(\frac{a}{p})^{\frac{w\varepsilon(a)}{4h}},$$

where $\varepsilon(a) = 1$ or -1 according as a is a quadratic residue (*i.e.*, congruent to a square), or a non-residue, modulo p; here w denotes the number of roots of unity (usually 2) in $\mathbb{Q}(\sqrt{-p})$, and h denotes the *class number* of this field. The formula for C in general is similar, with $\varepsilon(a)$ now denoting the value of the Dirichlet character associated to the quadratic field extension $\mathbb{Q}(\sqrt{-d})$ of $\mathbb{Q}$.

As a final topic in this section, we discuss briefly the modular curves, and their relation with *modular forms* (see the book [Sh] for more details). We have seen that the isomorphism class of an elliptic curve over $\mathbb{C}$ is determined by its j-invariant, and the set of isomorphism classes is thus identified with $\mathbb{C}$. In fact more is true; $\mathbb{C}$, considered as the affine line, is the coarse moduli space for elliptic curves (*i.e.*, given an 'algebraic family' of elliptic curves $f : X \to Y$ with a chosen 0-section, the map $y \mapsto j(f^{-1}(y)) \in \mathbb{C}$ is a morphism of algebraic varieties $Y \to \mathbb{C}$). From an analytic point of view, one views $\mathbb{C}$ as the quotient space $\mathbb{H}/\mathrm{SL}_2(\mathbb{Z})$, where $\mathbb{H}$ is the upper half plane, and $\mathrm{SL}_2(\mathbb{Z})$ acts in the usual way; the class of the Riemann surface $\mathbb{C}/\mathbb{Z}+\mathbb{Z}\tau$ is given by the image of $\tau \in \mathbb{H}$.

In a similar fashion, one can construct moduli spaces of pairs (X, G), where X is an ellliptic curve and G is some additional data, which may be (i) a cyclic subgroup of order N, or (ii) an element of order N, or (iii) a choice of two points of order N which form a $\mathbb{Z}/N\mathbb{Z}$ basis for the N-torsion subgroup of X. In each case, an analytic construction of the moduli space is given as $M(\Gamma) = \mathbb{H}/\Gamma$, where $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ is a subgroup containing the subgroup $\Gamma_N \subset \mathrm{SL}_2(\mathbb{Z})$ of all the matrices congruent to the identity modulo N (*i.e.*, Γ is a *congruence subgroup* of $\mathrm{SL}_2(\mathbb{Z})$). Each such moduli space $M(\Gamma)$ turns out to be a non-compact Riemann surface, which is naturally a branched covering space of the affine line $\mathbb{C}$; further, $M(\Gamma)$ has a compactification $\overline{M}(\Gamma)$ which is then a branched cover of $\mathbb{P}^1_{\mathbb{C}}$. The boundary points in $\overline{M}(\Gamma) - M(\Gamma)$ are called the *cusps* of $M(\Gamma)$ (or of the group Γ). The compact Riemann surfaces (or curves) $\overline{M}(\Gamma)$ obtained in this way are called *modular curves*. One can work out the ramification indices of $\overline{M}(\Gamma) \to \mathbb{P}^1_{\mathbb{C}}$, and hence compute invariants like the genus of $\overline{M}(\Gamma)$, for the various groups Γ of the above type.

Given a group Γ as above, one defines a *modular form of weight k* with respect to Γ to be a holomorphic function $f(z)$ on $\mathbb{H}$ such that (i) f has the transformation property

$$f(\frac{az+b}{cz+d}) = (cz+d)^k f(z), \ \ \forall \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma \subset \mathrm{SL}_2(\mathbb{Z}),$$

and (ii) $\mid f(z) \mid$ is bounded in any set of the form $\mid \mathrm{Re}(z) \mid \leq C$, $\mid \mathrm{Im}(z) \mid > D$ for any positive constants C, D. In particular, from (i), we see that $f(z+N) = f(z)$ if Γ contains Γ_N; hence f has a Fourier expansion

$$f(z) = \sum_{m\geq 0} a_m q_N^m,$$

where

$$q_N(z) = e^{\frac{2\pi i z}{N}}.$$

We also denote $q_1(z)$ by $q(z)$.

An example is given by the *Delta function*

$$\Delta(z) = q(z) \prod_{n=1}^{\infty} (1 - q(z)^n)^{24} = \sum_{n\geq 1} \tau(n) q(z)^n.$$

This is known to be a modular form of weight 12 with respect to $\Gamma = \mathrm{SL}_2(\mathbb{Z})$. The Fourier coefficients $\tau(n)$ are integers; the function $n \mapsto \tau(n)$ is called *Ramanujan's tau function.*

In general, the ratio of two modular forms of the same weight for Γ yields a meromorphic function on the modular curve $\overline{M}(\Gamma)$. In fact, one may regard the modular forms of a fixed weight as holomorphic sections of a suitable line bundle on $\overline{M}(\Gamma)$, and hence can compute the dimension of the space of modular forms of a given weight using the Riemann-Roch theorem. The modular forms of a sufficiently large (and perhaps sufficiently divisible) weight in fact determine a projective embedding of $\overline{M}(\Gamma)$; most interestingly, in this embedding, the defining equations have coefficients in a suitable algebraic number field.

Mazur's theorem on torsion subgroups of elliptic curves over $\mathbb{Q}$, which we mentioned earlier, is proved by studying geometric and number-theoretic properties of certain modular curves and their Jacobian varieties. Another application of the geometry and number theory of modular curves (and related objects) is Deligne's proof of the *Ramanujan conjecture*, that $\mid \tau(p) \mid \leq 2p^{11/2}$ for any prime number p. Finally, A. Wiles' recent proof of Fermat's Last Theorem is by relating certain elliptic curves over $\mathbb{Q}$ to modular forms.

The Fourier coefficients of modular forms are interesting for number-theoretic reasons. For example, one can show that if k is a positive integer, and

$$r_k(n) = \text{number of distinct ways of writing } n \text{ as a sum of } k \text{ squares},$$

then for any $k \geq 1$,

$$1 + 2\sum_{n=1}^{\infty} r_{4k}(n) q(z)^n$$

is a modular form of weight $2k$ for a suitable congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$. Techniques of modular form theory can then be used to prove (i) exact formulas for $r_4(n)$ and $r_8(n)$, and (ii) asymptotic formulas for $r_{4k}(n)$ for any $k > 0$. For examples of this reasoning, see Serre's book [Se], Chapter VII, and [Mu2], Chapter I, §15.

