COHOMOLOGY OF COHERENT SHEAVES

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The basic source for these lectures is Hartshorne's book [Ha], particularly Chapter III. Following the conventions in this book, we will assume that all schemes under consideration are Noetherian, unless specified otherwise. Our treatment more or less follows that in [Ha], except that we make use of spectral sequences to modify and "simplify" certain arguments.

1. Cohomology on affine schemes

The goal of this section is to discuss two basic results on cohomology for affine schemes. The first is the following vanishing theorem.

Theorem 1.1. *Let $X = \operatorname{Spec} A$ be a Noetherian affine scheme (i.e., A is a commutative Noetherian ring). Let $\mathcal{F}$ be a quasi-coherent sheaf of $\mathcal{O}_X$-modules on X. Then $H^i(X, \mathcal{F}) = 0$ for all $i > 0$.*

Proof. Recall that any quasi-coherent sheaf $\mathcal{F}$ on the affine scheme $X = \operatorname{Spec} A$ is of the form $\widetilde{M}$, where M is the A-module of global sections of $\mathcal{F}$. Also recall that the stalk at $x \in X$ of $\widetilde{M}$ is the localization $M_\wp$, where $\wp$ is the prime ideal in A corresponding to $x \in \operatorname{Spec} A$. A sequence

$$\mathcal{F} \to \mathcal{G} \to \mathcal{H}$$

of quasi-coherent sheaves is exact if and only if the corresponding sequence of A-modules

$$H^0(X, \mathcal{F}) \to H^0(X, \mathcal{G}) \to H^0(X, \mathcal{H})$$

is exact. We recall why: the sheaf sequence is exact precisely when its sequence of stalks is exact for each $x \in X$, which is the same as saying that the localizations at all primes $\wp$ of the sequence of global sections is exact; but for a sequence of A-modules, exactness at all such localizations is equivalent to exactness for the sequence itself.

The basic lemma needed to prove Theorem 1.1 is the following.

Lemma 1.2. *Let I be an injective A-module, $\widetilde{I}$ the associated quasi-coherent $\mathcal{O}_X$-module. Then $\widetilde{I}$ is flasque; in particular, $H^i(X, \widetilde{I}) = 0$ for all $i > 0$.*

Proof. We must show that for any open set $U \subset X = \operatorname{Spec} A$, the restriction $I = \Gamma(X, \widetilde{I}) \to \Gamma(U, \widetilde{I})$ is surjective. We first see that this is true for $U = D(f) = \operatorname{Spec} A_f$, for any $f \in A$.

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Suppose for simplicity that f is a non zero-divisor. Then applying the functor $\mathrm{Hom}_A(-, I)$ to the exact sequence

$$0 \to A \xrightarrow{f} A \to A/fA \to 0$$

the injectivity of I implies that multiplication by f on I is *surjective*. This immediately implies that $I \to I \otimes A_f = I_f$ is surjective.

In general, even if f is a zero divisor, there exist n such that $\mathrm{Ann}\,(f^n) = \mathrm{Ann}\,(f^{n+m})$ for all $m > 0$ (since A is Noetherian, and $\{\mathrm{Ann}\,(f^n)\}_{n\geq 1}$ is an ascending chain of ideals). Clearly f is a non zero-divisor on $A/\mathrm{Ann}\,(f^n)$. Hence multiplication by f is surjective on $I' = \mathrm{Hom}_A(A/\mathrm{Ann}\,(f^n), I) \subset I$, and so $I' \to I'_f$ is surjective. Since $f^n I \subset I' \subset I$, we have that $I'_f = I_f$, and so $I \to I_f$ is also surjective.

Now consider the case of an arbitrary open subset $U \subset \mathrm{Spec}\,A$. We use Noetherian induction on $Y = \overline{\mathrm{supp}\,I}$, where $\mathrm{supp}\,I$ is the subset of primes $\wp$ of A with $I_\wp \neq 0$ (the overbar denotes Zariski closure). If Y is a point, then $\widetilde{I}$ is a skyscraper sheaf; hence it is flasque. In general, we may as well assume $U \cap Y \neq \emptyset$ (or else $\Gamma(U, \widetilde{I}) = 0$, which is a trivial case). Then we can find $f \in A$ such that $D(f) = \mathrm{Spec}\,A_f \subset U$ has non-empty intersection with $U \cap Y$. If $s \in \Gamma(U, \widetilde{I})$, choose $t \in I$ such that $s \mid_{D(f)} = t \mid D(f)$ (we can do this because $I \to I_f = \Gamma(D(f), \widetilde{I})$ is surjective). Replacing s by $s - t \mid_U$, we may assume $s \mid_{D(f)} = 0$. Thus, if

$$J = \{a \in I \mid f^n a = 0 \text{ for some } n > 0\} = \ker(I \to I_f),$$

then $s \in \Gamma(U, \widetilde{J}) \subset \Gamma(U, \widetilde{I})$.

Now $Y' = \overline{\mathrm{supp}\,J}$ satisfies $Y' \cap D(f) = \emptyset$, so $Y' \subset Y$ is a proper subscheme. Thus, to complete the proof by induction, it suffices to observe that J is also an injective A-module. The proof of this, using the Artin-Rees lemma, is left as an exercise to the reader. □

We can now easily complete the proof of Theorem 1.1. Let $\mathcal{F} = \widetilde{M}$ be any quasi-coherent $\mathcal{O}_X$-module. Let

$$0 \to M \to I_0 \to I_1 \to \cdots \tag{1.1}$$

be a resolution of M by injective A-modules. Then we have an associated exact sequence of quasi-coherent $\mathcal{O}_X$-modules

$$0 \to \widetilde{M} \to \widetilde{I_0} \to \widetilde{I_1} \to \cdots \tag{1.2}$$

where the sheaves $\widetilde{I_j}$ are all flasque. Hence there are canonical isomorphisms between the cohomology module $H^j(X, \widetilde{M})$ and the cohomology modules of the complex

$$0 \to \Gamma(X, \widetilde{I_0}) \to \Gamma(X, \widetilde{I_1}) \to \cdots$$

which is just the complex

$$0 \to I_0 \to I_1 \to \cdots$$

This complex has H^0 equal to M, and vanishing H^j for $j > 0$, from the exact sequence (1.1). □

The second result is a characterization of Noetherian affine schemes, essentially due to Serre.

Theorem 1.3. *Let X be a Noetherian scheme. Then X is affine $\iff H^1(X, \mathcal{I}) = 0$ for any coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_X$.*

Proof. If X is affine, the higher cohomology of any quasi-coherent sheaf vanishes, in particular H^1 of any coherent ideal sheaf vanishes.

So assume $H^1(X, \mathcal{I}) = 0$ for any coherent ideal sheaf $\mathcal{I} \subset \mathcal{O}_X$. The goal is to prove $X = \operatorname{Spec} A$ where $A = \Gamma(X, \mathcal{O}_X)$. If $U = \operatorname{Spec} B$ is any affine open subset of X, the ring homomorphism $A = \Gamma(X, \mathcal{O}_X) \to \Gamma(U, \mathcal{O}_X) = B$ induces a morphism $U = \operatorname{Spec} B \to \operatorname{Spec} A$. These locally defined morphism are easily seen to patch together to give a morphism $X \to \operatorname{Spec} A$. We will show that this is an isomorphism.

We first show that X can be covered by open subsets of the form X_f, for suitable $f \in A = \Gamma(X, \mathcal{O}_X)$, where $X_f = \{x \in X \mid f(x) \neq 0\}$; here $f(x)$ denotes the image of f in the residue field $k(x)$. Let $x \in X$ be a closed point, and $U = \operatorname{Spec} B$ an affine open neighbourhood of x in X, with complement $Y = X \setminus U$ (with its reduced structure). Then $Y \cup \{x\}$ determines a reduced closed subscheme of X. There is an exact sequence of sheaves

$$0 \to \mathcal{I}_{Y \cup \{x\}} \to \mathcal{I}_Y \to (i_x)_* k(x) \to 0$$

where the first two terms are the respective ideal sheaves, and $(i_x)_* k(x)$ is a skyscraper sheaf at x with stalk $k(x)$. Since $H^1(X, \mathcal{I}_{Y \cup \{x\}}) = 0$, we can lift $1 \in k(x) = \Gamma(X, (i_x)_* k(x))$ to a global section $f \in \Gamma(X, \mathcal{I}_Y) \subset \Gamma(X, \mathcal{O}_X) = A$. In other words, we can find $f \in A = \Gamma(X, \mathcal{O}_X)$ with $f(x) = 1$, and so $x \in X_f$; further, f vanishes on Y, so $X_f \subset U$, *i.e.*, $X_f = U_f = \operatorname{Spec} B_f$.

We claim that the natural map $A_f \to B_f$, induced by $A \to \Gamma(X_f, \mathcal{O}_X) = B_f$, is an isomorphism (this will imply that $X \to \operatorname{Spec} A$ is an *open immersion*). First, suppose $a \in \ker(A \to B_f)$. Then for any affine open $\operatorname{Spec} C = V \subset X$, the image of a in $C_f = \Gamma(V \cap X_f, \mathcal{O}_X)$ vanishes, hence the image of a in C is annihilated by a power of f. Covering X by a finite number of such open sets V, we see that $a \in \Gamma(X, \mathcal{O}_X) = A$ is annihilated by a power of f; thus, $A_f \to B_f$ is injective. Also, similarly, if $W \subset X$ is any open subset, then any element of $\ker \Gamma(W, \mathcal{O}_X) \to \Gamma(W \cap X_f, \mathcal{O}_X)$ is annihilated by a power of f. Next, let $a \in \Gamma(X_f, \mathcal{O}_X) = B_f$. We claim that for some $n > 0$, af^n is in the image of $\Gamma(X, \mathcal{O}_X) = A \to B_f$. The restriction of a to $\Gamma(V \cap X_f, \mathcal{O}_X) = C_f$ is such that $f^n a$ lifts to $C = \Gamma(V, \mathcal{O}_X)$ for some n. A finite number of such open sets V_i cover X, so we may assume(after increasing n if needed) that $f^n a \mid_{V \cap X_f}$ lifts to $a_i \in \Gamma(V_i, \mathcal{O}_X)$ for each i. We have that

$$a_{ij} = a_i \mid_{V_i \cap V_j} - a_j \mid_{V_i \cap V_j} \in \Gamma(V_i \cap V_j, \mathcal{O}_X)$$

has zero restriction to $V_i \cap V_j \cap X_f$; hence we can find $m > 0$ with $f^m a_{ij} = 0$ for all i, j. Now the collection of sections $f^m a_i \in \Gamma(V_i, \mathcal{O}_X)$ patch up to a unique global section of $\mathcal{O}_X$ which lifts $f^{n+m}a$. We have now shown that $A_f \to B_f$ is surjective as well.

Thus X has an open cover by open subschemes of the form $X_f = \operatorname{Spec} A_f$, each of which is affine, and the natural map $X \to \operatorname{Spec} A$ is an open immersion whose image is the union of these open subsets $\operatorname{Spec} A_f$. Since X is Noetherian, this open cover of X has a finite subcover, say $X_{f_1}, \ldots, X_{f_r}$. Then the functions $f_1, \ldots, g_r$ have no common zeroes on X. So the sheaf map $\mathcal{O}_X^{\oplus r} \to \mathcal{O}_X$, given by $(g_1, \ldots, g_r) \mapsto g_1 f_1 + \cdots + g_r f_r$ for any local sections $g_1, \ldots, g_r$, is a surjection of sheaves, giving an exact sequence

$$0 \to \mathcal{F} \to \mathcal{O}_X^{\oplus r} \to \mathcal{O}_X \to 0.$$

The sheaf $\mathcal{O}_X^{\oplus r}$ has a filtration by the subsheaves $\mathcal{O}_X^{\oplus i}$ for $0 \le i \le r$, included through the first i summands. The induced filtration $\mathcal{F} \cap \mathcal{O}_X^{\oplus i}$ of $\mathcal{F}$ has graded pieces which are isomorphic to ideal sheaves. Hence by induction on i, we have $H^1(X, \mathcal{F} \cap \mathcal{O}_X^{\oplus i}) = 0$, and for $i = r$ this is just $H^1(X, \mathcal{F}) = 0$. Hence $\Gamma(X, \mathcal{O}_X^{\oplus r}) \to \Gamma(X, \mathcal{O}_X)$ is surjective, in other words, $A^{\oplus r} \to A$, $(a_1, \ldots, a_r) \mapsto \sum_i a_i f_i$, is surjective. Hence $f_1, \ldots, f_r$ generate the unit ideal in A, and so $\operatorname{Spec} A_{f_1}, \ldots, \operatorname{Spec} A_{f_r}$ cover $\operatorname{Spec} A$. □

2. Čech cohomology and the cohomology of projective space

We first review the basics of Čech cohomology. Recall that if $\mathcal{U} = \{U_0, \cdots, U_n\}$ is a finite open cover of a topological space X, then for any sheaf $\mathcal{F}$ of abelian groups on X, we have an associated Čech complex $\check{C}^\bullet(\mathcal{U}, \mathcal{F})$ with terms

$$\check{C}^p(\mathcal{U}, \mathcal{F}) = \prod_{0 \le i_0 < \cdots < i_p \le n} \mathcal{F}(U_{i_0} \cap \cdots \cap U_{i_p}),$$

and differential maps $\delta^p : \check{C}^p(\mathcal{U}, \mathcal{F}) \to \check{C}^{p+1}(\mathcal{U}, \mathcal{F})$ given by

$$(\delta^p \alpha)_{i_0, \ldots, i_{p+1}} = \sum_{j=0}^{p+1} (-1)^j \alpha_{i_0, \ldots, \widehat{i_j}, \ldots, i_{p+1}} |_{U_{i_0} \cap \cdots \cap U_{i_{p+1}}},$$

where $\widehat{i_j}$ means that the index i_j is omitted. Then $(\check{C}^\bullet(\mathcal{U}, \mathcal{F}), \delta^\bullet)$ is a complex, called the *Čech complex* of $\mathcal{F}$ with respect to $\mathcal{U}$, whose cohomology groups are called the *Čech cohomology* groups of $\mathcal{F}$ with respect to $\mathcal{U}$, and are denoted by $\check{H}^i(\mathcal{U}, \mathcal{F})$.

There is a natural map $\check{H}^i(\mathcal{U}, \mathcal{F}) \to H^i(X, \mathcal{F})$ for each i (compare [Ha], III, (4.4)). This is constructed as follows. For any open $V \subset X$, let $\mathcal{F}_V$ denote the sheaf $j_* j^{-1} \mathcal{F}$, where $j : V \hookrightarrow X$ is the inclusion (this is non-standard, temporary notation!). Then the same formula for the Čech differential gives rise to a complex of sheaves $\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F})$ with terms

$$\check{\mathcal{C}}^p(\mathcal{U}, \mathcal{F}) = \prod_{0 \le i_0 < \cdots < i_p \le n} \mathcal{F}_{U_{i_0} \cap \cdots \cap U_{i_p}}.$$

One also has a natural augmentation $\mathcal{F} \to \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F})$.

Lemma 2.1. *The complex of sheaves*

$$0 \to \mathcal{F} \to \check{\mathcal{C}}^0(\mathcal{U}, \mathcal{F}) \to \check{\mathcal{C}}^1(\mathcal{U}, \mathcal{F}) \to \cdots$$

is a resolution of $\mathcal{F}$, whose complex of global sections is the Čech complex $\check{C}^\bullet(\mathcal{U}, \mathcal{F})$.

Proof. The lemma is proved by showing that the complex of stalks at any point is chain contractible; the details are left to the reader as an exercise. □

Now if $0 \to \mathcal{F} \to \mathcal{I}^\bullet$ is any injective resolution of $\mathcal{F}$, the universal property of injectives gives rise to a map (unique up to chain homotopy) of complexes of sheaves $\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}) \to \mathcal{I}^\bullet$, over the identity map on $\mathcal{F}$. The induced maps between complexes of global sections gives rise to the desired maps $\check{H}^i(\mathcal{U}, \mathcal{F}) \to H^i(X, \mathcal{F})$. From the defining properties of a sheaf, one sees at once that $\check{H}^0(\mathcal{U}, \mathcal{F}) \to H^0(X, \mathcal{F})$ is always an isomorphism; with a little more work, one sees also that $\check{H}^1(\mathcal{U}, \mathcal{F}) \to H^1(X, \mathcal{F})$ is injective for any sheaf of abelian groups $\mathcal{F}$.

Proposition 2.2. (Leray's theorem) *Let $\mathcal{F}$ be a sheaf of abelian groups on X, and $\mathcal{U} = \{U_0, \dots, U_n\}$ an open covering such that $H^j(U_{i_0} \cap \cdots \cap U_{i_p}, \mathcal{F}) = 0$ for all $\{i_0, \dots, i_p\}$ and all $j > 0$. Then the natural maps $\check{H}^i(\mathcal{U}, \mathcal{F}) \to H^i(X, \mathcal{F})$ are isomorphisms.*

Proof. We give a proof using spectral sequences (other proofs are possible as well). Let $0 \to \mathcal{F} \to \mathcal{I}^\bullet$ be a flasque resolution. Then for any open $U \subset X$, the complex $\Gamma(U, \mathcal{I}^\bullet)$ computes $H^*(U, \mathcal{F}\mid_U)$.

