SHEAF THEORY

1. Adjoint functors and limits

Let $\mathcal{C}$ and $\mathcal{D}$ be categories and let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{C} \to \mathcal{D}$ be (covariant) functors.

Definition 1. 1. A *natural transformation* $T : F \to G$ is the following data: for each object A in $\mathcal{C}$ we have an arrow $T(A) : F(A) \to G(A)$ in $\mathcal{D}$, such that if $f : A \to B$ is an arrow in $\mathcal{C}$ then the following diagram commutes

$$\begin{array}{ccc} F(A) & \xrightarrow{F(f)} & F(B) \\ T(A)\downarrow & & \downarrow T(B) \\ G(A) & \xrightarrow{G(f)} & G(B) \end{array}$$

2. A *natural isomorphism* is a natural transformation $T : F \to G$ such that for all objects A of $\mathcal{C}$ the morphism $T(A) : F(A) \to G(A)$ is an isomorphism.
3. Two categories $\mathcal{C}$ and $\mathcal{D}$ are *equivalent* if there exists functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ such that both composites are natural isomorphisms to the respective identity functors. In this case F and G are called *equivalence* of categories.

Example Let R be a commutative ring and let $\mathbf{R-mod}$ be the category of R-modules. Let $G : \mathbf{R-mod} \to \mathbf{R-mod}$ be the functor defined by $G(M) = (M^*)^*$ where $M^* = Hom_R(M, R)$. Then there exists a natural transformation $T : Id_{\mathbf{R-mod}} \to G$ which is a natural isomorphism if $R = k$ is a field and if we restrict our attention to the subcategory of finitely generated R-modules.

Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ be two contravariant functors. Then we get two functors from $\mathcal{D}^{op} \times \mathcal{C}$ to the category of sets, **Sets** given by $(A, B) \mapsto Hom_{\mathcal{C}}(G(A), B)$ and $(A, B) \mapsto Hom_{\mathcal{D}}(A, F(B))$ where A is an object of $\mathcal{D}$ and B is an object of $\mathcal{C}$.

Definition 2. Let F, G be as in the above paragraph. (G, F) is said to form an *adjoint pair of functors* if there exists a natural isomorphism between the above two functors from $\mathcal{D}^{op} \times \mathcal{C}$ to **Sets**. In this case, G is said to be a *left adjoint* of F, and F is said to be a *right adjoint* of G.

Example Let R be a commutative ring and let $N \in \mathbf{R-mod}$. Then the functors $(. \otimes N, Hom_R(N, .))$ form an adjoint pair i.e. there exists a natural isomorphism $Hom_R(M \otimes N, P) \cong Hom_R(M, Hom_R(N, P))$ for any R-modules M and P.

Problem 1. 1. Show that any two left adjoints (or any two right adjoints) of a functor are isomorphic.
2. Let **Ab** be the category of abelian groups and let $F : \mathbf{Ab} \to \mathbf{Sets}$ be the forgetful functor. Then show that F has a left adjoint $G : \mathbf{Sets} \to \mathbf{Ab}$ given by $G(S) =$ Free abelian group on S.
3. Similarly, let **SAb** be the category of commutative semigroups and let $F : \mathbf{Ab} \to \mathbf{SAb}$ be the forgetful functor. Then show that F again has a left adjoint $K : \mathbf{SAb} \to \mathbf{Ab}$ where $K(A)$ is the Grothendieck group of A.

Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ be two functors such that (G, F) is an adjoint pair. If A is an object of $\mathcal{D}$, then there exists a bijection of sets

$$Hom_{\mathcal{C}}(GA, GA) \cong Hom_{\mathcal{D}}(A, FGA).$$

Let the identity map on GA,1_{GA}, correspond to $\eta_A : A \to FGA$ under the above bijection. It can be shown that η_A satisfies the following universal property: if $f : A \to FB$ is any morphism in $\mathcal{D}$ then there exists a unique morphism $g : GA \to B$ such that $f = Fg \cdot \eta_A$ (i.e. f factors through η_A in this sense). We have an analogous map $\psi_B : GFB \to B$ for any object B of $\mathcal{C}$ and ψ_B has a similar universal property.

Limits Let $\mathcal{I}$ be a *small* category (i.e. the objects of $\mathcal{I}$ form a set) and let $\mathcal{C}$ be any category. A *diagram* in $\mathcal{C}$ indexed by $\mathcal{I}$ is a functor $\mathcal{I} \to \mathcal{C}$. The diagrams in $\mathcal{C}$ indexed by $\mathcal{I}$ form a category $\mathcal{C}^{\mathcal{I}}$ whose morphisms are natural transformations between two functors from $\mathcal{I}$ to $\mathcal{C}$. The category $\mathcal{C}$ can be thought of as the subcategory of $\mathcal{C}^{\mathcal{I}}$ of constant functors.

Definition 3. Let $F \in \mathcal{C}^{\mathcal{I}}$. A *direct limit* of F is defined to be an object of A of $\mathcal{C}$ together with a morphism (i.e. a natural transformation) $\eta : F \to A$ in $\mathcal{C}^{\mathcal{I}}$ (where A is treated as the constant functor) which is universal among such morphisms i.e. given any other object B of $\mathcal{C}$ and any morphism $\theta : F \to B$ there exists a unique morphism $f : A \to B$ in $\mathcal{C}$ such that $f \cdot \eta = \theta$. We write this as $\varinjlim_{a \in \mathcal{I}} F(a) = A$. An *inverse limit* is defined as above by reversing all arrows.

Let $(\mathcal{I}, \leq)$ be a directed set. Then $\mathcal{I}$ can be thought of as a category as follows: the objects of $\mathcal{I}$ are the elements of the set $\mathcal{I}$ and for $a, b \in \mathcal{I}$ we define $Hom_{\mathcal{I}}(a, b)$ to consist of one morphism $a \to b$ if $a \leq b$ and empty otherwise. The definition of direct and inverse limit for this particular $\mathcal{I}$ is probably the one which the reader is more familiar with.

Examples to be given

Problem 2. Let $\mathcal{I}$ be a small category. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor which has a left adjoint. Then F preserves direct limits over $\mathcal{I}$. Similarly, if a functor has a right adjoint then it preserves inverse limits over $\mathcal{I}$. This gives another proof that tensoring is right exact while the Hom functor is left exact.

Presheaves and Sheaves

The concept of a sheaf provides a systematic way of keeping track of local algebraic data on a topological space.

Definition 4. Let X be a topological space. A *presheaf* $\mathcal{F}$ of abelian groups on X consists of the data

1. for every open subset $U \subseteq X$, an abelian group $\mathcal{F}(U)$ and,
2. for every inclusion $V \subseteq U$ of open subsets of X, a morphism of abelian groups $\rho_{UV} : \mathcal{F}(U) \to \mathcal{F}(V)$ (called *restriction maps*) satisfying,
   (a) $\mathcal{F}(\phi) = 0$, where ϕ is the empty set,
   (b) ρ_{UU} is the idenity map $\mathcal{F}(U) \to \mathcal{F}(U)$, and
   (c) if $W \subseteq V \subseteq U$ are three open subsets, then $\rho_{UW} = \rho_{VW} \cdot \rho_{UV}$.

The definition may be rephrased using the language of categories as follows. For any topological space X, let $\mathcal{T}op(X)$ denote the category whose objects are the open subsets of X and whose only morphisms are the inclusion maps. Hence, $Hom_{\mathcal{T}op(X)}(V, U)$ is empty if V is not a subset of U, and $Hom_{\mathcal{T}op(X)}(V, U)$ has just one element (the inclusion map) if $V \subseteq U$. Then, a presheaf may also be defined as a contravariant functor from the category $\mathcal{T}op(X)$ to the category **Ab** of abelian groups.

One can define a presheaf of rings, a presheaf of sets or, more generally, a presheaf with values in any category $\mathcal{C}$, by replacing the words "abelian group" in the above definition by "ring", "set", or "object of $\mathcal{C}$" respectively. We will usually stick to the case of abelian groups.

If $\mathcal{F}$ is a presheaf on X, then $\mathcal{F}(U)$ is referred to as the *group of sections* of the presheaf $\mathcal{F}$ over the open set U, and an element $s \in \mathcal{F}(U)$ is called a *section*. We sometimes use the notation $\Gamma(U, \mathcal{F})$ (and later, $H^0(U, \mathcal{F})$) to denote the group $\mathcal{F}(U)$. We also use the notation $s|_V$ for $\rho_{UV}(s)$ where $s \in \mathcal{F}(U)$.

A sheaf is roughly speaking a presheaf whose sections are determined by local data. The precise definition is as follows.

Definition 5. A presheaf $\mathcal{F}$ on a topological space X is a *sheaf* if it satisfies the following supplementary conditions

1. if U is an open set, if $\{V_i\}$ is an open covering of U, and if $s \in \mathcal{F}(U)$ is an element such that $s|_{V_i} = 0$ for all i, then $s = 0$ ("a section is globally zero if it is locally zero").
2. if U is an open set, if $\{V_i\}$ is an open cover of U, and if there exists elements $s_i \in \mathcal{F}(V_i)$ for each i with the property that for each i we have $s_i|_{V_i \cap V_j} = s_j|_{V_i \cap V_j}$, then there is an element $s \in \mathcal{F}(U)$ such that $s|_{V_i} = s_i$ for all i ("local sections patch to give a global section"). Note that the previous condition implies that this s is unique.

Definition 6. A presheaf is called a *monopresheaf* if it satisfies the first sheaf condition above i.e., a section which is locally zero must be globally zero.

Examples

1. Let X be a topological space. For any open set $U \subseteq X$, let $\mathcal{F}(U)$ be the ring of continuous real-valued functions on U, and for each $V \subseteq U$, let $\rho_{UV} : \mathcal{F}(U) \to \mathcal{F}(V)$ be the restriction map in the usual sense. Then $\mathcal{F}$ is a sheaf (of rings) as continuity is a local condition. More generally, we could have replaced $\mathbb{R}$ by any topological space Y and considered for each $U \subseteq X$ the set of continuous functions from U to Y, thus defining a sheaf of sets.
2. Let X, Y be differentiable manifolds. For an open $U \subseteq X$, define $\mathcal{F}(U)$ to be the set of differentiable functions from U to Y. The restriction maps are again the usual restriction maps. This is again a sheaf because differentiability is a local condition.
3. The sheaf of regular functions $\mathcal{O}_X$ (with usual restriction maps) on a variety X over a field k
4. The sheaf of holomorphic functions $\mathcal{H}_X$ (with usual restriction maps) on a complex manifold X.
5. Let X be a topological space and $\pi : E \to X$ be a real vector bundle on X. Define $\mathcal{F}(U)$ to be the ring of continuous *sections* on U with the usual restriction maps (in this case, a continuous section on U means a continuous map $s : U \to E|_U$ such that $\pi \cdot s = Id_U$). This is again a sheaf of rings called the *sheaf of sections of a vectorbundle*. We could have similarly considered vectorbundles on smooth manifolds, algebraic varieties etc.
6. Let X, Y be topological spaces and let $\mathcal{F}(U) =$ continuous maps from U to Y which have relatively compact image. This is a presheaf, in fact a monopresheaf, which is *not* a sheaf.
7. Let X be a topological space. For $U \subseteq X$ let $\mathcal{F}(U) =$ the vector space of locally constant real-valued functions on U modulo the constant functions on U. This is a presheaf which may not be a sheaf because if U is not connected then there may exist sections on U which are locally zero but *not* globally zero (hence $\mathcal{F}$ is not even a monopresheaf).
8. Let X be a topological space and G an abelian group. Define the *constant presheaf* associated to G on X to be the presheaf $\mathcal{F}(U) = G$ for all $U \neq \phi$, with restriction maps being the identity. The constant presheaf is in general *not* a sheaf.
9. Let X and G be as in the previous example. The *constant sheaf* G_X, associated to G on X is defined as follows. Give G the discrete topology. Let $G_X(U)$ be the group of continuous maps from U to G and the ρ_{UV} be the usual restriction maps. Then G_X is a sheaf. If U is a connected set then $G_X(U) \cong G$ hence the name "constant sheaf". If U is an open set whose connected components are open (which is true for a locally connected space) then $G_X(U)$ is a direct product of copies of G, one for each connected component of U. This is a special case of the first example.

Definition 7. Let $\mathcal{F}$ be a presheaf on X and P be a point on X. The *stalk* $\mathcal{F}_P$ of $\mathcal{F}$ at P is defined to be the direct limit of the groups $\mathcal{F}(U)$ for all open sets U containing P via the restriction maps ρ_{UV} i.e. $\mathcal{F}_P = \varinjlim_{P \in U} \mathcal{F}(U)$. Thus an element of $\mathcal{F}_P$ is represented by a pair (U, s) where U is an open neighbourhood of P and $s \in \mathcal{F}(U)$. Two such pairs (U, s) and (V, t) define the same element of $\mathcal{F}_P$ if and only if there is an open neighbourhood W of P with $W \subseteq U \cap V$ such that $s|_W = t|_W$. Thus elements of the stalk $\mathcal{F}_P$ are *germs* of sections of $\mathcal{F}$ at the point P.

Definition 8. If $\mathcal{F}$ and $\mathcal{G}$ are presheaves on X, a *morphism* of presheaves $\phi : \mathcal{F} \to \mathcal{G}$ consists of a morphism of abelian groups $\phi(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ for each open set U such that whenever $V \subseteq U$ is an inclusion, the following diagram

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{\phi(U)} & \mathcal{G}(U) \\ \rho_{UV} \downarrow & & \downarrow \rho'_{UV} \\ \mathcal{F}(V) & \xrightarrow{\phi(V)} & \mathcal{G}(V) \end{array}$$

commutes, where ρ and ρ' are the restriction maps for $\mathcal{F}$ and $\mathcal{G}$ respectively. If $\mathcal{F}$ and $\mathcal{G}$ are sheaves then a *morphism* of sheaves $\phi : \mathcal{F} \to \mathcal{G}$ is a morphism from $\mathcal{F}$ to $\mathcal{G}$ considering both $\mathcal{F}$ and $\mathcal{G}$ as presheaves. An *isomorphism* is a morphism which has a two-sided inverse.

Let $\mathbb{Z}_X$ denote the constant sheaf on X associated to $\mathbb{Z}$. Then for any sheaf $\mathcal{F}$ of abelian groups on X, the sheaf morphisms $Hom(\mathbb{Z}_X, \mathcal{F})$ can be indentified with the global sections $\mathcal{F}(X)$ of $\mathcal{F}$.

We can now talk of the category of presheaves, $\mathbf{PSh}(\mathbf{X})$, and the category of sheaves, $\mathbf{Sh}(\mathbf{X})$, on X, whose objects are presheaves (or sheaves) on X and whose morphisms are morphisms between presheaves (or sheaves). We will later see that $\mathbf{Sh}(\mathbf{X})$ is an *abelian* category.

A morphism $\phi : \mathcal{F} \to \mathcal{G}$ of presheaves on X induces a morphism $\phi_P : \mathcal{F}_P \to \mathcal{G}_P$ on the stalks for every point $P \in X$. The following proposition is *false* for presheaves and illustrates the local nature of sheaves.

Proposition 1. Let $\phi : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves on a topological space X. Then ϕ is an isomorphism if and only if the induced map on the stalks $\phi_P : \mathcal{F}_P \to \mathcal{G}_P$ is an isomorphism for every $P \in X$.

Proof. It is clear that ϕ_P is an isomorphism for every $P \in X$ if ϕ is an isomorphism. So we have to prove the converse. Assume that ϕ_P is an isomorphism for every $P \in X$. It is enough to show that for every open subset U of X, $\phi(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ is an isomorphism. We first prove that $\phi(U)$ is injective for every $U \subseteq X$. Suppose $s \in \mathcal{F}(U)$ is such that $\phi(U)(s) \in \mathcal{G}(U)$ is zero. Then, for every point $p \in U$, $\phi_P(s_P) \in \mathcal{G}_P$ which is the image of $\phi(U)(s)$ in $\mathcal{G}_P$ is zero. Since ϕ_P is injective for every P we have that $s_P = 0$ in $\mathcal{F}_P$ for each $P \in U$. $s_P = 0$ means that s and 0 have the same image in $\mathcal{F}_P$, hence there is an open neighbourhood $W_P \subseteq U$ of

P such that $s|_{W_P} = 0$. U is covered by the neighbourhoods W_P as we vary $P \in U$ and so by the first sheaf property we have that $s = 0$. This proves that $\phi(U)$ is injective for every U.

We now show that $\phi(U)$ is surjective for every $U \subseteq X$. Suppose that $t \in \mathcal{G}(U)$. Let $t_P \in \mathcal{G}_P$ be its germ at P. Since ϕ_P is surjective we can find $s_P \in \mathcal{F}_P$ such that $\phi_P(s_P) = t_P$. Let s_P be represented by a section $s(P)$ on a neighbourhood $V_P \subseteq U$ of P. Then $\phi(V_P)(s(P))$ and $t|_{V_P}$ are two elements of $\mathcal{G}(V_P)$ whose germs at P are the same. Hence replacing V_P by a smaller neighbourhood of P if necessary we may assume that $\phi(V_P)(s(P)) = t|_{V_P}$ in $\mathcal{G}(V_P)$. U is covered by V_P as we vary $P \in U$ and on each V_P we have a section $s(P) \in \mathcal{F}(V_P)$. IF P and Q are any two points of U then $s(P)|_{V_P \cap V_Q} = s(Q)|_{V_P \cap V_Q}$ since they are both sections of $\mathcal{F}(V_P \cap V_Q)$ which are sent by ϕ to $t|_{V_P \cap V_Q}$ and ϕ has already been proved to be injective. Now by the second sheaf property, there is a section $s \in \mathcal{F}$ such that $s|_{V_P} = s(P)$ for every $P \in U$. Now $\phi(s)$ and t are two sections of $\mathcal{G}(U)$ such that for each $P \in U$ we have $\phi(s)|_{V_P} = t|_{V_P}$. Hence again by the first sheaf property applied to $\phi(s) - t$ we get that $\phi(s) = t$. Hence $\phi(U)$ is surjective for all U and the proof is complete.

□

Definition 9. Let $\phi : \mathcal{F} \to \mathcal{G}$ be a morphism of presheaves. The *presheaf kernel of* ϕ, *presheaf cokernel of* ϕ and *presheaf image of* ϕ are defined to be the presheaves given by $U \mapsto ker(\phi(U))$, $U \mapsto coker(\phi(U))$ and $U \mapsto im(\phi(U))$ respectively.

We note that if $\mathcal{F}$ and $\mathcal{G}$ are sheaves then the presheaf kernel of ϕ is a sheaf. But the presheaf cokernel and the presheaf image need not be sheaves in general even if $\mathcal{F}$ and $\mathcal{G}$ are sheaves (why?). This leads us to the definition of the sheafification of a presheaf (also known as the sheaf associated to a presheaf).

Proposition 2. Given a presheaf $\mathcal{F}$ there is a sheaf $\mathcal{F}^+$ and a morphism $\theta : \mathcal{F} \to \mathcal{F}^+$ with the property that for any sheaf $\mathcal{G}$ and any morphism $\phi : \mathcal{F} \to \mathcal{G}$ there is a unique morphism $\psi : \mathcal{F}^+ \to \mathcal{G}$ such that $\phi = \psi.\theta$. Furthurmore the pair $(\mathcal{F}^+, \theta)$ is unique upto unique isomorphism.

Proof. For any open set $U \subseteq X$, let $\mathcal{F}^+(U)$ be the set of functions s from U to the union $\cup_{P \in U} \mathcal{F}_P$ of the stalks of $\mathcal{F}$ over the points of U such that

1. for each $P \in U$, $s(P) \in \mathcal{F}_P$
2. for each $P \in U$, there is a neighbourhood $V \subseteq U$ of P and an element $t \in \mathcal{F}(V)$ such that for all $Q \in V$ the germ t_Q of t at Q is equal to $s(Q)$.

$\mathcal{F}^+$ is a sheaf and there is a natural morphism $\theta : \mathcal{F} \to \mathcal{F}^+$ got by sending $s \mapsto \{P \mapsto s_P\}$ where $s \in \mathcal{F}(U)$. One can now show that θ has the universal property as described in the statement of the proposition. We note that for any point $P \in X$, $\mathcal{F}_P \cong \mathcal{F}^+_P$ via θ and that if $\mathcal{F}$ was a sheaf then $\mathcal{F}^+$ is isomorphic to $\mathcal{F}$ via θ. The uniqueness of $\mathcal{F}^+$ is a formal consequence of the universal property of $\mathcal{F}^+$. □

Definition 10. $\mathcal{F}^+$ is called the *sheaf associated to the presheaf* $\mathcal{F}$ or the *sheafification of the presheaf* $\mathcal{F}$.

$\mathcal{F}^+$ is the best possible sheaf that can be got from $\mathcal{F}$. To imagine how we got $\mathcal{F}^+$ from $\mathcal{F}$, note that in the proof of the above proposition we identified sections which have the same restrictions (by considering the function $P \mapsto s_P$ given by a section s) and then added all things which could be patched together thus making $\mathcal{F}^+$ satisfy both sheaf conditions. In our examples, the constant sheaf is the sheafification of the constant presheaf (associated to the same group). Also the sheafification of the presheaf in example 7 is the zero sheaf.