The *Langlands conjectures* form an exciting area of research today in number theory. In very rough terms, these conjectures relate representations of the Galois groups of algebraic number fields to modular forms. Modular curves, and their (étale) cohomology groups, provide interesting non-trivial examples of the Langlands correspondence. For an introduction to these ideas, see [Ge].

7 Vector bundles on Riemann surfaces

In this section, we discuss the topic of stable vector bundles on a compact Riemann surface of genus $g \geq 2$. We consider this topic for two reasons, apart from its intrinsic interest. One is that this is one area of current research in Riemann surface theory, motivated by problems and questions from physics (particularly gauge theory, and string theory). Another is that this is an area of research where significant contributions have been made by Indian mathematicians, particularly M. S. Narasimhan and C. S. Seshadri. There are at present a number of Indian mathematicians who are still actively doing research in this general area.

We begin by recalling the definition of a vector bundle on a Riemann surface. If X is a Riemann surface, a (complex) *vector bundle of rank* n on X is

(a) a (topological) space V and a continuous mapping $f : V \to X$, together with

(b) an open cover $\{U_i\}_{i\in I}$ and homeomorphisms $\varphi_i : f^{-1}(U_i) \to U_i \times \mathbb{C}^n$, such that

(i) for each i, we have
$$\varphi_i(y) = (f(y), \psi_i(y))$$
for some function $\psi_i : f^{-1}(U_i) \to \mathbb{C}^n$, and

(ii) for each $i \neq j$, the composite homeomorphism
$$\varphi_{ij} : (U_i \cap U_j) \times \mathbb{C}^n \to (U_i \cap U_j) \times \mathbb{C}^n,$$
$$\varphi_{ij} = \varphi_j \circ \varphi_i^{-1},$$
satisfies
$$\varphi_{ij}(x, v) = (x, g_{ij}(x)(v))$$
for an invertible matrix $g_{ij}(x) \in \mathrm{GL}_n(\mathbb{C})$.

Then in (ii) above, we see that $g_{ij} : U_i \cap U_j \to \mathrm{GL}_n(\mathbb{C})$ is a continuous mapping, with pointwise inverse given by $g_{ji}(x)$, and such that on $U_i \cap U_j \cap U_k$ (for distinct i, j, k), we have
$$g_{jk}(x) \cdot g_{ij}(x) = g_{ik}(x)$$
(here $\cdot$ is induced by composition of maps, *i.e.*, is given by matrix multiplication).

Here, another set of data consisting of an open covering $\{V_j\}_{j\in J}$ and homoemorphisms $\widetilde{\varphi}_j : f^{-1}(V_j) \to V_j \times \mathbb{C}^n$ are defined to give the same vector bundle on X if the open cover $\{U_i\}\cup\{V_j\}$ and the collection of homeomorphisms $\{\varphi_i\}\cup\{\widetilde{\varphi}_j\}$ define a vector bundle on X (this is analogous to the situation in the definition of a Riemann surface via an atlas of charts).

Conversely, suppose given an open cover $\{U_i\}$ and matrix valued functions $g_{ij} : U_i \cap U_j \to \mathrm{GL}_n(\mathbb{C})$, which satisfy the *cocycle conditions* $g_{ij}(x)^{-1} = g_{ji}(x)$ for $x \in U_i \cap U_j$, and $g_{jk} \cdot g_{ij} = g_{ik}$ on $U_i \cap U_j \cap U_k$ for all distinct i, j, k, then one can construct a vactor bundle $f : V \to X$ by 'glueing' (or 'patching'), as follows. Let V be the quotient of the disjoint union of the spaces $U_i \times \mathbb{C}^n$, modulo the equivalence relation generated by $(x, v) \sim (x, g_{ij}(x)v)$ for all i, j such that $x \in U_i \cap U_j$, and for all $v \in \mathbb{C}^n$. The cocycle conditions imply that this is an equivalence relation, such that the natural maps $U_i \times \mathbb{C}^n \to V$ are injective; clearly the projections $U_i \times \mathbb{C}^n \to U_i$ induce a map $V \to X$, giving a vector bundle on X.

The simplest example of a vector bundle of rank n is $f : V = X \times \mathbb{C}^n \to \mathbb{C}^n$, given by the first projection. This is called the *trivial bundle* of rank n. Of course, the whole point of the theory is that there are many interesting non-trivial vector bundles.

The reason for the term 'vector bundle' is because, via the maps φ_i, one can endow each fibre $V_x = f^{-1}(x)$ with the structure of a vector space of dimension n over $\mathbb{C}$. Thus, if $x \in U_i$, then to add 2 points $v_1, v_2 \in V_x$, we transport them over to $\{x\} \times \mathbb{C}^n$ using φ_i, add the coordinates in $\mathbb{C}^n$, and transport the result back to V_x via φ_i^{-1}. Scalar multiplication by complex numbes is defined similarly. If $x \in U_i \cap U_j$, then the vector space structures on V_x defined using φ_i and φ_j agree, since $g_{ij}(x)$ is an invertible linear transformation. We may thus view the collection $\{V_x\}$ as a 'continuously varying family of vector spaces paramterized by X'. From this point of view, it is easy to see that various constructions with vector spaces (like direct sums, tensor products, duals, exterior products, etc.) generalize in a natural way to yield analogous constructions on vector bundles. The trivial vector bundle can be thought of as a constant family of vector spaces.

In Riemann surface theory, one is more interested in a refinement of the above notion, that of a *holomorphic vector bundle* of rank n on X. This is analogous to the above notion, except that the 'transition matrix functions' $g_{ij} : U_i \cap U_j \to \mathrm{GL}_n(\mathbb{C}) \subset \mathbb{C}^{n^2}$ are required to be holomorphic (*i.e.*, the matrix entries of g_{ij} are holomorphic functions, in the usual sense). The constructions mentioned above (direct sum, etc.), when performed on holomorphic vector bundles, again yield holomorphic bundles; the trivial vector bundle is clearly holomorphic.