The Čech complexes of the $\mathcal{I}^q$ determine a double complex of abelian groups

$$C^{p,q} = \check{C}^p(\mathcal{U}, \mathcal{I}^q),$$

with augmentations $\check{C}^p(\mathcal{U}, \mathcal{F}) \to C^{p,0}$. From the hypotheses of Leray's theorem, it follows that for each p, the sequences

$$0 \to \check{C}^p(\mathcal{U}, \mathcal{F}) \to C^{p,0} \to C^{p,1} \to \cdots$$

are exact (the cohomology group of this complex in any degree $i > 0$ is

$$\prod_{0 \le i_0 < \cdots < i_p} H^i(U_{i_0} \cap \cdots \cap U_{i_p}, \mathcal{F}),$$

which is given to vanish). Hence, one of the spectral sequences for the double complex $C^{p,q}$ degenerates at E_2, and the natural maps $H^p(\check{C}^\bullet(\mathcal{U}, \mathcal{F})) \to H^p(Tot(C^{\bullet,\bullet}))$ are isomorphisms for all p.

On the other hand, there are augmented complexes

$$0 \to H^0(X, \mathcal{I}^q) \to C^{0,q} \to C^{1,q} \to \cdots$$

which may be viewed as obtained by taking global sections in the exact sequence (lemma 2.1) of sheaves

$$0 \to \mathcal{I}^q \to \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{I}^q).$$

By construction, $\check{\mathcal{C}}^p(\mathcal{U}, \mathcal{I}^q)$ is flasque for each $p \ge 0$. Hence all sheaves in the above sequence are flasque, so that the corresponding sequence of global sections is exact. This implies that the second spectral sequence for the double complex

$C^{p,q}$ degenerates at E_2, and the natural maps $H^p(H^0(X,\mathcal{I}^\bullet)) \to H^p(Tot(C^{\bullet,\bullet}))$ are also isomorphisms. Note that $H^p(H^0(X,\mathcal{I}^\bullet)) \cong H^p(X,\mathcal{F})$ by definition.

The resulting isomorphism between the Čech and derived functor cohomology of $\mathcal{F}$ is given by the natural map between these, since we may choose $\mathcal{I}^\bullet$ to be an injective resolution of $\mathcal{F}$, in which case $Tot(\check{\mathcal{C}}^\bullet(\mathcal{U},\mathcal{I}^\bullet))$ is also an injective resolution of $\mathcal{F}$, such that the augmentation map $\check{\mathcal{C}}^\bullet \to Tot(\check{\mathcal{C}}^\bullet(\mathcal{U},\mathcal{I}^\bullet))$ is a morphism of complexes lifting the identity map on $\mathcal{F}$. □

Corollary 2.3. *Let X be a Noetherian separated scheme, $\mathcal{U} = \{U_0, \ldots, U_n\}$ an open covering by affine open subsets $U_i = \operatorname{Spec} A_i$. Then for any quasi-coherent sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules, the canonical maps $\check{H}^i(\mathcal{U},\mathcal{F}) \to H^i(X,\mathcal{F})$ are isomorphisms for all i. In particular, $H^i(X,\mathcal{F}) = 0$ for all $i > n$ and all quasi-coherent sheaves $\mathcal{F}$.*

Proof. Since X is separated, the intersection of any finite number of affine open subsets of X is again an affine open set. Hence by Theorem 1.1, any quasi-coherent sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules satisfies the hypotheses of Leray's theorem (Theorem 2.2). The vanishing assertion is obvious for the Čech cohomology, since $\check{C}^i(\mathcal{U},\mathcal{F}) = 0$ for all $i > n$. □

For example, if A is a Noetherian ring, then on $\mathbb{P}^n_A$, any quasi-coherent sheaf has vanishing cohomology in degrees $> n$, since $\mathbb{P}^n_A$ has a covering by $n+1$ affine open subsets.

Apart from its theoretical significance, the above corollary is one of the basic tools in making concrete calculations with cohomology. We illustrate this by determining the cohomology on $\mathbb{P}^n_A$ of the sheaf $\mathcal{O}_{\mathbb{P}^n_A}(r)$, for any integer r. Recall that $\mathbb{P}^n_A = \operatorname{Proj} A[X_0, \ldots, X_n]$ is obtained from a polynomial algebra over A in $n+1$ variables, with the standard grading. Let $S = A[X_0, \ldots, X_n] = \oplus_{r\geq 0} S_r$, where S_r is the homogeneous component of degree r; we may view S as $\mathbb{Z}$-graded with vanishing homogeneous components of negative degree. For any $r \in \mathbb{Z}$, let $S(r)$ denote the ring S, considered as a graded S-module with $S(r)_t = S_{r+t}$; then $\mathcal{O}_{\mathbb{P}^n_A}(r)$ is the corresponding quasi-coherent (in fact coherent, even invertible) sheaf $\widetilde{S(r)}$. By construction, there is a canonical homomorphism $S_r \to \Gamma(\mathbb{P}^n_A, \mathcal{O}_{\mathbb{P}^n_A}(r))$. Recall that if M is a $\mathbb{Z}$-graded S-module, and $h \in S$ is a homogeneous element, then $M_{(h)}$ denotes the 0-th homogeneous component of the (also $\mathbb{Z}$-graded) localization M_h. Thus $D_+(\mathbb{P}^n_A) = \operatorname{Spec} S_{(h)}$ form a basis, consisting of affine open sets, for the Zariski topology on $\mathbb{P}^n_A = \operatorname{Proj} S$.

Theorem 2.4. *Let A be a Noetherian ring.*

(a) *The canonical homomorphism $S_r \to \Gamma(\mathbb{P}^n_A, \mathcal{O}_{\mathbb{P}^n_A}(r))$ is an isomorphism for all r. In particular, $\Gamma(\mathbb{P}^n_A, \mathcal{O}_{\mathbb{P}^n_A}(r)) = 0$ for $r < 0$, and is a free A-module of rank $\binom{n+r}{r}$ for $r \geq 0$.*

(b) *$H^i(\mathbb{P}^n_A, \mathcal{O}_{\mathbb{P}^n_A}(r)) = 0$ for all $0 < i < n$ and all $r \in \mathbb{Z}$.*

(c) *$H^n(\mathbb{P}^n_A, \mathcal{O}_{\mathbb{P}^n_A}(-n-1))$ is a free A-module of rank 1, and the natural maps*

$$H^n(\mathbb{P}^n_A, \mathcal{O}_{\mathcal{P}^n_A}(-n-1-r)) \otimes_A \Gamma(\mathbb{P}^n_A, \mathcal{O}_{\mathbb{P}^n_A}(r)) \to H^n(\mathbb{P}^n_A, \mathcal{O}_{\mathbb{P}^n_A}(-n-1)) \cong A$$

are perfect pairings between free A-modules.

Proof. The idea is the compute the Čech cohomology groups with respect to the standard affine open covering $\mathcal{U} = \{U_i\}_{i=0}^n$, where

$$U_i = D_+(X_i) = \operatorname{Spec} A[X_0/X_i, \dots, X_n/X_i], \quad 0 \le i \le n.$$

For $0 \le i_0 < i_1 < \dots < i_p \le n$, we have

$$U_{i_0} \cap \cdots \cap U_{i_p} = \operatorname{Spec} A\left[X_i/X_j \mid i \neq j, \text{ and } j \in \{i_0, \dots, i_p\}\right] = \operatorname{Spec} S_{(X_{i_0} \cdots X_{i_p})}.$$

Next, one identifies the sections of $\mathcal{O}_{\mathbb{P}^n_A}(r)$ on $U_{i_0} \cap \cdots U_{i_p}$ with the A-submodule of homogeneous elements of degree r in the $\mathbb{Z}$-graded A-algebra

$$S_{X_{i_0} \cdots X_{i_p}} = S[\prod_{j=0}^{p} X_{i_j}^{-1}].$$

Let $\widetilde{U}_i = \operatorname{Spec} S[X_i^{-1}]$, which is an open subscheme of $\operatorname{Spec} S = \mathbb{A}_A^{n+1}$. Consider the Čech complex for the structure sheaf $\mathcal{O}_X$ for the affine open cover $\widetilde{\mathcal{U}} = \{\widetilde{U}_i\}_{i=0}^n$ of $X = \operatorname{Spec} S \setminus V(X_0, \dots, X_n)$, where $V(X_0, \dots, X_n) = \operatorname{Spec} S/(X_0, \dots, X_n)$ is the closed subscheme determined by the ideal $S_+ = (X_0, \dots X_n)$. The terms in this complex are of the form

$$\check{C}^p(\widetilde{\mathcal{U}}, \mathcal{O}_X) = \prod_{0 \le i_0 < \cdots < i_p \le n} \Gamma(\widetilde{U_{i_0}} \cap \cdots \cap \widetilde{U_{i_p}}, \mathcal{O}_X) = \prod_{0 \le i_0 < \cdots < i_p \le n} S_{x_{i_0} \cdots x_{i_p}}.$$

Note that this is a complex whose terms are $\mathbb{Z}$-graded A-algebras, and the differentials are graded A-linear maps. The subcomplex of homogeneous elements of degree r is identified with $\check{C}^\bullet(\mathcal{U}, \mathcal{O}_{\mathbb{P}^n_A}(r))$, as noted earlier. There is also an augmentation

$$S \to \check{C}^0(\widetilde{\mathcal{U}}, \mathcal{O}_X) = \prod_{i=0}^{n} S_{X_i},$$

induced by the diagonal inclusion.

We claim that

$$0 \to S \to \check{C}^0(\widetilde{\mathcal{U}}, \mathcal{O}_X) \to \cdots \to \check{C}^n(\widetilde{\mathcal{U}}, \mathcal{O}_X) \to 0 \tag{2.1}$$

is exact, except for the surjectivity of $\check{C}^{n-1}(\widetilde{\mathcal{U}}, \mathcal{O}_X) \to \check{C}^n(\widetilde{\mathcal{U}}, \mathcal{O}_X)$. This will imply (a) and (b) of Theorem 2.4, on considering the homogeneous subcomplex of degree r, for each $r \in \mathbb{Z}$. In fact the above complex can be considered as a direct limit of the augmented Koszul complexes

(2.2)

$$K_m = K(X_0^m, \dots, X_n^m) = (0 \to S \to S^{\oplus n+1} \to \cdots \to S^{\oplus \binom{n+1}{p}} \to \cdots \to S \to 0)$$

where $K_m \to K_{m+1}$ is induced as follows:

$$(K_m)_p = S^{\oplus \binom{n+1}{p}} = \prod_{0 \le j_1 < \cdots < j_p \le n} S$$

and $(K_m)_p \to (K_{m+1})_p$ is multiplication by $X_{j_1} \cdots X_{j_p}$ on the $(j_1, \dots, j_p)$-component. One verifies that this commutes with the differentials in the Koszul complex.

Since $X_0^m, \dots, X_n^m$ form a *regular sequence* on the polynomial algebra S, each of the Koszul complexes is exact except at the right end point, and $H^{n+1}(K_m) =$

$S/(X_0^m, \ldots, X_n^m)$. By exactness of direct limits, we conclude that the complex (2.1) is exact except at the right end point (proving (a) and (b) of the theorem), and the cokernel of the last arrow is

$$\varinjlim_m H^{n+1}(K_m).$$

Going back to the Čech complex (2.1) itself, the last term is just $S_{X_0 \cdots X_n} = A[X_0, X_0^{-1}, \ldots, X_n, X_n^{-1}]$, the Laurent polynomial algebra, which is the free A-module with basis $\prod_{i=0}^n X_i^{a_i}$ for all $a_0, \ldots, a_n \in \mathbb{Z}$. The penultimate term of (2.1) is

$$\oplus_{i=0}^n S_{X_0 \cdots \widehat{X_i} \cdots X_n},$$

where '$\widehat{X_i}$' means that the variable X_i is omitted. The differential (the last map in (2.1)) is the alternating sum of the natural inclusion maps. Clearly the cokernel is a free A-module with basis $X_0^{-a_0} \cdots X_n^{-a_n}$ with all $a_i > 0$, which we can rewite as the set of all monomials $M(b_0, \ldots, b_n) = (X_0 \cdots X_n)^{-1} \cdot (\prod_{i=0}^n X_0^{b_0} \cdots X_n^{b_n})^{-1}$, where $b_0, \ldots, b_n$ range over all non-negative integers. Keeping track of the grading, $M(b_0, \ldots, b_n)$ is homogeneous of degree $-n-1-(\sum_i b_i)$. Thus the cohomology module $H^n(\mathbb{P}^n_A, \mathcal{O}_{\mathbb{P}^n_A}(-n-1-r))$ is a free A-module with basis consisting of all monomials $M(b_0, \ldots, b_n)$ satisfying $\sum_i b_i = r$ (in particular, the set of such monomials is non-empty precisely when $r \geq 0$).

The pairing

$$H^0(\mathbb{P}^n_A, \mathcal{O}_{\mathbb{P}^n_A}(r)) \otimes_A H^n(\mathbb{P}^n_A, \mathcal{O}_{\mathbb{P}^n_A}(-n-1-r)) \to H^n(\mathbb{P}^n_A, \mathcal{O}_{\mathbb{P}^n_A}(-n-1)) \tag{2.3}$$

is determined as follows: a global section of $\mathcal{O}_{\mathbb{P}^n_A}(r)$ determines a maps of sheaves $\mathcal{O}_{\mathbb{P}^n_A} \to \mathcal{O}_{\mathbb{P}^n_A}(r)$, and hence by tensoring a sheaf map

$$\mathcal{O}_{\mathbb{P}^n_A}(-n-1-r) \to \mathcal{O}_{\mathbb{P}^n_A}(-n-1),$$

and thus also an A-linear map on n-th cohomology modules

$$H^n(\mathbb{P}^n_A, \mathcal{O}_{\mathbb{P}^n_A}(-n-1-r)) \to H^n(\mathbb{P}^n_A, \mathcal{O}_{\mathbb{P}^n_A}(-n-1)).$$

Identifying a global section of $\mathcal{O}_{\mathbb{P}^n_A}(r)$ with a homogeneous polynomial of degree r, multiplication by this polynomial on the terms of the Čech complex (2.1) determines the action on the cohomology modules. Clearly $H^0(\mathbb{P}^n_A, \mathcal{O}_{\mathbb{P}^n_A}(r))$ has a basis of monomials $M'(b_0, \ldots, b_n) = X_0^{b_0} \cdots X_n^{b_n}$ where $b_i \geq 0$, $\sum_i b_i = r$. Identify $H^n(\mathbb{P}^n_A, \mathcal{O}_{\mathbb{P}^n_A}(-n-1))$ with A using the basis element $[\prod_{i=0}^n X_i^{-1}]$, the image of the corresponding monomial $M(0, \ldots, 0)$. Then $\{M'(b_0, \ldots, b_n)\}$ is visibly the dual basis, with respect to the pairing (2.3), to the monomials $\{M(b_0, \ldots, b_n)\}$, since

$$M(b_0, \ldots, b_n) \cdot M'(c_0, \ldots, c_n) = \begin{cases} [\prod_{i=0}^n X_i^{-1}] & \text{if } b_i = c_i \text{ for all } i, \\ 0 & \text{otherwise.} \end{cases}$$

□

3. Finite generation of cohomology and related results

In this section we discuss three basic results on cohomology of coherent sheaves: the finite generation theorem, the Serre vanishing theorem for ample invertible sheaves, and Serre's criterion for ampleness (a sort of converse to Serre's vanishing theorem).

All of these results depend on the following lemma on quasi-coherent sheaves, which has been established earlier in the lectures on schemes. Let X be a Noetherian scheme, $\mathcal{L}$ an invertible $\mathcal{O}_X$-module, $s \in \Gamma(X, \mathcal{L})$, and let $X_s = \{x \in X \mid s(x) \neq 0\}$, where $s(x)$ denotes the image of s in the 1-dimensional $k(x)$-vector space $\mathcal{L}_x \otimes_{\mathcal{O}_{X,x}} k(x)$. Then X_s is an open subset of X, which determines an open subscheme. The global section s gives an $\mathcal{O}_X$-linear map $\mathcal{L}^{-1} \to \mathcal{O}_X$, whose image is a coherent ideal sheaf, corresponding to a closed subscheme Y of X, which we call the scheme of zeroes of s. Then $X_s = X \setminus Y$. As usual we denote $\mathcal{L}^{\otimes n}$ by $\mathcal{L}^n$.

Lemma 3.1. *Let X, $\mathcal{L}$, $s \in \Gamma(X, \mathcal{L})$ be as above. Let $\mathcal{F}$ be a coherent sheaf on X.*

(a) *If $t \in \ker(\Gamma(X, \mathcal{F}) \to \Gamma(X_s, \mathcal{L}))$, then $s^n t = 0 \in \Gamma(X, \mathcal{F} \otimes \mathcal{L}^n)$.*
(b) *If $t \in \Gamma(X_s, \mathcal{F})$, then for some $n \geq 0$, the section $s^n t \in \Gamma(X_s, \mathcal{F} \otimes \mathcal{L}^n)$ lies in the image of $\Gamma(X, \mathcal{F} \otimes \mathcal{L}^n)$, i.e., $s^n t$ extends to a global section.*

Theorem 3.2. *Let X be a projective scheme over a Noetherian ring A, and $\mathcal{O}_X(1)$ a very ample invertible sheaf on X relative to* Spec A. *Let $\mathcal{F}$ be a coherent sheaf on X.*

(a) *For each $i \geq 0$, $H^i(X, \mathcal{F})$ is a finitely generated A-module.*
(b) (Serre's Vanishing Theorem) *There exists an $n_0 \geq 0$ (depending on $(X, \mathcal{O}_X(1))$ and $\mathcal{F}$) such that for any $n \geq n_0$, $i > 0$, we have $H^i(X, \mathcal{F}(n)) = 0$.*

Proof. Let $i : X \hookrightarrow \mathbb{P}^r_A$ be a projective embedding such that $\mathcal{O}_X(1) = i^*\mathcal{O}_{\mathbb{P}^r_A}(1)$. Then i_* is exact, and for each $j \geq 0$ we have

$$H^j(X, \mathcal{F}(n)) = H^j(\mathbb{P}^r_A, i_*(\mathcal{F}(n))) = H^j(\mathbb{P}^r_A, (i_*\mathcal{F})(n)).$$

Hence we reduce to proving the Theorem for coherent sheaves $\mathcal{F}$ on $X = \mathbb{P}^r_A$.