Sheafification can also be defined as an adjoint functor as follows. Let $\mathbf{PSh(X)}$ and $\mathbf{Sh(X)}$ denote the categories of presheaves and sheaves on X. Every sheaf is a presheaf hence we have an inclusion functor $i : \mathbf{Sh(X)} \to \mathbf{PSh(X)}$. Sheafification of a presheaf gives a functor $a : \mathbf{PSh(X)} \to \mathbf{Sh(X)}$ i.e. $a(\mathcal{F}) = \mathcal{F}^+$. Then the sheafification functor is the left adjoint of the inclusion functor because by definition we have $Hom_{\mathbf{PSh(X)}}(\mathcal{F}, i(\mathcal{G})) = Hom_{\mathbf{Sh(X)}}(a(\mathcal{F}), \mathcal{G})$ where $\mathcal{F}$ is a presheaf and $\mathcal{G}$ is a sheaf.

Definition 11.
1. A *subsheaf* of a sheaf $\mathcal{F}$ is a sheaf $\mathcal{F}'$ such that for every open set $U \subseteq X$, $\mathcal{F}'(U)$ is a subgroup of $\mathcal{F}(U)$ and the restriction maps of the sheaf $\mathcal{F}'$ are induced by those of $\mathcal{F}$. It follows that for every $P \in X$, $\mathcal{F}'_P$ is a subgroup $\mathcal{F}_P$.
2. If $\phi : \mathcal{F} \to \mathcal{G}$ is a morphism of sheaves then we define the *kernel sheaf of* ϕ*,* denoted by $Ker(\phi)$, to be the presheaf kernel of ϕ (which is a sheaf). Thus $Ker(\phi)$ is a subsheaf of $\mathcal{F}$.
3. A morphism of sheaves $\phi : \mathcal{F} \to \mathcal{G}$ is defined to be *injective* if $Ker(\phi) = 0$.

One can easily show that $\phi : \mathcal{F} \to \mathcal{G}$ is injective, if and only if, $\phi(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ is injective for every open set $U \subseteq X$, if and only if $\phi_P : \mathcal{F}_P \to \mathcal{G}_P$ is injective for every $P \in X$.

Definition 12. If $\phi : \mathcal{F} \to \mathcal{G}$ is a morphism of sheaves then we define the *image sheaf of* ϕ, denoted by $Im(\phi)$, to be the sheaf associated to the presheaf image of ϕ.

By the universal property of the sheafification, there is a natural map of sheaves $Im(\phi) \to \mathcal{G}$. This map is injective (since it is injective at the level of stalks) and hence $Im(\phi)$ can be identified with a subsheaf of $\mathcal{G}$.

Definition 13. A morphism $\phi : \mathcal{F} \to \mathcal{G}$ of sheaves is defined to be *surjective* if $Im(\phi) = \mathcal{G}$.

If $\phi : \mathcal{F} \to \mathcal{G}$ is a surjective morphism of sheaves then the maps on sections $\phi(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ may *not* be surjective for all open subsets $U \subseteq X$.

The converse is of course true. It is also true that $\phi : \mathcal{F} \to \mathcal{G}$ is a surjective morphism of sheaves, if and only if, $\phi_P : \mathcal{F}_P \to \mathcal{G}_P$ is surjective for all $P \in X$, if and only if, for every open set $U \subseteq X$ and for every $s \in \mathcal{G}(U)$ there is a covering $\{V_i\}$ of U and elements $t_i \in \mathcal{F}(U_i)$ such that $\phi(V_i)(t_i) = s|_{V_i}$ for all i (the last condition says that sections of $\mathcal{G}$ can be lifted over smaller open sets).

An example of a surjective morphism of sheaves $\phi : \mathcal{F} \to \mathcal{G}$ and an open set $U \subseteq X$ such that $\phi(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ is *not* surjective is as follows. Let X be the complex numbers with the complex topology. Let $\mathcal{H}_X$ be the sheaf of holomorphic functions on X and let $\mathcal{H}_X^*$ be the sheaf of invertible (nowhere zero) holomorphic functions on X. Let ϕ denote the exponential map defined from $\mathcal{H}_X \to \mathcal{H}_X^*$ by sending a holomorphic function f on U to $exp(f)$. Then this is a surjective map of sheaves as locally the exponential map has an inverse (the logarithm) but clearly the map on global sections is not surjective (on non-simply-connected open subsets).

It follows (by considering the maps at the level of stalks) that a morphism of sheaves is an isomorphism, if and only if, it is both injective and surjective.

Definition 14. A sequence of sheaves $\cdots \to \mathcal{F}^{i-1} \xrightarrow{\phi^{i-1}} \mathcal{F}^i \xrightarrow{\phi^i} \mathcal{F}^{i+1} \to \cdots$ is said to be *exact* if for every i, $Ker(\phi^i) = Im(\phi^{i-1})$.

Thus a sequence $0 \to \mathcal{F} \xrightarrow{\phi} \mathcal{G}$ is exact, if and only if, ϕ is injective. Similarly, a sequence $\mathcal{F} \xrightarrow{\phi} \mathcal{G} \to 0$ is exact, if and only if, ϕ is surjective. One can also show that, a sequence of sheaves is exact, if and only if, the corresponding sequence of stalks at P is exact for all $P \in X$.

Definition 15. Let $\mathcal{F}'$ be a subsheaf of $\mathcal{F}$. The *quotient sheaf* $\mathcal{F}/\mathcal{F}'$ is defined to be the sheaf associated to the presheaf $U \mapsto \mathcal{F}(U)/\mathcal{F}'(U)$. It follows that for every point $P \in U$ the stalk $(\mathcal{F}/\mathcal{F}')_P$ is the quotient $\mathcal{F}_P/\mathcal{F}'_P$.

If $\mathcal{F}'$ is a subsheaf of $\mathcal{F}$ then there is a short exact sequence of sheaves

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}/\mathcal{F}' \to 0.$$

Conversely, if $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is a short exact sequence of sheaves then $\mathcal{F}'$ is isomorphic to a subsheaf of $\mathcal{F}$ and $\mathcal{F}''$ is isomorphic to the quotient of $\mathcal{F}$ by this subsheaf.

Definition 16. Let $\phi : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves. The *cokernel sheaf of* ϕ, denoted by $Coker(\phi)$, is defined to be the sheaf associated to the cokernel presheaf of ϕ.

One can easily show that if $\phi : \mathcal{F} \to \mathcal{G}$ is a morphism of sheaves then $Im(\phi) \cong \mathcal{F}/Ker(\phi)$ and $Coker(\phi) \cong \mathcal{G}/Im(\phi)$.

All sheaves discussed so far have been on a single topological space. We now define some operations on sheaves associated with a continuous map from one topological space to another.

Definition 17. Let $f : X \to Y$ be a continuous map of topological spaces.

1. For any sheaf $\mathcal{F}$ on X, we define the *direct image sheaf* $f_*\mathcal{F}$ on Y by $f_*\mathcal{F}(V) = \mathcal{F}(f^{-1}(V))$ for any open set $V \subseteq Y$.
2. For any sheaf $\mathcal{G}$ on Y, we define the *inverse image sheaf* $f^{-1}\mathcal{G}$ on X to be the sheaf associated to the presheaf $U \mapsto \varinjlim_{f(U)\subseteq V} \mathcal{G}(V)$ where U is any open set of X and the limit is taken over all open sets V of Y containing $f(U)$.

f_* is a functor from $\mathbf{Sh(X)}$ to $\mathbf{Sh(Y)}$ and f^{-1} is a functor from $\mathbf{Sh(Y)}$ to $\mathbf{Sh(X)}$. For any sheaf $\mathcal{F}$ on X there is a natural map of sheaves $f^{-1}f_*\mathcal{F} \to \mathcal{F}$, and for any sheaf $\mathcal{G}$ on Y there is a natural (i.e. functorial) map of sheaves $\mathcal{G} \to f_*f^{-1}\mathcal{G}$. Using these maps one can show that there is a natural bijection of sets, for any sheaf $\mathcal{F}$ on X and any sheaf $\mathcal{G}$ on Y,

$$Hom_X(f^{-1}\mathcal{G}, \mathcal{F}) = Hom_Y(\mathcal{G}, f_*\mathcal{F})$$

This proves that f^{-1} is a left adjoint of f_* and that f_* is a right adjoint of f^{-1}.

Let $Z \subseteq X$ is a subspace of X and let $\mathcal{F}$ be a sheaf on X. Let i denote the inclusion map $Z \to X$. Then we denote the sheaf $i^{-1}\mathcal{F}$ by $\mathcal{F}|_Z$ and call it the *restriction* of $\mathcal{F}$ to Z.

Definition 18. Let $\mathcal{F}$, $\mathcal{G}$ be sheaves of abelian groups on X.

1. For any open set $U \subseteq X$ the set $Hom(\mathcal{F}|_U, \mathcal{G}|_U)$ of morphisms of the restricted sheaves has a natural structure of an abelian group. One can show that the presheaf $U \mapsto Hom(\mathcal{F}|_U, \mathcal{G}|_U)$ is a sheaf, which is defined to be the *sheaf hom* of $\mathcal{F}$ to $\mathcal{G}$. It is denoted by $\mathcal{H}om(\mathcal{F}, \mathcal{G})$.
2. The presheaf $U \mapsto \mathcal{F}(U) \oplus \mathcal{G}(U)$ is a sheaf. It is called the *direct sum* of $\mathcal{F}$ and $\mathcal{G}$ and is denoted by $\mathcal{F} \oplus \mathcal{G}$. It plays the role of direct sum and direct product in the category of sheaves of abelian groups on X.
3. Let $\{\mathcal{F}_i\}$ be a direct system of sheaves and morphisms on X. The *direct limit* of $\{\mathcal{F}_i\}$, denoted by $\varinjlim \mathcal{F}_i$, is defined to be the sheaf associated to the presheaf $U \mapsto \varinjlim \mathcal{F}_i(U)$. This is a direct limit in the category $\mathbf{Sh(X)}$ i.e., it has the following universal property: given a sheaf $\mathcal{G}$ and a collection of morphisms $\mathcal{F}_i \to \mathcal{G}$ compatible with the maps of the direct system, there exists a unique map $\varinjlim \mathcal{F}_i \to \mathcal{G}$ such that the original map $\mathcal{F}_i \to \mathcal{G}$ is obtained by composing $\mathcal{F}_i \to \varinjlim \mathcal{F}_i \to \mathcal{G}$. If X is a *noetherian* topological space, then the presheaf $U \mapsto \varinjlim \mathcal{F}_i(U)$ is already a sheaf (a topological space is *noetherian* if it satisfies the descending chain condition for closed subsets, this notion will be discussed in more detail later on). In particular, $\Gamma(X, \varinjlim \mathcal{F}_i) = \varinjlim \Gamma(X, \mathcal{F}_i)$.
4. Let $\{\mathcal{F}_i\}$ be an inverse system of sheaves on X. The presheaf $U \mapsto \varprojlim \mathcal{F}_i(U)$ is already a sheaf. It is called the *inverse limit* of the system $\{\mathcal{F}_i\}$ and is denoted by $\varprojlim \mathcal{F}_i$. It has the universal property of an inverse limit in the category $\mathbf{Sh(X)}$.
5. Let $\mathcal{F}$, $\mathcal{G}$ be sheaves of abelian groups on X. Then the sheaf associated to the presheaf $U \mapsto \mathcal{F}(U) \otimes_{\mathbb{Z}} \mathcal{G}(U)$ is called the *tensor product* of $\mathcal{F}$

and $\mathcal{G}$. It satisfies the universal property of a tensor product in the category $\mathbf{Sh(X)}$.

For any open subset $U \subseteq X$ the functor $\Gamma(U, .)$ from $\mathbf{Sh(X)}$ to the category of abelian groups, $\mathbf{Ab}$, is a left exact functor. This means that if $0 \to \mathcal{F}^{'} \to \mathcal{F} \to \mathcal{F}^{''}$ is an exact sequence of sheaves then $0 \to \Gamma(U, \mathcal{F}^{'}) \to \Gamma(U, \mathcal{F}) \to \Gamma(U, \mathcal{F}^{''})$ is an exact sequence of groups. The functor $\Gamma(U, .)$ as has been shown earlier need not be exact (as surjectivity may fail). We also mention in this context, that the functors $Hom(\mathcal{F}, \cdot)$ and f_* from the category of sheaves to itself are left exact whereas the tensor functor $\mathcal{F} \otimes \cdot$ from the category of sheaves to itself is right exact.

Definition 19. A sheaf $\mathcal{F}$ on a topological space X is said to be *flasque* (or *flabby*) if for every inclusion $V \subseteq U$ of open sets, the restriction map $\mathcal{F}(U) \to \mathcal{F}(V)$ is surjective. A constant sheaf on an irreducible topological space is flasque.

Some of the properties of flasque sheaves are given below. If $\mathcal{F}^{'}$ is a flasque sheaf and there is a short exact sequence of sheaves $0 \to \mathcal{F}^{'} \to \mathcal{F} \to \mathcal{F}^{''} \to 0$ then for any open set $U \subseteq X$ the sequence of abelian groups $0 \to \mathcal{F}^{'}(U) \to \mathcal{F}(U) \to \mathcal{F}^{''}(U) \to 0$ is also exact. If $\mathcal{F}^{'}$ and $\mathcal{F}$ are flasque sheaves in the above short exact sequence then so is $\mathcal{F}^{''}$. Finally, if $f : X \to Y$ is a continuous map then it is trivial to observe that $f_*\mathcal{F}$ is flasque if $\mathcal{F}$ is flasque. All these properties are easy to prove except the first one. To prove the first one note that the only thing we have to prove is that the map $\mathcal{F}(U) \to \mathcal{F}^{''}(U)$ is surjective. Let $\alpha \in \mathcal{F}^{''}(U)$ be a section over U. We know that sections can be locally lifted so let $V \subseteq U$ be an open subset and $\beta \in \mathcal{F}(V)$ such that β maps to $\alpha|_V$. If $V = U$ we are done so assume $V \neq U$. Let $p \in U$ be a point such that p is not in V. Then there exists an open subset $W \subseteq U$ such that $p \in W$ and a section $\gamma \in \mathcal{F}(U)$ which lifts α over W. If we can show that β and γ can be modified so that they patch over $V \cap W$ and still map to α then we have extended the lift of the section α to a bigger set and we are done by using Zorn's Lemma. So it remain to show that β and γ can be modified to patch over $V \cap W$ and this is where we use the fact that $\mathcal{F}^{'}$ is flasque. Consider the section $\delta = \beta - \gamma$ over $V \cap W$. This can be considered as a section of $\mathcal{F}^{'}(V \cap W)$ and hence can be lifted by the flasque property to a section θ over $\mathcal{F}^{'}(W)$. Now consider the two sections $\beta \in \mathcal{F}(V)$ and $(\gamma + \theta) \in \mathcal{F}(W)$. It is easy to see that they patch and map to α so we are done.

Definition 20. Let $\mathcal{F}$ be a sheaf on X. The *sheaf of discontinuous sections* of $\mathcal{F}$, denoted by $\mathcal{C}^0(\mathcal{F})$, is defined as follows. For each open set $U \subseteq X$ $\mathcal{C}^0(\mathcal{F})(U)$ is the set of maps $s : U \to \cup_{p \in U} \mathcal{F}_P$ such that for each $P \in U$, $s(P) \in \mathcal{F}_P$. There is a natural injective map $\mathcal{F} \to \mathcal{C}^0(\mathcal{F})$. It is also clear that $\mathcal{C}^0(\mathcal{F})$ is a flasque sheaf.

Definition 21. Given a sheaf $\mathcal{F}$ on X there exists a canonical resolution of $\mathcal{F}$ by flasque sheaves called the *Godemont resolution.* In other words, there exists an exact sequence to sheaves $0 \to \mathcal{F} \to \mathcal{C}^0(\mathcal{F}) \to \mathcal{C}^1(\mathcal{F}) \to ...$ where $\mathcal{C}^i(\mathcal{F})$, for each i, is a flasque sheaf. The Godemont resolution is defined as follows. $\mathcal{C}^0(\mathcal{F})$ is the sheaf of discontinuous sections of $\mathcal{F}$ as defined above. Define $\mathcal{C}^1(\mathcal{F})$ to be the sheaf of discontinuous sections of the sheaf $\mathcal{G}^0$ where $\mathcal{G}^0$ is the cokernel of the morphism $\mathcal{F} \to \mathcal{C}^0(\mathcal{F})$. Note that there is natural map $\mathcal{C}^0(\mathcal{F}) \to \mathcal{C}^1(\mathcal{F})$ which is the composition of the maps $\mathcal{C}^0(\mathcal{F}) \to \mathcal{G}^0 \to \mathcal{C}^1(\mathcal{F})$. We now define $\mathcal{C}^i(\mathcal{F})$ inductively by the same procedure. Assume that $\mathcal{C}^j(\mathcal{F})$ has been defined for all $j < i$. Let $\mathcal{G}^{i-1}$ be the cokernel of the map $\mathcal{C}^{i-2}(\mathcal{F}) \to \mathcal{C}^{i-1}(\mathcal{F})$. Then $\mathcal{C}^i(\mathcal{F})$ is defined to be the sheaf of discontinuous sections of the sheaf $\mathcal{G}^{i-1}$ and there is a natural map $\mathcal{C}^{i-1}(\mathcal{F}) \to \mathcal{C}^i(\mathcal{F})$. The way the sequence is constructed it is obvious that it is exact at every stage. It is also clear by construction that each $\mathcal{C}^i(\mathcal{F})$ is a flasque sheaf.

Definition 22. Let $\mathcal{F}$ be a sheaf on X. Let $s \in \mathcal{F}(U)$ be a section over an open set U. The *support* of s, denoted by $Supp(s)$, is defined to be the set $\{P \in U | s_P \neq 0\}$ where s_P denotes the germ of s in the stalk $\mathcal{F}_P$. $Supp(s)$ is a closed subset of U (because $s_P = 0$ is by definition an open condition). We define the *support* of $\mathcal{F}$, denoted by $Supp(\mathcal{F})$, to be the set $\{P \in U | \mathcal{F}_P \neq 0\}$. This need not be a closed subset of X.

We define skyscraper sheaves to illustrate the notion of support. Skyscraper sheaves are so named because they are supported only on the closure of a point.

Definition 23. Let X be a topological space, let P be a point, and let G be an abelian group. The *skyscraper sheaf* on X associated to G and supported on $\bar{P}$ (the closure of the point P) is denoted by $i_P(G)$ and is defined by $i_P(G)(U) = G$ if $P \in U$ and 0 otherwise. The stalk of $i_P(G)$ is G at every point $Q \in \bar{P}$ and is 0 elsewhere. If $i : \bar{P} \to X$ denotes the inclusion map then the sheaf $i_P(G)$ could also have been defined as $i_*(G_P)$ where G_P denotes the constant sheaf on P associated to the group G.

We can also talk about sections having support in a closed subset. This will later lead to the definition of cohomology with supports.

Definition 24. Let Z be a closed subset of X and let $\mathcal{F}$ be a sheaf on X. We define $\Gamma_Z(X, \mathcal{F})$ (also denoted by $H^0_Z(X, \mathcal{F})$ to be the subgroup of $\Gamma(X, \mathcal{F})$ cosisting of all sections whose support is contained in Z. The presheaf $V \mapsto \Gamma_{V \cap Z}(V, \mathcal{F}|_V)$ is actually a sheaf. It is called *the subsheaf of $\mathcal{F}$ with supports in Z* and is denoted by $\mathcal{H}^0_Z(\mathcal{F})$.

If $U = X - Z$ and $j : U \to X$ denotes the inclusion map then there is an exact sequence of sheaves on X

$$0 \to \mathcal{H}^0_Z(\mathcal{F}) \to \mathcal{F} \to j_*(\mathcal{F}|_U).$$

Furthurmore, if $\mathcal{F}$ is flasque then the map $\mathcal{F} \to j_*(\mathcal{F}|_U)$ is surjective.

We now introduce the concept of extending a sheaf by zero.

Definition 25. Let X be a topological space, let Z be a closed subspace of X and let $U = X - Z$ be the complement of Z. Let $i : Z \to X$ and $j : U \to X$ denote the two inclusion maps.

1. Let $\mathcal{F}$ be a sheaf on Z. We call the sheaf $i_*\mathcal{F}$ the sheaf obtained by *extending* $\mathcal{F}$ *by zero* outside Z. This is because the stalk $(i_*\mathcal{F})_P$ is $\mathcal{F}_P$ if $P \in Z$ and 0 if P is not in Z.
2. Let $\mathcal{F}$ be a sheaf on U. Define $j_!(\mathcal{F})$ to be the sheaf on X associated to the presheaf $V \mapsto \mathcal{F}(V)$ if $V \subseteq U$ and $V \mapsto 0$ otherwise. We call $j_!(\mathcal{F})$ to be the sheaf obtained by *extending* $\mathcal{F}$ *by zero* outside U. Again, the stalk $(j_!(\mathcal{F}))_P$ is equal to $\mathcal{F}_P$ if $P \in U$ and is 0 if P is not in U.