Holomorphic vector bundles of rank 1 are also called *line bundles*. The isomorphism class of line bundles on X forms an abelian group under the tensor product, with identity element given by the trivial line bundle $X \times \mathbb{C} \to X$. One can associate a line bundle $f : \mathbb{L}(D) \to X$ to any divisor D on X, as follows. Suppose for simplicity that X is compact, and $D = \sum_{i=1}^r n_i P_i$. Choose local coordinate neighbourhoods (U_i, z_i) near P_i, with $z_i(P_i) = 0$, such that $U_i \cap U_j = \emptyset$ for $i \neq j$. Take $U_0 = X - \{P_1, \ldots, P_r\}$. To define a line bundle on X by glueing, with respect to the given open cover $\{U_0, U_1, \ldots, U_r\}$, notice that one only needs to define holomorphic transition functions $g_{0i} : U_0 \cap U_i \to \mathbb{C}^*$ for each $1 \leq i \leq r$, since we must then have $g_{i0}(x) = 1/g_{0i}(x)$, and all other intersections $U_i \cap U_j$ are empty; further, there are no non-empty triple intersections, so the second cocycle condition holds vacuously. We now define $g_{0i}(x) = z_i(x)^{-n_i}$. The resulting line bundle is defined to be $\mathbb{L}(D)$. It turns out that (i) every line bundle on a compact Riemann surface arises in this way, and (ii) $\mathbb{L}(D) \to X$ and

$\mathbb{L}(D') \to X$ are isomorphic if and only if D, D' are linearly equivalent. Thus the group of isomorphism classes of line bundles is isomorphic to the familiar invariant $\mathrm{Cl}\,(X)$.

Given a holomorphic vector bundle, one can associate to it the underlying topological vector bundle; equivalently, we can view the given data as putting a holomorphic structure on a given topological vector bundle. Just as in the case of moduli theory for compact Riemann surfaces, one can fix a topological vector bundle $f : V \to X$ on a given compact Riemann surface X, and ask if there is a parameter space (or parameter variety) for all possible holomorphic structures on V, upto isomorphism.

For line bundles, the degree map on divisors yields an integer-valued invariant. It turns out that two line bundles are topologically isomorphic precisely when their degrees are equal. Fixing the topological type, *i.e.*, the degree, the possible holomorphic structures are then parametrized by (a coset of) $J(X)$, the Jacobian variety. This has a rich structure, as seen earlier; it is then natural to hope for some similar theory associated to vector bundles of higher rank. It turns out that in order to get a good answer in the new situation, one should further restrict the types of possible holomorphic bundles.

One interesting way to construct a holomorphic vector bundle on a Riemann surface X is as follows. Let $\rho : \pi_1(X, x) \to \mathrm{GL}_n(\mathbb{C})$ be a representation of the fundamental group of X (with a chosen base point $x \in X$). Let $\alpha : \widetilde{X} \to X$ be the universal covering space of X, so that $\widetilde{X}$ has a unique Riemann surface structure such that α is holomorphic. As remarked earlier, $\pi_1(X, x)$ acts on the Riemann surface $\widetilde{X}$ through holomorphic automorphisms, with quotient space X. There is an induced action of $\pi_1(X, x)$ on the product space $\widetilde{X} \times \mathbb{C}^n$, given by

$$\gamma \cdot (\widetilde{x}, v) = (\gamma \cdot \widetilde{x}, \rho(\gamma)(v)).$$

The quotient space

$$V_\rho = (\widetilde{X} \times \mathbb{C}^n)/\pi_1(X, x)$$

maps naturally via the first projection to $\widetilde{X}/\pi_1(X, x) = X$. Then $f : V_\rho \to X$ is easily seen to be a holomorphic vector bundle on X; two representations $\rho_i : \pi_1(X, x) \to \mathrm{GL}_n(\mathbb{C})$ which are conjugate by an element of $\mathrm{GL}_n(\mathbb{C})$ yield isomorphic holomorphic vector bundles on X. The converse is false in general, however.

An equivalent way of considering the above bundles is via differential geometry: any vector bundle as above has an integrable (holomorphic) *connection.* We explain what this means. If $f : V \to X$ is a vector bundle of rank n, a *section* of V on an open subset $U \subset X$ is a continuous map $s : U \to V$ such that the composite $f \circ s : U \to X$ is the inclusion map. Then, if $\{(U_i, \varphi_i)\}$ are the data giving the structure of a vector bundle, then

$$s_i = \varphi_i \circ s : U_i \cap U \to (U_i \cap U) \times \mathbb{C}^n$$

has the form

$$s_i(x) = (x, h_i(x)),$$

where $h_i : U_i \cap U \to \mathbb{C}^n$ is a continuous function. The local vector-valued functions h_i are related by

$$h_j(x) = g_{ij}(x)h_i(x),$$

where $g_{ij} : U_i \cap U_j \to \mathrm{GL}_n(\mathbb{C})$ are the (matrix valued) transition functions. Thus, if $f : V \to X$ is a holomorphic vector bundle, it makes sense to speak of a section s being C^∞, or holomorphic, in a neighbourhood of a point $x \in U$: if $x \in U \cap U_i$, this means that the corresponding function h_i is C^∞, or holomorphic, respectively. Since $g_{ij}(x)$ has holomorphic matrix entries, these notions do not depend on the choice of the index i such that $x \in U_i$. Thus sections of a vector bundle of rank n are natural generalizations of vector valued functions.

Now a connection on $f : V \to X$ is a rule for differentiating sections of the bundle on any open set U with respect to local coordinate functions on the base space X. It is convenient to state this using sheaf theory; thus, let $\mathcal{V}^\infty$ denote the sheaf (on X) of C^∞ sections of $f : V \to X$, so that $\mathcal{V}^\infty(U)$ is the $\mathbb{C}$-vector space of sections on U, and the restriction homomorphisms $\mathcal{V}^\infty(U) \to \mathcal{V}^\infty(U')$ (for open sets $U' \subset U$) are given by restriction of functions. The holomorphic sections of the bundle clearly yield a subsheaf $\mathcal{V}$ of $\mathcal{V}^\infty$. Clearly $\mathcal{V}^\infty$ is a sheaf of modules over the sheaf $\mathcal{A}_X$ of C^∞ (complex valued) functions on X; hence it is also a module over the sheaf $\mathcal{O}_X$ of holomorphic functions, such that $\mathcal{V}$ is an $\mathcal{O}_X$-submodule.