We first show that for any coherent sheaf $\mathcal{F}$ on $\mathbb{P}^r_A$, we have that $H^r(\mathbb{P}^r_A, \mathcal{F})$ is finitely generated, and $H^r(\mathbb{P}^r_A, \mathcal{F}(n)) = 0$ for all large n. Now $\mathcal{F}(m)$ is generated by its global sections, for any sufficiently large m. Fix such a value of m. Since $\mathbb{P}^r_A$ is Noetherian and $\mathcal{F}(m)$ is coherent, a finite number of global sections suffice to generate $\mathcal{F}(m)$, giving rise to a surjection of sheaves $\mathcal{O}_{\mathbb{P}^r_A}^{\oplus N} \to \mathcal{F}(m)$. This is turn induces a surjection $\mathcal{O}_{\mathbb{P}^r_A}(-m)^{\oplus N} \to \mathcal{F}$, say with kernel $\mathcal{G}$, giving rise to an exact sequence

$$0 \to \mathcal{G}(n) \to \mathcal{O}_{\mathbb{P}^r_A}(n - m) \to \mathcal{F}(n) \to 0$$

for each $n \in \mathbb{Z}$. Now $H^i(\mathbb{P}^r_A, \mathcal{H}) = 0$ for all $i > r$ and all quasi-coherent sheaves $\mathcal{H}$ (Corollary 2.3). Hence from the exact cohomology sequence

$$\cdots \to H^r(\mathbb{P}^r_A, \mathcal{O}_{\mathbb{P}^r_A}(n - m)^{\oplus N}) \to H^r(\mathbb{P}^r_A, \mathcal{F}(n)) \to H^{r+1}(\mathbb{P}^r_A, \mathcal{G}(n)) \to \cdots$$

we conclude using Theorem 2.4 that $H^r(\mathbb{P}^r_A, \mathcal{F}(n))$ is a finitely generated A-module for all n, and that for sufficiently large n (in fact $n \geq m - r$), we have $H^r(\mathbb{P}^r_A, \mathcal{F}(n)) = 0$.

Now we proceed by descending induction on i to show that for any coherent $\mathcal{F}$, we have that $H^i(\mathbb{P}^r_A, \mathcal{F})$ is finitely generated, and if $i > 0$, then $H^i(\mathbb{P}^r_A, \mathcal{F}(n)) = 0$ for all sufficiently large n. We have already dealt with the case $i = r$. Applying this to $\mathcal{G}$, we conclude using the earlier part of the same cohomology exact sequence

$$\cdots \to H^{i-1}(\mathbb{P}^r_A, \mathcal{O}_{\mathbb{P}^r_A}(n-m)^{\oplus N}) \to H^{i-1}(\mathbb{P}^r_A, \mathcal{F}(n)) \to H^i(\mathbb{P}^r_A, \mathcal{G}(n)) \to \cdots$$

combined with Theorem 2.4, that $H^{i-1}(\mathbb{P}^r_A, \mathcal{F}(n))$ is always finitely generated, and if $i - 1 > 0$, vanishes for all large enough n. □

Corollary 3.3. *Let X be an integral projective scheme over and algebraically closed field k. Then $H^0(X, \mathcal{O}_X) = k$, or in other words, any global regular function on X is a constant.*

Proof. Since X is an integral projective k-scheme, $R = H^0(X, \mathcal{O}_X)$ is an integral domain, containing k. By Theorem 3.2, R is a finite dimensional vector space. Hence R must be a finite extension field of k, and since $k = \bar{k}$ this means $R = k$. □

Definition 3.4. X be a projective scheme over a field k, or more generally an Artinian ring A. Then for any coherent sheaf $\mathcal{F}$ on X, we can define its *Euler characteristic*

$$\chi(X, \mathcal{F}) = \chi(\mathcal{F}) = \sum_{i \geq 0} (-1)^i \ell(H^i(X, \mathcal{F})) \in \mathbb{Z},$$

where $\ell(M)$ denotes the length of any finite A-module.

Note that if

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$

is any exact sequence of coherent sheaves, then

$$\chi(X, \mathcal{F}) = \chi(X, \mathcal{F}') + \chi(X, \mathcal{F}''),$$

from the long exact sequence of cohomology modules

$$0 \to H^0(X, \mathcal{F}') \to H^0(X, \mathcal{F}) \to H^0(X, \mathcal{F}'') \to H^1(X, \mathcal{F}') \to \cdots \to H^n(X, \mathcal{F}'') \to 0,$$

where $n = \dim X$, which is a finite exact sequence of modules of finite length, for which the alternating sum of the lengths must vanish.

If $\mathcal{O}_X(1)$ is a very ample invertible sheaf on X relative to A, then we can also consider the numerical function

$$H(X, \mathcal{F})(n) = \chi(X, \mathcal{F}(n)), \ \forall\, n \in \mathbb{Z}.$$

Theorem 3.5. *Let $\mathcal{F}$ be a coherent sheaf on X, which is projective over* $\operatorname{Spec} A$, *where A is an Artinian local ring. Let $\mathcal{O}_X(1)$ be a very ample invertible sheaf on X. We then have the following.*

(a) *There is a polynomial* $P(t) \in \mathbb{Q}[t]$ *such that*

$$H(X, \mathcal{F})(n) = P(n) \ \ \forall\, n \in \mathbb{Z}.$$

We call $P(t)$ *the* Hilbert polynomial *of the sheaf* $\mathcal{F}$*. The degree of* P *equals the dimension of* $\operatorname{supp}(\mathcal{F})$*.*

(b) *If* $i : X \hookrightarrow \mathbb{P}^N_k$*, and* $\mathcal{O}_X(1)$ *is the resulting very ample sheaf, then* $P(t)$ *is the Hilbert polynomial of the homogeneous coordinate ring* $R(X)$ *of* X*; more generally, if* M *is any finite graded* $R(X)$*-module, and* $\mathcal{F} = \widetilde{M}$*, then* $P(t)$ *coincides with the Hilbert polynomial of* M *(that is,* $H(X, \mathcal{F})(n) = \ell(M_n)$ *for all large* n*.)*

Proof. If

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$

is an exact sequence of coherent sheaves, then $H(X, \mathcal{F})(n) = H(X, \mathcal{F}')(n) + H(X, \mathcal{F}'')(n)$ for all $n \in \mathbb{Z}$. Hence if (a) of the Theorem holds for any two of $\mathcal{F}$, $\mathcal{F}'$, $\mathcal{F}''$, then it holds for the third. If $\mathcal{F} = \widetilde{M}$ for a finite graded $R(X)$-module M, then $M_n \cong H^0(X, \mathcal{F}(n))$ for all large enough n, since we know that

$$(\oplus_{n \geq 0} H^0(X, \mathcal{F}))^{\sim} = \mathcal{F},$$

and any two finite, graded $R(X)$-modules determining the same sheaf $\mathcal{F}$ must coincide in large enough degrees. Hence $H(X, \mathcal{F})(n) = P(n)$ holds for all large enough n, where $P(t)$ is the Hilbert polynomial of any such M (in particular, the Hilbert polynomials of different possible such modules M coincide).

We now show that $H(X, \mathcal{F})(n) = P(n)$ for all n, by Noetherian induction on $\operatorname{supp} \mathcal{F}$, and then by descending induction on n. If $\operatorname{supp} \mathcal{F}$ is a point, then $\mathcal{F}$ is a skyscraper sheaf, so that $H(X, \mathcal{F})(n)$ is a constant function, which coincides with a polynomial function $P(n)$ for large n; then $P(t)$ must be a constant, and $H(X, \mathcal{F})(n) = P(n)$ holds for all n.

In general, let $s \in R(X)_1 \subset \Gamma(X, \mathcal{O}_X(1))$ be a section which is non-zero in the stalk of $\mathcal{O}_X(1)$ at every associated point of the sheaf $\mathcal{F}$ (recall that the associated points of $\mathcal{F}$ are points of the subset $S \subset X$ such that, for any affine open $U = \operatorname{Spec} A$, with $\mathcal{F} \mid_U = \widetilde{M}$, the set $S \cap U$ consists of the associated primes of M). Then multiplication by s defines an injective sheaf homomorphism $\mathcal{F}(-1) \to \mathcal{F}$, with cokernel $\mathcal{G}$ say. From the exact sequence $0 \to \mathcal{F}(-1) \to \mathcal{F} \to \mathcal{G} \to 0$, it follows that $H(X, \mathcal{F})(n-1) = H(X, \mathcal{F})(n) - H(X, \mathcal{G})(n)$. Since $\mathcal{G}$ is not supported at any associated point of $\mathcal{F}$, which includes the generic points of $\operatorname{supp} \mathcal{F}$, we have that $\operatorname{supp} \mathcal{G} \subsetneq \operatorname{supp} \mathcal{F}$ is a proper closed subscheme. Hence $H(X, \mathcal{G})(n) = Q(n)$ for all $n \in \mathbb{Z}$, for a suitable polynomial $Q(t)$. Since $H(X, \mathcal{F})(n) = P(n)$ for all large n, we must have $Q(t) = P(t) - P(t-1)$. Hence by descending induction on n, we conclude that $H(X, \mathcal{F})(n) = P(n)$ for all $n \in \mathbb{Z}$. □

Corollary 3.6. *In the above situation,*

$$\ell(H^0(X, \mathcal{F}(n))) - P(n) = \sum_{i \geq 1} (-1)^{i-1} \ell(H^i(X, \mathcal{F}(n))).$$

Thus the non-agreement of the Hilbert function and Hilbert polynomial is accounted for by non-vanishing higher cohomology groups.

Remark 3.7. If X is a non-singular projective variety over a field k, the *Riemann-Roch theorem* of Grothendieck-Hirzebruch gives a formula for the polynomial $P(t)$ using intersection theory on X, in terms of the Chern classes of $\mathcal{F}$ and of the sheaf of Kähler differentials $\Omega^1_{X/k}$.

Theorem 3.8. *Let X be proper over a Noetherian affine scheme* $\operatorname{Spec} A$*, and $\mathcal{L}$ an invertible sheaf on X. Then $\mathcal{L}$ is ample on X if and only if, for each coherent sheaf $\mathcal{F}$ on X, we have $H^i(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^n) = 0$ for all $i > 0$ and all sufficiently large n.*

Proof. If $\mathcal{L}$ is ample, choose m so that $\mathcal{L}^m$ is very ample, and apply Serre's vanishing theorem to $\mathcal{F} \otimes \mathcal{L}^i$ for $0 \leq i \leq m-1$. We deduce that $H^i(X, \mathcal{F} \otimes \mathcal{L}^n) = 0$ for all $i > 0$ for all sufficiently large n.

Conversely, suppose the above cohomology vanishing assertion holds. By the definition of ampleness, we must show that, for any coherent $\mathcal{F}$, the sheaf $\mathcal{F} \otimes \mathcal{L}^n$ is globally generated for all large enough n. Let $x \in X$ be a closed point, $\mathcal{M}_x \subset \mathcal{O}_X$ its maximal ideal sheaf. We have an exact sheaf sequence

$$0 \to \mathcal{M}_x \mathcal{F} \to \mathcal{F} \to \mathcal{F} \otimes k(x) \to 0,$$

where $\mathcal{F} \otimes k(x)$ denotes the skyscraper sheaf at x associated to the finite dimensional $k(x)$-vector space $\mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} k(x)$. Tensoring wih $\mathcal{L}^n$ and taking cohomology, we see that for large n,

$$H^0(X, \mathcal{F} \otimes \mathcal{L}^n) \to \mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{L}^n_x \otimes_{\mathcal{O}_{X,x}} k(x)$$

is surjective, since $H^1(X, \mathcal{M}_x \mathcal{F} \otimes \mathcal{L}^n) = 0$. By Nakayama's lemma, this means that global sections of $\mathcal{F} \otimes \mathcal{L}^n$ generate the $\mathcal{O}_{X,x}$-module $\mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{L}^n_x$ at the chosen point x, for all large n (how large may depend on x). Thus, the quotient of $\mathcal{F} \otimes \mathcal{L}^n$ by the $\mathcal{O}_X$-submodule generated by global sections is not supported at x. Since the support of this quotient is closed, the global sections of $\mathcal{F} \otimes \mathcal{L}^n$ generate the sheaf in an open neighbourhood of x, for all sufficiently large n (how large may depend on x, and the neighbourhood may depend on the chosen n).

Apply this to $\mathcal{F} = \mathcal{O}_X$; then there is an n_0 and a neighbourhood V of x on which $\mathcal{L}^{n_0}$ is generated by its global sections. Choose n_1 and neighbourhoods U_i of x, $0 \leq i < n_0$, such that global sections of $(\mathcal{F} \otimes \mathcal{L}^i) \otimes \mathcal{L}^{n_1}$ generate this sheaf on U_i. Then global sections of $\mathcal{F} \otimes \mathcal{L}^m$ generate this sheaf on $U_x = V \cap U_0 \cap \cdots \cap U_{n_0-1}$, for all $m \geq n_1$, since we may write any such m as $qn_0 + i + n_1$ for some $q \geq 0$ and $0 \leq i < n_0$, so that $\mathcal{F} \otimes \mathcal{L}^m = (\mathcal{F} \otimes \mathcal{L}^{i+n_1}) \otimes (\mathcal{L}^{n_0})^q$ is a tensor product of two sheaves, each of whose global sections generate it on U_x.

Now since X is Noetherian, a finite number of such U_x cover X. □

4. Serre Duality

We begin with a discussion of Ext groups and sheaves. We let **Ab** denote the category of abelian groups. If X is a scheme, (or more generally any ringed space

$(X, \mathcal{O}_X))$, and $\mathcal{F}$ is any $\mathcal{O}_X$-module, we have two left exact exact functors defined on the category $\mathcal{M}od(X)$ of $\mathcal{O}_X$-modules, namely

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, -) : \mathcal{M}od(X) \to \mathbf{Ab}, \;\; \mathcal{G} \mapsto \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}),$$

and

$$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, -) : \mathcal{M}od(X) \to \mathcal{M}od(X), \quad \mathcal{G} \mapsto \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}).$$

The first functor is obtained from the second by composition with the global section functor $\Gamma(X, -)$. We define

$$\mathrm{Ext}^i(\mathcal{F}, -) : \mathcal{M}od(X) \to \mathbf{Ab}$$

to be the i-th right derived functor of $\mathrm{Hom}_X(\mathcal{F}, -)$. Similarly define

$$\mathcal{E}xt^i_{\mathcal{O}_X}(\mathcal{F}, -) : \mathcal{M}od(X) \to \mathcal{M}od(X)$$

to be the i-th right derived functor of $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, -)$.

Lemma 4.1. *For any open set $U \subset X$,*

$$\mathcal{E}xt^i_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \mid_U = \mathcal{E}xt^i_{\mathcal{O}_U}(\mathcal{F} \mid_U, \mathcal{G} \mid_U).$$

Proof. For fixed $\mathcal{F}$, both sides (considered as functors in $\mathcal{G}$) give δ-functors $\mathcal{M}od(X) \to \mathcal{M}od(U)$, which coincide for $i = 0$. The functor $\mathcal{G} \mapsto \mathcal{E}xt^i_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \mid_U$ vanishes if $\mathcal{G}$ is injective, since $\mathcal{E}xt^i_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) = 0$ in that case. Hence it is a universal δ-functor.

On the other hand, we claim that if $\mathcal{G}$ is injective, then $\mathcal{G} \mid_U$ is an injective object of $\mathcal{M}od(U)$. To see this, let $j : U \to X$ be the inclusion; if $0 \to \mathcal{Q} \to \mathcal{R}$ is an exact sequence in $\mathcal{M}od(U)$, and $f : \mathcal{Q} \to \mathcal{G} \mid_U$ is given, then $j_!(f) : j_!\mathcal{Q} \to j_!(\mathcal{G} \mid_U)$ composed with the inclusion $j_!(\mathcal{G} \mid_U) \hookrightarrow \mathcal{G}$ gives rise to a map $\widetilde{f} : j_!\mathcal{Q} \to \mathcal{G}$. Since $0 \to j_!\mathcal{Q} \to j_!\mathcal{R}$ is exact in $\mathcal{M}od(X)$, and $\mathcal{G}$ is injective, we can extend $\widetilde{f}$ to a map $j_!\mathcal{R} \to \mathcal{G}$. the restriction of this map to U is the desired extension $\mathcal{R} \to \mathcal{G} \mid_U$ of f.

As a consequence, we also have $\mathcal{E}xt^i_{\mathcal{O}_U}(\mathcal{F} \mid_U, \mathcal{G} \mid_U) = 0$ for injective $\mathcal{G}$. Hence it is also a universal δ-functor. □

Lemma 4.2. *For any $\mathcal{F} \in \mathcal{M}od(X)$, we have:*

(a) $\mathcal{E}xt^0_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{F}) = \mathcal{F}$
(b) $\mathcal{E}xt^i_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{F}) = 0$ *for all* $i > 0$
(c) $\mathrm{Ext}^i_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{F}) = H^i(X, \mathcal{F})$.