If $\mathcal{F}$ is a sheaf on X and Z, U are as above then there is a short exact sequence of sheaves on X,

$$0 \to j_!(\mathcal{F}|_U) \to \mathcal{F} \to i_*(\mathcal{F}|_Z) \to 0$$

This follows by considering the corresponding sequence of stalks and using the properties given in the definition of extension by 0.

We now define the notion of a family of supports, this more general notion will be used when we define cup products in a later section.

Definition 26. Let X be a topological space. A *family of supports* for X is a collection Φ of subsets of X such that

1. if $A \in \Phi$ then A is a closed subset of X,
2. if $A \in \Phi$, $B \subseteq A$ and B is closed then $B \in \Phi$,
3. if $A B \in \Phi$ then $A \cup B \in \Phi$.

ϕ is said to be a *paracompactifying family of supports* if in addition Φ satisfies

1. if $A \in \Phi$ then A is paracompact,
2. if $A \in \Phi$ then there exists $B \in \Phi$ such that $A \subset B^\circ$ where $B^\circ =$ the interior of B.

Examples 1) $\Phi =$ all closed subsets of X.

2) Let X be a Hausdorff space. Then let $\Phi =$ all compact subsets of X. If X is locally compact then Φ is also paracompactifying.

3) Let $Z \subseteq X$ be a closed subset. Let $\Phi = \Phi_Z =$ all closed subsets of Z.

Definition 27. Let X be a topological space, $\mathcal{F}$ be a sheaf of abelian groups on X and let Φ be a family of supports on X. Then the group of sections of $\mathcal{F}$ with *supports in* Φ, $\Gamma_\Phi(X, \mathcal{F})$ (also denoted by $H^0_\Phi(X, \mathcal{F})$) is defined to be the set $\{s \in \mathcal{F}(X) | supp(s) \in \Phi\}$.

Lemma 1. *The functor* $\Gamma_\Phi : \mathbf{Sh(X)} \to \mathbf{Ab}$ *is left exact.*

Proof. □

We will now discuss affine schemes and then ringed spaces. Affine schemes (and varieties) are examples of a somewhat bizarre but important class of spaces called noetherian topological spaces. Since we will be considering mainly such spaces in this workshop we will give the relevant definitions.

Definition 28. A topological space X is called *noetherian* if it satisfies the em descending chain condition for closed subsets: for any sequence $Y_1 \supseteq Y_2 \supseteq \cdots$ of closed subsets of X, there is an integer r such that $Y_r = Y_{r+1} = \cdots$.

Let k be a field and let $\mathbb{A}_k^n = k^n$ be the affine n-space over k. Let $A = k[x_1, \cdots, x_n]$ be the polynomial ring in n variables. A polynomial in n variables can be thought of as a function from $\mathbb{A}_k^n$ to k. A subset Y of $\mathbb{A}_k^n$ is said to be an *algebraic set* if there exists a subset $T \subseteq A$ such that $Y = \{p \in \mathbb{A}_k^n | f(p) = 0 \forall f \in T\}$. We define the *Zariski topology*on $\mathbb{A}_k^n$ by taking the open subsets to be the complements of algebraic sets, it can be easily checked that this forms a toplogy. $\mathbb{A}_k^n$ is a noetherian topological space (this depends on the fact that A is a noetherian ring). It also happens to be the first example of an *affine variety.* A nonempty subset Y of a topological space X is said to be *irreducible* if it cannot be expressed as the union of $Y = Y_1 \cup Y_2$ of two proper subsets, each of which is closed in Y (the empty set by definition is not considered to be irreducible). Now an *affine variety* is an irreducible closed subset of $\mathbb{A}_k^n$ with the induced Zariski topology. All affine varieties are also noetherian topological spaces. It can be easily shown that every open cover of a noetherian topological space has a finite subcover. Noetherian topological spaces are usually never Hausdorff, in fact it is easy to see that a Hausdorff, noetherian toplogical space must be a finite set with the discrete topology. Noetherian topological spaces also satisfy the following property which we write in the form of a proposition.

Proposition 3. In a noetherian topological space X, every nonempty closed subset Y can be expressed as a finite union $Y = Y_1 \cup \cdots \cup Y_r$ of irreducible closed subsets Y_i. If we require that no Y_i contains any other Y_j in this decomposition then the Y_i are uniquely determined. They are called the *irreducible components* of Y.

This immediately yields the following corollary,

Corollary 1. *Every algebraic set in $\mathbb{A}_k^n$ can be expressed uniquely as a union of varieties, no one containing another.*

We lastly define the notion of dimension of a topological space which will be appear in the statement of Grothendieck's theorem.

Definition 29. If X is a topological space, we define the *dimension* of X (denoted by dim X) to be the supremum of all integers n such that there exists a chain $Z_0 \subset Z_1 \subset \cdots \subset Z_n$ of distinct irreducible closed subsets of X. We define the dimension of an affine variety to be it dimension as a topological space.

One can show that the dimension of $\mathbb{A}_k^n$ is n. This follows from the fact that the dimension of an algebraic set Y is the same as the dimension of it's affine coordinate ring $A(Y)$ (to be defined in a moment) and the affine coordinate ring of $\mathbb{A}_k^n$ is $k[x_1, \cdots, x_n]$ whose dimension is n. Now for the definition of the affine coordinate ring of an algebraic set Y. By definition $Y \subset \mathbb{A}^n_k$ for some n. Define the ideal of Y, $I(Y) = \{f \in k[x_1, \cdots, x_n] | f(p) = 0 \forall p \in Y\}$. We define the *affine coordinate ring* of Y to be the ring $k[x_1, \cdots, x_n]/I(Y)$, it can be shown to be independent of the embedding of Y in $\mathbb{A}^n_k$. The affine coordinate ring of an algebraic set is a special kind of a ring, namely it is a finitely generated k-algebra with no nilpotent elements. The affine coordiante ring of an affine variety is in addition an integral domain. The converse of both these statements are also true, i.e. any such ring is the affine coordinate ring of an algebraic set or an affine variety. Given a finitely generated k algebra with no nilpotent elements and under the assumption that k is an algebraically closed field, we can recover the affine algebraic set by considering the maximal ideals of the coordinate ring, these correspond to the points on the variety. We have the notion of a regular function on an affine algebraic set which forms a sheaf of rings on it (as discussed in the examples of sheaves). All these notions will be discussed in full detail in other lectures of this workshop, hence we have been very brief. Now by considering the prime ideals of a general ring (instead of maximal ideals) we are led to our next topic, affine schemes.

We now discuss another example of a noetherian topological space, namely the spectrum of a ring A. We will then define a sheaf of rings on this space and call it an affine scheme and denote it by $\text{Spec}(A)$. This construction parallels the construction of affine varieties (as has been loosely mentioned in the previous paragraph). Let A be a commutative ring with identity. As a set, define $\text{Spec}(A)$ to be the set of all prime ideals of A. Let I be any ideal of A, then we define the subset $V(I) \subseteq \text{Spec}(A)$ to be the subset of all prime ideals which contain I. Next we define a topology on $\text{Spec}(A)$ by taking the subsets of the form $V(I)$ to be the closed subsets. It can again be checked easily that these sets satisfy the axioms of a topology. We now define a sheaf of rings $\mathcal{O}$ on $\text{Spec}(A)$. For every prime ideal $\mathfrak{p} \subseteq A$, let $A_\mathfrak{p}$ be the localization of A at $\mathfrak{p}$. For any open set $U \subseteq \text{Spec}(A)$, we define $\mathcal{O}(U)$ to be the set of functions $s : U \to \coprod_{\mathfrak{p} \in U} A_\mathfrak{p}$ such that $s(\mathfrak{p}) \in A_\mathfrak{p}$ for each $\mathfrak{p}$ and (again for each $\mathfrak{p} \in U$) there is a neighbourhood V of $\mathfrak{p}$ contained in U and elements $a, f \in A$ such that for each $\mathfrak{q} \in V$ we have f not in $\mathfrak{q}$ and $s(\mathfrak{q}) = a/f$ in $A_\mathfrak{q}$ (i.e. s is locally a quotient of elements of A). It is clear then that this is a sheaf of rings on $\text{Spec}(A)$. We give a far cleaner definition of $\mathcal{O}$ by just considering a basis of open sets as follows. To give the data fo a sheaf $\mathcal{F}$ on a space X it is enough to specify $\mathcal{F}(U)$ and restriction maps where U ranges over a basis $\mathcal{B}$ of open subsets of X. The sheaf axioms then force the vales of $\mathcal{F}$ on other open sets and other restriction maps. Now we use the basis of open sets $D(f)$ for $f \in A$ where $D(f)$ consists of those prime ideals of A which do not contain the element f. It can be easily checked that this forms a basis of $\text{Spec}(A)$. Now define $\mathcal{O}(D(f)) = A_f$. If $D(f) \subseteq D(g)$ for $f, g \in A$ then $f^n \in (g)$ for some n, hence there is a natural map $\mathcal{O}(D(g)) = A_g \to A_f = \mathcal{O}(D(f))$ which is our rstriction map. This finishes the cleaner definition of $\mathcal{O}$.

The *spectrum* of a ring A is defined to be the pair consisting of the topological space $\operatorname{Spec}(A)$ together with the sheaf of rings $\mathcal{O}$ defined above. We also note that $\Gamma(\operatorname{Spec}(A), \mathcal{O}) = A$.

Definition 30. 1. A *ringed space* is a pair $(X, \mathcal{O}_X)$ consisting of a topological space X and a sheaf of rings $\mathcal{O}_X$ on X. A *morphism of ringed spaces* from $(X, \mathcal{O}_X)$ to $(Y, \mathcal{O}_Y)$ is a pair $(f, f^{\#})$ of a continuous map $f : X \to Y$ and a map $f^{\#} : \mathcal{O}_Y \to f_*\mathcal{O}_X$ of sheaves of rings on Y.
2. A ringed space $(X, \mathcal{O}_X)$ is a *locally ringed space* if for each point $P \in X$ the stalk $\mathcal{O}_{X,P}$ is a local ring. A *morphism of locally ringed spaces* is a morphism $(f, f^{\#})$ of ringed spaces such that for each $P \in X$ the induced map (discussed below) $f_P^{\#} : \mathcal{O}_{Y,f(P)} \to \mathcal{O}_{X,P}$ is a *local homomorphism* of local rings. The map $f_P^{\#}$ is defined as follows, given $P \in X$ the morphism $f^{\#} : \mathcal{O}_Y \to f_*\mathcal{O}_X$ induces a homomorphism of rings $\mathcal{O}_Y(V) \to \mathcal{O}_X(f^{-1}(V))$ for every open set $V \subseteq Y$. As V ranges over all the open neighbourhoods of $f(P)$, $f^{-1}(V)$ ranges over a subset of the neighbourhoods of P. Taking direct limits we get a map
$$\mathcal{O}_{Y,f(P)} = \varinjlim{}_V \mathcal{O}_Y(V) \to \varinjlim{}_V \mathcal{O}_X(f^{-1}(V))$$
and the latter limit maps to the stalk $\mathcal{O}_{X,P}$. Thus we have an induced map $f_P^{\#} : \mathcal{O}_{Y,f(P)} \to \mathcal{O}_{X,P}$.

We have the following proposition

Proposition 4. 1. If A is a ring, then $(\operatorname{Spec} A, \mathcal{O})$ is a locally ringed space.
2. If $\phi : A \to B$ is a homoorphism of rings, then ϕ induces a natural morphism of locally ringed spaces
$$(f, f^{\#}) : (\operatorname{Spec} B, \mathcal{O}_{\operatorname{Spec}(B)}) \to (\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec}(A)}).$$
3. If A and B are rings, then any morphism of locally ringed spaces from $\operatorname{Spec} B$ to $\operatorname{Spec} A$ is induced by a homomorphism of rings $\phi : A \to B$ as above.

The morphism $f : \operatorname{Spec} B \to \operatorname{Spec} A$ mentioned in the proposition is given by $f(\mathfrak{p}) = \phi^{-1}(\mathfrak{p})$ (for each $\mathfrak{p} \in \operatorname{Spec} B$) which is easily checked to be continuous. Also we can localize ϕ to get local homomorphism of local rings $\phi_{\mathfrak{p}} : A_{\phi^{-1}(\mathfrak{p})} \to B_{\mathfrak{p}}$ which is precisely the map $f^{\#}$ at the level of stalks.

Definition 31. Let $(X, \mathcal{O}_X)$ be a ringed space. A *sheaf of $\mathcal{O}_X$-modules* or an *$\mathcal{O}_X$-module* is a sheaf $\mathcal{F}$ on X, such that for every open set $U \subseteq X$ the group $\mathcal{F}(U)$ is an $\mathcal{O}_X(U)$-module and for each inclusion of open sets $V \subseteq U$ the restriction homomorphism $\mathcal{F}(U) \to \mathcal{F}(V)$ is compatible with the module structures via the ring homomorphism $\mathcal{O}_X(U) \to \mathcal{O}_X(V)$. A *morphism of $\mathcal{O}_X$-modules* $\mathcal{F} \to \mathcal{G}$ is a morphism of sheaves such that for every open set $U \subseteq X$ the map $\mathcal{F}(U) \to \mathcal{G}(U)$ is a homomorphism of $\mathcal{O}_X(U)$-modules. Hence we have defined a category, denoted by $\mathcal{M}od(X)$ of $\mathcal{O}_X$-modules and morphisms between them. $\mathcal{M}od(X)$ will also turn out to be an *abelian* category. A sheaf of abelian groups can be considered as an $\mathcal{O}_X$-module by taking $\mathcal{O}_X$ to be the constant sheaf $\mathbb{Z}_X$.

As an example of sheaves of modules we briefly discuss the case of sheaves of modules on Spec(A). Let A be a commutative ring with identity as before and let M be an A-module. One can define a sheaf of modules on $\operatorname{Spec} A$, $\tilde{M}$, which is associated to M. This is done as follows. For each prme ideal $\mathfrak{p}$ of A let $M_{\mathfrak{p}}$ be the localization of M at $\mathfrak{p}$ which will be the stalks of $\tilde{M}$ at $\mathfrak{p}$. For any basic open set of $\operatorname{Spec} A$ of the form $D(f)$ for $f \in A$ define $\tilde{M}(D(f) = M_f$ the localization of M at the multiplicative set $\{1, f, f^2, \cdots\}$. This is enough to specify the sheaf of modules $\tilde{M}$ by an earlier comment. In particular, $\Gamma(\operatorname{Spec} A, \tilde{M}) = M$.

The kernel, cokernel and image of a morphism of a sheaf of $\mathcal{O}_X$-modules is again a $\mathcal{O}_X$-module. If $\mathcal{F}^{'}$ is a subsheaf of $\mathcal{O}_X$-modules of an $\mathcal{O}_X$-module $\mathcal{F}$ then the quotient sheaf $\mathcal{F}/\mathcal{F}^{'}$ is an $\mathcal{O}_X$-module. Any direct sum, direct product, direct limit or inverse limit of $\mathcal{O}_X$-modules is an $\mathcal{O}_X$-module. If $\mathcal{F}$ and $\mathcal{G}$ are two $\mathcal{O}_X$-modules then the group of $\mathcal{O}_X$-module morphisms from $\mathcal{F}$ to $\mathcal{G}$ is denoted by $Hom_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$.

Definition 32. 1. A sequence of $\mathcal{O}_X$-modules and morphisms is *exact* if it is exact as a sequence of sheaves of abelian groups.

2. If $\mathcal{F}$ and $\mathcal{G}$ are two $\mathcal{O}_X$-modules then the presheaf $U \mapsto Hom_{\mathcal{O}_X|_U}(\mathcal{F}|_U, \mathcal{G}|_U)$ is a sheaf called the *sheaf* $\mathcal{H}om$ and is denoted by $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$. This is also an $\mathcal{O}_X$-module.
3. The *tensor product* of two $\mathcal{O}_X$-modules is defined to be the sheaf associated to the presheaf $U \mapsto \mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U)$ and is denoted by $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$.
4. An $\mathcal{O}_X$-module $\mathcal{F}$ is said to be *free* if it is isomorphic to a direct sum of copies of $\mathcal{O}_X$. $\mathcal{F}$ is said to be *locally free* if X can be covered by open sets U such that $\mathcal{F}|_U$ is a free $\mathcal{O}_X|_U$-module. If $\mathcal{F}$ is locally free then the *rank* of $\mathcal{F}$ on such an open set is the number of copies of $\mathcal{O}_X$'s needed (finite or infinite). If X is connected then the rank of a locally free sheaf is constant. A locally free sheaf of rank one is called *invertible*.
5. A *sheaf of ideals* on X is defined to be a sheaf of $\mathcal{O}_X$-modules $\mathcal{I}$ which is a subsheaf of $\mathcal{O}_X$. Hence for every open set U, $\mathcal{I}(U)$ is an ideal of $\mathcal{O}_X(U)$.

Definition 33. 1. Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. If $\mathcal{F}$ is an $\mathcal{O}_X$-module, then $f_*\mathcal{F}$ is an $f_*\mathcal{O}_X$-module. Since there exists a morphism $f^{\#} : \mathcal{O}_Y \to f_*\mathcal{O}_X$ of sheaves of rings on Y this gives $f_*\mathcal{F}$ a natural structure of an $\mathcal{O}_Y$-module. This is called the *direct image* of $\mathcal{F}$ by the morphism f.

2. Let $\mathcal{G}$ be a sheaf of $\mathcal{O}_Y$-modules. Then $f^{-1}\mathcal{G}$ is an $f^{-1}\mathcal{O}_Y$-module. By the adjointness property of f^{-1} there exists a morphism $f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$ of sheaves of rings on X. The *inverse image* of $\mathcal{G}$, $f^*\mathcal{G}$ is defined to be the tensor product sheaf $f^{-1}\mathcal{G} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X$. Thus $f^*\mathcal{G}$ is an $\mathcal{O}_X$-module.

One can show that the functors f^* and f_* are adjoint functors between the categories of $\mathcal{O}_X$-modules and $\mathcal{O}_Y$-modules. Hence for any $\mathcal{O}_X$-module $\mathcal{F}$ and for any $\mathcal{O}_Y$-module $\mathcal{G}$, there is a natural isomorphism of groups

$$Hom_{\mathcal{O}_X}(f^*\mathcal{F}, \mathcal{G}) \cong Hom_{\mathcal{O}_Y}(\mathcal{F}, f_*\mathcal{G}).$$

We have the following proposition about sheaves of modules on $\operatorname{Spec} A$.

Proposition 5. Let $A \to B$ be a homomorphism of rings and let $f : \operatorname{Spec} B \to \operatorname{Spec} A$ be the corresponding map at the level of Specs. Then,

1. the map $M \to \tilde{M}$ is an exact, fully faithful functor from the category of A-modules to the category of $\mathcal{O}_{\operatorname{Spec}(A)}$ modules,
2. if M and N are two A-modules, then $(M \otimes_A N)\tilde{} \cong \tilde{M} \otimes_{\mathcal{O}_{\operatorname{Spec}(A)}} \tilde{N}$
3. if $\{M_i\}$ is any family of A-modules, then $(\oplus M_i)\tilde{} \cong \oplus \tilde{M}_i$,
4. for any B-module N we have $f_*(\tilde{N}) \cong ({}_AN)\tilde{}$ where ${}_AN$ means N considered as an A-module,
5. for any A-module M we have $f^*(\tilde{M}) \cong (M \otimes_A B)\tilde{}$.

Cohomology of Sheaves

We first state some results from homological algebra before going into the definition of sheaf cohomology.

Definition 34. An *exact* category $\mathcal{C}$ is a category with zero objects, kernels, cokernels and such that the natural map $Coim(f) \to Im(f)$, for f any morphism between two objects of $\mathcal{C}$, is an isomorphism.

Definition 35. An *additive* category $\mathcal{C}$ is a category with a zero object such that

1. for any pair (X, Y) of objects of $\mathcal{C}$, $Hom_{\mathcal{C}}(X, Y)$ has a structure of an abelian group and the composite law is bilinear with respect to this group structure.
2. $\mathcal{C}$ has direct sums.

Definition 36. A functor $F : \mathcal{C} \to \mathcal{D}$ between two additive categories $\mathcal{C}$ and $\mathcal{D}$ is said to be *additive* if for any pair of objects (X, Y) of $\mathcal{C}$ the map induced by F on the morphisms, $Hom_{\mathcal{C}}(X, Y) \to Hom_{\mathcal{D}}(F(X), F(Y))$ is a homomorphism of groups.

Definition 37. A category $\mathcal{C}$ is said to be *abelian* if it is both exact and additive.

Examples The following are abelian categories

1. The category of abelian groups, **Ab**.
2. The category of modules over a fixed ring A, denoted by $\mathcal{M}od(A)$.
3. The category of sheaves of abelian groups, $\mathbf{Sh(X)}$, on a topological space X.
4. The category of sheaves of $\mathcal{O}_X$-modules, $\mathcal{M}od(X)$, on a ringed space $(X, \mathcal{O}_X)$

Definition 38. Let $\mathcal{C}$ be an abelian category. A *complex* $A^{\cdot}$ in $\mathcal{C}$ is a collection of objects A^i, $i \in \mathbb{Z}$ and morphisms $d^i : A^i \to A^{i+1}$ such that $d^{i+1} \cdot d^i = 0$ for all i. If the objects A^i are specified only in a certain range then we set $A^i = 0$ for all other i. A *morphism* of complexes $f : A^{\cdot} \to B^{\cdot}$ is a set of morphisms $f^i : A^i \to B^i$ for each i, which commute with the coboundary maps d^i.