Now a C^∞ connection on V is defined to be a sheaf homomorphism

$$\nabla : \mathcal{V}^\infty \to \mathcal{A}^1_X \otimes_{\mathcal{A}_X} \mathcal{V}^\infty,$$

where $\mathcal{A}^1_X$ is the sheaf of C^∞ 1-forms, such that the formula (called the *Leibniz Rule*)

$$\nabla(f\, s) = f\nabla(s) + s \otimes df$$

holds, for any section $s \in \mathcal{V}^\infty(U)$, and any C^∞ function f on U. Here df denotes the exterior derivative of f, which is a C^∞ 1-form; if $z = x + iy$ is a local holomorphic coordinate, then $df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy)$. The connection ∇ is said to be a *holomorphic connection* if it restricts to a sheaf homomorphism

$$\mathcal{V} \to \Omega^1_X \otimes_{\mathcal{O}_X} \mathcal{V}$$

(recall Ω^1_X is the sheaf of holomorphic 1-forms).

Let $\{(U_i, \varphi_i)\}$ be data giving the local trivialization of the bundle, and let $s_1, \ldots, s_n$ be the sections of $f : V \to X$ corresponding under φ_i to (the constant vector-valued functions determined by) the standard basis of $\mathbb{C}^n$. Then an arbitrary section s of the bundle on any open set $U \subset U_i$ is uniquely expressible as a linear combination

$$s = \sum_{j=1}^n f_j s_j \mid_U .$$

Here f_j are continuous functions on U, which are C^∞, or holomorphic, if s is a C^∞ section, or a holomorphic section, respectively. From the Leibniz rule, $\nabla(s)$ is determined by the values of $\nabla(s_j)$, $1 \leq j \leq n$. We can further uniquely write

$$\nabla(s_j) = \sum_{k=1}^{n} \omega_{jk} \otimes s_k,$$

where ω_{jk} are C^∞ 1-forms on U_i; then ∇ is completely determined by the $n \times n$ matrix of 1-forms $\omega = [\omega_{jk}]$, called the *connection matrix* of ∇ with respect to the chosen trivialization of V on U_i. Clearly ∇ is a holomorphic connection precisely when the ω_{jk} are holomorphic 1-forms.

It is a simple computation with the Leibniz rule to show that if $\nabla_1, \ldots, \nabla_r$ are C^∞ connections on a bundle $f : V \to X$, and $f_1, \ldots, f_r$ are C^∞ functions with $f_1 + \cdots + f_r = 1$, then $\nabla = f_1\nabla_1 + \cdots + f_r\nabla_r$ is also a C^∞ connection. Now using partitions of unity, one deduces that any C^∞ vector bundle has a C^∞ connection.

A connection ∇ on a vector bundle $f : V \to X$ is called *flat* if one can find local trivializations $\{U_i, \varphi_i)\}$ such that the connection matrices $[\omega_{jk}]$ are all 0. This is easily seen to be equivalent to the condition that the matrix entries of the transition functions $g_{ij}(x)$ are locally constant (*i.e.*, constant on each connected component of U_i). From the theory of covering spaces, one then sees that such a structure on $f : V \to X$ is equivalent to the bundle being of the form V_ρ, arising from a representation of the fundamental group $\rho : \pi_1(X, x) \to \mathrm{GL}_n(\mathbb{C})$, as described earlier.

The condition of flatness, as described, depends on the choice of the special local trivializations. More intrinsically, it can be described using the *curvature*. First, we can define an action of ∇ on $\mathcal{A}^1_X \otimes_{\mathcal{A}_X} \mathcal{V}^\infty$ by

$$\nabla(\omega \otimes s) = d\omega \otimes s - \omega \wedge \nabla(s).$$

Here, $d\omega$ is the exterior derivative of ω, and if locally we have an expression $\nabla(s) = \sum_j \eta_j \otimes s_j$, then $\omega \wedge \nabla(s)$ denotes $\sum_j (\omega \wedge \eta_j) \otimes s_j$. One verifies that $\omega \wedge \nabla(s)$ is well-defined by these local expressions.

For any C^∞ connection ∇, define its curvature F_∇ by

$$F_\nabla : \mathcal{V}^\infty \to \mathcal{A}^2_X \otimes_{\mathcal{A}_X} \mathcal{V}^\infty,$$

$$F_\nabla(s) = \nabla \circ \nabla(s).$$

One verifies by using the Leibniz rule that for any C^∞ function f on an open set U, and any section s on U, we have

$$F_\nabla(fs) = fF_\nabla(s).$$

This means that F_∇ determines a '2-form valued endomorphism' of $\mathcal{V}^\infty$.

Clearly, $F_\nabla = 0$ if ∇ is flat. Conversely, Frobenius' theorem on 'total differential equations' (or 'integrable distributions in the tangent bundle') implies that if $F_\nabla = 0$, then ∇ is flat. The idea is that if $s_1, \ldots, s_n$ is a basis of sections of the bundle on some open set U, then we can change the trivialization on U, giving a new basis $t_j = \sum_k a_{jk}(x)s_k$, $1 \leq j \leq n$, for any function $[a_{jk}(x)] : U \to \mathrm{GL}_n(\mathbb{C})$. One obtains a differential equation expressing the condition on the functions $a_{ij}(x)$ such that with respect to the new basis t_j, the connection matrix vanishes; now the integrability condition, under which Frobenius' theorem guarantees (at least) a local solution $a_{jk}(x)$, boils down to the given condition $F_\nabla = 0$.

We remark here that, locally, F_∇ is a matrix of 2-forms. The trace of this local matrix yields a 2-form, which one can show is independent of the choice of the local trivialization (this reduces to the fact that the trace of a matrix equals that of any conjugate matrix). Hence we can associate to ∇ the invariant

$$\frac{i}{2\pi}\int_X \mathrm{Tr}\,(F_\nabla).$$

Remarkably, this turns out to be an *integer*, called the (first) *Chern class* of the vector bundle $f : V \to X$.

Another remark is that for a holomorphic connection ∇, the curvature F_∇ would be given locally by a matrix of holomorphic 2-forms, *i.e.*, is 0, on a Riemann surface. Hence such a vector bundle is associated to a representation of the fundamental group. Conversely, any vector bundle V_ρ associated to a representation of $\pi_1(X, x)$ is holomorphic, and carries an obvious holomorphic (flat) connection.