Proof. Since $\mathcal{E}xt^0$ coincides with $\mathcal{H}om$, (a) is clear. Since the identity functor is exact, its derived functors vanish, which gives (b). Since $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_X, -)$ coincides with the global sections functor $\Gamma(X, -)$, the derived functors of these two functors must coincide as well. □

Lemma 4.3. *If $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is a short exact sequence in $\mathcal{M}od(X)$, then for any $\mathcal{G} \in \mathcal{M}od(X)$ there are long exact sequences, functorial in $\mathcal{G}$,*

$$0 \to \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}'', \mathcal{G}) \to \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \to \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}', \mathcal{G}) \to \mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{F}'', \mathcal{G})$$
$$\to \mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \to \cdots$$

$$0 \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}'', \mathcal{G}) \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}', \mathcal{G}) \to \mathcal{E}xt^1_{\mathcal{O}_X}(\mathcal{F}'', \mathcal{G}) \to \mathcal{E}xt^1_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \to \cdots$$

Proof. Since $\mathrm{Hom}_{\mathcal{O}_X}(-, \mathcal{I})$ is exact for any injective sheaf $\mathcal{I}$, we see that if $0 \to \mathcal{G} \to \mathcal{I}^\bullet$ is an injective resolution, then we have a short exact sequence of complexes

$$0 \to \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}'', \mathcal{I}^\bullet) \to \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{I}^\bullet) \to \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}', \mathcal{I}^\bullet) \to 0.$$

The corresponding long exact sequence in cohomology gives the desired exact sequence of Ext groups. To get the analogous sequence of $\mathcal{E}xt$ sheaves, we remark that if $\mathcal{I}$ is injective, then so is $\mathcal{I}\mid_U$ for any open U (see the proof of lemma 4.1), so that $\mathcal{H}om_{\mathcal{O}_X}(-, \mathcal{I})$ is an exact functor, and a similar argument applies again. □

Lemma 4.4. *Let* $\cdots \to \mathcal{E}_2 \to \mathcal{E}_1 \to \mathcal{E}_0 \to \mathcal{F} \to 0$ *be a resolution of an* $\mathcal{O}_X$*-module* $\mathcal{F}$ *by locally free* $\mathcal{O}_X$*-modules* $\mathcal{E}_j$ *of finite rank. Then for any* $\mathcal{O}_X$*-module* $\mathcal{G}$*, we have isomorphisms of functors*

$$\mathcal{E}xt^i_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \cong H^i(\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}_\bullet, \mathcal{G}))$$

(on the right, H^i *denotes the* i*-th cohomology sheaf of the complex of* $\mathcal{O}_X$*-modules).*

Proof. Both sides give δ-functors (in the variable $\mathcal{G}$) on $\mathcal{M}odX$, which clearly agree for $i = 0$, by the left exactness of $\mathcal{H}om$. If $\mathcal{G}$ is injective, the left side vanishes, and because $\mathcal{H}om_{\mathcal{O}_X}(-, \mathcal{G})$ is exact in this case, the right side vanishes as well. Hence both δ-functors are universal, and must coincide. □

For example, on a scheme X which is quasi-projective over a Noetherian ring, any coherent sheaf is a quotient of a localy free sheaf of finite rank, and so we can always find such a resolution for any coherent sheaf $\mathcal{F}$.

Lemma 4.5. *Let* $\mathcal{E}$ *be a locally free* $\mathcal{O}_X$*-module, and let* $\mathcal{E}^\vee = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{O}_X)$*. Then for any* $\mathcal{O}_X$*-modules* $\mathcal{F}$*,* $\mathcal{G}$ *we have natural isomorphisms for all* $i \geq 0$

$$\mathrm{Ext}^i_{\mathcal{O}_X}(\mathcal{F} \otimes \mathcal{E}, \mathcal{G}) \cong \mathrm{Ext}^i(\mathcal{F}, \mathcal{E}^\vee \otimes_{\mathcal{O}_X} \mathcal{G}),$$

$$\mathcal{E}xt^i_{\mathcal{O}_X}(\mathcal{F} \otimes \mathcal{E}, \mathcal{G}) \cong \mathcal{E}xt^i(\mathcal{F}, \mathcal{E}^\vee \otimes_{\mathcal{O}_X} \mathcal{G}) \cong \mathcal{E}xt^i_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \otimes_{\mathcal{O}_X} \mathcal{E}^\vee.$$

Proof. First consider the claim for Ext. Both sides of the formula define δ-functors which agree for $i = 0$, and for injective $\mathcal{G}$, the left side vanishes for all $i > 0$. On the other hand, the functor $\mathcal{H}om_{\mathcal{O}_X}(-, \mathcal{E}^\vee \otimes \mathcal{G})$ is isomorphic to $\mathcal{H}om_{\mathcal{O}_X}(- \otimes_{\mathcal{O}_X}, \mathcal{E}, \mathcal{G})$, which is exact (since $\mathcal{G}$ is injective); hence $\mathcal{E}^\vee \otimes_{\mathcal{O}_X} \mathcal{G}$ is also injective, and the right side of the formula is 0 as well. Thus both sides of the formula define universal δ-functors, which must hence coincide.

For $\mathcal{E}xt$, the argument is very similar, again using the fact that the three δ-functors agree for $i = 0$ and all vanish for $i > 0$ when $\mathcal{G}$ is injective (for the middle sheaf, this is because $\mathcal{E}^\vee \otimes_{\mathcal{O}_X} \mathcal{G}$ is also injective as noted above). □

The above lemmas have all been in the situation of an arbitrary ringed space. We now prove a lemma which is valid in a more geometric situation. It allows one to make connections between the Ext groups and sheaves and notions in commutative algebra, like depth, projective dimension, etc.

Lemma 4.6. *Let X be a Noetherian scheme and $\mathcal{F}$ a coherent sheaf on X. Then for any $\mathcal{O}_X$-module $\mathcal{G}$, and any point $x \in X$, we have isomorphisms for all $i \geq 0$*

$$\mathcal{E}xt^{i}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})_x \cong \operatorname{Ext}^{i}_{\mathcal{O}_{X,x}}(\mathcal{F}_x, \mathcal{G}_x).$$

Proof. The question is local, so we may assume $X = \operatorname{Spec} A$ is affine, where A is a Noetherian ring, and $\mathcal{F} = \widetilde{M}$ for some finitely generated A-module M. Let $A_x = \mathcal{O}_{X,x}$. Let $E_\bullet \to M \to 0$ be a free resolution of M, so that $\widetilde{E_\bullet} \to \widetilde{M} \to 0$ is a locally free $\mathcal{O}_X$-resolution of $\widetilde{M}$. Let $E_{i,x} = E_i \otimes_A A_x$, $M_x = M \otimes_A A_x$. We have a natural isomorphism

$$\mathcal{H}om_{\mathcal{O}_X}(\widetilde{\mathbb{E}}_i, \mathcal{G})_x \cong \operatorname{Hom}_{A_x}(E_{i,x}, \mathcal{G}_x),$$

since E_i is a free A-module of finite rank. Since a free A_x-resolution of M_x may be used to compute the Ext groups $\operatorname{Ext}^{i}_{A_x}(M_x, -)$, the lemma follows using lemma 4.5. □

Exercise 4.7. Let $X = \operatorname{Spec} A$, where A is Noetherian, and let M, N be A modules with M finitely generated. Then

$$\operatorname{Ext}^{i}_{\mathcal{O}_X}(\widetilde{M}, \widetilde{N}) = \operatorname{Ext}^{i}_{A}(M, N)$$

and

$$\mathcal{E}xt^{i}_{\mathcal{O}_X}(\widetilde{M}, \widetilde{N}) = \widetilde{\operatorname{Ext}^{i}_{A}(M, N)}.$$

In particular, $\mathcal{E}xt^{i}_{\mathcal{O}_X}(\widetilde{M}, \widetilde{N})$ is quasi-coherent, and is coherent if N is also finitely generated. This implies that on any Noetherian scheme, if $\mathcal{F}$ is coherent and $\mathcal{G}$ is quasi-coherent, then $\mathcal{E}xt^{i}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ is quasi-coherent, and is coherent whenever $\mathcal{G}$ is coherent.

Proposition 4.8. *Let X be a projective scheme over a Noetherian ring A, and $\mathcal{O}_X(1)$ a very ample invertible sheaf on X. Let $\mathcal{F}$, $\mathcal{G}$ be coherent $\mathcal{O}_X$-modules. Then there exists n_0 (depending on $\mathcal{F}$, $\mathcal{G}$ and i) such that for any $n \geq n_0$, we have*

$$\operatorname{Ext}^{i}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}(n)) = \Gamma(X, \mathcal{E}xt^{i}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}(n))).$$

Proof. Fix $\mathcal{F}$. Now $\operatorname{Hom}_{\mathcal{O}_X}(\mathcal{F}, -) = \Gamma(X, -) \circ \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, -)$ is a composition of functors, and for injective $\mathcal{G}$, one verifies easily that $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ is flasque, hence $\Gamma(X, -)$-acyclic. Hence there is a Grothendieck spectral sequence

$$E_2^{p,q} = R^p\Gamma(X, -)\,(R^q\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, -)(\mathcal{G})) \implies R^{p+q}\operatorname{Hom}_{\mathcal{O}_X}(\mathcal{F}, -)(\mathcal{G}),$$

which we unravel to be

$$E_2^{p,q} = H^p(X, \mathcal{E}xt^{q}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})) \implies \operatorname{Ext}^{p+q}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}).$$

Apply this to $\mathcal{F}$, and the twist $\mathcal{G}(n)$ of $\mathcal{G}$. We compute that, by lemma 4.5,

$$\mathcal{E}xt^{q}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}(n)) \cong \mathcal{E}xt_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})(n).$$

Hence from Serre's vanishing theorem (Theorem 3.2(b)), we have that for all sufficiently large n (how large depends on $\mathcal{F}, \mathcal{G}$ and q)

$$H^p(X, \mathcal{E}xt^{q}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}(n))) = 0 \;\; \forall\, p > 0.$$

Thus, fixing i, we can choose n_0 such that for all $n \geq n_0$, we have $E_2^{p,q} = 0$ for all $p > 0$ and $q \leq i$. Now the relevant part of the spectral sequence degenerates at E_2, giving isomorphisms

$$E_2^{0,i} \cong E_\infty^{0,i} \cong \operatorname{Ext}^i_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}(n)).$$

But $E_2^{0,i} = \Gamma(X, \mathcal{E}xt^i_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}(n)))$. □

We recall also some results from commutative algebra.

Lemma 4.9. *Let A be a ring, and M be a finite A-module.*

(a) *M is projective $\iff \operatorname{Ext}^1_A(M, N) = 0$ for all modules N.*
(b) *M has projective dimension $\leq n \iff \operatorname{Ext}^i_A(M, N) = 0$ for all $i > n$, for all modules N.*
(c) *If $(A, \mathbf{m})$ is a regular local ring of dimension n, then every finite A-module M has projective dimension $\leq n$, and $A/\mathbf{m}$ has projective dimension exactly n. In general, the projective dimension of M is $n - \operatorname{depth} M$. We have $\operatorname{pd}_A M \leq n \iff \operatorname{Ext}^i_A(M, A) = 0$ for all $i > n$.*

Proof. Let

$$0 \to N \to F \to M \to 0 \tag{4.1}$$

be exact with F a free A-module. If $\operatorname{Ext}^1_A(M, N) = 0$, then this sequence is split exact, and M is projective. Conversely, if M is projective, $\operatorname{Hom}_A(M, -)$ is exact, and so its derived functors $\operatorname{Ext}^i_A(M, -)$ vanish. This proves (a). The proof of (b) is by induction on the projective dimension of M; if M is not projective, then in the above exact sequence (4.1), the projective dimension of N is $\operatorname{pd}_A M - 1$. If we already know the vanishing of $\operatorname{Ext}^i(N, -)$ for $i > \operatorname{pd} M - 1$, then from the long exact sequence of Ext's, we deduce that $\operatorname{Ext}^i(M, -) = 0$ for $i > \operatorname{pd} M$.

We now discuss the proof of (c). Recall that if $(A, \mathbf{m})$ is a regular local ring of dimension n, then $\mathbf{m}$ is generated by n elements, say $a_1, \ldots, a_n$. Then the associated graded ring $\operatorname{gr}_{\mathbf{m}} A = \oplus_{i \geq 0} \mathbf{m}^i/\mathbf{m}^{i+1}$ is a quotient of the symmetric algebra on $\mathbf{m}/\mathbf{m}^2$, which is a polynomial algebra over $k = A/\mathbf{m}$ on the n elements which are the images of $a_1, \ldots, a_n$ in $\mathbf{m}/\mathbf{m}^2$. It is easy to see that for any quotient of a polynomial algebra $k[X_1, \ldots, X_n]$ by a non-zero homogenous ideal, the Hilbert polynomial has degree $< n$ (it suffices to consider the quotient by a homogenous principal ideal). Since from dimension theory, $\dim A = n$ is the degree of the Hilbert polynomial of $\operatorname{gr}_{\mathbf{m}} A$, we conclude that $\operatorname{gr}_{\mathbf{m}} A \cong k[X_1, \ldots, X_n]$ is a polynomial algebra over k. Using this, one can prove (left to the reader) that $a_1, \ldots, a_n$ form a regular sequence on A, so that the residue field $k = A/\mathbf{m}$ has a finite, free resolution by the Koszul complex $K_A(a_1, \ldots, a_n)$. Using this complex, we compute at once that $\operatorname{Tor}^A_i(k, k) = 0$ for $i > n$, and is $\cong k$ if $i = n$. In particular, k has projective dimension n. Hence $\operatorname{Tor}^A_i(M, k) = 0$ for all $i > n$, for any finite A-module M. Considering a minimal free resolution $F_\bullet \to M$, we have that $\operatorname{rank} F_i = \dim_k \operatorname{Tor}^A_i(M, k)$, and so conclude that $\operatorname{pd}_A M \leq n$.

Now we show $\operatorname{depth} M + \operatorname{pd}_A M = n$ for any finite A-module M. We do this by induction on $\operatorname{depth} M$. If $\operatorname{depth} M = 0$, there is an exact sequence

$$0 \to k \to M \to M' \to 0$$

and since $\operatorname{pd} M' \leq n$, we have $\operatorname{Tor}^A_{n+1}(M', k) = 0$, and from the long exact sequence of Tor's we see that $\operatorname{Tor}^A_n(M, k) \neq 0$. Hence $\operatorname{pd}_A M \geq n$, and so it must be equal to n. If $\operatorname{depth} M > 0$, then choose a non zero-divisor $a \in n\mathbf{m}$ on M to get an exact sequence

$$0 \to M \xrightarrow{\cdot a} M \to M/aM \to 0$$

Then $\operatorname{depth} M/aM = \operatorname{depth} M - 1$, by standard properties of depth. If $\operatorname{pd}_A M/aM = r$ (necessarily > 0), then $\operatorname{Tor}^A_{r+1}(M/aM, k) = 0$ and $\operatorname{Tor}^A_r(M/aM, k) \neq 0$, as seen by considering a minimal free resolution of M/aM. We have an exact sequence

$$\operatorname{Tor}^A_{r+1}(M/aM, k) \to \operatorname{Tor}^A_r(M, k) \xrightarrow{\cdot a} \operatorname{Tor}^A_r(M, k) \to \operatorname{Tor}_r(M/aM, k)$$
$$\to \operatorname{Tor}^A_{r-1}(M, k) \xrightarrow{\cdot a} \operatorname{Tor}^A_{r-1}(M, k)$$

Since multiplication by a is the zero map on $\operatorname{Tor}^A_i(-, k)$, we deduce that $\operatorname{Tor}^A_r(M, k) = 0$, and $\operatorname{Tor}^A_{r-1}(M, k) \neq 0$. Hence $\operatorname{pd}_A M = r - 1$.

Finally, we show that if $\operatorname{Ext}^i_A(M, A) = 0$ for $i > n$, then $\operatorname{pd}_A M \leq n$. Let N be any finite A-module. We see easily by induction on $\operatorname{pd}_A N$ that $\operatorname{Ext}^i_A(M, N) = 0$ for all $i > n$. Now the criterion in (b) applies. □

We now prove the Serre Duality Theorem for coherent sheaves on a projective scheme over a field. We first consider the case of projective space itself; then we reduce the general case to this case. If V is a k-vector space, let $V^\vee$ denotes the dual vector space $\operatorname{Hom}_k(V, k)$.

Theorem 4.10. *Let k be a field, $X = \mathbb{P}^n_k = \operatorname{Proj} k[X_0, \ldots, X_n]$. Let $\omega_X = \bigwedge^n \Omega^1_{X/k}$.*

(a) *$H^n(X, \omega_X) \cong k$; we fix such an isomorphism.*

(b) *For any coherent sheaf $\mathcal{F}$ on X, the natural pairing*

$$\operatorname{Hom}_{\mathcal{O}_X}(\mathcal{F}, \omega_X) \otimes_k H^n(X, \mathcal{F}) \to H^n(X, \omega_X) \cong k$$

is a perfect pairing between finite dimensional vector spaces.

(c) *For each $i \geq 0$ there is a natural isomorphism, functorial in $\mathcal{F}$,*

$$\operatorname{Ext}^i_{\mathcal{O}_X}(\mathcal{F}, \omega_X) \cong H^{n-i}(X, \mathcal{F})^\vee,$$

which for $i = 0$ is the pairing in (b).