Definition 39. The ith *cohomology object* $h^i(A^{\cdot})$ of the complex $A^{\cdot}$ is defined to be $ker(d^i)/im(d^{i-1})$.

If $f : A^{\cdot} \to B^{\cdot}$ is a morphism of complexes then f induces a natural map $h^i(f) : h^i(A^{\cdot}) \to h^i(B^{\cdot})$. If $0 \to A^{\cdot} \to B^{\cdot} \to C^{\cdot} \to 0$ is a short exact sequence of complexes then there are natural functorial maps $\delta^i : h^i(C^{\cdot}) \to h^{i+1}(A^{\cdot})$ (called *connecting homomorphisms*) giving rise to a long exact sequence

$$\cdots \to h^i(A^{\cdot}) \to h^i(B^{\cdot}) \to h^i(C^{\cdot}) \xrightarrow{\delta^i} h^{i+1}(A^{\cdot}) \to \cdots$$

Definition 40. Two morphisms of complexes $f, g : A^{\cdot} \to B^{\cdot}$ are said to be *homotopic* (written as $f\ g$) if there is a collection of morphisms $k^i : A^i \to B^{i-1}$ for each i such that $f - g = dk + kd$. The collection of morphisms $k = (k^i)$ is called a *homotopy operator.*

Lemma 2. *Two homotopic morphisms f and g induce the same morphism on the cohomology objects $h^i(A^{\cdot}) \to h^i(B^{\cdot})$ for every i.*

Definition 41. A covariant functor $F : \mathcal{C} \to \mathcal{D}$ from an abelian category to another is said to be *left exact* if F is an additive functor and if for every short exact sequence

$$0 \to A^{'} \to A \to A^{''} \to 0$$

in $\mathcal{C}$, the sequence

$$0 \to FA^{'} \to FA \to FA^{''}$$

is exact in $\mathcal{D}$. If we write a 0 on the right instead of the left we say that F is *right exact.* If F is both left and right exact, then we say that it is *exact.* If only the middle part $FA^{'} \to FA \to FA^{''}$ is exact then we say that F is *exact in the middle.* For a contravariant functor, we have analogous definitions. For example, a contravariant functor $F : \mathcal{C} \to \mathcal{D}$ is *left exact* if F is additive and for every short exact sequence as above, the sequence

$$0 \to FA^{''} \to FA \to FA^{'}$$

is exact in $\mathcal{D}$.

Examples If $\mathcal{C}$ is an abelian category and A is a fixed object of $\mathcal{C}$ then the functor $Hom(A, .)$ defined by $B \mapsto Hom(A, B)$ is a covariant left exact functor from $\mathcal{C}$ to **Ab** and the functor $Hom(., A)$ defined by $B \mapsto Hom(B, A)$ is a contravariant left exact functor from $\mathcal{C}$ to $\mathcal{A}b$.

Definition 42. An object I in an abelian category $\mathcal{C}$ is said to be *injective* if the functor $Hom(., I)$ is an exact functor. An *injective resolution* of an object A of $\mathcal{C}$ is a complex $I^{\cdot}$, defined in degrees $i \geq 0$, together with a morphism $\epsilon : A \to I^0$, such that I^i is an injective object of $\mathcal{C}$ for each $i \geq 0$ and such that the sequence

$$0 \to A \xrightarrow{\epsilon} I^0 \to I^1 \to \cdots$$

is exact.

Definition 43. An abelian category $\mathcal{C}$ is said to have *enough injectives* if every object of $\mathcal{C}$ is isomorphic to a subobject of an injective object.

Problem 3.
1. Let **Ab** be the category of abelian groups. Show that an abelian group A if and only if A is a divisible abelian group. Show that **Ab** has enough injectives.
2. Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor between two abelian categories. Assume that F has a left adjoint G such that G is an exact functor. Then show that F preserves injectives.
3. In the previous situation assume that $\mathcal{A}$ has enough injectives. Assume furthur that for each $B \in \mathcal{B}$ the natural map $B \to FGB$ is a monomorphism. Then prove that $\mathcal{B}$ has enough injectives.

If $\mathcal{C}$ has enough injectives then every object has an injective resolution. Furthurmore, it can be proved that any two injective resolutions are homotopy equivalent.

Proposition 6. Let $0 \to A \to C.$ be a resolution of A and let $0 \to B \to I_0 \to I_1 \to \cdots$ be a complex with I_j injective for all j (here all objects belong to an underlying abelian category $\mathcal{C}$). Then any map $f : A \to B$ extends to a chain map $f. : C. \to I.$ which is unique upto homotopy.

By a complex in the following corollary we mean one which is bounded below.

Corollary 2.
1. *Any map from an exact complex to a complex of injectives is* null-homotopic *(i.e. is homotopic to the zero morphism). In particular, an exact complex of injectives is* contractible *(i.e. the identity and zero maps are homotopic).*
2. *Any two injective resolutions of an object are homotopy equivalent.*
3. *If $0 \to A \to I.$ and $0 \to B \to J.$ are injective resolutions of A and B respectively and if $f : A \to B$ then there exists $f. : I. \to J.$ which is compatible with f and which is unique upto homotopy.*

Definition 44. Let $\mathcal{C}$ be an abelian category with enough injectives and let $F : \mathcal{C} \to \mathcal{D}$ be a covariant left exact functor to another abelian category. We define the *right derived functors* R^iF (for $i \geq 0$) of F as follows. For each object A of $\mathcal{F}$ choose and fix an injective resolution $I^\cdot$ of A. Then we define $R^iF(A) = h^i(F(I^\cdot))$.

Theorem 1. *Let $\mathcal{C}$ be an abelian category with enough injectives and let $F : \mathcal{C} \to \mathcal{D}$ be a covariant left exact functor to another abelian category $\mathcal{D}$. Then,*

1. *For each $i \geq 0$, R^iF as defined above is an additive functor from $\mathcal{C}$ to $\mathcal{D}$. Furthurmore, it is independent (upto natural isomorphisms of functors) of the choices of the injective resloutions made.*
2. *There is a natural isomorphism $F \cong R^0F$.*
3. *For each short exact sequence in $\mathcal{C}$, $0 \to A \to B \to C \to 0$ and for each $i \geq 0$ there is a natural morphism $\delta^i : R^iF(C) \to R^{i+1}F(A)$ (again called* connecting homomorphisms*) such that we obtain a long exact sequence*

$$\cdots \to R^iF(A) \to R^iF(B) \to R^iF(C) \xrightarrow{\delta^i} R^{i+1}F(A) \to \cdots$$

4. *Given a morphism between two short exact sequences in $\mathcal{C}$*

$$\begin{array}{ccccccccc} 0 & \to & A & \to & B & \to & C & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & A' & \to & B' & \to & C' & \to & 0 \end{array}$$

the δ's give a commutative diagram

$$\begin{array}{ccc} R^iF(C) & \xrightarrow{\delta^i} & R^{i+1}F(A) \\ \downarrow & & \downarrow \\ R^iF(C') & \xrightarrow{\delta^i} & R^{i+1}F(A') \end{array}$$

5. *For each injective object I of $\mathcal{C}$ and for each $i > 0$ we have that $R^iF(I) = 0$.*

Proof. For every object A of $\mathcal{C}$ choose and fix an injective resolution $A \to I^{\cdot}_A$ of A. As above we define $R^iF(A) = h^i(F(I^{\cdot}_A))$. If A, B are objects of $\mathcal{C}$ and $f : A \to B$ a morphism then there exists a chain map $f^{\cdot} : I^{\cdot}_A \to I^{\cdot}_B$ which is compatible with f and unique upto homotopy. Hence $F(f^{\cdot}) : F(I^{\cdot}_A) \to F(I^{\cdot}_B)$ is welldefined upto homotopy and we thus have a map $R^iF(f) : R^iF(A) \to R^iF(B)$. It is easy to check that R^i is an additive functor for every i. It is independent of the choices made follows from the properties of injective resolutions stated before. Also it is easy to see that there exists a natural transformation $F \to R^0F$ which is an isomorphism by using the fact that F is left exact. Next, if J is an injective object of $\mathcal{C}$ then $0 \to J \to I^{\cdot}_J$ is an exact complex of injectives and hence is contractible. This means that $R^iF(J) = 0$ for all $n > 0$. So to finish the proof of the thoerem we have to show that if $0 \to A \to B \to C \to 0$ is a short excat sequence in $\mathcal{C}$ then there exists a functorial long exact sequence

$$0 \to R^0F(A) \to R^0F(B) \to R^0F(C) \to R^1F(A) \to \cdots$$

To prove this we first claim that there exists the following commutative diagram with exact rows and columns.

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & A & \to & I_A^0 & \to & I_A^1 & \to & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & B & \to & I_A^0 \oplus I_C^0 & \to & I_A^1 \oplus I_C^1 & \to & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & C & \to & I_C^0 & \to & I_C^1 & \to & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

We see that all the columns of this diagram except the left most one are split exact. Hence there exists an exact sequence of complexes

$$0 \to F(I_A^\cdot) \to F(I_B^\cdot) \to F(I_C^\cdot) \to 0.$$

This yields a long exact sequence of cohomology objects which we claim is the long exact sequence in the statement of this theorem. To see this we observe that there exists a chain homotopy equivalence

$$I_A^\cdot \oplus I_C^\cdot \to I_B^\cdot$$

such that the $I_A^\cdot \to I_A^\cdot \oplus I_C^\cdot \to I_B^\cdot$ is compatible with the map $A \to B$

and there exists another chain homotopy equivalence

$$I_B^\cdot \to I_A^\cdot \oplus I_C^\cdot$$

such that $I_B^\cdot \to I_A^\cdot \oplus I_C^\cdot \to I_C^\cdot$ is compatible with the map $B \to C$.

Hence we have the long exact sequence

$$0 \to R^0F(A) \to R^0F(B) \to R^0F(C) \to R^1F(A) \to \cdots$$

The whole process is functorial hence given a commutative diagram of two short exact sequences, the induced maps commute with the boundary maps. This finishes the proof of the theorem. □

Definition 45. With $F : \mathcal{C} \to \mathcal{D}$ as in the previous theorem, an object J of $\mathcal{C}$ is called *F-acyclic* if $R^iF(J) = 0$ for all $i > 0$. The last conclusion of the theorem states that injective objects are F-acyclic.

Proposition 7. Let $F : \mathcal{C} \to \mathcal{D}$ as in the theorem. Suppose there is an exact sequence

$$0 \to A \to J^0 \to J^1 \to \cdots$$

where each J^i is F-acyclic ($J^\cdot$ is defined to an *F-acyclic resolution* of A). Then, for each $i > 0$ there is a natural isomorphism $R^iF(A) \cong h^i(F(J^\cdot))$.

There are analogous definitions of *projective objects*, *projective resolutions*, *an abelian category with enough projectives* and the *left derived functors* of a covariant right exact functor. One can also talk about the derived functors of a contravariant functor - the right derived functors of a left exact contravariant functor using projective resolutions and the left derived functors of a right exact contravariant functor using injective resolutions.

We next discuss the notion of a δ-functor and a universal property of derived functors.

Definition 46. Let $\mathcal{C}$ and $\mathcal{D}$ be abelian categories. A *(covariant) δ-functor* from $\mathcal{C}$ to $\mathcal{D}$ is a collection of functors $T = (T^i)_{i\geq 0}$ together with a morphism $\delta^i : T^i(A^{''}) \to T^{i+1}(A^{'})$ for each short exact sequence $0 \to A^{'} \to A \to A^{''} \to 0$ in $\mathcal{C}$ and for each $i \geq 0$ such that:

1. For each short exact sequence in $\mathcal{C}$ as above there is long exact sequence

$$0 \to T^0(A^{'}) \to T^0(A) \to T^0(A^{''}) \xrightarrow{\delta^0} T^1(A^{'}) \to \cdots \to T^i(A) \to T^i(A^{''}) \xrightarrow{\delta^i} T^{i+1}(A^{'}) \to \cdots$$

2. For each morphism of one short exact sequence (as above) into another $0 \to B^{'} \to B \to B^{''} \to 0$, the δ's give a commutative diagram

$$\begin{array}{ccc} T^i(A^{''}) & \xrightarrow{\delta^i} & T^{i+1}(A^{'}) \\ \downarrow & & \downarrow \\ T^i(B^{''}) & \xrightarrow{\delta^i} & T^{i+1}(B^{'}) \end{array}$$

Definition 47. A δ-functor $T = (T^i) : \mathcal{C} \to \mathcal{D}$ is said to be *universal* if given any other δ-functor $S = (S^i) : \mathcal{C} \to \mathcal{D}$ and given any morphism of functors $f^0 : T^0 \to S^0$, there exists a unique sequence of morphisms $f^i : T^i \to S^i$ for each $i \geq 0$ starting with the given f^0 which commute with the δ^i for each short exact sequence. Note that by definition if $F : \mathcal{C} \to \mathcal{D}$ is a covariant additive functor then there can exist at most one (upto unique isomorphism) universal δ-functor T with $T^0 = F$. If such a T exists then the T^i are called the *right satellite* functors of F. A δ-functor $T = (T^i)$ is *augmented over F* if it is a δ-functor $T = (T^i)$ together with a natural transformation $\epsilon : F \to T^0$.

Definition 48. An additive functor $F : \mathcal{C} \to \mathcal{D}$ is said to be *effaceable* if for each object A of $\mathcal{C}$ there is a monomorphism $u : A \to M$ for some M such that $F(u) = 0$. F is said to be *coeffaceable* if for each A there exists an epimorphism $u : N \to A$ such that $F(u) = 0$.

Theorem 2. *Let $T = (T^i)_{i\geq 0}$ be a covariant δ-functor from $\mathcal{C}$ to $\mathcal{D}$. If T^i is effaceable for each $i > 0$ then T is universal.*

Proof. Suppose $(S_i)_{i\geq 0}$ is another δ-functor from $\mathcal{C}$ to $\mathcal{D}$ and $F_0 : T_0 \to S_0$ is a natural transformation. We first construct for any object A of $\mathcal{C}$ a morphism $T_1(A) \to S_1(A)$. Choose a monomorphism $i : A \to M$ such that $T_1(i) = 0$. Then we have the follwoing diagram whose top row is exact and bottom row is a complex.

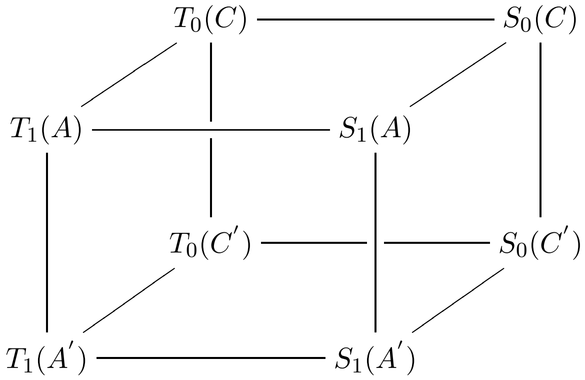
$$\begin{array}{ccccccc} T_0(A) & \to & T_0(M) & \to & T_0(Coker(i)) & \xrightarrow{\delta_0} & T_1(A) \to 0 \\ F_0(A)\downarrow & & F_0(M)\downarrow & & \downarrow F_0(coker(i)) & & \\ S_0(A) & \to & S_0(M) & \to & S_0(coker(i)) & \to & S_1(A) \end{array}$$

Therefore we have a map $T_1(A) \to S_1(A)$ making the above diagram commutative. We now show that this map is independent of the choices made in defining it. Suppose we have a diagram of exact rows

$$\begin{array}{ccccccccc} 0 & \to & A & \xrightarrow{i} & B & \to & C & \to & 0 \\ & & f\downarrow & & \downarrow & & \downarrow g & & \\ 0 & \to & A^{'} & \xrightarrow{i^{'}} & B^{'} & \to & C^{'} & \to & 0 \end{array}$$

where $T_1(i) = T_1(i^{'}) = 0$

Then we have the following commutative diagram (cube2) where all faces except the front face commute.



Since the map $T_0(C) \to T_1(A)$ is onto, hence the front face commutes too. In particular if we take $A = A^{'}$ and $f = Id_A$, then this proves that the two maps $T_1(A) \to S_1(A)$ induced from the two sequences are the same. Finally if $i : A \to B$ and $i^{'} : A \to B^{'}$ are two maps with $T_1(i) = T_1(i^{'}) = 0$ then these two maps can both be compared to the third map $(i, i^{'}) : A \to B \oplus B^{'}$ to prove that the map $T_1(A) \to S_1(A)$ is independent of the choices made in the defining it. Call the map $T_1(A) \to S_1(A)$ as $F_1(A)$.

We next show the functoriality of the map $F_1(A)$. Let $f : A \to A^{'}$ be a map, we want to show that the following diagram (*) commutes

$$\begin{array}{ccc} T_1(A) & \xrightarrow{F_1(A)} & S_1(A) \\ T_1(f)\downarrow & & \downarrow S_1(f) \\ T_1(A^{'}) & \xrightarrow{F_1(A^{'})} & S_1(A^{'}) \end{array}$$

Choose monomorphisms $i : A \to B$ and $i^{'} : A^{'} \to B^{'}$ such that $T_1(i) = T_1(i^{'}) = 0$. We have the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & A & \xrightarrow{(i,i^{'}\cdot f)} & B \oplus B^{'} & \to & coker(i, i^{'} \cdot f) & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & A^{'} & \to & B^{'} & \to & coker(i^{'}) & \to & 0 \end{array}$$

The rows of the above diagram are clearly exact and we also have that $T_1(i, i^{'} \cdot f) = 0$ alongwith $T_1(i^{'}) = 0$. Hence by the earlier computation our claim is true i.e. the diagram (*) commutes.

We now show compatibility with δ, i.e. if $0 \to A \to B \to C \to 0$ is a short exact sequence then we want to show that the following diagram commutes

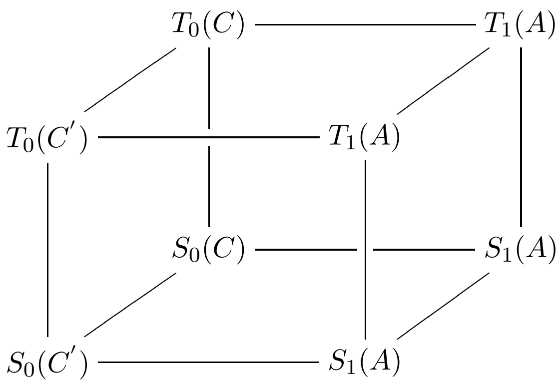
$$\begin{array}{ccc} T_0(C) & \xrightarrow{\delta_0} & T_1(A) \\ F_0(C)\downarrow & & \downarrow F_1(A) \\ S_0(C) & \xrightarrow{\delta_1} & S_1(A) \end{array}$$

Choose a monomorphism $i : B \to B^{'}$ such that $T_1(i) = 0$. Then there exists a commutaive diagram

$$\begin{array}{ccccccccc} 0 & \to & A & \xrightarrow{f} & B & \to & C & \to & 0 \\ & & || & & \downarrow i & & \downarrow & & \\ 0 & \to & A & \xrightarrow{i \cdot f} & B^{'} & \to & C^{'} & \to & 0 \end{array}$$

with exact rows and such that $T_1(i \cdot f) = 0$.

Consider the following diagram (cube3) all of whose faces commute apart from the back face.



Since the map $S_1(A) \to S_1(A)$ is the identity and in particular injective the back face commutes too. Hence we have proved compatibility with δ. We are done by induction as F_i are now constructed step by step. □

Corollary 3. *Assume that $\mathcal{C}$ has enough injectives. Then for any left exact functor $F : \mathcal{C} \to \mathcal{D}$ the derived functors $(R^i F)_{i \geq 0}$ form a universal δ-functor with $F \cong R^0 F$. Conversely, if $T = (T^i)_{i \geq 0}$ is any universal δ-functor then T^0 is left exact, and the T^i are isomorphic to $R^i T^0$ for each $i \geq 0$.*

We now apply all these abstract homological algebra results to define sheaf cohomology. We first show that the categories $\mathcal{M}od(X)$ and $\mathbf{Sh(X)}$ have enough injectives.

Proposition 8. If A is a ring (not necessarily commutative) then every A-module is isomorphic to a submodule of an injective A-module.

Proof. Let $G : \mathbf{A-mod} \to \mathbf{Ab}$ be the forgetful functor. We have earlier mentioned that G is exact and has a right adjoint (exercise) denoted by $F : \mathbf{Ab} \to \mathbf{A-mod}$ where $FM = Hom_{\mathbf{Ab}}(A, M)$ with the A-module structure given by $(af)(s) = f(sa)$. Hence, the category $\mathbf{A-mod}$ has enough injectives. □

Proposition 9. Let $(X, \mathcal{O}_X)$ be a ringed space. Then the category $\mathcal{M}od(X)$ has enough injectives.