Now suppose $n = 1$, so that we are dealing with line bundles, and with characters of $\pi_1(X, x)$. The group of characters of $\pi_1(X, x)$ is seen to be isomorphic to $H^1(X, \mathbb{C}^*) = H^1(X, \mathbb{C})/H^1(X, \mathbb{Z})$. From the exact sequence (2.3), there is a natural surjection $H^1(X, \mathbb{C}) \to H^1(X, \mathcal{O}_X)$, with kernel $H^0(X, \Omega^1_X)$, and hence an exact sequence

$$0 \to H^0(X, \Omega^1_X) \to H^1(X, \mathbb{C}^*) \to J(X) \to 0.$$

The map $H^1(X, \mathbb{C}^*) \to J(X) \hookrightarrow H^1(X, \mathcal{O}^*_X)$ (induced by $\mathbb{C}^* \subset \mathcal{O}^*_X$, as the subsheaf of constant functions) associates to a flat line bundle the corresponding holomorphic line bundle. From the exponential sequence, there is a boundary map

$$H^1(X, \mathcal{O}^*_X) \to H^2(X, \mathbb{Z}) = \mathbb{Z}$$

corresponding to the degree map on divisors, with kernel $J(X)$.

We may interpret this as follows: every line bundle of degree 0 on X is associated to a character, and hence, by what we mentioned earlier, carries a holomorphic (flat) connection. This is not unique, however; one can change the connection by adding to it a holomorphic 1-form. This is consistent with the fact that any two preimages in $H^1(X, \mathbb{C}^*)$ of a given point of $J(X)$ differ by the addition of a holomorphic 1-form.

One way to make the character of $\pi_1(X, x)$ associated to such a line bundle unique is to choose it to be *unitary*, that is, taking values in the unitary group $\mathrm{U}(1)$, which is the unit circle in $\mathbb{C}^*$. Now the group of unitary characters of $\pi_1(X, x)$ is the subgroup $H^1(X, \mathrm{U}(1)) \subset H^1(X, \mathbb{C}^*)$; clearly the composite

$$H^1(X, \mathrm{U}(1)) \to H^1(X, \mathbb{C}^*) \to J(X)$$

is an isomorphism of groups.

This suggests one way to determine a distinguished class of vector bundles of rank n, namely the bundles which are determined by a representation of $\pi_1(X, x)$ into the unitary group $\mathrm{U}(n) \subset \mathrm{GL}_n(\mathbb{C})$. Note that when X has genus 0 or 1, the fundamental group is either trivial or abelian, so there are no irreducible unitary representations of $\pi_1(X, x)$ for $n > 1$.

The celebrated theorem of M. S. Narasimhan and C. S. Seshadri (see [NS]) gives an algebraic characterization of those bundles which arise in this way: they are precisely the *stable vector bundles* of rank n and degree 0, in the sense of the geometric invariant theory of D. Mumford. In particular, they show that the (coarse) moduli space $M(n)$ of stable bundles (of rank n and degree 0) is identified with the space of irreducible representations of π_1 into $\mathrm{U}(n)$; in particular, the space of representations, which depends only on the fundamental group, hence the underlying topological space of X, in fact has a structure of a Zariski open subset of a projective algebraic variety, induced by the Riemann surface structure of X.

We recall briefly the definition of a stable vector bundle. If $f : V \to X$ is a holomorphic vector bundle of rank n, define its *degree* to be that of the line bundle $\overset{n}{\wedge} V$, and its *slope* to be $\mu(V) = \deg(V)/n$. Then V is *stable* if for any non-zero proper sub-bundle $W \subset V$ (this notion has the obvious meaning) we have $\mu(W) < \mu(V)$. If $\deg(V) = 0$, then V is stable $\Leftrightarrow$ any proper subbundle has negative degree. In passing, we remark here that for any holomorphic vector bundle V, we have $\deg(V) = c_1(V) \in \mathbb{Z}$; this perhaps motivates why in general, the complex number c_1, defined as an integral, should give an integer valued invariant of a bundle, independent of the chosen connection.

One proof[12] of the Narasimhan-Seshadri theorem is by finding a special connection on any stable vector bundle $f : V \to X$. Recall that the curvature F_∇ is, with respect to any local trivialization, a matrix of holomorphic 2-forms. From the uniformization theorem, the Riemann surface X (which has genus ≥ 2) is covered by the unit disk, and hence supports a unique 2-form ω which pulls back to a constant multiple the volume form on the disk for the Poincaré metric. This 2-form is naturally associated to the unique metric with constant negative curvature on X. Now one may try to find connections ∇ on X whose curvature

[12] This approach, using 'gauge theory', is due to S. Donaldson [Do1], motivated by ideas of Atiyah and Bott.

F_∇ is locally given by a 'scalar' matrix, all of whose diagonal entries are $C\omega$, for some constant C, and with vanishing off-diagonal entries (then by computing c_1, one sees that $C = (2\pi i)\mu(V)$). When $\deg(V) = 0$, such a connection is clearly flat (since we must have $C = 0$).

To find this distinguished connection (which is also required to satisfy certain additional conditions[13]), one first chooses an arbitrary one, then tries to modify it by adding a section of $\mathcal{A}^1_X \otimes_{\mathcal{A}_X} (\mathcal{V}^\infty)^* \otimes \mathcal{V}^\infty$, which is locally just a matrix of 1-forms; then the condition we want to achieve is, locally, a differential equation for the entries of this matrix of 1-forms. This equation is thought of as analogous to Einstein's field equations in general relativity; hence, the connection sought is now called an *Einstein-Hermitian connection*. Subsequently, there has been a lot of work on finding such connections[14] on bundles in other contexts (for example, S. Donaldson has constructed such connections on appropriate vector bundles on complex projective manifolds of arbitrary dimension [Do2], and analogous results for compact Kähler manifolds have been obtained by Uhlenbeck and Yau [UY]).

It is interesting to ask if the variety $M(n)$ has properties analogous to the Jacobian $J(X)$. The detailed geometry of $M(n)$ is still under investigation, though many interesting results are known. For example, $M(n)$ is non-compact except for $n = 1$. However, one can find a projective compactification $\overline{M}(n)$, such that the boundary points have the following description. Define a vector bundle V to be *semi-stable* if for any proper subbundle W, we have $\mu(W) \leq \mu(V)$ (this corresponds to a semi-stable point as in [GIT]). Then any semi-stable V has a canonically defined filtration $\{F^iV\}$ by sub-bundles, such that $\mu(V) = \mu(F^iV) = \mu(F^iV/F^{i+1}V)$, and each graded piece $F^iV/F^{i+1}V$ is stable. Define 2 semi-stable bundles V, V' to be S-equivalent if their associated graded bundles (with respect to the canonical filtrations) are isomorphic. Seshadri proved that boundary points of $M(n)$ correspond to S-equivalence classes of semi-stable bundles of rank n and degree 0.