Proof. One has the "Euler exact sequence"

$$0 \to \Omega^1_{X/k} \to \mathcal{O}_X(-1) \otimes_k H^0(X, \mathcal{O}_X(1)) \to \mathcal{O}_X \to 0,$$

from which, by taking determinants, one obtains

$$\omega_X = \bigwedge^n \Omega^1_{X/k} \cong \bigwedge^{n+1} (\mathcal{O}_X(-1) \otimes_k H^0(X, \mathcal{O}_X(1)) \cong \mathcal{O}_X(-n-1).$$

Now (a) follows from Theorem 2.4. Since an $\mathcal{O}_X$-linear map $\mathcal{F} \to \omega_X$ induces a k-linear map $H^n(X, \mathcal{F}) \to H^n(X, \omega_X)$, we have a pairing as described in (b). To see that it is perfect, first note that if $\mathcal{F} = \mathcal{O}_X(r)$ for some r, or a direct sum of such sheaves, then Theorem 2.4 gives the perfection of the pairing. Now we may write $\mathcal{F}$ as a cokernel of a map $\oplus_i \mathcal{O}_X(n_i) \to \oplus_j \mathcal{O}_X(m_j)$ for suitable integers n_i, m_j (why?). We also have $H^i(X, \mathcal{G}) = 0$ for any coherent sheaf $\mathcal{G}$ and any $i > n$

(this follows from Corlooary 2.3), so that $H^n(X,-)$ is right exact, and we have a presentation

$$\oplus_i H^n(X,\mathcal{O}_X(n_i)) \to \oplus_j H^n(X,\mathcal{O}_X(m_j)) \to H^n(X,\mathcal{F}) \to 0$$

This gives rise to a commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 \longrightarrow & H^n(X,\mathcal{F})^\vee & \longrightarrow & \oplus_j H^n(X,\mathcal{O}_X(m_j))^\vee & \longrightarrow & \oplus_i H^n(X,\mathcal{O}_X(n_i))^\vee \\ & \uparrow & & \uparrow & & \uparrow \\ 0 \longrightarrow & \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{F},\omega_X) & \longrightarrow & \oplus_j \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{O}_X(m_j),\omega_X) & \longrightarrow & \oplus_i \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{O}_X(n_i),\omega_X) \end{array}$$

The middle and right hand vertical arrows are isomorphism, and hence so is the left vertical arrow. This proves (b).

To prove (c), note that both sides define contravariant δ-functors on the abelian category of coherent sheaves on X, which coincide in degree 0, by (b). Now for any $r >> 0$, we have (using lemmas 4.2 and 4.5, and the Serre Vanishing theorem)

$$\operatorname{Ext}^i_{\mathcal{O}_X}(\mathcal{O}_X(-r),\omega_X) \cong H^i(X,\omega_X(r)) = 0 \ \forall\ i > 0.$$

We can find a surjection $\mathcal{O}_X(-r)^{\oplus N} \twoheadrightarrow \mathcal{F}$ for any sufficiently large r (this just means $\mathcal{F}(r)$ is globally generated). Also note that $H^{n-i}(X,\mathcal{O}_X(-r)) = 0$ for all $i > 0$ and $r > 0$. This implies that both of our contravariant δ-funcotrs are co-effaceable, and hence both are universal, and thus isomorphic. □

We now consider the general case. We begin with a definition.

Definition 4.11. Let k be a field, and let X be a proper scheme of dimension n over $\operatorname{Spec} k$. A *dualizing sheaf* on X is a coherent sheaf ω°_X, together with a *trace map* $t : H^n(X,\omega^\circ_X) \to k$, such that for all coherent sheaves $\mathcal{F}$ on X, the pairing

$$\operatorname{Hom}_{\mathcal{O}_X}(\mathcal{F},\omega^\circ_X) \otimes_k H^n(X,\mathcal{F}) \to H^n(X,\omega^\circ_X) \xrightarrow{t} k$$

is a perfect pairing between finite dimensional vector spaces.

Since (ω°_X, t) is defined by a universal property, it is easy to see that it is unique up to unique isomorphism.

Proposition 4.12. *Let $i : X \hookrightarrow \mathbb{P}^N_k$ be a closed subscheme of codimension r. Then*

$$\omega^\circ_X = i^*\mathcal{E}xt^r_{\mathcal{O}_{\mathbb{P}^N_k}}(i_*\mathcal{O}_X,\omega_{\mathbb{P}^N_k})$$

is a dualizing sheaf for X.

Proof. We first show that for any coherent $\mathcal{O}_X$-module $\mathcal{F}$, we have

$$\mathcal{E}_s = \mathcal{E}xt^s_{\mathcal{O}_{\mathbb{P}^N_k}}(i_*\mathcal{F},\omega_{\mathbb{P}^N_k}) = 0 \ \ \forall\ s < r.$$

It suffices to prove that $\mathcal{E}_s(m)$ has no global sections for all large enough m. From lemma 4.8, this is equivalent to showing that $\operatorname{Ext}^s_{\mathbb{P}^N_k}(i_*\mathcal{F}(-m),\omega_{\mathbb{P}^N_k}) = 0$ for $s < r$. By Serre duality for $\mathbb{P}^N_k$ (Theorem 4.10), this amounts to showing $H^{N-s}(\mathbb{P}^N_k, i_*\mathcal{F}(-m)) = 0$. But $H^j(\mathbb{P}^N_k, i_*\mathcal{F}(-m)) \cong H^j(X,\mathcal{F}(-m))$ for all j, and $N - s > N - r = \dim X$, so $H^{N-s}(X,\mathcal{F}(-m)) = 0$ as desired.

Now we note that for any $\mathcal{O}_X$-module $\mathcal{F}$ and $\mathcal{O}_{\mathbb{P}^N_k}$-module $\mathcal{G}$, we have

$$\operatorname{Hom}_{\mathcal{O}_X}(\mathcal{F}, i^*\mathcal{H}om_{\mathcal{O}_{\mathbb{P}^N_k}}(i_*\mathcal{O}_X, \mathcal{G})) = \operatorname{Hom}_{\mathcal{O}_{\mathbb{P}^N_k}}(i_*\mathcal{F}, \mathcal{G}).$$

Also if $\mathcal{G}$ is injective in $\mathcal{M}od(\mathbb{P}^n_k)$, then $i^*\mathcal{H}om_{\mathcal{O}_{\mathbb{P}^N_k}}(i_*\mathcal{O}_X, \mathcal{G})$ is injective in $\mathcal{M}od(X)$. Hence there is a Grothendieck spectral sequence

$$E_2^{p,q} = \operatorname{Ext}^p_{\mathcal{O}_X}(\mathcal{F}, i^*\mathcal{E}xt^q_{\mathcal{O}_{\mathbb{P}^N_k}}(i_*\mathcal{O}_X, \mathcal{G})) \implies \operatorname{Ext}^{p+q}_{\mathbb{P}^N_k}(i_*\mathcal{F}, \mathcal{G}).$$

Applying this to $\mathcal{G} = \omega_{\mathbb{P}^N_k}$ and using that $E_2^{p,q} = 0$ for all $q < r$ and all p (since $i^*\mathcal{E}xt^q_{\mathcal{O}_{\mathbb{P}^N_k}}(i_*\mathcal{O}_X, \omega_{\mathbb{P}^N_k}) = 0$), we deduce that

$$E_2^{0,r} \cong E_\infty^{0,r} \cong \operatorname{Ext}^r_{\mathcal{O}_{\mathbb{P}^N_k}}(i_*\mathcal{F}, \omega_{\mathbb{P}^N_k})$$

$$\cong H^{N-r}(\mathbb{P}^N_k, i_*\mathcal{F})^\vee \cong H^{N-r}(X, \mathcal{F})^\vee.$$

On the other hand, we have

$$E_2^{0,r} = \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{F}, i^*\mathcal{E}xt^r_{\mathcal{O}_{\mathbb{P}^N_k}}(i_*\mathcal{O}_X, \omega_{\mathbb{P}^N_k}) = \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{F}, \omega_X^\circ).$$

Hence we have constructed an isomorphism of (contravariant, vector space valued) functors in $\mathcal{F}$ between $H^{N-r}(X, \mathcal{F})^\vee$ and $\operatorname{Hom}_{\mathcal{O}_X}(\mathcal{F}, \omega_X^\circ)$. □

We now give a more concrete description of ω_X° is some cases.

Proposition 4.13. *Let $i : X \hookrightarrow \mathbb{P}^N_k$ be a closed, local complete intersection suscheme which is purely of codimension r, and let $\mathcal{I}$ be the ideal sheaf of X in $\mathbb{P}^N_k$, so that $\overset{r}{\wedge}\, \mathcal{I}/\mathcal{I}^2$ is an invertible $\mathcal{O}_X$-module. Then*

$$\omega_X^\circ \cong (\overset{r}{\wedge}\, \mathcal{I}/\mathcal{I}^2)^{-1} \otimes_{\mathcal{O}_X} i^*\omega_{\mathbb{P}^N_k}.$$

In particular, if X is non-singular (i.e., smooth over k) of dimension $n = N - r$, then

$$\omega_X^\circ \cong \Omega^n_{X/k} = \overset{n}{\wedge}\, \Omega^1_{X/k}.$$

Proof. If A is a Noetherian ring, $I \subset A$ a complete intersection of height r, generated by a regular sequence $x_1, \ldots, x_r$, then the Koszul complex $K_A(x_1, \ldots, x_r)$ gives an A-free resolution of A/I, such that the dual complex (obtained by applying $\operatorname{Hom}_A(-, A)$) is isomorphic to $K_A(x_1, \ldots, x_n)$, up to reindexing suitably. In particular, we compute that $\operatorname{Ext}^i_A(A/I, A) = 0$ for $i \neq r$, and $\operatorname{Ext}^r_A(A/I, A) \cong A/I$. One computes immediately that, identifying $\overset{r}{\wedge}\, I/I^2$ with A/I using the generator $x_1 \wedge \cdots \wedge x_r$, the resulting isomorphism $\operatorname{Ext}\operatorname{Hom}_{A/I}(\overset{r}{\wedge}\, I/I^2, A/I)$ is independent of the choice of the regular sequence $x_1, \ldots, x_r$ generating I.

Thus, if $X \hookrightarrow \mathbb{P}^N_k$ is a local complete intersection subscheme purely of codimension r, then (using 4.7) we obtain an isomorphism

$$\mathcal{E}xt^r_{\mathcal{O}_{\mathbb{P}^N_k}}(i_*\mathcal{O}_X, \mathcal{O}_{\mathbb{P}^N_k}) \cong i_*\mathcal{H}om_{\mathcal{O}_X}(\overset{r}{\wedge}\, \mathcal{I}/\mathcal{I}^2, \mathcal{O}_X).$$

Now $\omega_X^\circ = \mathcal{E}xt^r_{\mathcal{O}_{\mathbb{P}^N_k}}(i_*\mathcal{O}_X, \omega_{\mathbb{P}^N_k})$ has the description stated in the Proposition, by lemma 4.5.

If X is non-singular of dimension $n = N - r$, then $\Omega^1_{X/k}$ is a locally free $\mathcal{O}_X$-module of rank r. We have an exact sequence of Kähler differentials

$$0 \to \mathcal{I}/\mathcal{I}^2 \to i^*\Omega^1_{\mathbb{P}^N_k/k} \to \Omega^1_{X/k} \to 0.$$

Taking determinants, and noting that the determinant of $\Omega^1_{\mathbb{P}^N_k/k}$ is just $\omega_{\mathbb{P}^N_k}$, we deduce that

$$\Omega^n_{X/k} \cong (\overset{r}{\wedge}\, \mathcal{I}/\mathcal{I}^2)^{-1} \otimes_{\mathcal{O}_X} i^*\omega_{\mathbb{P}^N_k} \cong \omega^\circ_X.$$

□

Now we prove the Serre duality theorem for more general projective k-schemes.

Theorem 4.14. *Let X be a projective scheme over an of dimension n over a field k, and let (ω°_X, t) be a dualizing sheaf and trace map for X. Let $\mathcal{O}_X(1)$ be a very ample invertible sheaf on X.*

(a) *For all $j \geq 0$, there are natural tranformations of contravariant functors (in $\mathcal{F}$)*

$$\theta_j : \operatorname{Ext}^j(\mathcal{F}, \omega^\circ_X) \to H^{n-j}(X, \mathcal{F})^\vee,$$

such that θ_0 is given by the dualizing property of ω°_X.

(b) *the following are equivalent:*
 (i) *X is Cohen-Macaulay and equidimensional,*
 (ii) *for any locally free $\mathcal{E}$ on X, we have $H^j(X, \mathcal{E}(-q)) = 0$ for all $j < n$ and $q >> 0$,*
 (iii) *the natural transformations θ_j are isomorphisms.*

Proof. For each j, we may identify $H^{n-j}(X, \mathcal{F})^\vee$ with

$$H^{n-j}(\mathbb{P}^N_k, i_*\mathcal{F})^\vee \cong \operatorname{Ext}^{j+r}_{\mathcal{O}_{\mathbb{P}^n_k}}(i_*\mathcal{F}, \omega_{\mathbb{P}^n_k}).$$

In the spectral sequence

$$E_2^{p,q} = \operatorname{Ext}^p_{\mathcal{O}_X}(\mathcal{F}, i^*\mathcal{E}xt^q_{\mathcal{O}_{\mathbb{P}^N_k}}(i_*\mathcal{O}_X, \omega_{\mathbb{P}^N_k})) \implies \operatorname{Ext}^{p+q}_{\mathcal{O}_{\mathbb{P}^N_k}}(i_*\mathcal{F}, \omega_{\mathbb{P}^N_k}),$$

we have $E_2^{p,q} = 0$ if $q < r = N - n = \operatorname{codim} X$, as seen in the proof of Proposition 4.12. The natural transformations in (a) are then just the edge homomorphisms

$$\operatorname{Ext}^j_{\mathcal{O}_X}(\mathcal{F}, \omega^\circ_X) = E_2^{j,r} \twoheadrightarrow E_\infty^{j,r} \hookrightarrow \operatorname{Ext}^{j+r}_{\mathcal{O}_{\mathbb{P}^n_k}}(i_*\mathcal{F}, \omega_{\mathbb{P}^n_k}).$$

Now we show that any of the three conditions in (b) is equivalent to

$$\mathcal{E}xt^q_{\mathcal{O}_{\mathbb{P}^N_k}}(i_*\mathcal{O}_X, \omega_{\mathbb{P}^N_k}) = 0 \ \ \forall \ \ q \neq r. \tag{4.2}$$

From the spectral sequence, and Serre duality for $\mathbb{P}^N_k$, the above vanishing statement (4.2) does imply (iii), since we will have $E_2^{p,q} = 0$ for $q \neq r$, giving isomorphisms (where $r = N - n$)

$$\operatorname{Ext}^j_{\mathcal{O}_X}(\mathcal{F}, \omega^\circ_X) \cong \operatorname{Ext}^{r+j}_{\mathbb{P}^N_k}(i_*\mathcal{F}, \omega_{\mathbb{P}^N_k}).$$

Next, if the duality assertion (iii) holds, then for any locally free $\mathcal{E}$, $H^j(X, \mathcal{E}(-q))$ is dual to

$$\operatorname{Ext}^{n-j}_{\mathcal{O}_X}(\mathcal{E}(-q), \omega^\circ_X) \cong H^{n-j}(X, \mathcal{E}^\vee \otimes \omega^\circ_X(q)).$$

By the Serre vanishing theorem, we deduce (ii). Now, assuming (ii), we have that for any locally free $\mathcal{E}$ on X, any integer $j < n$, and $q >> 0$, we have

$$0 = H^j(X, \mathcal{E}(-q))^\vee \cong H^j(\mathbb{P}^N_k, i_*\mathcal{E}(-q))^\vee \cong \operatorname{Ext}^{N-j}_{\mathcal{O}_{\mathbb{P}^N_k}}(i_*\mathcal{E}(-q), \omega_{\mathbb{P}^N_k})$$

$$\cong \operatorname{Ext}^{N-j}_{\mathcal{O}_{\mathbb{P}^N_k}}(i_*\mathcal{E}, \omega_{\mathbb{P}^N_k}(q)) \cong H^0(\mathbb{P}^N_k, \mathcal{E}xt^{n-j}(i_*\mathcal{E}, \omega_{\mathbb{P}^N_k})(q)).$$

Now any coherent sheaf $\mathcal{G}$ on $\mathbb{P}^N_k$ with $H^0(\mathbb{P}^N_k, \mathcal{G}(q)) = 0$ for $q >> 0$ must satisfy $\mathcal{G} = 0$. Taking $\mathcal{E} = \mathcal{O}_X$, we deduce that the vanishing assertion (4.2) must hold. Finally, we note that the stalk of $\mathcal{E}xt^q_{\mathcal{O}_{\mathbb{P}^N_k}}(i_*\mathcal{O}_X, \omega_{\mathbb{P}^N_k})$ at any point $x \in X$ is isomorphic to $\operatorname{Ext}^q_{\mathcal{O}_{\mathbb{P}^N_k,x}}(\mathcal{O}_{X,x}, \mathcal{O}_{\mathbb{P}^N_k,x})$ (since $\omega_{\mathbb{P}^N_k}$ is an invertible sheaf). By lemma 4.9(c), we deduce that (4.2) is equivalent to

$$\operatorname{pd}_{\mathcal{O}_{\mathbb{P}^N_k,x}} \mathcal{O}_{X,x} = r,$$

which in turn is equivalent to

$$\operatorname{depth}_{\mathcal{O}_{\mathbb{P}^N_k,x}} \mathcal{O}_{X,x} = \dim \mathcal{O}_{\mathbb{P}^N_k,x} - r \geq \dim \mathcal{O}_{X,x}.$$

But we always have $\operatorname{depth} \mathcal{O}_{X,x} \leq \dim \mathcal{O}_{X,x}$. Hence this last condition can hold only if

$$\dim \mathcal{O}_{\mathbb{P}^N_k,x} - r = \dim \mathcal{O}_{X,x},$$

(which is the same as saying that X is equidimensional), and all the local rings $\mathcal{O}_{X,x}$ have depth equal to their dimension, *i.e.*, are Cohen-Macaulay. □

Corollary 4.15. *Let X be an equidimensional Cohen-Macaulay projective k-scheme of dimension n and $\mathcal{E}$ a locally free sheaf on X of finite rank. Then there are isomorphisms*

$$H^i(X, \mathcal{E})^\vee \cong H^{n-i}(X, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \omega^\circ_X)) \cong H^{n-i}(X, \mathcal{E}^\vee \otimes_{\mathcal{O}_X} \omega^\circ_X).$$

Proof. This is Theorem 4.14 combined with Lemma 4.5. □

Remark 4.16. From (b) of the above Theorem, if X is purely n-dimensional and Cohen-Macaulay, we have a perfect duality between $H^i(X, \mathcal{F})$ and $\operatorname{Ext}^{n-i}_{\mathcal{O}_X}(\mathcal{F}, \omega^\circ_X)$. Even without the Cohen-Macaulay hypothesis, one can get a similar duality assertion between $H^i(X, \mathcal{F})$ and $\mathbb{E}xt^{n-i}(\mathcal{F}, \omega^\bullet_X)$, where $\omega^\bullet_X$ is a certain complex of injective $\mathcal{O}_X$-modules, and $\mathbb{E}xt^i$ are the hyper-ext groups (hyperderived functors of Hom) of pairs of complexes. One takes

$$\omega^\bullet_X = i^*\mathcal{H}om_{\mathcal{O}_{\mathbb{P}^N_k}}(i_*\mathcal{O}_X, \mathcal{J}^\bullet),$$

where $\mathcal{J}^\bullet$ is an injective resolution of $\omega_{\mathbb{P}^N_k}$. Thus $\omega^\bullet_X$ is a complex of $\mathcal{O}_X$-injectives, which is well-defined up to homotopy (because $\mathcal{J}^\bullet$ is), and whose cohomology sheaves are the sheaves $i^*\mathcal{E}xt^j_{\mathcal{O}_{\mathbb{P}^N_k}}(i_*\mathcal{O}_X, \omega_{\mathbb{P}^N_k})$. The spectral sequence used in the proofs of Proposition 4.12 and Theorem 4.14 gets re-interpreted as a spectral sequence for hyper-ext groups. Such a complex $\omega_X^\bullet$ is called a *dualizing complex* for X.