Proof. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. For each point $x \in X$ choose an injection $\mathcal{F}_x \to I_x$ where I_x is an injective $\mathcal{O}_{X,x}$-module. Let $\mathcal{I}$ be the sheaf $\prod_{x\in X} j_*(I_x)$ where j denotes the inclusion x in X for each point x and I_x is considered as a sheaf on the one-point space $\{x\}$. We claim that there exists a natural injective map $\mathcal{F} \hookrightarrow \mathcal{I}$ and the sheaf $\mathcal{I}$ is an injective $\mathcal{O}_X$-module. This is because, for any sheaf $\mathcal{G}$ of $\mathcal{O}_X$-modules, we have $Hom_{\mathcal{O}_X}(\mathcal{G}, \mathcal{I}) = \prod Hom_{\mathcal{O}_X}(\mathcal{G}, j_*(I_x)) \cong \prod Hom_{\mathcal{O}_{X,x}}(\mathcal{G}_x, I_x)$. The first equality follows from the definition of the direct product and the second one because j_* is a right adjoint of j^*. Hence we have a natural map $\mathcal{F} \to \mathcal{I}$ by putting together all the maps $\mathcal{F}_x \to I_x$ and this map is clearly injective. Secondly, the functor $Hom_{\mathcal{O}_X}(\cdot, \mathcal{I})$ as shown above is the compositon of two functors, the first one being the functor $\mathcal{G} \mapsto \prod_{x\in X} \mathcal{G}_x$ (which is an exact functor) followed by the functor $Hom_{\mathcal{O}_{x,X}}(\cdot, I_x)$ which too is exact as I_x is an injective $\mathcal{O}_{x,X}$-module. Thus the functor $Hom_{\mathcal{O}}(\cdot, \mathcal{I})$ is exact which is the same as saying $\mathcal{I}$ is an injective $\mathcal{O}_X$-module. This finishes the proof. □

By considering $\mathcal{O}_X$ to be the constant sheaf of rings $\mathbb{Z}$ we have the following corollary.

Corollary 4. *If X is any topological space, then the category* $\mathbf{Sh(X)}$ *of sheaves of abelian groups on X has enough injectives.*

We now define sheaf cohomology.

Definition 49. Let X be any topological space. Let $\Gamma(X, \cdot)$ be the global section functor from $\mathbf{Sh(X)}$ to $\mathbf{Ab}$. We define the *cohomology functors* $H^i(X, \cdot)$ to be the right derived functors of $\Gamma(X, \cdot)$. For any sheaf $\mathcal{F}$, the groups $H^i(X, \mathcal{F})$ are called the *cohomology groups* of $\mathcal{F}$ (this justifies the notation $H^0(X, \mathcal{F})$ made at the beginning for $\mathcal{F}(X)$). Even if X and $\mathcal{F}$ have some additional structure (for example if X is a ringed space and $\mathcal{F}$ is an $\mathcal{O}_X$-module) we always take cohomology groups in this sense regarding $\mathcal{F}$ as a sheaf of abelian groups on a topological space X.

Recall that a sheaf $\mathcal{F}$ on X is said to be *flasque* if for every inclusion of open sets $V \subseteq U$, the restriction maps $\mathcal{F}(U) \to \mathcal{F}(V)$ is surjective.

Lemma 3. *If $(X, \mathcal{O}_X)$ is a ringed space, then any injective $\mathcal{O}_X$ module is flasque.*

Proof. For any open subset $U \subseteq X$ let $\mathcal{O}_U$ denote the sheaf $j_!(\mathcal{O}_X|_U)$ which is the restriction of $\mathcal{O}_X$ to U and then extended by zero outside U to X. Let $\mathcal{I}$ be an injective $\mathcal{O}_X$-module and let $V \subseteq U$ be an inclusion of open sets. We want to show that $\mathcal{I}(U) \to \mathcal{I}(V)$ is surjective. We have an inclusion $\mathcal{O}_V \hookrightarrow \mathcal{O}_U$ of sheaves of $\mathcal{O}_X$-modules. Since $\mathcal{I}$ is injective we have a surjection $Hom(\mathcal{O}_U, \mathcal{I}) \twoheadrightarrow Hom(\mathcal{O}_V, \mathcal{I})$. We are done with the observation that $Hom(\mathcal{O}_U, \mathcal{I}) = \mathcal{I}(U)$ and similarly for V. □

The next proposition proves that flasque sheaves are acyclic for the functor $\Gamma(X, \cdot)$. Hence, for example, we could have used the Godemont resolution of a sheaf to calculate it's cohomology.

Proposition 10. If $\mathcal{F}$ is a flasque sheaf on a topological space X, then $H^i(X, \mathcal{F}) = 0$ for all $i > 0$.

Proof. Let $\mathcal{F}$ be a flasque sheaf of abelian groups. Since $\mathbf{Sh(X)}$ has enough injectives, there exists an injective sheaf $\mathcal{I}$ and an injection $\mathcal{F} \hookrightarrow \mathcal{I}$. Let $\mathcal{G}$ be the quotient, hence we have a short exact sequence of sheaves,

$$0 \to \mathcal{F} \to \mathcal{I} \to \mathcal{G} \to 0.$$

$\mathcal{G}$ is a flasque sheaf since $\mathcal{F}$ and $\mathcal{I}$ are flasque. Since $\mathcal{F}$ is flasque we have a short exact sequence

$$0 \to \Gamma(X, \mathcal{F}) \to \Gamma(X, \mathcal{I}) \to \Gamma(X, \mathcal{G}) \to 0.$$

Also, $H^i(X, \mathcal{I}) = 0$ for $i > 0$ since $\mathcal{I}$ is injective. Now using the long exact sequence of cohomology groups we have, $H^1(X, \mathcal{F}) = 0$ and $H^{i+1}(X, \mathcal{F}) \cong H^i(X, \mathcal{G})$ for $i \geq 1$. Since $\mathcal{G}$ is also flasque we are done by induction. □

If we take an injective resolution of an $\mathcal{O}_X$-module in the category $\mathcal{M}od(X)$ then each term is flasque and hence acyclic (by the above propositions) and so this resolution computes the usual cohomology groups (meaning treating the sheaf as an element of $\mathbf{Sh(X)}$). We write this in the form of a proposition.

Proposition 11. Let $(X, \mathcal{O}_X)$ be a ringed space. Then the derived functors of the functors of the functor $\Gamma(X, \cdot)$ from $\mathcal{M}od(X)$ to $\mathbf{Ab}$ coincide with the cohomology functors $H^i(X, \cdot)$.

Theorem 3. *(Grothendieck's theorem) Let X be a noetherian topological space of dimension n. Then for all $i > n$ and for all sheaves of abelian groups $\mathcal{F}$ on X, we have $H^i(X, \mathcal{F}) = 0$.*

This is analogous to the theorem that singular cohomology on a (real) manifold of dimension n vanishes in degrees $i > n$. To prove the theorem we need some other results which we prove first.

Lemma 4. *Let X be a noetherian topological space. Then, a direct limit of flasque sheaves on X is flasque.*

Proof. Let $\mathcal{F}_i$ be a directed system of sheaves. Since, for each i and for any inclusion of open sets $V \subseteq U$, $\mathcal{F}_i(U) \to \mathcal{F}_i(V)$ is surjective and $\varinjlim$ is an exact functor, we have that $\varinjlim \mathcal{F}_i(U) \to \varinjlim \mathcal{F}_i(V)$ is also surjective. Since X is noetherian, $\varinjlim \mathcal{F}_i(U) = (\varinjlim \mathcal{F}_i)(U)$ for any open set, and so $\varinjlim \mathcal{F}_i$ is a flasque sheaf. $\square$

Proposition 12. Let X be a noetherian topological space and let $(\mathcal{F}_i)$ be a direct system of sheaves of abelian groups. Then there are natural isomorphisms for each $p \geq 0$,

$$\varinjlim H^p(X, \mathcal{F}_i) \to H^p(X, \varinjlim \mathcal{F}_i)$$

i.e. cohomology commutes with direct limits on a noetherian topological space.

Proof. For each i, there is natural map $\mathcal{F}_i \to \varinjlim \mathcal{F}_i$, which induces a map on cohomology and by taking the direct limit of these maps we get the map $\varinjlim H^p(X, \mathcal{F}_i) \to H^p(X, \varinjlim \mathcal{F}_i)$. For $p = 0$ this result has already been proved. We prove the general case by considering the category $\mathcal{I}nd_I(\mathbf{Sh(X)})$, consisting of all directed systems of sheaves of abelian groups on X indexed by I. This is an abelian category. We have a natural tranformation for each p of δ-functors from $\mathcal{I}nd_I(\mathbf{Sh(X)})$ to $\mathbf{Ab}$

$$\varinjlim H^p(X, \cdot) \to H^p(X, \varinjlim \cdot)$$

(They are δ-functors as $\varinjlim$ is an exact functor).

Both of these functors agree for $p = 0$ so to prove that they are the same it is enough to show that both are effaceable δ-functors. So let $\mathcal{F}_i$ be a directed sytem of sheaves of abelian groups on X. For each i, let $\mathcal{C}^0(\mathcal{F}_i)$ be the sheaf of discontinuos sections of $\mathcal{F}_i$. Then for each i there is a natural inclusion $\mathcal{F}_i \hookrightarrow \mathcal{C}^0(\mathcal{F}_i)$ and $\mathcal{C}^0(\mathcal{F}_i)$ is a flasque sheaf. The $\mathcal{C}^0(\mathcal{F}_i)$ also forms a direct system indexed by the same set I and we obtain a monomorphism $u : (\mathcal{F}_i) \to (\mathcal{C}^0(\mathcal{F}_i))$ as objects in the category $\mathcal{I}nd_I(\mathbf{Sh(X)})$. Since $\mathcal{C}^0(\mathcal{F}_i)$ are all flasque sheaves, we have that $H^p(X, \mathcal{C}^0(\mathcal{F}_i)) = 0$ for $p > 0$ and so $\varinjlim H^p(X, \mathcal{C}^0(\mathcal{F}_i)) = 0$ for $p > 0$ which proves the functor on the left hand side is effaceable for $p > 0$. Also, $\varinjlim \mathcal{C}^0(\mathcal{F}_i)$ is a flasque sheaf (as a direct limit of flasque sheaves is flasque on a noetherian topological space) and so $H^p(X, \varinjlim \mathcal{C}^0(\mathcal{F}_i)) = 0$ for all $p > 0$ which proves that the functor on the right hand side is also effaceable. This proves the proposition.

$\square$

The above result shows that in particular cohomology commutes with infinite direct sums. If $Y \subseteq X$ is a closed subset and $\mathcal{F}$ is a sheaf of abelian groups on Y then by definition $H^0(Y, \mathcal{F}) = H^0(X, j_*\mathcal{F})$ where $j : Y \to X$ denotes the inclusion. Now, if $\mathcal{J}^\cdot$ is a flasque resolution of $\mathcal{F}$ on Y then so is $j_*\mathcal{J}^\cdot$ on X. Also for each i, $\Gamma(Y, \mathcal{J}^i) = \Gamma(X, j_*\mathcal{J}^i)$. Hence we have the following proposition.

Proposition 13. Let $Y \subset X$ be a closed subset, $\mathcal{F}$ a sheaf of abelian groups on Y and let $j : Y \hookrightarrow X$ denote the inclusion map. Then for any $i \geq 0$ we have $H^i(Y, \mathcal{F}) = H^i(X, j_*\mathcal{F})$.

We sometimes abuse notation and write $j_*\mathcal{F}$ as $\mathcal{F}$ as sheaf on X. We just showed that there is no difference as far as cohomology groups are concerned.

Proof. (of Grothendieck's Theorem) We will prove the theorem by induction on $n = dim(X)$ in several steps. First note that if $j : Y \hookrightarrow X$ is a closed subset and $i : U \hookrightarrow X$ is the complement of Y then we have a short exact sequence of sheaves (as discussed before)

$$0 \to \mathcal{F}|_U \to \mathcal{F} \to \mathcal{F}|_Y \to 0$$

where $\mathcal{F}|_U = i_!(\mathcal{F}|_U)$ and $\mathcal{F}|_Y = j_*(\mathcal{F}|_Y)$.

Step 1: We may reduce the theorem to the case when X is irreducible. Suppose X is reducible and let Y be one of it's irreducible components. Then by the short exact sequence above, it will be sufficient to prove that for $i \geq n$, both $H^i(X, \mathcal{F}|_Y)$ and $H^i(X, \mathcal{F}|_U)$ are zero. This is true for Y because we are assuming the thoerem to be true for irreducible spaces. It is true for U as $\mathcal{F}$ can be considered as a sheaf on $\bar{U}$ and $\bar{U}$ has one less irreducible component than X (and so by induction on the number of irreducible components and the pervious proposition).

Step 2: The theorem is true for the case when X is irreducible of dimension 0. In this case X is a point and $\Gamma(X, \cdot)$ induces an equivalence of categories between $\mathbf{Sh(X)}$ and $\mathbf{Ab}$. In particular, $\Gamma(X, \cdot)$ is an exact functor and so $H^i(X, \mathcal{F}) = 0$ for all $i > 0$ and for all $\mathcal{F}$.

Step 3: Let X be an irreducible space of dimension n. Let $\mathcal{F}$ be a sheaf on X. Let $B = \cup_{U \subseteq X} \mathcal{F}(U)$ and let A be the set of all finite subsets of B. Then for each $i \in A$, let $\mathcal{F}_i$ be the subsheaf of $\mathcal{F}$ generated by the sections in i over various open subsets of X. A is a directed set and $\mathcal{F} = \varinjlim \mathcal{F}_i$. Hence to prove the theorem for $\mathcal{F}$ it will be enough to prove the theorem for each $\mathcal{F}_i$ as cohomology commutes with direct limits in this situation. If j is a subset of i we have a short exact sequence of sheaves

$$0 \to \mathcal{F}_j \to \mathcal{F}_i \to \mathcal{G} \to 0$$

where $\mathcal{G}$ is sheaf generated by $\#(i - j)$ sections over suitable open sets of X. Now, by induction on $\#(i)$ and the long exact sequence of cohomology we may reduce the case of proving the theorem for $\mathcal{F}$ when it is generated by a single section over some open set $U \subseteq X$. In such a case, $\mathcal{F}$ is a quotient of $\mathbb{Z}_U$, the constant sheaf $\mathbb{Z}$ on U and then extended by zero outside U. Hence we again have a short exact sequence of sheaves

$$0 \to \mathcal{R} \to \mathbb{Z}_U \to \mathcal{F} \to 0$$

where $\mathcal{R}$ is the kernel sheaf. Again using the long exact sequnce for cohomology, it is enough to prove the theorem for $\mathcal{R}$ and $\mathbb{Z}_U$ as above.

Step 4: We prove the theorem for $\mathcal{R}$ as in the previous step. If $\mathcal{R} = 0$ then go to the next step. Else consider $\mathcal{R}_x$ which for each $x \in U$ is a subgroup of

$\mathbb{Z}$. Let d be the least positive integer that occurs in any of these groups $\mathcal{R}_x$. Then, there is a nonempty open subset $V \subseteq U$ such that $\mathcal{R}|_V \cong d \cdot \mathbb{Z}|_V$ as a subsheaf of $\mathbb{Z}|_V$. Thus $\mathcal{R}_V \cong \mathbb{Z}_V$ and there exists a short exact sequence of sheaves

$$0 \to \mathbb{Z}_V \to \mathcal{R} \to \mathcal{R}/\mathbb{Z}_V \to 0.$$

The quotient $\mathcal{R}/\mathbb{Z}_V$ is supported on the closed subet $\bar{U} - V$ of X which has dimension strictly less than n since X is irreducible. So by the induction hypothesis we have $H^i(X, \mathcal{R}/\mathbb{Z}_V) = 0$ for $i \geq n$. So to prove the theorem for $\mathcal{R}$ we have to prove it for $\mathbb{Z}_V$ and use the long exact sequence for cohomology.

Step 5: We prove that $H^i(X, \mathbb{Z}_U) = 0$ for $i > n$ and for any open subset $U \subseteq X$. This will finish the proof of the theorem. Let $Y = X - U$. We have an exact sequence of sheaves

$$0 \to \mathbb{Z}_U \to \mathbb{Z} \to \mathbb{Z}_Y \to 0.$$

Since X is irreducible, $dim(Y) < dim(X)$ and so by the induction hypothesis we have that, $H^i(X, \mathbb{Z}_Y) = 0$ for $i \geq n$. Also, $\mathbb{Z}$ is a flasque sheaf on X as it is a constant sheaf on an irreducible space so $H^i(X, \mathbb{Z}) = 0$ for $i > 0$. Hence using the long exact sequence for cohomology we have the result.

□

Let Z be a closed subset of the topological space X and let $\mathcal{F}$ be a sheaf of abelian groups on X. Recall that $\Gamma_Z(X, \mathcal{F})$ was defined to be the subgroup of sections of $\Gamma(X, \mathcal{F})$ whose support lie in Z. More generally, let Φ be a family of supports on X and recall that we defined $\Gamma_\phi(X, \mathcal{F})$ to be those sections whose support is contained in some element of Φ. We now define the cohomology groups with support in Z and and more generally in Φ. Note that it is easy to see that $\Gamma_Z(X, \cdot)$ and $\Gamma_\Phi(X, \cdot)$ are left exact functors from **Sh(X)** to **Ab**.

Definition 50. The right derived functors of $\Gamma_Z(X, \cdot)$ are denoted by $H^i_Z(X, \cdot)$ and are defined to be the *cohomology groups of X with supports in Z* and coefficients in a given sheaf. Similarly the right derived functors of $\Gamma_\Phi(X, \cdot)$ are defined to be the *cohomology groups with support in* Φ and are denoted by $H^i_\Phi(X, \cdot)$. Note that if $\Phi = \Phi_Z =$ all closed subsets of Z, then $H^i_{\Phi_Z}(X, \cdot) = H^i_Z(X, \cdot)$.

It is true that if $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is a short exact sequence of sheaves and $\mathcal{F}'$ is flasque then

$$0 \to \Gamma_Z(X, \mathcal{F}') \to \Gamma_Z(X, \mathcal{F}) \to \Gamma_Z(X, \mathcal{F}'') \to 0$$

is an exact sequence. More generally, if $\mathcal{F}$ is flasque then $H^i_Z(X, \mathcal{F}) = 0$ for all $i > 0$. Also if $U = X - Z$ then for any sheaf $\mathcal{F}$ on X, then there exists a long exact sequence of cohomology groups

$$0 \to H^0_Z(X, \mathcal{F}) \to H^0(X, \mathcal{F}) \to H^0(U, \mathcal{F}|_U) \to H^1_Z(X, \mathcal{F}) \cdots$$

Other properties of the cohomology groups with supports are excision and the existence of the Mayer-Vietoris sequence. Excision says that if V is an open subset of X containing Z then there are functorial isomorphisms $H^i_Z(X,\mathcal{F}) \cong H^i_Z(V,\mathcal{F}|_V)$ for every i and every $\mathcal{F}$. The Mayer Vietoris sequence is the following long exact sequence:

$$\cdots \to H^i_{Y\cap Z}(X,\mathcal{F}) \to H^i_Y(X,\mathcal{F})\oplus H^i_Z(X,\mathcal{F}) \to H^i_{Y\cup Z}(X,\mathcal{F}) \to H^{i+1}_{Y\cap Z}(X,\mathcal{F})\cdots$$

where Y and Z are two closed subsets of X.

We now define the higher direct images of a sheaf.

Definition 51. Let $f : X \to Y$ be a continuous map of topological spaces. Then the *higher direct image* functors $R^i f_* : \mathbf{Sh(X)} \to \mathbf{Sh(Y)}$ are defined to be the right derived functors of f_*. This makes sense as f_* is left exact and $\mathbf{Sh(X)}$ has enough injectives.

The higher direct image sheaves $R^i f_*\mathcal{F}$ of a sheaf $\mathcal{F}$ is related to the cohomology along the fibres of f.

Proposition 14. For each $i \geq 0$ and each $\mathcal{F} \in \mathbf{Sh(X)}$, $R^i f_*\mathcal{F}$ is the sheaf associated to the presheaf

$$V \mapsto H^i(f^{-1}(V),\mathcal{F}|_{f^{-1}(V)})$$

on Y.

Proof. We need the following lemma.

Lemma 5. *If $\mathcal{I}$ is an injective object of $\mathcal{M}od(X)$ then for any open set $U \subseteq X$, $\mathcal{I}|_U$ is an injective object of $\mathcal{M}od(U)$.*

Assuming the lemma we proceed as follows. Denote the sheaf associated to the above presheaf by $\mathcal{H}^i(X,\mathcal{F})$. The functors $\mathcal{H}^i(X,\cdot)$ form a δ-functor from $\mathbf{Sh(X)}$ to $\mathbf{Sh(Y)}$ since sheafification is an exact functor. So both $\mathcal{H}^i(X,\cdot)$ and $R^i f_*(\cdot)$ are δ-functors from $\mathbf{Sh(X)}$ to $\mathbf{Sh(Y)}$ which agree at $i = 0$ since we have $f_*\mathcal{F} = \mathcal{H}^0(X,\mathcal{F})$ by definition. If we could prove both are universal δ-functors then they would be naturally isomorphic proving the proposition. $R^i f_*$ is a derived functor hence universal. So it is enough to show that $\mathcal{H}^i(X,\cdot)$ is an effaceable (and hence universal) δ-functor. Let $\mathcal{I}$ be an injective sheaf on X. For any open $V \subseteq Y$ we have by the lemma that $\mathcal{I}|_{f^{-1}(V)}$ is an injective sheaf on $f^{-1}(V)$ (we apply the lemma by thinking of X as a ringed space with the constant sheaf $\mathbb{Z}_X$).Hence $\mathcal{H}^i(X,\mathcal{I}) = 0$ for $i > 0$ which proves it is effaceable. So we are through modulo the proof of the lemma.