Another interesting point is that, unlike in the case of line bundles, the moduli spaces $M(n, d)$ of stable vector bundles of a fixed rank n and varying degree d can be non-isomorphic, for different d, in general. Narasimhan and Seshadri characterize stable vector bundles of rank n and degree d as arising from certain particular classes of irreducible representions of $\pi_1(X - \{x_0\}, x) \to \mathrm{U}(n)$; using this, they show for example that $M(n, d)$ is compact if n and d are relatively prime. Ramanan and Narasimhan showed that in this case, $M(n, d)$ is a non-singular projective algebraic variety.

One topic which has attracted much interest recently is a formula (called the

[13]Basically, compatibility with a Hermitian metric, *i.e.*, one wants a *unitary* connection.

[14]What we have implicitly described above is a *projectively flat* unitary connection; this is the same as an Einstein-Hermitian connection for bundles on a Riemann surface, but in higher dimensions, is a weaker notion, which seems to be the correct generalization. There is a criterion in terms of Chern classes which characterizes bundles supporting a projectively flat connections among those supporting Einstein-Hermitian ones; see [K].

Verlinde formula) for the dimensions of spaces of 'generalized theta functions', which may be described as analogues for $M(n)$, $M(n,d)$ and certain other similar spaces, of the 'classical' theta functions associated to Jacobians and abelian varieties. The formula for the dimensions was predicted by Verlinde, a theoretical physicist, from conformal field theory (in physics). It has been proved now in many cases; for an introduction to these ideas, see [So].

We mention another interesting development, which has also had generalizations in the higher dimensional theory. N. Hitchin (using also some ideas of Donaldson) found an appropriate algebraic way of characterizing arbitrary irreducible representations of $\pi_1(X,x)$ into $\mathrm{GL}_n(\mathbb{C})$. His result is also (apparently) motivated by considerations from physics.

Hitchin considers pairs consisting of a holomorphic vector bundle $f : V \to X$, and a holomorphic section of $\Theta \in \mathcal{V}^* \otimes_{\mathcal{O}_X} \mathcal{V} \otimes_{\mathcal{O}_X} \Omega^1_X$, which we may equivalently regard as an $\mathcal{O}_X$-linear sheaf map $\mathcal{V} \to \mathcal{V} \otimes_{\mathcal{O}_X} \Omega^1_X$. (Unlike a holomorphic connection, we stress that this map is $\mathcal{O}_X$-linear, and so does not satisfy the Leibniz rule.) The section Θ is called a *Higgs field*, so a pair is also called a *Higgs bundle*.

Hitchin defines such a pair (V,Θ) to be a *stable pair* if any non-zero proper sub-bundle W of V, which is also Θ-invariant, has strictly smaller slope. One the one hand, when $\Theta = 0$, this reduces to the usual notion of stability. On the other hand, if Θ is 'complicated', there may be no proper sub-bundle invariant under Θ, and so the condition is trivially satisfied. At any rate, Hitchin proves that there is a natural bijection (inducing a homeomorphism between the corresponding coarse moduli spaces) between stable pairs of rank n and degree 0, and irreducible representations $\pi_1(X,x) \to \mathrm{GL}_n(\mathbb{C})$, such that if ρ is the representation associated to (V,Θ), then $V \cong V_\rho$. The distinct ρ giving rise to the same V correspond to the different choices of Θ.

Hitchin's work has been generalized by C. Simpson and others to study stable pairs, and representations of the fundamental group, for higher dimensional varieties, leading to many interesting results. One conclusion from Simpson's work which is easily stated is the following: $\mathrm{SL}_n(\mathbb{Z})$ cannot be (isomorphic to) the fundamental group of a projective algebraic manifold, for any $n \geq 3$. A recent paper, giving background and other references, is [Sim].

A Sheaves

For this section, a good reference is [W].

Definition: A *sheaf* of abelian groups on a topological space X is a rule $\mathcal{F}$, which associates (i) to each open set $U \subset X$, an abelian group $\mathcal{F}(U)$ (called the group of *sections* of $\mathcal{F}$ over U), with $\mathcal{F}(\emptyset) = 0$, and (ii) to every pair of open sets $V \subset U$, a *restriction homomorphism* $\rho_{UV} : \mathcal{F}(U) \to \mathcal{F}(V)$, such that for $W \subset V \subset U$, we have $\rho_{VW} \circ \rho_{UV} = \rho_{UW}$. Further, if $U \subset X$ is open, and $\{U_i\}_{i \in I}$ is an open cover of U, then the following properties must hold:

(a) if $s \in \mathcal{F}(U)$ with $\rho_{UU_i}(s) = 0$ for all i, then $s = 0$ (*i.e.*, "a section which vanishes locally is 0");

(b) for $i \neq j$, let $U_{ij} = U_i \cap U_j$; then given $s_i \in \mathcal{F}(U_i)$ for all i, such that $\rho_{U_iU_{ij}}(s_i) = \rho_{U_jU_{ij}}(s_j)$ for all i, j with $U_{ij} \neq \emptyset$, then there exists $s \in \mathcal{F}(U)$ with $\rho_{UU_i}(s) = s_i$ (*i.e.*, , "locally defined sections, which agree with each other on the overlaps, patch up").

A *morphism* of sheaves $\mathcal{F} \to \mathcal{G}$ is a homomorphism $\mathcal{F}(U) \to \mathcal{G}(U)$ for each open set U, which is compatible with the restriction homomorphisms for the two sheaves.

In a similar way, we may define sheaves of sets, sheaves of rings, sheaves of modules over a sheaf of rings, etc. As a matter of notation, we may write "$s\mid_V$" in place of "$\rho_{UV}(s)$".

We mention a basic motivating example of a sheaf of abelian groups: if Y is any abelian topological group, define $\mathcal{F}(U)$ to be the set of continuous maps $f : U \to Y$, with ρ_{UV} being given by restriction of mappings. Taking Y to be $\mathbb{R}$ or $\mathbb{C}$, with the Euclidean topology, we get the sheaves of continuous $\mathbb{R}$-valued or $\mathbb{C}$-valued functions, respectively. On the other hand, taking Y to be a group A with the discrete topology, we obtain the *constant sheaf* A_X, with $A_X(U) = A$ for any non-empty connected open set U.