Remark 4.17. In fact Serre duality is true for equidimenional Cohen-Macaulay *proper* k-schemes. The proof in this case is more difficult; see [Ha3].

Proposition 4.18. (Lemma of Enriques-Severi-Zariski) *Let X be an integral normal projective scheme over k of dimension ≥ 2, and $\mathcal{O}_X(1)$ a very ample invertible sheaf on X. Then for any locally free $\mathcal{O}_X$-module $\mathcal{E}$, we have $H^1(X, \mathcal{E}(-q)) = 0$ for all $q >> 0$.*

Proof. As in the proof of Theorem 4.14(b), the normality of X implies that $\mathcal{E}xt^j_{\mathcal{O}_{\mathbb{P}^N_k}}(i_*\mathcal{O}_X, \omega_{\mathbb{P}^N_k}) = 0$ for $j \geq N-1$. From the spectral sequence, we then obtain (for $q >> 0$, using Serre vanishing)

$$\mathrm{Ext}^{N-1}_{\mathcal{P}^N_k}(i_*\mathcal{O}_X(-q), \omega_{\mathbb{P}^N}) \cong H^0(X, i^*\mathcal{E}xt^j_{\mathcal{O}_{\mathbb{P}^N_k}}(i_*\mathcal{O}_X, \omega_{\mathbb{P}^N_k})(q)) = 0.$$

By duality on $\mathbb{P}^N_k$ we obtain the desired vanishing statement for H^1. □

Corollary 4.19. *Let X be an integral normal projective scheme of dimension ≥ 2 over an algebraically closed field k, and let $D \subset X$ be the support of an effective ample divisor on X. Then D is connected.*

Proof. We have $H^0(X, \mathcal{O}_X) = k$. It suffices to show that, for a suitable scheme structure on D, we have $H^0(D, \mathcal{O}_D) = k$. Choose a scheme structure on D so that D is an effective Cartier divisor, which is a sufficiently high multiple of a very ample divisor on X – first choose a strcuture D_0 which is an ample Cartier divisor, then a multiple $D_1 = mD_0$ which is very ample, then a sufficiently high multiple $D = qD_1$. Then by Proposition 4.18, we have $H^1(X, \mathcal{O}_X(-D)) = H^1(X, \mathcal{O}_X(-qD_1)) = 0$. Hence from the long exact sequence in cohomology for the exact sheaf sequence

$$0 \to \mathcal{O}_X(-D) \to \mathcal{O}_X \to i_*\mathcal{O}_D \to 0$$

(where $i : D \hookrightarrow X$) we get that $k = H^0(X, \mathcal{O}_X) \to H^0(D, \mathcal{O}_D)$ is surjective. □

Corollary 4.20. *Let X be an integral normal projective variety of dimension ≥ 2 over an algebraically closed field k, and let D be the support of an ample divisor on X. Then*

$$\pi_1^{alg}(D) \to \pi_1^{alg}(X)$$

is surjective.

Proof. From the definition of the algebraic fundamental group, it suffices to prove that if $f : Y \to X$ is a connected algebraic covering space (*i.e.*, Y is conencted, and f is finite, flat and unramified), then for the induced pull-back covering space $D \times_X Y \to D$, $D \times_X Y$ is connected. But Y is again an irreducible normal k-scheme of finite type, and the pull-back to Y of any ample Cartier divisor on X is ample on Y. Hence Y is also projective, and $D \times_X Y \subset Y$ is the support of an ample divisor, hence by Corollary 4.19 it is connected. □

We use the Serre duality theorem to discuss the Riemann-Roch theorem for non-singular projective curves. Recall that a non-singular projective curve X over an algebraically closed field k is an integral projective k-scheme of dimension 1. The rational functions $k(X)$ on X form a field, which is a finitely generated extension field of k of transcendence degree 1, and the points of X are in bijection with the discrete valuation rings of $k(X)$ which contain k (see [Ha], Chapter 1, §6), where the point x is associated to the discrete valuation ring $\mathcal{O}_{X,x}$ – we regard $k(X)$ as $\mathcal{O}_{X,\eta}$ where η is the generic point of x; since x lies in the closure of η, there is a natural homomorphism $\mathcal{O}_{X,x} \to \mathcal{O}_{X,\eta}$, which is an inclusion $\mathcal{O}_{X,x} \hookrightarrow k(X)$ as a discrete valuation subring containing k, with quotient field $k(X)$.

A *divisor* on X is an element of the free abelian group $\mathrm{Div}\,(X)$ on closed points of X; the degree of a divisor $D = \sum_i n_i x_i$ (with $n_i \in \mathbb{Z}$, $x_i \in X$) is the integer $\deg D = \sum_i n_i$. If $f \in k(X) \setminus \{0\}$, then we can associate to it the divisor $(f)_X = \sum_x \mathrm{ord}\,_x(f)x$, where $\mathrm{ord}\,_x$ denotes the normalized discrete valuation on $k(X)$ associated to the closed point x (only finitely many of the integers $\mathrm{ord}\,_x(f)$ are non-zero, so the sum does define a divisor).

We have an invertible sheaf $\mathcal{O}_X(D)$ associated to any divisor D on X, and this induces an isomorphism between the Picard group of isomorphism classes of invertible sheaves on X, and the divisor class group of linear equivalence classes of divisors (a divisor is linearly equivalent to 0 if it is of the form (f) for some $f \in k(X) \setminus \{0\}$).

If $\omega \in \Omega^1_{k(X)/k}$ is non-zero, one can similarly associate to it a divisor as follows: for each closed point $x \in X$, choose a generator ω_x of the free $\mathcal{O}_{X,x}$-module $\Omega^1_{\mathcal{O}_{X,x}/k}$. Since $\mathcal{O}_{X,x}$ has quotient field $k(X)$, we have

$$\Omega^1_{k(X)/k} = \Omega^1_{\mathcal{O}_{X,x}/k} \otimes_{\mathcal{O}_{X,x}} k(X) \supset \Omega^1_{\mathcal{O}_{X,x}/k},$$

and so $\omega = f_x \omega_x$ for a unique $f_x \in k(X)$; if we make a different choice of ω_x, the coefficient f_x is replaced by uf_x for some unit $u \in \mathcal{O}_{X,x}$. Define $K_X = \mathrm{div}\,(\omega) = \sum_x \mathrm{ord}\,_x(f_x)$. Clearly $\mathrm{div}\,(f\omega) = (f)_X + \mathrm{div}\,(\omega)$ for any non-zero $f \in k(X)$; hence the linear equivalence class of K_X is clearly well-defined. It is called the *canonical divisor class* on X. Any such divisor K_X is called a canonical divisor on X; it determines an isomorphism $\mathcal{O}_X(K_X) \cong \Omega^1_{X/k} = \omega_X$.

Theorem 4.21. *Let X be a non-singular projective curve over an algebraically closed field k, and K_X a canonical divisor. Then for any divisor D on X, we have*

$$\dim_k H^0(X, \mathcal{O}_X(D)) - \dim_k H^0(X, \mathcal{O}_X(K_X - D)) = \deg D + 1 - g,$$

where $g = \dim_k H^0(X, \Omega^1_{X/k})$ is the genus *of X.*

Proof. We have $\mathcal{O}_X(K_X) \cong \omega_X$, and so $\mathcal{O}_X(K_X - D) \cong \mathcal{O}_X(D)^{-1} \otimes \omega_X$. By Serre duality applied to $\mathcal{O}_X$ and $\mathcal{O}_X(D)$, we then have $g = \dim_k H^1(X, \mathcal{O}_X)$, and $\dim_k H^0(X, \mathcal{O}_X(K_X - D)) = \dim_k H^1(X, \mathcal{O}_X(D))$. Thus, the Theorem is equivalent to the statement:

$$\chi(\mathcal{O}_X(D)) = \dim_k H^0(X, \mathcal{O}_X(D)) - \dim_k H^1(X, \mathcal{O}_X(D)) \tag{4.3}$$

$$= \deg D + 1 - g = \deg D + \chi(\mathcal{O}_X).$$

Let $D = \sum_i n_i x_i$ be any divisor, and define $||D|| = \sum_i |n_i|$. We prove (4.3) by induction on $||D||$. If $||D|| = 0$, then $D = 0$, and $\mathcal{O}_X(D) = \mathcal{O}_X$, and $\deg D = 0$, so (4.3) holds trivially. For any D_1, let $D_2 = D_1 + x$, and consider the exact sheaf sequence

$$0 \to \mathcal{O}_X(D_1) \to \mathcal{O}_X(D_2) \to i_{x*}k(x) \to 0,$$

where $i_{x*}k(x)$ is the skyscraper sheaf with stalk $k(x)$ at x. The associated exact sequence in cohomology gives

$$\chi(\mathcal{O}_X(D_2)) = \chi(\mathcal{O}_X(D_1)) + \chi(i_{x*}k(x)) = \chi(\mathcal{O}_X(D_1)) + 1.$$

Hence (4.3) holds for $D_1 \iff$ it holds for D_2. Now choosing x suitably and taking $D = D_1$ or $D = D_2$, we deduce (4.3) by induction on $||D||$. □

Corollary 4.22. *If X is as above,* $\deg K_X = 2g - 2$ *and* $\dim_k H^1(X, \Omega^1_{X/k}) = 1$. *For any non-zero $f \in k(X)$, we have* $\deg(f)_X = 0$.

Proof. By Serre duality for $\Omega^1_{X/k}$ we have $\dim_k H^1(X, \Omega^1_{X/k}) = \dim_k H^0(X, \mathcal{O}_X) = 1$. Since $\dim_k H^0(X, \Omega^1_{X/k}) = g$, the Riemann-Roch theorem for $\mathcal{O}_X(K_X)$ implies $\deg K_X = 2g-2$. Finally, if $D = (f)_X$, then $\mathcal{O}_X(D) = \mathcal{O}_X$, so the Riemann-Roch theorem implies $\deg D = \deg 0 = 0$. □

5. Higher direct images

If $f : X \to Y$ is a morphism, and $\mathcal{F}$ is a coherent sheaf on X, then we may want to study the cohomology of the restrictions of $\mathcal{F}$ to the fibers of f. Ideally, one would like to have a quasi-coherent sheaf $\mathcal{F}_i$ on Y, such that for any point $y \in Y$, if $X_y = X \times_Y \operatorname{Spec} k(y)$ is the scheme theoretic fiber, then $(\mathcal{F}_i)_y \otimes k(y) = H^i(X_y, \mathcal{F} \otimes \mathcal{O}_{X_y})$.

In general, it is impossible to find such a sheaf $\mathcal{F}_i$ (one situation where this is possible, called the base change theorem, is discussed later). In general, the nearest substitute is the i-th higher direct image sheaf $R^i f_* \mathcal{F}$, mentioned in the sheaf theory lectures.

Definition 5.1. Let $f : X \to Y$ be a continuous map of topological spaces, and $\mathcal{F}$ a sheaf of abelian groups on X. The i-th direct image sheaf $R^i f_* \mathcal{F}$ is the value on $\mathcal{F}$ of the i-th derived functor of the left exact functor $f_* : \mathcal{M}od(\mathbb{Z}_X) \to \mathcal{M}od(\mathbb{Z}_Y)$.

Thus, if we have an injective resolution $0 \to \mathcal{F} \to \mathcal{I}^\bullet$, then $R^i f_* \mathcal{F}$ is the i-th cohomology sheaf of the complex

$$0 \to f_* \mathcal{I}^0 \to f_* \mathcal{I}^1 \to \cdots$$

For example, if Y is a point, so that sheaves of abelian groups on Y are identified with abelian groups, then f_* becomes identified with the global section functor, and $R^i f_*$ become the i-th cohomology functor $H^i(X, -)$.

Lemma 5.2. *If $\mathcal{F}$ is any sheaf of abelian groups on X, then $R^if_*\mathcal{F}$ is the sheaf on Y associated to the presheaf*

$$U \mapsto H^i(f^{-1}(U), \mathcal{F}\mid_{f^{-1}(U)}).$$

Proof. Let $\mathcal{H}^i(X, \mathcal{F})$ denote the sheaf associated to the above presheaf. Both $\{R^if_*\}_{i\geq 0}$ and $\{\mathcal{H}^i(X,-)\}_{i\geq 0}$ form δ-functors $\mathcal{M}od(\mathbb{Z}_X) \to \mathcal{M}od(\mathbb{Z}_Y)$, which agree for $i = 0$. If $\mathcal{I}$ is injective, then $R^if_*\mathcal{I} = 0$ for $i > 0$ by the definition of derived functors, while the above presheaf is 0, because $\mathcal{I}\mid_{f^{-1}(U)}$ is flasque, for all U. Hence both δ-functors are universal, and must coincide. □

Corollary 5.3. *If $V \subset Y$ is open, then*

$$R^if_*\mathcal{F}\mid_V = R^i(f\mid_V)_*\mathcal{F}\mid_{f^{-1}(V)}.$$

Corollary 5.4. *If $\mathcal{F}$ is flasque, then $R^if_*\mathcal{F} = 0$ for $i > 0$. Hence R^if_* may be computed using flasque resolutions (e.g. the Godement resolution).*

Corollary 5.5. *If $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a morphism of ringer spaces, then the i-th derived functor of $f_* : \mathcal{M}od(\mathcal{O}_X) \to \mathcal{M}od(\mathcal{O}_Y)$ coincides with R^if_*, restricted to $\mathcal{O}_X$-modules.*

Proof. Any injective resolution in $\mathcal{M}od(\mathcal{O}_X)$ is a flasque resolution as well. □

The following is slightly weaker than a similar result in [Ha] but is sufficient for our purpose.

Proposition 5.6. *Let $f : X \to Y$ be a morphism of between Noetherian separated schemes.*

1. *If $Y = \operatorname{Spec} A$ is affine, then for any quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$, we have*

$$R^if_*\mathcal{F} = \widetilde{H^i(X, \mathcal{F})}.$$

2. *In general, if $\mathcal{F}$ is quasi-coherent, then $R^if_*\mathcal{F}$ is quasi-coherent.*

Proof. The case with Y affine implies the general case. So assume $Y = \operatorname{Spec} A$. There is in any case a map $\widetilde{H^i(X, \mathcal{F})} \to R^if_*\mathcal{F}$. For $h \in A$, let $U = \operatorname{Spec} A_h$ be the corrersponding basic open set. We claim $H^i(f^{-1}(U), \mathcal{F}\mid_{f^{-1}(U)}) = H^i(X, \mathcal{F}) \otimes_A A_h$. Assuming the claim, lemma 5.2 implies that $\widetilde{H^i(X, \mathcal{F})} \to R^if_*\mathcal{F}$ is an isomorphism on stalks, hence an isomorphism.

To prove the claim, let $\mathcal{U} = \{U_0, \ldots, U_n\}$ be an affine open cover of X, where $U_i = \operatorname{Spec} B_i$, and consider the Čech complex $\check{C}^\bullet(\mathcal{U}, \mathcal{F})$, which computes the cohomology groups $H^i(X, \mathcal{F})$. Then $V_j = U_j \times_Y U = \operatorname{Spec} B_i \otimes_A A_h$ determines an affine open cover $\mathcal{V} = \{V_0, \ldots, V_n\}$ of $V = f^{-1}(U)$, and the corresponding Čech complex for $\mathcal{F}\mid_{f^{-1}(V)}$ is just

$$\check{C}(\mathcal{V}, \mathcal{F}\mid_V) \cong \check{C}(\mathcal{U}, \mathcal{F}) \otimes_A A_h.$$

Since localization is exact, the cohomology groups of $\check{C}(\mathcal{U}, \mathcal{F}) \otimes_A A_h$ are just the localizations $H^i(X, \mathcal{F}) \otimes_A A_h$. □

As a corollary to the proof, we also get the following.