Proof. (of lemma) Let $j : U \to X$ denote the inlcusion. Then given an inclusion $\mathcal{F} \hookrightarrow \mathcal{G}$ in $\mathcal{M}od(U)$ and given a map $\mathcal{F} \to \mathcal{I}|_U$ we get an inclusion $j_!\mathcal{F} \hookrightarrow j_!\mathcal{G}$ and a map $j_!\mathcal{F} \to j_!(\mathcal{I}|_U)$ where $j_!$ is the extension by zero functor. $j_!(\mathcal{I}|_U)$ is a subsheaf of $\mathcal{I}$ hence by compostion we have a map $j_!\mathcal{F} \to \mathcal{I}$. $\mathcal{I}$ injective implies that this map extends to a map $j_!\mathcal{G} \to \mathcal{I}$. Restricting to U now gives the required extension $\mathcal{G} \to \mathcal{I}|_U$. □

□

The proof of the following corollary is obvious.

Corollary 5. *If $V \subseteq Y$ is an open subset, then*

$$R^i f_*(\mathcal{F})|_V = R^i f'_*(\mathcal{F}|_{f^{-1}(V)})$$

where $f' : f^{-1}(V) \to V$ is the restricted map.

The restriction of a flasque sheaf on an open subset is again flasque so we have the following

Corollary 6. *If $\mathcal{F}$ is a flasque sheaf on X, then $R^i f_*(\mathcal{F}) = 0$ for all $i > 0$.*

Proposition 15. Let $f : X \to Y$ be a morphism of ringed spaces. Then the functors $R^i f_*$ can be calculated on $\mathcal{M}od(X)$ as the derived functors of $f_* : \mathcal{M}od(X) \to \mathcal{M}od(Y)$.

Proof. To calculate the derived functors of f_* in the category $\mathcal{M}od(X)$ one uses resolutions by injective objects in $\mathcal{M}od(X)$. An injective object in $\mathcal{M}od(X)$ is flasque and hence f_*-acyclic so we are done. □

Cup products

We now discuss the notion of cup products on the cohomology of sheaves. Let Φ_1 and Φ_2 be two families of supports in X. Then $\Phi_1 \cap \Phi_2$ is also a family of supports. If $\mathcal{F}$ and $\mathcal{G}$ are sheaves on X then the natural map

$$\mathcal{F}(X) \otimes \mathcal{G}(X) \to (\mathcal{F} \otimes \mathcal{G})(X)$$

induces a map

$$\Gamma_{\Phi_1}(\mathcal{F}) \otimes \Gamma_{\Phi_2}(\mathcal{G}) \to \Gamma_{\Phi_1 \cap \Phi_2}(\mathcal{F} \otimes \mathcal{G})$$

(since $supp(s \otimes t) \subseteq supp(s) \cap supp(t)$). We first define cup products in a general category theoretic set-up. Let $\mathcal{C}$ and $\mathcal{D}$ be abelian categories admitting bilinear maps and tensor products (we do not give the exact definition of *abelian tensor categories* but one can take $\mathcal{C}$ to be $\mathbf{Sh(X)}$ and $\mathcal{D}$ to be $\mathbf{Ab}$). Let (S^i), (T^i) and (U^i) be δ-functors from $\mathcal{C}$ to $\mathcal{D}$. Assume that for every pair of objects (A, B) of the category $\mathcal{C}$ there is a natural map $\eta_{A,B} : S^0(A) \otimes T^0(B) \to U^0(A \otimes B)$. By natural we mean that if (C, D) is another pair of objects of $\mathcal{C}$ and if there are morphisms in $\mathcal{C}$, $f : A \to C$ and $g : B \to D$ then the following diagram commutes

$$\begin{array}{ccc} S^0(A) \otimes T^0(B) & \xrightarrow{\eta_{A,B}} & U^0(A \otimes B) \\ S^0(f) \otimes T^0(g) \downarrow & & \downarrow U^0(f \otimes g) \\ S^0(C) \otimes T^0(D) & \xrightarrow{\eta_{C,D}} & U^0(C \otimes D) \end{array}$$

Definition 52. Let $\mathcal{C}$ and $\mathcal{D}$ be abelian tensor categories and let (S^i), (T^i) and (U^i) be δ-functors from $\mathcal{C}$ to $\mathcal{D}$. Assume the existence of $\eta_{A,B}$ as in the previous paragraph. A *cup product* for these functors and η is a natural map for each pair (A, B) of objects of $\mathcal{C}$ and each pair of non negative integers (p, q), $\eta^{p,q}_{A,B} : S^p(A) \otimes T^q(B) \to U^{p+q}(A \otimes B)$ satisfying the following axioms

1. $\eta^{0,0} = \eta$
2. Let $0 \to A' \to A \to A'' \to 0$ be a short exact sequence in $\mathcal{C}$. Let B be another object of $\mathcal{C}$. Consider the commutative diagram

$$\begin{array}{ccccccc} & A' \otimes B & \to & A \otimes B & \to & A'' \otimes B & \to 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \to & C' & \to & C & \to & C'' & \to 0 \end{array}$$

where $0 \to C' \to C \to C'' \to 0$ is another short exact sequence of objects in $\mathcal{C}$. Then the following diagram is commutative

$$\begin{array}{ccc} S^p(A'') \otimes T^q(B) & \xrightarrow{\delta\otimes 1} & S^{p+1}(A') \otimes T^q(B) \\ \downarrow & & \downarrow \\ U^{p+q}(C'') & \xrightarrow{\delta} & U^{p+q+1}(C') \end{array}$$

where δ denotes the connecting homomorphisms and the vertical maps are cupproducts composed with the maps due to the commutative diagram between the sequences.
3. Let $0 \to B' \to B \to B'' \to 0$ be an exact sequence of objects in $\mathcal{C}$ and let A be a fixed object of $\mathcal{C}$. Consider the commutative diagram

$$\begin{array}{ccccccc} & A \otimes B' & \to & A \otimes B & \to & A \otimes B'' & \to 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \to & C' & \to & C & \to & C'' & \to 0 \end{array}$$

Let w stand for the map $(S^i) \to (S^i)$ which is $(-1)^p$ in dimension p. Then we want the following diagram to commute

$$\begin{array}{ccc} S^p(A) \otimes T^q(B'') & \xrightarrow{w\otimes \delta} & S^p(A) \otimes T^{q+1}(B') \\ \downarrow & & \downarrow \\ U^{p+q}(C'') & \xrightarrow{\delta} & U^{p+q+1}(C') \end{array}$$

where again δ denotes the connecting homomorphisms and the vertical arrows are the compositions of the cupproducts and the maps arising from the maps between the exact sequences.

We now want to prove a theorem which implies that if cup products exist then they are unique but for that we need to assume some extra hypothesis for the category $\mathcal{C}$. We assume that for any object A of $\mathcal{C}$ and the functors (S^i) and (T^i) there is a short exact sequence

$$0 \to A \to P \to Q \to 0$$

in $\mathcal{C}$ (the short exact sequences for (S^i) and (T^i) may be different i.e. there need not exist a common one) such that

1. $S^j(P) = 0$ for $j > 0$ and $T^j(P) = 0$ for $j > 0$ (for the two corresponding sequences, one for the S's and the other for the T's).
2. for any B an object of $\mathcal{C}$ the sequence $0 \to A\otimes B \to P\otimes B \to Q\otimes B \to 0$ is exact.

Note that this assumption is true for $\mathbf{Sh(X)}$ when both S and T are the global section functor.

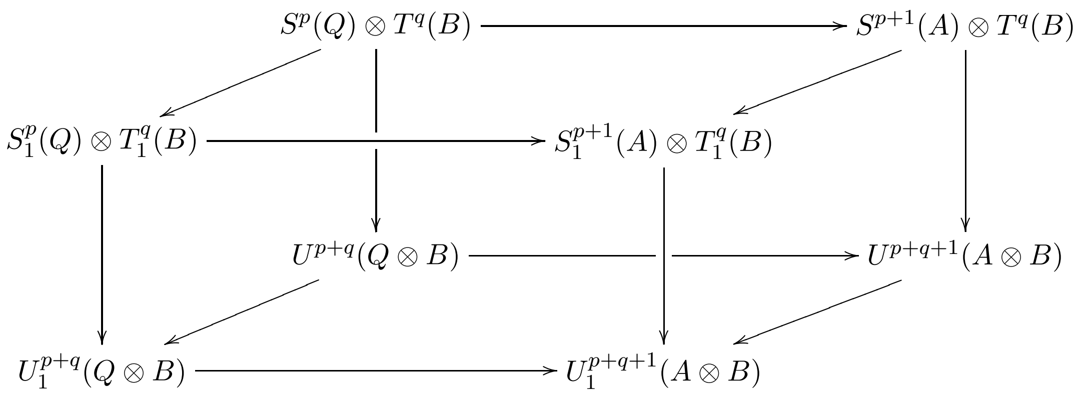
Theorem 4. *Assume that the category $\mathcal{C}$ satisfies the hypothesis in the previous paragraph. Let (S^i), (T^i) and (U^i) be δ-functors (as before) admitting cup products. Let (S_1^i), (T_1^i) and (U_1^i) be another set of δ-functors admitting cup products (denoted by $\zeta_{A,B}$). Let $a : S \to S_1$, $b : T \to T_1$ and $c : U \to U_1$ be natural transformations of functors such that the following diagram commutes (for every pair (A, B) of objects in $\mathcal{C}$),*

$$\begin{array}{ccc} S(A) \otimes T(B) & \xrightarrow{\eta_{A,B}} & U(A \otimes B) \\ \downarrow a \otimes b & & \downarrow c \\ S_1(A) \otimes T_1(B) & \xrightarrow{\zeta_{A,B}} & U_1(A \otimes B \end{array}$$

Then the unique extensions of a, b and c, $(a^i) : (S^i) \to (S_1^i)$, $(b^i) : (T^i) \to (T_1^i)$ and $(c^i) : (U^i) \to (U_1^i)$ preserve cup products i.e. the following diagram commutes (for every pair of objects (A, B) of $\mathcal{C}$)

$$\begin{array}{ccc} S^p(A) \otimes T^q(B) & \to & U^{p+q}(A \otimes B) \\ \downarrow & & \downarrow \\ S_1^p(A) \otimes T_1^q(B) & \to & U_1^{p+q}(A \otimes B) \end{array}$$

Proof. We use induction to prove this result. Let (p, q) denote the commutativity of the diagram of the theorem for the case p, q. $(0, 0)$ is true by hypothesis. We now show that $(p, q) \Rightarrow (p+1, q)$ and $(p, q) \Rightarrow (p, q+1)$. Let $0 \to A \to P \to Q \to 0$ be such that $S^j(P) = 0$ for $j > 0$ and for any object B of $\mathcal{C}$ the sequence $0 \to A \otimes B \to P \otimes B \to Q \otimes B \to 0$ is exact. We thus obtain the following diagram



where we must show that the right-hand face commutes. By our assumptions and by naturality all the other faces do. Therefore the diagram commutes as far as paths starting from $S^p(Q) \otimes T^q(B)$ are concerned. But since $S^j(P) = 0$ and $\otimes$ is right exact we have that the map $\delta \otimes Id$ is onto. Therefore the right-hand face commutes. This proves the case $(p, q) \Rightarrow (p, q + 1)$. The argument for the other case is almost exactly the same except that we must take care of the signs and the map $w \otimes \delta$ replaces $\delta \otimes Id$. Hence the theorem is proved.

□

Definition 53. Let $\mathcal{C}$ be an abelian tensor category and let $0 \to A_0 \xrightarrow{d_0} A_1 \xrightarrow{d^1} \cdots$ and $0 \to B_0 \xrightarrow{d_0'} B_1 \xrightarrow{d_1'} \cdots$ be complexes in $\mathcal{C}$. The *tensor product* of two such complexes $A_.$ and $B_.$ is defined to be the complex $A_. \otimes B_.$ where $(A_. \otimes B_.)_n = \oplus_{p+q=n} A_p \otimes B_q$ and the differential $\delta_n : (A_. \otimes B_.)_n \to (A_. \otimes B_.)_{n+1}$ is defined by $\delta_n|_{A_p \otimes B_q}(a \otimes b) = d_p(a) \otimes b + (-1)^p a \otimes d_q'(b)$.

Let $\mathcal{F}$ be a sheaf on X and let

$$0 \to \mathcal{F} \to \mathcal{C}^0(\mathcal{F}) \to \mathcal{C}^1(\mathcal{F}) \to \cdots$$

be the Godemont resolution of $\mathcal{F}$ by flasque sheaves. We have the following results about this resolution which will be used to define cup products on sheaves.

Lemma 6. *For any $\mathcal{F}$ on X as above and for any $x \in X$ the map $\mathcal{F}_x \to \mathcal{C}^0(\mathcal{F})_x$ is a split inclusion.*

Corollary 7. *For any $\mathcal{F}$ on X as above and for any $x \in X$, $0 \to \mathcal{F}_x \to \mathcal{C}^{\cdot}(\mathcal{F})_x$ is contractible.*

We now return to the genral situation and state the following lemma.

Lemma 7. *Let $\mathcal{C}$ be an abelian tensor category and let $0 \to A \xrightarrow{\epsilon} A_0 \xrightarrow{d_0} \cdot$ and $0 \to B \xrightarrow{\epsilon'} B_0 \xrightarrow{d_0'} \cdot$ be contractible complexes. Then $0 \to A \otimes B \xrightarrow{\epsilon \otimes \epsilon'} A_. \otimes B_.$ is also contractible.*

Proof. □

The previous results now give us the following

Corollary 8. *If $\mathcal{F}_1$ and $\mathcal{F}_2$ are two sheaves on X then $0 \to \mathcal{F}_1 \otimes \mathcal{F}_2 \to \mathcal{C}^{\cdot}(\mathcal{F}_1) \otimes \mathcal{C}^{\cdot}(\mathcal{F}_2)$ is a resolution.*

Now we define cup products for sheaves. Let $\mathcal{C}$ be the category of sheaves $\mathbf{Sh(X)}$ on X and $\mathcal{D}$ be the category $\mathcal{A}b$ of abelian groups. Let Φ_1 and Φ_2 be two families of supports and let $S^i = H^i_{\Phi_1}(X, \cdot)$, $T^i = H^i_{\Phi_2}(X, \cdot)$ and $U^i = H^i_{\Phi_1 \cap \Phi_2}(X, \cdot)$. As before if $\mathcal{F}$ and $\mathcal{G}$ are sheaves then there is a natural pairing $\Gamma_{\Phi_1}(X, \mathcal{F}) \otimes \Gamma_{\Phi_2}(X, \mathcal{G}) \to \Gamma_{\Phi_1 \cap \Phi_2}(X, \mathcal{F} \otimes \mathcal{G})$. Hence cup products would now be pairings

$$H^p_{\Phi_1}(X, \mathcal{F}) \otimes H^q_{\Phi_2}(X, \mathcal{G}) \to H^{p+q}_{\Phi_1 \cap \Phi_2}(X, \mathcal{F} \otimes \mathcal{G}).$$

The last theorem shows that if cup products exist then they are unique since the assumption in the statement of the thoerem is satisfied for the category $\mathbf{Sh(X)}$. This is because we may take the short exact sequence

$$0 \to \mathcal{F} \to \mathcal{C}^0(\mathcal{F}) \to \mathcal{G}^0 \to 0$$

where (as before) $\mathcal{G}^0$ is the quotient sheaf. The sheaf of discontinuous sections is flabby hence satisfies the first condition in the hypothesis. Also $\mathcal{F}_x$ is a direct summand of $\mathcal{C}^0(\mathcal{F})_x$ for all points $x \in X$ hence tensoring this short exact sequence with any other sheaf gives another short exact sequence. We now prove the main theorem which states

Theorem 5. *There are cup products over the above functors and maps.*

Proof. Choose an injective resolution of $\mathcal{F}_1 \otimes \mathcal{F}_2$

$$0 \to \mathcal{F}_1 \otimes \mathcal{F}_2 \to \mathcal{I}.$$

Then there exists a map of complexes

$$\mathcal{C}^{\cdot}(\mathcal{F}_1) \otimes \mathcal{C}^{\cdot}(\mathcal{F}_2) \to \mathcal{I}.$$

which is unique upto homotopy and which is compatible with the identity map on $\mathcal{F}_1 \otimes \mathcal{F}_2$. We observed earlier that there exists map of complexes

$$\Gamma_{\Phi_1}(\mathcal{C}^{\cdot}(\mathcal{F}_1)) \otimes \Gamma_{\Phi_2}(\mathcal{C}^{\cdot}(\mathcal{F}_2)) \to \Gamma_{\Phi_1 \cap \Phi_2}(\mathcal{C}^{\cdot}(\mathcal{F}_1) \otimes \mathcal{C}^{\cdot}(\mathcal{F}_2))$$

Therefore by composition we get a map of complexes

$$\Gamma_{\Phi_1}(\mathcal{C}^{\cdot}(\mathcal{F}_1)) \otimes \Gamma_{\Phi_2}(\mathcal{C}^{\cdot}(\mathcal{F}_2)) \to \Gamma_{\Phi_1 \cap \Phi_2}(\mathcal{I}.)$$

This induces maps

$$H^p_{\Phi_1}(X, \mathcal{F}_1) \otimes H^q_{\Phi_2}(X, \mathcal{F}_2) \to H^{p+q}_{\Phi_1 \cap \Phi_2}(X, \mathcal{F}_1 \otimes \mathcal{F}_2)$$

We claim that this is a cup product i.e. we will now show that

1. if $0 \to \mathcal{F}_1^{'} \to \mathcal{F}_1 \to \mathcal{F}_1^{''} \to 0$ is an exact sequence of sheaves and there is a commutative diagram

$$\begin{array}{ccccccc} & \mathcal{F}_1^{'} \otimes \mathcal{F}_2 & \to & \mathcal{F}_1 \otimes \mathcal{F}_2 & \to & \mathcal{F}_1^{''} \otimes \mathcal{F}_2 & \to 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \to & \mathcal{H}^{'} & \to & \mathcal{H} & \to & \mathcal{H}^{''} & \to 0 \end{array}$$

with exact rows, then the following diagram commutes

$$\begin{array}{ccc} H^p_{\Phi_1}(X, \mathcal{F}^{''}) \otimes H^q_{\Phi_2}(X, calF_2) & \to & H^{p+1}_{\Phi_1}(X, \mathcal{F}_1^{'}) \otimes H^q_{\Phi_2}(X, \mathcal{F}_2) \\ \downarrow & & \downarrow \\ H^{p+q}_{\Phi_1 \cap \Phi_2}(X, \mathcal{H}^{''}) & \to & H^{p+q+1}_{\Phi_1 \cap \Phi_2}(X, \mathcal{H}^{'}) \end{array}$$

2. A similar functorial property in the second variable upto sign of $(-1)^p$.

We will now prove the first property and end the proof of the theorem by saying that the proof of the second property above is analogous. We prove the first property as follows. Let

$$0 \to \mathcal{F}_1^{'} \otimes \mathcal{F}_2 \to \mathcal{I}^{'}_{.}$$

$$0 \to \mathcal{F}_1 \otimes \mathcal{F}_2 \to \mathcal{I}.$$

$$0 \to \mathcal{F}_1^{''} \otimes \mathcal{F}_2 \to \mathcal{I}^{''}_{.}$$

$$0 \to \mathcal{H}^{'} \to \mathcal{J}^{'}_{.}$$

$$0 \to \mathcal{H} \to \mathcal{J}.$$

$$0 \to \mathcal{H}^{''} \to \mathcal{J}^{''}_{\cdot}$$

be injective resolutions. We then have the following commutative diagram whose top and bottom rows are exact and which commutes upto homotopy.

$$\begin{array}{ccccccc} & \mathcal{C}^{\cdot}(\mathcal{F}_1^{'}) \otimes \mathcal{C}^{\cdot}(\mathcal{F}_2) & \to & \mathcal{C}^{\cdot}(\mathcal{F}_1) \otimes \mathcal{C}^{\cdot}(\mathcal{F}_2) & \to & \mathcal{C}^{\cdot}(\mathcal{F}_1^{''}) \otimes \mathcal{C}^{\cdot}(\mathcal{F}_2) & \to \\ & \downarrow & & \downarrow & & \downarrow & \\ & \mathcal{I}^{'}_{\cdot} & \to & \mathcal{I}_{\cdot} & \to & \mathcal{I}^{''}_{\cdot} & \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \to & \mathcal{J}^{'}_{\cdot} & \to & \mathcal{J}_{\cdot} & \to & \mathcal{J}^{''}_{\cdot} & \to 0 \end{array}$$

Therefore we have the following diagram with exact rows which is homotopy commutative,

$$\begin{array}{ccccccc} & \Gamma_{\Phi_1}(\mathcal{C}^{\cdot}(\mathcal{F}_1^{'})) \otimes \Gamma_{\Phi_2}(\mathcal{C}^{\cdot}(\mathcal{F}_2)) & \to & \Gamma_{\Phi_1}(\mathcal{C}^{\cdot}(\mathcal{F}_1)) \otimes \Gamma_{\Phi_2}(\mathcal{C}^{\cdot}(\mathcal{F}_2)) & \to & \Gamma_{\Phi_1}(\mathcal{C}^{\cdot}(\mathcal{F}_1^{''})) \otimes \Gamma_{\Phi_2}(\mathcal{C}^{\cdot}(\mathcal{F}_2)) & \to 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ & \Gamma_{\Phi_1 \cap \Phi_2}(\mathcal{I}^{'}_{\cdot}) & \to & \Gamma_{\Phi_1 \cap \Phi_2}(\mathcal{I}_{\cdot}) & \to & \Gamma_{\Phi_1 \cap \Phi_2}(\mathcal{I}^{''}_{\cdot} & \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \to & \Gamma_{\Phi_1 \cap \Phi_2}(\mathcal{J}^{'}_{\cdot}) & \to & \Gamma_{\Phi_1 \cap \Phi_2}(\mathcal{J}_{\cdot}) & \to & \Gamma_{\Phi_1 \cap \Phi_2}(\mathcal{J}^{''}_{\cdot}) & \to 0 \end{array}$$

This gives the required commutative diagram in the first property. □

The Leray spectral sequence

Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be abelian categories each having enough injectives. Let $G : \mathcal{A} \to \mathcal{B}$ and $F : \mathcal{B} \to \mathcal{C}$ be left exact functors. Recall that an object B of $\mathcal{B}$ is said to be F-acyclic if $R^i F(B) = 0$ for $i > 0$.