If X is a differentiable manifold, we can similarly form the sheaf $\mathcal{A}_X$ of C^∞ complex-valued functions, or the sheaves $\mathcal{A}^k_X$ of complex valued C^∞ differential k-forms, for $0 \leq k \leq \dim X$. More or less by definition, if X is a Riemann surface, then we are given a sheaf $\mathcal{O}_X$ of holomorphic functions on X; similarly, one has the sheaf Ω^1_X of holomorphic 1-forms on X, which is an $\mathcal{O}_X$-module.

We need the important notion of an *exact sequence* of sheaves. A sequence of sheaves $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ on a space X is said to be exact if (i) for each open set U in X,

$$0 \to \mathcal{F}(U) \to \mathcal{G}(U) \to \mathcal{H}(U)$$

is an exact sequence of abelian groups, and (ii) for each open U, and each $s \in \mathcal{H}(U)$, there exists an open cover $\{U_i\}_{i \in I}$ of U and elements $s_i \in \mathcal{G}(U_i)$ with $s_i \mapsto \rho_{UU_i}(s)$ for all i (*i.e.*, , any s is 'locally liftable' to a section of $\mathcal{G}$). This somewhat complicated definition of the surjectivity of the map of sheaves $\mathcal{G} \to \mathcal{H}$ is necessary, in order that, for example, the cokernel of a morphism of sheaves is again a sheaf. It is a fact that the sheaves of abelian groups on a topological space form an abelian category (*i.e.*, direct sums of sheaves exist, and any morhism has a well-defined kernel and cokernel, and factorizes uniquely as a composition of a surjection followed by an inclusion). From the definitions, one can show that exactness at $\mathcal{B}$ of a sequence of sheaves

$$\mathcal{A} \to \mathcal{B} \to \mathcal{C}$$

means: the composite map $\mathcal{A} \to \mathcal{C}$ is 0, and for any $s \in \mathcal{B}(U)$ with $s \mapsto 0 \in \mathcal{C}(U)$, there exists an open cover $\{U_i\}$ of U and $s_i \in \mathcal{A}(U_i)$ with $s_i \mapsto s\mid_{U_i} \in \mathcal{B}(U_i)$. Thus exactness may be verified locally on the space X.

One can perform certain operations on sheaves, analogous to those on abelian groups or modules. For example, as mentioned above, we can define the direct sum $\mathcal{F} \oplus \mathcal{G}$ by the standard universal property, which reduces to $(\mathcal{F} \oplus \mathcal{G})(U) = \mathcal{F}(U) \oplus \mathcal{G}(U)$. Next, $\mathcal{H}om\,(\mathcal{F}, \mathcal{G})(U) = \mathrm{Hom}\,(\mathcal{F}\mid_U, \mathcal{G}\mid_U)$ defines a sheaf, where $\mathcal{F}\mid_U$ denotes the restriction of $\mathcal{F}$ to an open set $U \subset X$ (since any open subset of U is also open in X, this restriction makes sense as a sheaf on U). Similarly, if $\mathcal{A}$ is a sheaf of rings, and $\mathcal{F}$, $\mathcal{G}$ are $\mathcal{A}$-modules, then one can define sheaves $\mathcal{H}om_{\mathcal{A}}(\mathcal{F}, \mathcal{G})$ and $\mathcal{F} \otimes_{\mathcal{A}} \mathcal{G}$, with the standard properties (for example, if $\mathcal{A}$ is commutative, then the Hom-sheaf and the tensor product are again $\mathcal{A}$-modules); the definition of the tensor product is a bit subtle (like the definition of a surjective map of sheaves), but sections of $\mathcal{F} \otimes_{\mathcal{A}} \mathcal{G}$ are locally expressible as sums $\sum_i a_i \otimes b_i$ with a_i sections of $\mathcal{F}$, and b_i sections of $\mathcal{G}$.

If $\mathcal{F}$ is a sheaf of abelian groups on a topological space X, and $\mathcal{U} = \{U_i\}_{i \in I}$ is an open cover of X, then we can define the *Čech complex* $\check{C}^*(\mathcal{U}, \mathcal{F})$, which has terms

$$\check{C}^p(\mathcal{U}, \mathcal{F}) = \prod_{(i_0, \ldots, i_p) \in I^p} \mathcal{F}(U_{i_0} \cap \cdots \cap U_{i_p}),$$

for $p \geq 0$, and a differential

$$\delta^p : \check{C}^p(\mathcal{U}, \mathcal{F}) \to \check{C}^{p+1}(\mathcal{U}, \mathcal{F}),$$

given by

$$\delta^p(\{a_{i_0, \ldots, i_p}\}) = \{b_{j_0, \ldots, j_{p+1}}\},$$

$$b_{j_0, \ldots, j_{p+1}} = \sum_{k=0}^{p+1} (-1)^k a_{j_0, \ldots, j_{k-1}, j_{k+1}, \ldots, j_{p+1}} \mid_{U_{j_0} \cap \cdots \cap U_{j_{p+1}}}$$

(we verify at once that $\delta^p \circ \delta^{p-1} = 0$ for $p > 0$). Finally, we may define the groups of Čech p-cocycles $\check{Z}^p(\mathcal{U}, \mathcal{F}) = \ker \delta^p$, p-coboundaries $\check{B}^p(\mathcal{U}, \mathcal{F}) = \mathrm{im}\, \delta^{p-1}$, and the p-th *Čech cohomology group*

$$\check{H}^p(\mathcal{U}, \mathcal{F}) = \frac{\check{Z}^p(\mathcal{U}, \mathcal{F})}{\check{B}^p(\mathcal{U}, \mathcal{F})}\ .$$

If $\mathcal{V} = \{V_j\}_{j \in J}$ is another open cover which refines $\mathcal{U}$, so that there is a map $f : J \to I$ such that $V_j \subset U_{f(j)}$ for each $j \in J$, then there is a map of complexes

$$f^* : \check{C}^*(\mathcal{U}, \mathcal{F}) \to \check{C}^*(\mathcal{V}, \mathcal{F}).$$

The induced homomorphism

$$\check{H}^p(\mathcal{U}, \mathcal{F}) \to \check{H}^p(\mathcal{V}, \mathcal{F})$$

can be shown to be independent of the choice of the map $f : J \to I$ on index sets. This allows one to take the direct limit over all open covers of X, to define the (Čech) *cohomology groups*

$$H^i(X, \mathcal{F}) = \lim_{\overrightarrow{\mathcal{U}}} \check{H}^p(\mathcal{U}, \mathcal{F}).$$

Note that a 0-cochain for $\mathcal{F}$ with respect to the covering $\mathcal{U} = \{U_i\}_{i\in I}$ is a collection of sections $a_i \in \mathcal{F}(U_i)$, for $i \in I$; it is a cocycle precisely when the patching conditions $a_i \mid_{U_i \cap U_j} = a_j \mid_{U_i \cap U_j}$, for all i, j, are satisfied. Thus $\check{H}^0(\mathcal{U}, \mathcal{F}) \cong \mathcal{F}(X)$.