Corollary 5.7. *Let $f : X \to Y$ a morphism of separated Noetherian schemes, and $\mathcal{U} = \{U_0, \dots, U_n\}$ be an affine open cover of X. Then for any quasi-coherent sheaf $\mathcal{F}$ on X,*

$$R^i f_* \mathcal{F} = i\text{-th cohomology sheaf of } f_* \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}).$$

Proof. Using the fact that, since X, Y are separated, $f^{-1}(V) \cap U$ is affine whenever $U \subset X$, $V \subset Y$ are affine open subsets, the proof is reduced to the case Y is affine, which we treat as above. □

Another corollary is as follows, whose conclusion is expressed in words by saying that "cohomology commutes with flat base change".

Corollary 5.8. *Suppose we have a pull-back square of separated Noetherian schemes*

$$\begin{array}{ccc} W & \xrightarrow{q} & X \\ g \downarrow & & \downarrow f \\ Z & \xrightarrow{p} & Y \end{array}$$

where f is proper and g is flat. Then for any coherent sheaf $\mathcal{F}$ on X, there is are canonical isomorphisms

$$p^* R^i f_* \mathcal{F} \cong R^i g_* q^* \mathcal{F}, \ \ \forall\, i \geq 0.$$

Proof. The proof reduces at once to the case when $Y = \operatorname{Spec} A$, $Z = \operatorname{Spec} B$ are affine. Now if $\mathcal{U} = \{U_0, \dots, \mathcal{U}_n\}$ is an affine open cover of X, then $\mathcal{V} = \{V_0, \dots, V_n\}$ is one of W, where

$$V_i = Z \times_Y U_i = q^{-1}(U_i) \cong U_i \times_{\operatorname{Spec} A} \operatorname{Spec} B.$$

The corollary amounts to the assertion that $H^i(W, q^* F) \cong H^i(X, \mathcal{F}) \otimes_A B$. This is true because $\check{C}^\bullet(\mathcal{V}, q^*\mathcal{F}) = \check{C}^\bullet(\mathcal{U}, \mathcal{F}) \otimes_A B$, and B is flat over A. □

One of the basic results on higher direct images is the following coherence theorem, which is an elaboration of Theorem 3.2.

Theorem 5.9. *Let $f : X \to Y$ be a proper morphism of Noetherian schemes. Let $\mathcal{O}_X(1)$ be a very ample invertible sheaf on X relative to Y, and let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module.*

(a) *For all $n >> 0$, the natural map $f^* f_* \mathcal{F}(n) \to \mathcal{F}(n)$ is surjective.*
(b) *For all $i \geq 0$, $R^i f_* \mathcal{F}$ is a coherent sheaf on Y.*
(c) *For all $i > 0$ and all $n >> 0$, we have $R^i f_* \mathcal{F}(n) = 0$.*

Proof. The result is local on Y, since Y is Noetherian, hence quasi-compact. So we may assume Y is affine. Now the result follows from Proposition 5.6 and Theorem 3.2. □

Exercise 5.10. (Projection Formula) Let $f : X \to Y$ be a morphism of ringer spaces, $\mathcal{F}$ an $\mathcal{O}_X$-module, and $\mathcal{E}$ a locally free $\mathcal{O}_Y$-module of finite rank. Prove that there are natural isomorphisms for all $i \geq 0$

$$R^i f_*(f^* \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{F}) \cong \mathcal{E} \otimes_{\mathcal{O}_Y} R^i f_* \mathcal{F}.$$

We now briefly discuss the Leray spectral sequence.

Theorem 5.11. (Leray Spectral Sequence) *Let $f : X \to Y$ be a continuous map between topological spaces, and $\mathcal{F}$ a sheaf of abelian groups on X. Then there is a spectral sequence*

$$E_2^{p,q} = H^p(Y, R^q f_*\mathcal{F}) \implies H^{p+q}(X, \mathcal{F}).$$

In particular, we have a functorial five term exact sequence *of low degree terms*

$$0 \to H^1(Y, f_*\mathcal{F}) \to H^1(X, \mathcal{F}) \to \Gamma(Y, R^1 f_*\mathcal{F}) \to H^2(Y, f_*\mathcal{F}) \to H^2(X, \mathcal{F}).$$

Proof. If $\mathcal{I}$ is an injective sheaf on X, it is flasque, and hence so is $f_*\mathcal{I}$ (in fact $f_*\mathcal{I}$ is injective), hence $\Gamma(Y, -)$-acyclic. Hence there is a Grothendieck spectral sequence of composite functors, which is just the Leray spectral sequence described above. To get the five term exact sequence, note that $E_2^{1,0} = E_\infty^{1,0}$, while there is an exact sequence

$$0 \to E_\infty^{0,1} \to E_2^{0,1} \to E_2^{2,0} \to E_\infty^{2,0} \to 0$$

(in other words, $E_3^{p,q} = E_\infty^{p,q}$ for $(p,q) = (0,1)$ or $(2,0)$). Since the limit of the spectral sequence is the sequence of groups $H^i(X, \mathcal{F})$, the group $H^1(X, \mathcal{F})$ fits into an exact sequence

$$0 \to E_\infty^{1,0} \to H^1(X, \mathcal{F}) \to E_\infty^{0,1} \to 0,$$

and there is a 3-step filtration

$$0 \subset F^2H^2(X, \mathcal{F}) \subset F^1H^2(X, \mathcal{F}) \subset F^0H^2(X, \mathcal{F}) = H^2(X, \mathcal{F})$$

where $F_2H^2(X, \mathcal{F}) = E_\infty^{2,0}$, $F^1/F^2 \cong E_\infty^{1,1}$ and $F^0/F^1 \cong E_\infty^{0,2}$. In particular, there is an inclusion $E_\infty^{2,0} \hookrightarrow H^2(X, \mathcal{F})$, giving a map $E_2^{2,0} \to H^2(X, \mathcal{F})$. Combining the above, we get an exact sequence

$$0 \to E_2^{1,0} \to H^1(X, \mathcal{F}) \to E_2^{0,1} \to E_2^{2,0} \to H^2(X, \mathcal{F}),$$

which is the five term exact sequence. □

Corollary 5.12. *Let $f : X \to Y$ be an affine morphism between Noetherian schemes, and let $\mathcal{F}$ be quasi-coherent on X. Then*

(a) *$R^i f_*\mathcal{F} = 0$ for all $i > 0$, and*
(b) *there are natural isomorphisms $H^i(Y, f_*\mathcal{F}) \cong H^i(X, \mathcal{F})$.*

Proof. We first prove (a). From lemma 5.2,it suffices to observe that if $U \subset Y$ is affine, then since $f^{-1}(U) = V$ is also affine (since f is affine) and so $H^i(V, \mathcal{F}\mid_V) = 0$. This proves (a). For (b), note that the Leray spectral sequence has $E_2^{p,q} = 0$ for $q > 0$, and hence degenerates at E_2 giving isomorphisms

$$H^i(Y, f_*\mathcal{F}) = E_2^{i,0} \cong E_\infty^{i,0} \cong H^i(X, \mathcal{F}).$$

□

Remark 5.13. In a similar way, if $f : X \to Y$ is any continous map, and $\mathcal{F}$ a sheaf with $R^i f_*\mathcal{F} = 0$ for $i > 0$, the Leray spectral sequence implies that $H^i(Y, f_*\mathcal{F}) \cong H^i(X, \mathcal{F})$.

One application of the Leray Spectral Sequence is to the extension of the Coherence Theorem 3.2(b) to proper morphisms. The proof uses the *Chow Lemma*, which states that for any proper morphism $g : X \to S$ of Noetherian schemes, there exists a projective morphism $h : Z \to S$, and a morphism of S-schemes $p : Z \to X$, such that for some open dense subscheme $U \subset X$, the morphism $p^{-1}(U) \to U$ is an isomorphism. The following proof is prototypical of proofs of results for proper morphisms, by reduction to the case of projective morphisms.

Theorem 5.14. *Let $f : X \to Y$ be a proper morphism between Noetherian schemes, and $\mathcal{F}$ a coherent sheaf on X. Then $R^i f_*\mathcal{F}$ is coherent on Y for all $i \geq 0$.*

Proof. Since the property is local on Y, we reduce to the case $Y = \operatorname{Spec} A$ is affine. Now we are reduced to proving $H^i(X, \mathcal{F})$ is a finite A-module, for any coherent $\mathcal{F}$. We do this by Noetherian induction on $\operatorname{supp}(\mathcal{F})$, the closed subscheme of X determined by the (coherent) annihilator ideal sheaf of $\mathcal{F}$. If $\operatorname{supp}(\mathcal{F})$ is a closed point x, then the residue field $k(x)$ is a finitely generated A-algebra, hence a finite A-module; since $\mathcal{F}$ is a skyscraper sheaf whose stalk is a finite dimensional $k(x)$-vector space, we have that $H^0(X, \mathcal{F})$ is a finite A-module, and $H^i(X, \mathcal{F}) = 0$ for $i > 0$. Next, if $i : \operatorname{supp}(\mathcal{F}) \to X$ is the inclusion, then $\mathcal{F} = i_*i^*\mathcal{F}$, and we have $H^i(X, \mathcal{F}) = H^i(\operatorname{supp}(\mathcal{F}), i^*\mathcal{F})$. So we may assume $X = \operatorname{supp}(\mathcal{F})$.

By Chow's Lemma, we can find a morphism $p : Z \to X$ such that the composition $g = p \circ f : Z \to Y$ is projective, and for an open dense $U \subset X$, the morphism $p^{-1}(U) \to U$ is an isomorphism. From lemma 5.2, the natural map $\mathcal{F} \to p_*p^*\mathcal{F}$ has kernel an cokerel supported on the complement of U, and the higher direct images $R^i p_*p^*\mathcal{F}$ are also supported on the complement of U. By Noetherian induction, we thus know that this kernel and cokernel, as well as the higher direct images $R^i p_*p^*\mathcal{F}$, have cohomology modules on X which are finite A-modules. In particular, from suitable long exact sequences of cohomology modules, we are reduced to proving that $H^i(X, p_*p^*\mathcal{F})$ is a finite A-module for all i.

In the Leray spectral sequence

$$E_2^{s,t} = H^s(X, R^t p_*p^*\mathcal{F}) \implies H^{s+t}(Z, p^*\mathcal{F}),$$

the limit terms $H^n(Z, p^*\mathcal{F})$ are finite A-modules, since $Z \to Y = \operatorname{Spec} A$ is projective, and $p^*\mathcal{F}$ is coherent. Hence all the $E_\infty^{s,t}$ terms are finite A-modules; hence $E_r^{s,t}$ is a finite A-module for each s, t and all sufficiently large r. Further, we are given (the Noetherian induction hypothesis) that $E_2^{s,t} = H^s(X, R^t p_*p^*\mathcal{F})$ is a finite A-module whenever $t > 0$. From the exact sequences

$$E_r^{i-r-1,r} \to E_r^{i,0} \to E_{r+1}^{i,0} \to 0$$

we conclude by descending induction on r that $E_r^{i,0}$ is a finite A-module for all $i \geq 0$ and all $r \geq 2$. The finite generation of $E_2^{i,0} = H^i(X, p_*p^*\mathcal{F})$ is the desired conclusion. □

6. Some relations between coherent sheaf cohomology and local cohomology

If A is a Noetherian ring, I an ideal, and M an A-module, let

$$H^0_I(M) = \{m \in M \mid I^n m = 0 \text{ for some } n > 0\}.$$

Recall that the *local cohomology module* $H^i_I(M)$ is defined to be the value on M of the i-th derived functor of $M \mapsto H^0_I(M)$. By definition, this means that for any injective resolution $0 \to M \to J^\bullet$,

$$H^i_I(M) = \ i\text{-th cohomology of } H^0_I(J^\bullet).$$

By definition, the local cohomology depends only on the radical of I.

The category of A-modules is equivalent to the category of quasi-coherent sheaves on $X = \operatorname{Spec} A$. If $Z \subset X$ is the subscheme defined by the ideal I (or equivalently the coherent ideal sheaf $\widetilde{I}$), then we have a natural identification

$$\Gamma_Z(X, \widetilde{M}) \cong H^0_I(M).$$

Lemma 6.1. *Let A, I be as above. For any A-module M, we have natural identifications for all $i \geq 0$*

$$H^i_Z(X, \widetilde{M}) \cong H^i_I(M).$$

Proof. If $0 \to M \to J^\bullet$ is an injective resolution, then by lemma 1.2, $0 \to \widetilde{M} \to \widetilde{J^\bullet}$ is a flasque resolution, and flasque sheaves are acyclic for cohomology with support. Hence

$$H^i_Z(X, \mathcal{F}) \cong H^i(\Gamma_Z(X, \widetilde{J^\bullet})) \cong H^i(H^0_I(J^\bullet)) \cong H^i_I(M).$$

□

Corollary 6.2. *Let $U = \operatorname{Spec} A \setminus \operatorname{Spec}(A/I)$. There is an exact sequence*

$$0 \to H^0_I(M) \to M \to H^0(U, \widetilde{M}\mid_U) \to H^1_I(M) \to 0,$$

and natural isomorphisms

$$H^i(U, \widetilde{M}\mid_U) \cong H^{i+1}_I(M) \ \ \forall \ \ i > 0.$$

Proof. There is a long exact sequence of cohomology with supports

$$0 \to H^0_Z(X, \widetilde{M}) \to H^0(X, \widetilde{M}) \to H^0(U, \widetilde{M}\mid_U) \to H^1_Z(X, \widetilde{M}) \to H^1(X, \widetilde{M})$$
$$\to H^1(U, \widetilde{M}\mid_U) \to H^2_Z(X, \widetilde{M}) \to \cdots$$

from which the corollary follows immediately, since on the affine scheme X, we have $H^0(X, \widetilde{M}) = M$ and $H^i(X, \widetilde{M}) = 0$ for $i > 0$ (Theorem 1.1). □

The above corollary provides the basic link between local cohomology modules defined in commutative algebra and the cohomology theory of quasi-coherent sheaves. We will illustrate how this is useful in one important situation.

Let $R = \oplus_{n \geq 0} R_n$ be a graded ring, where $A = R_0$ is Noetherian, each R_n is a finite A-module, and R is a finitely generated A-algebra. Let $R_+ \subset R$ be the "irrelevant graded ideal", *i.e.*, $R_+ = \oplus_{n>0} R_n = \ker R \to A$. Let $X = \operatorname{Proj} R$ be the associated projective A-scheme. If $M = \oplus_{n\in\mathbb{Z}} M_n$ is a graded R-module, there is an associated quasi-coherent sheaf $\widetilde{M}$ on X. If $\mathcal{F} = \widetilde{M}$, we let $\mathcal{F}(n)$ denote the sheaf $\widetilde{M(n)}$, where $M(n)$ is the graded R-module with underlying R-module M and shifted grading $M(n)_r = M_{n+r}$. If R is generated by R_1 as an A-algebra, then $\mathcal{O}_X(1)$ is an invertible sheaf on X, and $\mathcal{F}(n) \cong \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(1)^{\otimes n}$.

Theorem 6.3. *In the above situation, there is a natural exact sequence*

$$0 \to H^0_{R_+}(M) \to M \to \oplus_{n\in\mathbb{Z}} H^0(X, \mathcal{F}(n)) \to H^1_{R_+}(M) \to 0,$$

and there are natural isomorphisms

$$H^{i+1}_{R_+}(M) \cong \oplus_{n\in\mathbb{Z}} H^i(X, \mathcal{F}(n)).$$

Proof. Let $Y = \operatorname{Spec} R$, $U = \operatorname{Spec} R \setminus \operatorname{Spec} A$, where we regard $\operatorname{Spec} A$ as the subscheme defined by the ideal R_+. There is a canonical morphism of A-schemes $f : U \to X$, given on points by associating to a prime ideal $\wp \subset A$ the prime ideal $\widetilde{\wp}$ generated by all homogenous elements in $\wp$; provided $R_+ \not\subset \wp$ (*i.e.*, if $\wp \in U \subset Y$), this homogeneous prime ideal does not contain R_+ and is hence a point of $\operatorname{Proj} R$.

At the level of schemes, the morphism f may be defined as follows: if $h \in R_+$ is a homogeneous element, we have an affine open subset $Y_h = \operatorname{Spec} R_h \subset Y$, as well as an affine open subset $D_+(h) = \operatorname{Spec} R_{(h)}$, where $R_{(h)}$ is the subring of elements of degree 0 in the $\mathbb{Z}$-graded ring localization R_h. Note that $Y_h \subset U$, and U is covered by such open subsets. The morphism $Y_h \to D_+(h)$ is that determined by the incusion $R_{(h)} \subset R_h$ as the subring of elements of degree 0.

One sees easily that the diagrams

$$\begin{array}{ccc} Y_{h_1h_2} & \to & Y_{h_i} \\ \downarrow & & \downarrow \\ D_+(h_1h_2) & \hookrightarrow & D_+(h_i) \end{array}$$

commute for each i, where $Y_{h_1h_2} = Y_{h_1} \cap Y_{h_2}$, and so these locally defined morphisms patch together to define $f : U \to X$. One verifies further that $f^{-1}(D_+(h) = Y_h$ (this just says $h \notin \widetilde{\wp} \iff h \notin \wp$), so f is an affine morphism.