Theorem 6. *(Grothendieck spectral sequence) With the notation as in the above paragraph, assume that G sends injective objects of $\mathcal{A}$ to F-acyclic objects of $\mathcal{B}$. Then there exists a first quadrant (cohomological) spectral sequence (called the Grothendieck spectral sequence) for each A in $\mathcal{A}$:*

$$E_2^{pq} = (R^p F)(R^q G)(A) \Rightarrow R^{p+q}(F \cdot G)(A).$$

The Leray spectral sequence is a special case of the Grothendieck spectral sequence. Let $f : X \to Y$ be a continuous map of topological spaces. The direct image sheaf functor $f_* : \mathbf{Sh}(\mathbf{X}) \to \mathbf{Sh}(\mathbf{Y})$ has a left adjoint f^{-1} which is exact. This imples that f_* is left exact (seen before) and preserves injectives (this is because $Hom(\cdot, f_*\mathcal{I}) = Hom(f^{-1}(\cdot), \mathcal{I})$ and hence is exact as it is the composition of an exact functor f^{-1} with the functor $Hom(\cdot, \mathcal{I})$). We now apply the Grothendieck spectral sequence to the following situation: $\mathcal{A} = \mathbf{Sh}(\mathbf{X})$, $\mathcal{B} = \mathbf{Sh}(\mathbf{Y})$, $\mathcal{C} = \mathbf{Ab}$, $G = f_*$ and $F = \Gamma(Y, f_*(\cdot))$. Note that the composite in this case $G \cdot F = \Gamma(X, \cdot)$. Thus we have the following

Corollary 9. *(Leray spectral sequence) Let $f : X \to Y$ be a continuous map of topological spaces. Then there exists a spectral sequence (called the Leray spectral sequence)*

$$E_2^{pq} = H^p(Y, R^q f_* \mathcal{F}) \Rightarrow H^{p+q}(X, \mathcal{F}).$$

For the sake of completeness, we develop the the theory of spectral sequences upto the point where we can prove the above theorem. This topic will also be done in much greater detail in other lectures in this workshop. We start with the definition of an exact couple which is one way of defining spectral sequences.

Definition 54. Let $\mathcal{C}$ be an abelian category and let D_1, E_1 be objects of $\mathcal{C}$. Let $a_1 : E_1 \to D_1$, $b_1 : D_1 \to D_1$ and $c_1 : D_1 \to E_1$ be morphisms in $\mathcal{C}$ (the reasons for choosing the subscript will soon be clear). The data $(D_1, E_1, a_1, b_1, c_1)$ is said to be an *exact couple* if the sequence

$$E_1 \xrightarrow{a_1} D_1 \xrightarrow{b_1} D_1 \xrightarrow{c_1} E_1 \xrightarrow{a_1} D_1$$

is exact.

We denote an exact couple by the following diagram

$$\begin{array}{ccc} D_1 & \xrightarrow{b_1} & D_1 \\ & a_1 \nwarrow \swarrow\!\!\nearrow c_1 & \\ & E_1 & \end{array}$$

Given an exact couple as above we get a new exact couple

$$\begin{array}{ccc} D_2 & \xrightarrow{b_2} & D_2 \\ & a_2 \nwarrow \swarrow\!\!\nearrow c_2 & \\ & E_2 & \end{array}$$

as follows. First define $d_1 : E_1 \to E_1$ by

$$d_1 = c_1 \cdot a_1.$$

Then, $d_1 \cdot d_1 = c_1 \cdot (a_1 \cdot c_1) \cdot a_1 = 0$. Now define $E_2 = Ker(d_1)/Im(d_1)$ and $D_2 \subseteq D_1$ by $D_2 = Im(b_1)$. Hence, $D_2 = Ker(c_1) \cong D_1/Ker(b_1) = D/Im(a_1)$. We define $b_2 : D_2 \to D_2$ by $b_2 = b_1|_{D_2}$. To define a_2 we first note that $a_1|_{Im(d_1)} = 0$. This is because $a_1(Im(d)) = Im(d_1 \cdot a_1) = 0$. We now define $a_2 : E_2 \to D_2$ to be the map induced $a_1|_{Ker(d_1)}$ (which can be defined by the previous remark and which lands up in D_2 because $a_1(Ker(d_1)) \subset Im(b_1) = Ker(c_1)$). We finally define $c_2 : D_2 \to E_2$ by $c_2(b_1(x)) = c_1(x) + Im(d_1)$. This is well-defined because $b_1(x) = 0$, if and only if, $x \in Im(a_1)$, if and only if, $c_1(x) \in Im(c_1 \cdot a_1) = Im(d_1)$. One can now check that,

Proposition 16. $(D_2, E_2, a_2, b_2, c_2)$ forms a new exact couple.

We skip the proof.

Definition 55. The new exact couple

$$\begin{array}{ccc} D_2 & \xrightarrow{b_2} & D_2 \\ & a_2 \nwarrow \swarrow\!\!\nearrow c_2 & \\ & E_2 & \end{array}$$

is called the *derived couple* of the original one.

Let

$$\begin{array}{ccc} D_r & \xrightarrow{b_r} & D_r \\ & a_r \nwarrow \swarrow\!\!\nearrow c_r & \\ & E_r & \end{array}$$

be the $(r-1)$th derived couple of the original one. Let $d_r = c_r \cdot a_r : E_r \to E_r$ and as before $d_r \cdot d_r = 0$ and $H(E_r, d_r) = E_{r+1}$.

Definition 56. Let the notation be as above. The collection of differential objects $\{(E_r, d_r)\}$ is called the *spectral sequence* of the exact couple $(D_1, E_1, a_1, b_1, c_1)$ that we started out with.

Problem 4. Show that

1. $D_r = Im(b_1^{r-1}) \cong D/Ker(b_1^{r-1})$.
2. $E_r = a_1^{-1}(Im(b_1^{r-1}))/c_1(Ker(b_1^{r-1}))$.
3. $a_r(x + c_1(Ker(b_1^{r-1}))) = a_1(x)$ if $x \in a_1^{-1}(Im(b_1^{r-1}))$.
4. $b_r(y) = b_1(y)$ if $y \in Im(b_1^{r-1})$.
5. $c_r(b_1^{r-1}(y)) = c_1(y) + c_1(Ker(b_1^{r-1}))$ for all $y \in D_1$.
6. $d_r(x + c(Ker(b_1^{r-1})))$ is computed as follows: $a_1(x) \in Im(b_1^{r-1})$ implies $a_1(x) = b_1^{r-1}(x)$ for some $y \in D_1$. Then, $d_r(x + c_1(Ker(b_1^{r-1}))) = c_1(y) + c_1(Ker(b_1^{r-1}))$ for that y.
7. Denote $a_1^{-1}(Im(b_1^{r-1}))$ by Z_r and $c_1(Ker(b_1^{r-1}))$ by B_r. Then,
$$E_1 = Z_1 \supseteq \cdots \supseteq Z_r \supseteq Z_{r+1} \supseteq B_{r+1} \supseteq B_r \supseteq \cdots B_1 = 0$$
We define, $Z_\infty = \cap_r Z_r$, $B_\infty = \cup_r B_r$ and the *limit* $E_\infty = Z_\infty / B_\infty$.

We now discuss cohomological spectral sequences. Let $\{A^{m,n}\}_{(m,n)\in\mathbb{Z}^2}$ and $\{E^{m,n}\}_{(m,n)\in\mathbb{Z}^2}$ be families of objects in an abelian category $\mathcal{C}$ such that for each $p \in \mathbb{Z}$ there are exact sequences

$$\cdots \xrightarrow{h_{p,q-1}} A^{p+1,q-1} \xrightarrow{f_{p,q}} A^{p,q} \xrightarrow{g_{p,q}} E^{p,q} \xrightarrow{h_{p,q}} A^{p+1,q} \xrightarrow{f_{p+1,q}} A^{p,q+1} \xrightarrow{g_{p+1,q}} \cdots$$

Define $D_1 = \oplus_{p,q} A^{p,q}$ and $E_1 = \oplus_{p,q} E^{p,q}$. Also define $a_1 : E_1 \to D_1$ by $a_1 = \oplus_{p,q} h_{p,q}$, $b_1 : D_1 \to D_1$ by $b_1 = \oplus_{p,q} f_{p-1,q+1}$ (meaning it is $f_{p-1,q+1}$ on it's $(p,q)^{th}$ component) and finally $c_1 : D_1 \to E_1$ by $c_1 = \oplus_{p,q} g_{p,q}$. One can check that the above family of sequences yield an exact couple $(D_1, E_1, a_1, b_1, c_1)$ as defined above. We note that in this case the objects D_1 and E_1 are bigraded, and the maps a_1, b_1, c_1 are morphisms which respect this bigrading. We note down the bidegrees of these maps,

$$bideg(a_1) = (1, 0),$$
$$bideg(b_1) = (-1, 1)$$

and

$$bideg(c_1) = (0, 0).$$

Hence B_r and Z_r, as defined in the problem above, are bigraded submodules of E_1 for all r (this is because they are defined in terms of a_1, b_1 and c_1 as in the problem) and $E_r = Z_r/B_r$ has a natural induced bigrading inherited from E_1. Similarly, $D_r = Im(b_1^{r-1}) \subseteq D_1$ is bigraded too, and the maps a_r, b_r, c_r, d_r are compatible with the bigradings with the following bidegrees:

$$bideg(a_r) = (1, 0)$$

$$bideg(b_r) = (-1, 1)$$

$$bideg(c_r) = (r - 1, 1 - r)$$

$$bideg(d_r) = (r, 1 - r)$$

So d_r is a map $d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}$.

Definition 57. A *cohomological spectral sequence* is a bigraded spectral sequence with the above bigrading.

Example Let $C^\cdot$ be a complex (in $\mathcal{C}$) which comes equipped with a decreasing filtration $\{F^pC^\cdot\}$ of subcomplexes. Let $Gr_F^pC^\cdot = F^pC^\cdot/F^{p+1}C^\cdot$. Then there exists a short exact sequence of complexes

$$0 \to F^{p+1}C^\cdot \to F^pC^\cdot \to Gr_F^pC^\cdot \to 0.$$

This gives a long exact sequence of cohomology groups

$$\cdots \to H^{p+q}(F^{p+1}C^\cdot) \to H^{p+q}(F^pC^\cdot) \to H^{p+q}(Gr_F^pC^\cdot) \to H^{p+q+1}(F^{p+1}C^\cdot) \to \cdots$$

Let

$$A^{p,q} = H^{p+q}(F^pC^\cdot)$$

and

$$E^{p,q} = H^{p+q}(Gr_F^pC^\cdot)$$

One can check that this gives us an example of a cohomological spectral sequence.

We now discuss the notions of convergence and limits of a (cohomological) spectral sequence. Given the exact sequences of a cohomological spectral sequence, we have the following sequence of objects and morphisms

$$\cdots \xrightarrow{f_{n-q,q}} A^{n-q,q} \xrightarrow{f_{n-q-1,q+1}} A^{n-q-1,q+1} \xrightarrow{f_{n-q-2,q+2}} A^{n-q-2,q+2} \to \cdots$$

Define,

$$A^n = \varinjlim_q A^{n-q,q}$$

Then there exists a decreasing filtration $F^pA^n = Im(A^{p,n-p} \to A^n)$.

Assume that for each $n \in \mathbb{Z}$ we have,

1. there exists $q_1(n) \in \mathbb{Z}$ such that $f_{n-q,q} : A^{n-q+1,q-1} \to A^{n-q,q}$ is an isomorphism for all $q \geq q_1(n)$,
2. there exists $q_0(n) \in \mathbb{Z}$ such that $A^{n-q,q} = 0$ for all $q \leq q_0(n)$.

The above are called the *boundedness conditions* of a cohomological spectral sequence.

Under these assumptions we have the following lemma.

Lemma 8. *For each $(p, q) \in \mathbb{Z}^2$ there exists a positive integer $r_0(p, q)$ such that for all $r \geq r_0 = r_0(p, q)$ we have*

1.
$$Z_r^{p,q} = Z_{r+1}^{p,q} = \cdots = Z_\infty^{p,q}$$
$$B_r^{p,q} = B_{r+1}^{p,q} = \cdots B_\infty^{p,q}$$
$$d_r^{p,q} = 0, d_r^{p-r,q+r-1} = 0$$

Hence
$$E_r^{p,q} \cong E_\infty^{p,q}$$

2. *There exists a natural isomorphism*
$$E_\infty^{p,q} \cong F^pA^{p+q}/F^{p+1}A^{p+q} = Gr_F^pA^{p+q}$$

3. $d_1^{p,q} : E_1^{p,q} \to E_1^{p+1,q}$ *is given by the slanted arrow in the following diagram*
$$\begin{array}{ccccccccc} \cdots \to & A^{p,q} & \to & E^{p,q} & \to & A^{p+1,q} & \to \cdots \\ & & & & \searrow & & \\ \cdots \to & A^{p+2,q-1} & \to & A^{p+1,q} & \to & E^{p+1,q} & \to & \cdots \end{array}$$

Definition 58. 1. A cohomological spectral sequence is said to be *bounded* if there are only finitely many nonzero terms $E_r^{p,q}$ for every fixed r (note that the boundedness conditions above imply this).

2. A bounded cohomological spectral sequence is said to *converge* to $A^\cdot = \{A^n\}_n$ if we are given a family of objects $\{A^n\}_n$ of $\mathcal{C}$ each having a finite decreasing filtration $F^\cdot A^n$ as above such that, $E_\infty^{p,q} = Gr_F^pA^{p+q}$. One denotes a bounded convergence of a cohomological spectral sequence by the following notation:
$$E_r^{p,q} \Rightarrow A^{p+q}.$$

Example Given a filtered complex $\{F^pC^\cdot\}_{n\in\mathbb{Z}}$ assume that for each $n \in \mathbb{Z}$ there exists $p_0 > p_1$ with $F^{p_0}C^n = 0$ and $F^{p_1}C^n = C^n$. Then one can check that the boundedness assumptions for the cohomological spectral sequence hold. One can then show that
$$A^n = H^n(C^\cdot),$$
$$F^pA^n = Im(H^n(F^pC^\cdot) \to H^n(C^\cdot)),$$
$$E_\infty^{p,q} = Gr_F^pH^{p+q}(C^\cdot)$$
Since,
$$E_1^{p,q} = H^{p+q}(Gr_F^pC^\cdot)$$
it is said that spectral sequences interchange "the grade" in cohomology.

We now disuss double complexes and spectral sequences arising out of them.

Definition 59. 1. Let $\mathcal{C}$ be an abelian category as usual. A *double complex* in $\mathcal{C}$ is a collection of objects $\{C^{m,n}\}_{(m,n)\in\mathbb{Z}^2}$ together with maps
$$d_1^{m,n} : C^{m,n} \to C^{m+1,n}$$
$$d_2^{m,n} : C^{m,n} \to C^{m,n+1}$$
such that $d_1^2 = 0$, $d_2^2 = 0$ and $d_1 \cdot d_2 = d_2 \cdot d_1$.

2. The *total complex* of a double complex $C^{\cdot,\cdot}$ as above is defined to be a complex, $Tot^{\cdot}(C^{\cdot,\cdot})$, whose nth term is defined as $Tot^n(C^{\cdot,\cdot}) = \oplus_{r+s=n}C^{r,s}$ and whose differential $\delta : Tot^n(C^{\cdot,\cdot}) \to Tot^{n+1}(C^{\cdot,\cdot})$ is given by $\delta|_{C^{r,s}} = d_1^{r,s} + (-1)^r d_2^{r,s}$ (one can check that $\delta^2 = 0$).

Double complexes with objects and morphisms from $\mathcal{C}$ form an abelian category (with an obvious notion of morphism). If $\mathcal{C}$ is closed under direct sums then $Tot^{\cdot}$ is an exact functor from the category of double complexes to the category of complexes in $\mathcal{C}$.

We have the following boundedness condition: for each $p \in \mathbb{Z}$, $C^{n,p-n} = 0$ for all but finitely many n. This is equivalent to the condition that $Tot^n(C^{r,s})$ is a finite direct sum for every n.

Now consider the following two decreasing filtrations on $C^{r,s}$ by sub-double complexes.

1. $F_I^p(C^{r,s}) = C^{r,s}$ if $r \geq p$ and is 0 otherwise.
2. $F_{II}^p(C^{r,s}) = C^{r,s}$ if $s \geq p$ and is 0 otherwise.

Hence we get induced filtrations on $Tot^{\cdot}(C^{r,s})$

1. $F_I^p(Tot^{\cdot}(C^{r,s})) = \oplus_{r=p}^{\infty} C^{r,\cdot-r}$
2. $F_{II}^p(Tot^{\cdot}(C^{r,s})) = \oplus_{s=p}^{\infty} C^{\cdot-s,s}$

The filtrations F_I and F_{II} give finite decreasing filtrations on $Tot^n(C^{r,s})$ for each $n \in \mathbb{Z}$ (under the boundedness assumption).

We now analyse the spectral sequences arising from these filtrations on the $Tot^{\cdot}$ complex. We have, $Gr_{F_I}^p(Tot^n(C^{r,s})) \cong C^{p,n-p}$. The differential δ' on $Gr_{F_I}^p(Tot^n)$ is induced by the one on Tot^n. One can easily see that the map $Gr_{F_I}^p(Tot^n) \xrightarrow{\delta'} Gr_{F_I}^p(Tot^{n+1})$ is the map $(-1)^p d_2^{p,n-p} : C^{p,n-p} \to C^{p,n-p+1}$. Thus, F_I induces a spectral sequence whose E_1 terms are

$$E_1^{p,q} = H^{p+q}(C^{p,\cdot}, (-1)^{\cdot} d_2^{p,\cdot}).$$

Consider the complex $(C^{p,n}, (-1)^p d_2^{p,n})$ where $C^{p,n}$ is the *term in degree* n. The E_1 term is the cohomology of this complex obtained by *shifting this complex by* p i.e. $C^{p,n}$ is now the term in degree $p+n$. Thus with the natural indexing (by n) on $(C^{p,n}, (-1)^p d_2^{p,n})$ we have that

$$E_1^{p,q} = H^q(C^{p,\cdot}, d_2^{p,\cdot}).$$

Lemma 9. *With the notation as above the E_1 differentials are computed as as follows :*

$$E_1^{p,q} = H^q(C^{p,\cdot}, d_2^{p,\cdot}) \to H^q(C^{p+1,\cdot}, d_2^{p+1,\cdot}) = E_1^{p+1,q}$$

where the map on the cohomology groups is induced by $d_1 : C^{p,\cdot} \to C^{p+1,\cdot}$. Hence the filtration F_I gives rise to a spectral sequence

$$E_2^{p,q} = H^p(H^q(C^{r,s}, d_2), d_1) \Rightarrow H^{p+q}(Tot^{\cdot}(C^{r,s})).$$

Similarly F_{II} gives rise to a spectal sequence

$$E_2^{p,q} = H^p(H^q(C^{r,s}, d_1), d_2) \Rightarrow H^{p+q}(Tot^{\cdot}(C^{r,s}))$$

Example *Dolbeault Double Complex*

Let X be a complex manifold and let $\mathcal{E}_X^{p,q}$ be the sheaf of C^∞ forms of type (p,q) on X. We have maps $\delta : \mathcal{E}_X^{p,q} \to \mathcal{E}_X^{p+1,q}$ and $\overline{\delta} : \mathcal{E}_X^{p,q} \to \mathcal{E}_X^{p,q+1}$. The Cauchy-Riemann equations imply that $\overline{\delta}$ is $\mathcal{O}_X$-linear. The *exterior derivative* d is defined by $d = \delta + \overline{\delta}$. We then have,

Lemma 10. *(Dolbeault's Lemma) The sheaf sequence*

$$0 \to \mathcal{O}_X \to \mathcal{E}_X^{0,0} \xrightarrow{\overline{\delta}} \mathcal{E}_X^{0,1} \xrightarrow{\overline{\delta}} \cdots$$

is an exact sequence of sheaves.