Similarly, a 1-cochain is a collection of sections $a_{ij} \in \mathcal{F}(U_i \cap U_j)$. The 1-cochain $\{a_{ij}\}$ is a 1-cocycle $\Leftrightarrow$ (i) $a_{ii} = 0$ for all i, (ii) $a_{ij} = -a_{ji}$ for all $i \neq j$, and (iii) for all distinct indices i, j, k, we have

$$a_{ij} \mid_{U_i \cap U_j \cap U_k} + a_{jk} \mid_{U_i \cap U_j \cap U_k} = a_{ik} \mid_{U_i \cap U_j \cap U_k} .$$

These are very similar to the conditions satisfied by transition functions for line bundles; in fact, if $\mathcal{A}^*_X$ is the sheaf of C^∞ complex valued invertible functions (the group operation being multiplication), then $H^1(X, \mathcal{A}^*_X)$ is identified with the group of (isomorphism classes of) C^∞ line bundles on X, with tensor product of line bundles as group operation; similarly, $H^1(X, \mathbb{C}^*_X)$ is identified with the group of flat line bundles, and $H^1(X, \mathcal{O}^*_X)$ is identified with the group of holomorphic line bundles.

One of the important technical results from sheaf theory is the following.

Lemma A.1 *Let X be a paracompact Hausdorff space. Then for any short exact sequence of sheaves of abelian groups*

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$$

on X, there are boundary homomorphisms

$$\delta : H^i(X, \mathcal{H}) \to H^{i+1}(X, \mathcal{F}), \ \ \forall\, i \geq 0,$$

and a long exact sequence of Čech cohomology groups

$$0 \to H^0(X, \mathcal{F}) \to H^0(X, \mathcal{G}) \to H^0(X, \mathcal{H}) \xrightarrow{\delta} H^1(X, \mathcal{F}) \to \cdots$$
$$\cdots \to H^i(X, \mathcal{F}) \to H^i(X, \mathcal{G}) \to H^i(X, \mathcal{H}) \xrightarrow{\delta} H^{i+1}(X, \mathcal{F}) \to H^{i+1}(X, \mathcal{G}) \to \cdots .$$

Suppose further that X is locally contractible. Then for any abelian group A, the sheaf cohomology groups $H^i(X, A_X)$, with coefficients in the constant sheaf A_X, are naturally isomorphic to the (singular) cohomology groups $H^i(X, A)$ of Algebraic Topology.

Next, one has the notion of a *fine sheaf*: a sheaf $\mathcal{F}$ of abelian groups on X is called fine if for any open cover $\mathcal{U} = \{U_i\}_{i\in I}$ of X, there are sheaf maps $\psi_i : \mathcal{F} \to \mathcal{F}$ such that, if $\mathrm{supp}\,(\psi_i)$ is the closure of the set of points $\{x \in X \mid (\psi_i)_x : \mathcal{F}_x \to \mathcal{F}_x \text{ is non-zero}\}$, then

(i) $\mathrm{supp}\,(\psi_i) \subset U_i$ for each i, and the collection of subsets $\{\mathrm{supp}\,(\psi_i)\}_{i\in I}$ forms a locally finite family of subsets of X

(ii) we have
$$\sum_{i\in I} \psi_i = 1_{\mathcal{F}},$$
where $1_{\mathcal{F}}$ denotes the identity endomorphism of $\mathcal{F}$.

The usefulness of fine sheaves stems from the following result.

Lemma A.2 *If $\mathcal{F}$ is a fine sheaf, then $\check{H}^i(\mathcal{U},\mathcal{F}) = 0$ for all $i > 0$, for any open cover $\mathcal{U}$ of X; hence $H^i(X,\mathcal{F}) = 0$ for all $i > 0$.*

Corollary A.3 *If*
$$0 \to \mathcal{F} \to \mathcal{A}^0 \to \mathcal{A}^1 \to \cdots$$
is an exact sequence of sheaves, where $\mathcal{A}^i$ is a fine sheaf for each $i \geq 0$, then there are natural isomorphisms
$$H^i(X,\mathcal{F}) \cong \frac{\ker\left(\mathcal{A}^i(X) \to \mathcal{A}^{i+1}(X)\right)}{\text{image}\ (\mathcal{A}^{i-1}(X) \to \mathcal{A}^i(X))},$$
where the denominator is defined to be the trivial group if $i = 0$.

A basic example of a fine sheaf is the sheaf $\mathcal{A}_X$ of C^∞ functions on a Riemann surface (or more generally, on any C^∞ differential manifold). The fineness follows immediately from the existence of C^∞ partitions of unity subordinate to any covering of X. Since this is a sheaf of rings, we see also that any sheaf of modules over $\mathcal{A}_X$ is also fine, since multiplication by elements of a partition of unity will give the desired endomorphisms ψ_i. This means, for example, that sheaves of C^∞ differential forms are fine.

This leads to a quick proof of the de Rham theorem.

Theorem A.4 (de Rham) *If X is a C^∞ manifold, then there are natural isomorphisms*
$$H^i(X,\mathbb{C}) \cong \frac{\text{closed } C^\infty \ i\text{-forms}}{\text{exact } C^\infty \ i\text{-forms}}.$$
The idea is to use the *de Rham complex* of sheaves
$$0 \to \mathbb{C}_X \to \mathcal{A}_X \xrightarrow{d} \mathcal{A}^1_X \xrightarrow{d} \mathcal{A}^2_X \to \cdots \to \mathcal{A}^n_X \to 0,$$
where $\mathcal{A}^i_X$ is the sheaf of C^∞ i-forms, $n = \dim X$ and d is the exterior derivative. The local exactness of this sequence, *i.e.*, the exactness of any closed form on a disk in $\mathbb{R}^d$, is called the *Poincaré lemma*; it may be proved using an explicit integral formula. Now the Theorem becomes a particular case of Corollary A.3.

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