Corollary 10. *If $\mathcal{F}$ is a locally free $\mathcal{O}_X$-module then,*

$$0 \to \mathcal{F} \to \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{E}_X^{0,0} \xrightarrow{1\otimes\overline{\delta}} \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{E}_X^{0,1} \xrightarrow{1\otimes\overline{\delta}} \cdots$$

is a fine *resolution of sheaves.*

We form the double complex of sheaves $\mathcal{E}_X^{m,n}$ with the two differentials δ and $(-1)^n\overline{\delta}$. We get an associated double complex of $\mathbb{C}$ vector spaces on applying the global section functor $\Gamma(X,\cdot)$. Consider the spectral sequence in this case for the filtration F_I. We have that $Tot^n(\Gamma(\mathcal{E}_X^{r,s})) = \oplus_{r+s=n}\Gamma(\mathcal{E}_X^{r,s})$ = space of sections of C^∞ n-forms on X with values in $\mathbb{C}$. Therefore by De Rham's theorem we have that,

$$H^n(Tot^{\cdot}(\Gamma(\mathcal{E}_X^{r,s}))) = H^n(X,\mathbb{C}).$$

Thus the spectral sequence is

$$E_1^{p,q} = H^q(\Gamma(\mathcal{E}_X^{p,\cdot}),\overline{\delta}) \Rightarrow H^{p+q}(X,\mathbb{C}).$$

But by the corollary to Dolbeault's lemma above tells us that $(\mathcal{E}_X^{p,\cdot},\overline{\delta})$ is a fine resolution of Ω_X^p = sheaf of holomorphic p-forms on X (note that, $\mathcal{E}_X^{p,q} = \Omega_X^p \otimes_{\mathcal{O}_X} \mathcal{E}_X^{0,q}$). Putting these together we have the *Hodge to De Rham spectral sequence* (also known as the *Fröhlicher spectral sequence*),

$$E_1^{p,q} = H^q(X,\Omega_X^p) \Rightarrow H^{p+q}(X,\mathbb{C}).$$

The induced filtration on $H^n(X,\mathbb{C})$ is called the *Hodge filtration.* If X is a compact (complex manifold), we have that $H^q(X,\Omega_X^p)$ is a finite dimensional vector space over $\mathbb{C}$ for all p,q and hence

$$\sum_{p+q=n} dim_{\mathbb{C}}(H^q(X,\Omega_X^p)) \geq dim_{\mathbb{C}}(H^n(X,\mathbb{C}))$$

If X is a compact, Kähler complex manifold then Hodge theory implies that the above inequality is actually an equality.

Let $\mathcal{A}$ be an abelian category and let $\mathcal{C}(\mathcal{A})$ be the category of complexes in $\mathcal{A}$. Let $\mathcal{C}^+(\mathcal{A}) \subset \mathcal{C}(\mathcal{A})$ be the subcategory of complexes that are bounded below. We have the following description of injective objects in $\mathcal{C}(\mathcal{A})$.

Proposition 17. A complex $\mathcal{I}^{\cdot}$ is an injective object in $\mathcal{C}(\mathcal{A})$, if and only if, it has the following form:

$$\mathcal{I}^n = \mathcal{I}_0^n \oplus \mathcal{I}_1^n$$

for every n where $\mathcal{I}_0^n$ and $\mathcal{I}_1^n$ are injective objects of $\mathcal{A}$ and $d^n : \mathcal{I}^n \to \mathcal{I}^{n+1}$ can be factored as

$$\mathcal{I}^n \twoheadrightarrow \mathcal{I}_1^n \hookrightarrow \mathcal{I}_0^n.$$

Thus, $\mathcal{I}^{\cdot} = \oplus_{n\in\mathbb{Z}}(\mathcal{I}_1^n \hookrightarrow \mathcal{I}_0^{n+1})$, where the first term is in degree n and the second term is in degree $n+1$.

Lemma 11. *If $\mathcal{A}$ has enough injectives then so does the category $\mathcal{C}^+(\mathcal{A})$. In fact, if $0 \to C^n \to C^{n+1} \to C^{n+2} \to \cdots$ is an object of $\mathcal{C}^+(\mathcal{A})$ then there exists an object $\mathcal{I}^{\cdot}$ of $\mathcal{C}^+(\mathcal{A})$ which is injective in $\mathcal{C}(\mathcal{A})$ together with a monomorphism $C^{\cdot} \to \mathcal{I}^{\cdot}$ such that the induced maps $H^n(C^{\cdot}) \to H^n(\mathcal{I}^{\cdot})$ are also monomorphisms. We also have that $\mathcal{I}^j = 0$ for all $j < n$.*

Corollary 11. *If $C^{\cdot}$ is an object of $\mathcal{C}^+(\mathcal{A})$ then there exists a resolution*

$$0 \to C^{\cdot} \to \mathcal{I}^{0,\cdot} \to \mathcal{I}^{1,\cdot} \to \mathcal{I}^{2,\cdot} \to \cdot$$

where $\mathcal{I}^{j,\cdot}$ are injectives and such that for each n,

$$0 \to H^n(C^{\cdot}) \to H^n(\mathcal{I}^{j,\cdot})$$

is an injective resolution.

Proof. □

Definition 60. An injective resolution of a complex $0 \to C^{\cdot} \to \mathcal{I}^{j,\cdot}$ satisfying the properties of the above corollary is called a *Cartan-Eilenberg resolution.*

Consider the double complex $\{\mathcal{I}^{\cdot,\cdot}\} = \{\mathcal{I}^{m,n}\}$ associated to a Cartan-Eilenberg resolution. Then there exists two spectral sequences

$${}_IE_1^{p,q} = H^q(\mathcal{I}^{p,\cdot}) \Rightarrow H^{p+q}(Tot^{\cdot}(\mathcal{I}^{m,n}))$$

and

$${}_{II}E_1^{p,q} = H^q(\mathcal{I}^{\cdot,p}) \Rightarrow H^{p+q}(Tot^{\cdot}(\mathcal{I}^{m,n})).$$

Consider the first spectral sequence. After fixing q the complexes of E_1 terms look like

$$\begin{array}{ccccccc} 0 \to & E_1^{0,q} & \to & E_1^{1,q} & \to & E_1^{2,q} & \to \cdots \\ & || & & || & & || & \\ 0 \to & H^q(\mathcal{I}^{0,\cdot}) & \to & H^q(\mathcal{I}^{1,\cdot}) & \to & H^q(\mathcal{I}^{2,\cdot}) & \to \cdots \end{array}$$

which is an injective resolution of $H^q(C^{\cdot})$. Thus,
$E_2^{p,q} = H^q(C^{\cdot})$ if $p = 0$ and is 0 otherwise,
and hence,
$E_\infty^{p,q} = H^q(C^{\cdot})$ if $p = 0$ and is 0 otherwise.
Therefore, $H^q(Tot^{\cdot}(\mathcal{I}^{m,n})) \cong E_\infty^{0,q} \cong H^q(C^{\cdot})$.

We claim that this isomorphism is induced from a map of complexes $C^{\cdot} \to Tot^{\cdot}(\mathcal{I}^{m,n})$. In fact, we may regard $C^{\cdot}$ as a double complex concentrated in the column $p = 0$. So the natural map of complexes $C^{\cdot} \to \mathcal{I}^{0,\cdot}$ can be thought of a map of double complexes $C^{\cdot} \to \mathcal{I}^{\cdot,\cdot}$. Hence we get an induced map on the total complexes $C^{\cdot} = Tot^{\cdot}(C^{\cdot}) \to Tot^{\cdot}(\mathcal{I}^{\cdot,\cdot})$. This is compatible with a morphism between their I^{th} spectral sequences. Using the fact that $E_1^{p,q}(C^{\cdot}) = H^q(C^{\cdot})$ if $p = 0$ and is 0 otherwise, we get that the map $E_1^{p,q}(C^{\cdot}) \to E_1^{p,q}(\mathcal{I}^{\cdot,\cdot})$ is just the map $H^q(C^{\cdot}) \to H^q(\mathcal{I}^{0,\cdot})$. Thus, $E_2^{p,q}(C^{\cdot}) \cong E_2^{p,q}(\mathcal{I}^{\cdot,\cdot})$ and we have proved our claim.

We now define hyperderived functors.

Definition 61. Let $\mathcal{A}$ be an abelian category with enough injectives and let $\mathcal{B}$ be another abelian category. Let $F : \mathcal{A} \to \mathcal{B}$ be a left exact (covariant) functor. Let $C^{\cdot}$ be an object of $\mathcal{C}^+(\mathcal{A})$. The *n-th hyperderived functor* of F, $\mathbb{R}^n F(\cdot)$, is a functor from $\mathcal{C}^+(\mathcal{A})$ to $\mathcal{B}$ which is defined as follows:

$$\mathbb{R}^n F(C^{\cdot}) = H^n(F(Tot^{\cdot}(\mathcal{I}^{\cdot,\cdot}))) = H^n(Tot^{\cdot}(F(\mathcal{I}^{\cdot,\cdot})))$$

where $C^{\cdot} \to \mathcal{I}^{\cdot,\cdot}$ is a Cartan-Eilenberg resolution of $C^{\cdot}$.

If $f^{\cdot} : C^{\cdot} \to D^{\cdot}$ is a morphism in $\mathcal{C}^+(\mathcal{A})$ and

$$0 \to C^{\cdot} \to \mathcal{I}^{0,\cdot} \to \mathcal{I}^{1,\cdot} \to \cdots$$

and,

$$0 \to D^{\cdot} \to \mathcal{J}^{0,\cdot} \to \mathcal{J}^{1,\cdot} \to \cdots$$

are Cartan-Eilenberg resolutions of $C^{\cdot}$ and $D^{\cdot}$ respectively, then in particular they are injective resolutions in (the abelian category) $\mathcal{C}^+(\mathcal{A})$ and so there exists a map of complexes $I^{p,\cdot} \to J^{p,\cdot}$ which is well defined upto homotopy.

Lemma 12. *Let $f^{\cdot,\cdot} : A^{\cdot,\cdot} \to B^{\cdot,\cdot}$ and $g^{\cdot,\cdot} : A^{\cdot,\cdot} \to B^{\cdot,\cdot}$ be two morphisms of double complexes in $\mathcal{A}$. Consider these as morphisms of complexes in $\mathcal{C}^+(\mathcal{A})$, $A^{p,\cdot} \to B^{p,\cdot}$ where $A^{p,\cdot}$ and $B^{p,\cdot}$ are objects of $\mathcal{C}^+(\mathcal{A})$. Now suppose f, g are homotopic. Then, $Tot^{\cdot}(f^{\cdot,\cdot})$ and $Tot^{\cdot}(g^{\cdot,\cdot})$ are also homotopic.*

Corollary 12. $\mathbb{R}^n F(C^{\cdot})$ *is independent of the Cartan-Eilenberg resolution and gives a well defined additive functor from $\mathcal{C}^+(\mathcal{A})$ to $\mathcal{B}$.*

We now discuss the two spectral sequences for hyperderived functors $\mathbb{R}^n F(C^{\cdot}) = H^n(Tot^{\cdot}(F(\mathcal{I}^{\cdot,\cdot})))$. The E_1 terms for the spectral sequence with the II^{th} filtration is is ${}_{II}E_1^{p,q} = H^q(F(\mathcal{I}^{\cdot}, p), d_1)$. We have that $\mathcal{I}^{\cdot,p}$ is an injective resolution for C^p and so $H^n(F(I^{\cdot,p}), d_1) = R^n F(C^p)$. Therefore we have the following spectral sequence

$$E_1^{p,q} = R^q F(C^p) \Rightarrow \mathbb{R}^{p+q} F(C^{\cdot}).$$

This is called the *first spectral sequence for hyperderived functors.*

The other spectral sequence is

$${}_{I}E_1^{p,q} = H^q(F(\mathcal{I}^{p,\cdot}), d_2) \Rightarrow \mathbb{R}^n F(C^{\cdot}).$$

$(\mathcal{I}^{p,\cdot}, d_2)$ has the following form:

$$\mathcal{I}^{p,q} = \mathcal{I}_0^{p,q} \oplus \mathcal{I}_1^{p,q} \oplus \mathcal{I}_2^{p,q}$$

and d_2 has the following form,

$$d_2|_{\mathcal{I}_0^{p,q} \oplus \mathcal{I}_1^{p,q}} = 0$$

and

$$d_2 : \mathcal{I}_2^{p,q} \to \mathcal{I}_0^{p,q+1}$$

is an isomorphism.

$(F(\mathcal{I}^{p,\cdot}), d_2)$ has a similar description. In particular,

$$H^q(F(\mathcal{I}^{p,\cdot}), d_2) \cong F(\mathcal{I}_1^{p,q}) = F(H^q(\mathcal{I}^{p,\cdot}), d_2).$$

But $\mathcal{I}^{\cdot,\cdot}$ is a Cartan-Eilenberg resolution and so $H^q(\mathcal{I}^{p,\cdot}, d_2)$ is an injective resolution of $H^q(C^\cdot)$. Also the E_1 differentials are just the differentials of the complexes $F(H^q(C^\cdot))$. Therefore, we have a spectral sequence,

$$_IE_2^{p,q} = \mathbb{R}^pF(H^q(C^\cdot)) \Rightarrow \mathbb{R}^{p+q}F(C^\cdot)$$

which is called the *second spectral sequence for hyperderived functors*. We will use these two spectral sequences for hyperderived functors to prove the Grothendieck spectral sequence theorem. But before that we make some more remarks and definitions.

Definition 62. Let $\mathcal{A} = \mathbf{Sh(X)}$ be the category of sheaves on a topological space X, let $\mathcal{B} = \mathbf{Ab}$ be the category of abelian groups and let $F = \Gamma(X, \cdot)$ be the global section functor. Then the *nth hypercohomology group of $\mathcal{F}^\cdot$* is defined to be the group $\mathbb{R}^nF(\mathcal{F}^\cdot)$ and is denoted by $\mathbb{H}^n(X, \mathcal{F}^\cdot)$. The two hypercohomology spectral sequences translate to the following two spectral sequences in this case.

$$E_1^{p,q} = H^q(X, \mathcal{F}^p) \Rightarrow \mathbb{H}^{p+q}(X, \mathcal{F}^\cdot)$$

and

$$E_2^{p,q} = H^P(X, \mathcal{H}^q(\mathcal{F}^\cdot)) \Rightarrow \mathbb{H}^{p+q}(X, \mathcal{F}^\cdot).$$

Definition 63. Let $f : X \to Y$ be a map of topological spaces, let $\mathcal{A} = \mathbf{Sh(X)}$, $\mathcal{B} = \mathbf{Sh(Y)}$ and $F = f_*$. Then the *nth hyperdirect image sheaf* of $\mathcal{F}^\cdot$ is defined to be the sheaf $\mathbb{R}^nf_*\mathcal{F}^\cdot$. The hypercohomology group is a special case of the hyperdirect image sheaf, when $Y =$ point. In this case, we again have two spectral sequences

$$E_1^{p,q} = R^qf_*\mathcal{F}^p \Rightarrow \mathbb{R}^{p+q}f_*\mathcal{F}^\cdot$$

and,

$$E_2^{p,q} = R^pf_*\mathcal{H}^q(\mathcal{F}^\cdot) \Rightarrow \mathbb{R}^{p+q}f_*\mathcal{F}^\cdot.$$

The two spectral sequences for hyperderived functors are functorial for maps of complexes. We have the following proposition.

Proposition 18. 1. If $f^\cdot : C^\cdot \to D^\cdot$ is a morphism of complexes such that $R^pF^q : R^pF(C^q) \to R^pF(D^q)$ is an isomorphism for all p, q then $\mathbb{R}^nf : \mathbb{R}^nF(C^\cdot) \to \mathbb{R}^nF(D^\cdot)$ is an isomorphism for all n.

2. If $f^{\cdot} : C^{\cdot} \to D^{\cdot}$ is an isomorphism on cohomology objects ($f^{\cdot}$ is then said to be a *quasi-isomorphism*) then, $\mathbb{R}^n f : \mathbb{R}^n F(C^{\cdot}) \to \mathbb{R}^n F(D^{\cdot})$ is an isomorphism for all n.
3. If $f^{\cdot}$ and $g^{\cdot}$ are maps $C^{\cdot} \to D^{\cdot}$ which are homotopic then $\mathbb{R}^n f = \mathbb{R}^n g$.
4. If $0 \to C^{\cdot} \to \mathcal{I}^{0,\cdot} \to \mathcal{I}^{1,\cdot} \to \cdots$ is an injective resolution of $C^{\cdot}$ in $\mathcal{C}^+(\mathcal{A})$ then there exists a natural isomorphism $\mathbb{R}^n F(C^{\cdot}) \cong H^n(Tot^{\cdot}(F(\mathcal{I}^{\cdot,\cdot})))$.
5. If $0 \to C_1^{\cdot} \to C_2^{\cdot} \to C_3^{\cdot} \to 0$ is an exact sequence in $\mathcal{C}^+(\mathcal{A})$ then there exists a functorial long excat sequence

$$\cdots \to \mathbb{R}^n F(C_1^{\cdot}) \to \mathbb{R}^n F(C_2^{\cdot}) \to \mathbb{R}^n F(C_3^{\cdot}) \to r^{n+1} F(C_1^{\cdot}) \to \cdot$$

We now prove the existence of the Grothendieck spectral sequence as stated in the theorem i.e., there exists a spectral sequence

$$E_2^{p,q} = R^p G(R^q F(C)) \Rightarrow R^{p+q}(G \cdot F)(C)$$

which is functorial for C in $\mathcal{A}$ where $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{C}$ are left exact functors and we assume that for any injective object I of $\mathcal{A}$ we have that $R^p G(F(I)) = 0$ for all $p > 0$. This will, as earlier remarked, prove the existence of the Leray spectral sequence.

Proof. Let $0 \to C \to I^{\cdot}$ be an injective resolution. Therefore, $R^n(G{\cdot}F)(C) = H^n(G \cdot F(I^{\cdot}))$. Consider the two spectral sequences for the hyperderived functor $\mathbb{R}^n G(F(I^{\cdot}))$. The first spectral sequence in this case is

$$E_1^{p,q} = R^q G(F(I^p)) \Rightarrow \mathbb{R}^n G(F(I^{\cdot})).$$

By assumption, we have that $E_1^{p,q} = 0$ if $q \neq 0$ since $F(I^p)$ is G-acyclic for all $p > 0$. For the nonzero E_1 terms the differentials are the obvious maps $E_1^{p,0} = G(F(I^p)) \to G(F(I^{p+1})) = E_1^{p+1,0}$. Therefore,

$$R^n(G \cdot F)(C) = H^n(G \cdot F(I^{\cdot})) = E_2^{n,0} = E_\infty^{n,0} = \mathbb{R}^n(G \cdot F)(I^{\cdot})$$

The other E_2 and hence E_∞ terms (for $q \neq 0$ are 0. Now the second spectral sequence for hyperderived functors is

$$E_2^{p,q} = R^p G(R^q F(C)) = \mathbb{R}^p G(H^q(F(I^{\cdot}))) \Rightarrow \mathbb{R}^{p+q} G(F(I^{\cdot})) = R^{p+q}(G{\cdot}F)(C)$$

by the above step. This is what we wanted to prove. □

We also have the Leray spectral sequence with supports. Let $f : X \to Y$, Ψ be a family of supports on Y and let $\Phi = f^{-1}(\Psi) = \{Z \subseteq X | Z \text{ is closed and } f(Z) \subseteq W \text{ for some } W \in \Psi\}$. Let $F = f_*$ be as before and let $G = \Gamma_\Psi$. Then $G \cdot F = \Gamma_\phi$ and hence there exists a spectral sequence

$$E_2^{p,q} = H_\Psi^p(Y, R^q f_* \mathcal{F}) \Rightarrow H_\Phi^{p+q}(X, \mathcal{F}).$$

Similarly, let $f : X \to Y$ and $g : Y \to Z$ be maps between topological spaces. Let $\mathcal{A} = \mathbf{Sh}(\mathbf{X})$, $\mathcal{B} = \mathbf{Sh}(\mathbf{Y})$ and $\mathcal{C} = \mathbf{Sh}(\mathbf{Z})$. Then there exists a spectral sequence of sheaves,

$$E_2^{p,q} = R^p g_*(R^q f_* \mathcal{F}) \Rightarrow R^{p+q}(g \cdot f)_*(\mathcal{F})$$

for all $\mathcal{F}$ in $\mathbf{Sh}(\mathbf{X})$ and functorial in $\mathcal{F}$.

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