Now let $\widetilde{\mathcal{F}}$ denote the quasi-coherent sheaf on Y associated to the R-module M (ignoring the grading). The R_h-module of sections of $\widetilde{\mathcal{F}}$ on Y_h is just M_h, which is a $\mathbb{Z}$-graded module over the $\mathbb{Z}$-graded ring R_h. Notice that by definition of $\mathcal{F}(n)$, the $R_{(h)}$ submodule of homogeneous elements of degree n in M_h is naturally identified with the $R_{(h)}$-module of sections of $\mathcal{F}(n)$ on $D_+(h)$. Thus, we have a canonical isomorphism of quasi-coherent sheaves on X,

$$f_*(\widetilde{\mathcal{F}} \mid_U) \cong \oplus_{n\in\mathbb{Z}} \mathcal{F}(n).$$

Since f is an affine morphsim, $R^i f_* \widetilde{\mathcal{F}} = 0$ for all $i > 0$, and so by the Leray spectral sequence for $f : U \to X$ and the sheaf $\widetilde{\mathcal{F}}$, we have a canonical identification

$$H^i(U, \widetilde{\mathcal{F}} \mid_U) \cong H^i(X, f_*\widetilde{\mathcal{F}} \mid_U) \cong \oplus_{n\in\mathbb{Z}} H^i(X, \mathcal{F}(n)).$$

Now the Theorem follows from Corollary 6.2. □

Note that the local cohomology modules $H^i_{R_+}(M)$ carry natural gradings, for example as a consequence of the above Theorem.

Corollary 6.4. *In the above situation, we have an exact sequence of finite A-modules for each $n \in \mathbb{Z}$*

$$0 \to H^0_{R_+}(M)_n \to M_n \to H^0(X, \mathcal{F}(n)) \to H^1_{R_+}(M)_n \to 0,$$

and isomorphisms

$$H^{i+1}_{R_+}(M)_n \cong H^i(X, \mathcal{F}(n)).$$

Example 6.5. Let $X \subset \mathbb{P}^N_k$ be a projective scheme, and $R = \oplus_{n\geq 0} R_n$ the homogeneous coordinate ring of X. Then from Theorem 6.3, we have

$$H^0_{R_+}(R) = 0,$$

$$H^1_{R_+}(R) \cong \operatorname{coker}\left(R \hookrightarrow \oplus_{n\in\mathbb{Z}} H^0(X, \mathcal{O}_X(n))\right),$$

$$H^{i+1}_{R_+}(R) \cong \oplus_{n\in\mathbb{Z}} H^i(X, \mathcal{O}_X(n)).$$

Thus, if X is a normal projective variety, so that the affine cone $\operatorname{Spec} R$ is normal except perhaps at the vertex $V(R_+)$, then the normality of the cone (*i.e.*, of R) is equivalent to the vanishing of $H^1_{R_+}(R)$, or equivalently, to the surjectivity for each n of the natural maps

$$H^0(\mathbb{P}^N_k, \mathcal{O}_{\mathbb{P}^N_k}(n)) \to H^0(X, \mathcal{O}_X(n)),$$

whose image is just $R_n \subset H^0(X, \mathcal{O}_X(n))$.

On the other hand, the condition that R is Cohen-Macaulay is equivalent to this surjectivity, together with the vanishing result

$$H^i(X, \mathcal{O}_X(n)) = 0 \ \ \forall\, i < \dim X, \ \ \forall\, n \in \mathbb{Z}.$$

For example, let E be a smooth projective plane curve of degree 3 (an elliptic curve), and $X = E \times E$ its Segre embedding in $\mathbb{P}^8_k$. The homogeneous coordinate ring of X is then easily seen to be normal, but it is not Cohen-Macaulay, since $H^1(X, \mathcal{O}_X) \neq 0$.

Lemma 6.6. *In the situation of Theorem 3.5, suppose M is a finite graded $R(X)$-module, and $\mathcal{F}$ the corresponding coherent sheaf on X. Let $P(t)$ be the Hilbert polynomial of M (and hence also of $\mathcal{F}$). Then*

$$P(n) - \ell(M_n) = \sum_{i\geq 0} (-1)^{i-1} \ell(H^i_{R_+}(M)_n).$$

Proof. Since $P(n) = \chi(\mathcal{F}(n)) = \sum_{i\geq 0}(-1)^i \ell(H^i(X, \mathcal{F}(n))$, the lemma follows from Corollary 6.4. □

The above lemma deals with the case of sheaves on a projective scheme over an Artinian ring. There is another situation where one has a siimilar result, which is proved by reduction to the Artinian case. Let $(A, \mathbf{m})$ be a Noetherian local ring, I an $\mathbf{m}$-primary ideal. Let $R(I) = \oplus_{n\geq 0} I^n$ be the Rees algebra of I, and let $X = \operatorname{Proj} R(I)$ be the blow-up scheme of I. If M is any finite A-module, we can associate to it a finite graded $R(I)$-module $M^* = \oplus_{n\geq 0} I^n M$, and hence a coherent sheaf $\mathcal{F} = \widetilde{M^*}$ on X. Since $X \to \operatorname{Spec} A$ is an isomorphism over the punctured spectrum, we deduce that for each $n \geq 0$, we have

(i) the kernel and cokernel of the natural map $M \to H^0(X, \mathcal{F})$ are supported at $\mathbf{m}$, *i.e.*, have finite length, and
(ii) the A-modules $H^i(X, \mathcal{F}(n))$ for $i > 0$ are supported at $\mathbf{m}$.

We also have, associated to M, a polynomial $P(t) \in \mathbb{Q}[t]$ (the *Hilbert-Samuel polynomial*) with the property that $P(n) = \ell(M/I^nM)$ for all $n >> 0$. Since

$$\ell(M/I^nM) = \sum_{j=0}^{n-1} \ell(I^jM/I^{j+1}M),$$

the existence of a Hilbert-Samuel polynomial as above follows from the existence of one for the finite graded module $\oplus_{n\geq 0} I^nM/I^{n+1}M$ over the graded ring $R(I)/R(I)_+ = \oplus_{n\geq 0} I^n/I^{n+1}$ (to which Theorem 3.5 applies).

Theorem 6.7. (Johnston-Verma-Trivedi) *For any finite A-module M, with Hilbert-Samuel polynomial $P(t) \in \mathbb{Q}[t]$, we have*

$$P(n) - \ell(M/I^nM) = \sum_{i\geq 0} (-1)^{i-1} \ell(H^i_{R(I)_+}(M^*)_n) \ \ \forall\, n \geq 0.$$

Proof. Let $Q(n) = \sum_{i\geq 0} (-1)^{i-1} \ell(H^i_{R(I)_+}(M^*)_n)$. For $n >> 0$, we have $Q(n) = 0$, from Corollary 6.4, because

(i) the graded $R(I)$-modules M^* and $\oplus_{n\geq 0} H^0(X, \mathcal{F}(n))$ coincide in high enough degrees (both graded modules determine the same coherent sheaf $\mathcal{F}$ on X); this implies the terms in $Q(n)$ for $i = 0, 1$ vanish for $n >> 0$
(ii) $H^i(X, \mathcal{F}(n)) = 0$ for $i \geq 1$ and $n >> 0$ by Serre vanishing, so the terms in $Q(n)$ for $i \geq 2$ vanish.

On the other hand, $\ell(M/I^nM) = P(n)$ for large n, since $P(t)$ is the Hilbert-Samuel polynomial of M relative to I. Hence the formula in the Theorem is valid for $n >> 0$.

Let $\operatorname{gr}_I M = \oplus_{n\geq 0} I^nM/I^{n+1}M$, which is a finite graded module over $\operatorname{gr}_I A = R(I)/R(I)_+$. There is an exact sequence of graded $R(I)$-modules

$$0 \to N \to M^* \to \operatorname{gr}_I M \to 0,$$

where $N = \oplus_{n\geq 0} I^{n+1}M$ is a graded submodule of $M^*(1)$ such that the graded quotient $M^*(1)/N$ is concentrated in degree -1, and vanishes in other degrees. Hence $\widetilde{N} = \widetilde{M^*(1)} = \mathcal{F}(1)$, and we have an associated exact sequence of coherent sheaves on X

$$0 \to \mathcal{F}(1) \to \mathcal{F} \to \mathcal{G} \to 0,$$

where $\mathcal{G} = \widetilde{\operatorname{gr}_I M}$. The sequence of graded modules gives rise to a long exact sequence of (graded) local cohomology modules

$$0 \to H^0_{R_+}(N) \to H^0_{R_+}(M^*) \to H^0_{R_+}(\operatorname{gr}_I M) \to H^1_{R_+}(N) \to \cdots .$$

Restricting to the homogeneous components of degree n, where $n \geq 0$, we get a corresponding bounded exact sequence of A-modules of finite length. Hence we obtain a formula

$$Q(n) - Q(n+1) = \sum_{i \geq 0} (-1)^i \ell(H^i_{R_+}(\mathrm{gr}_I M))_n).$$

On the other hand, by lemma 6.6, we have

$$\chi(\mathcal{G}(n)) - \ell(\mathrm{gr}^n_I M) = \sum_{i \geq 0} (-1)^{i-1} \ell(H^i_{R_+}(\mathrm{gr}_I M))_n) = Q(n+1) - Q(n).$$

But $\ell(\mathrm{gr}^n_I M) = \ell(R/I^{n+1}M) - \ell(R/I^n M) = P(n+1) - P(n)$, for large n. Hence $\chi(\mathcal{G}(n)) = P(n+1) - P(n)$ for *all* $n \in \mathbb{Z}$ (in other words, $P(t+1) - P(t)$ is the Hilbert polynomial for $\mathcal{G}$). This means

$$P(n+1) - \ell(M/I^{n+1}M) - P(n) + \ell(M/I^n M) = Q(n+1) - Q(n).$$

Hence if the Theorem holds for $n+1$, then it also holds for n. □

7. The formal function theorem

In this section we discuss the Formal Function Theorem and some applications.

Theorem 7.1. *Let $f : X \to Y$ be a proper morphism, and $\mathcal{F}$ a coherent sheaf on X. Let $y \in Y$, and let $X_n = X \times_Y \operatorname{Spec} \mathcal{O}_{Y,y}/\mathbf{m}_y^n$, where $\mathbf{m}_y \subset \mathcal{O}_{Y,y}$ is the maximal ideal; let $i_n : X_n \hookrightarrow X$ be the given closed immersion. Then there is a canonical isomorphism*

$$(R^i f_* \mathcal{F})_y \otimes_{\mathcal{O}_{Y,y}} \widehat{\mathcal{O}}_{Y,y} \cong \lim_{\substack{\leftarrow \\ n}} H^i(X_n, i_n^* \mathcal{F}),$$

where $\widehat{\mathcal{O}}_{Y,y}$ is the $\mathbf{m}_y$-adic completion of $\mathcal{O}_{Y,y}$.

Proof. We will consider only the case of projective morphisms, though the result holds more generally for proper morphisms (the proper case can be reduced to the projective case, as in the proof of Theorem 5.14.

We first reduce to the case $Y = \operatorname{Spec} A$, where A is Noetherian local, and y is the closed point. Then $R^i f_* \mathcal{F} = \widetilde{H^i(X, \mathcal{F})}$. We may further make the (flat) base change to $\operatorname{Spec} \widehat{A}$; if $\widehat{X} = X \times_{\operatorname{Spec} A} \operatorname{Spec} \widehat{A}$, $\pi : \widehat{X} \to X$, and $\widehat{f} : \widehat{X} \to \operatorname{Spec} \widehat{A}$, then (Corollary 5.8) we have

$$H^i(\widehat{X}, \pi^* \mathcal{F}) \cong H^i(X, \mathcal{F}) \otimes_A \widehat{A},$$

and also $\widehat{X}_n \cong X_n$. Hence we further reduce the proof of the Theorem to the case when $A = \widehat{A}$. Now we must show that

$$H^i(X, \mathcal{F}) \to \lim_{\substack{\leftarrow \\ n}} H^i(X_n, i_n^* \mathcal{F})$$

is an isomorphism.

Since X is projective over A, we may embed it as a closed subscheme of some $\mathbb{P}^N_A$. Let $A_n = A/\mathbf{m}^n$, where $\mathbf{m}$ is the maximal ideal of A. Then $X \cap \mathbb{P}^N_{A_n} = X_n$ is realized as a closed subscheme of $\mathbb{P}^N_{A_n}$. Replacing $\mathcal{F}$ and $i_n^* \mathcal{F}$ by their direct images on $\mathbb{P}^N_A$ and $\mathbb{P}^N_{A_n}$ respectively, which doesn't change the corresponding cohomology modules, we may assume without loss of generality that $X = \mathbb{P}^N_A$.

Now we prove the Theorem by descending induction on i. First, the Theorem clearly holds for the sheaves $\mathcal{O}_{\mathbb{P}^N_A}(r)$ for all $r \in \mathbb{Z}$, by Theorem 2.4 applied to $\mathbb{P}^N_A$ and $\mathbb{P}^N_{A_n}$.

Next, we show that the Theorem holds for $i = N$ for any coherent $\mathcal{F}$. We have a presentation

$$\mathcal{O}_{\mathbb{P}^N_A}(-a)^{\oplus p} \to \mathcal{O}_{\mathbb{P}^N_A}(-b)^{\oplus q} \to \mathcal{F} \to 0,$$

which remains exact on applying i_n^*. This gives corresponding presentations

$$H^N(\mathbb{P}^N_A, \mathcal{O}_{\mathbb{P}^N_A}(-a))^{\oplus p} \to H^N(\mathbb{P}^N_A, \mathcal{O}_{\mathbb{P}^N_A}(-b))^{\oplus q} \to H^n(\mathbb{P}^N_A, \mathcal{F}) \to 0,$$

and an inverse system of similar presentations for $H^N(\mathbb{P}^N_{A_n}, i_n^*\mathcal{F})$, for each n. Note that the cohomology modules $H^i(\mathbb{P}^N_{A_n}, i_n^*\mathcal{F})$ are modules of finite length, for any coherent $\mathcal{F}$, any $i \geq 0$ and any $n > 0$. We now make use of the following lemma (the proof is left as an exercise!).

Lemma 7.2. *Let*

$$0 \to A_n \to B_n \to C_n \to 0$$

be an inverse system of exact sequences of A-modules of finite length. Then

$$0 \to \varprojlim_n A_n \to \varprojlim_n B_n \to \varprojlim_n C_n \to 0$$

is exact.

Thus, the functor $\varprojlim_n$ is exact on the category of inverse systems of A-modules of finite length. Applying this lemma to the inverse system of presentations for $H^N(\mathbb{P}^N_{A_n}, i_n^*\mathcal{F})$, we see that the Theorem is valid for $\mathcal{F}$, for $i = N$.

Now we proceed by descending induction on i. Choose an exact sequence

$$0 \to \mathcal{G} \to \mathcal{O}_{\mathbb{P}^N_k}(-a)^{\oplus r} \to \mathcal{F} \to 0$$

with $a > 0$. To simplify notation, let $X = \mathbb{P}^N_A$, and $X_n = \mathbb{P}^N_{A_n}$. Let $\mathcal{F}_n = i_n^*\mathcal{F}$, $\mathcal{G}_n = i_n^*\mathcal{G}$. If the functors i_n^* were exact, the proof would now be fairly simple: we would have an exact sequence

$$0 \to H^{N-1}(X_n, \mathcal{F}_n) \to H^N(X_n, \mathcal{G}_n) \to H^N(X_n, \mathcal{O}_{X_n}(-a))^{\oplus r} \to H^N(X_n, \mathcal{F}_n) \to 0,$$

and isomorphisms

$$H^i(X_n, \mathcal{F}_n) \cong H^{i+1}(X_n, \mathcal{G}_n) \ \ \forall\, i < N-1.$$

We have (unconditionally) an analogous exact sequence and isomorphism on X. Taking inverse limits over n, we would get an exact sequence

$$(7.1) \quad 0 \to \varprojlim_n H^{N-1}(X_n, \mathcal{F}_n) \to \varprojlim_n H^N(X_n, \mathcal{G}_n) \to \varprojlim_n H^N(X_n, \mathcal{O}_{X_n}(-a))^{\oplus r}$$

$$\to \varprojlim_n H^N(X_n, \mathcal{F}_n) \to 0,$$

and isomorphisms

$$\lim_{\substack{\longleftarrow \\ n}} H^i(X_n, \mathcal{F}_n) \cong \lim_{\substack{\longleftarrow \\ n}} H^{i+1}(X_n, \mathcal{G}_n) \;\; \forall\, i < N-1. \tag{7.2}$$

Now the result for H^N for $\mathcal{G}$, $\mathcal{F}$ and $\mathcal{O}_X(-a)$ implies it for H^{N-1}, for any coherent $\mathcal{F}$. Similarly, for any $i < N-1$, the result for H^{i+1} for all coherent sheaves implies it for H^i for all coherent sheaves.

So our proof will be complete if, inspite of the failure of exactness of i_n^*, we still have (7.1) and (7.2). For this, we need to make use of a lemma.

Lemma 7.3. *Let R be a Noetherian ring, I an ideal in R, and M a finite R-module. For each $n > 0$, there exists $m > n$ such that the natural map*

$$\mathrm{Tor}_1^R(M, R/I^m) \to \mathrm{Tor}_1^R(M, R/I^n)$$

is zero.

Proof. Choose a presentation

$$0 \to N \to F \to M \to 0$$

where F is free of finite rank. Then $\mathrm{Tor}_1^R(M, R/J) = (JF \cap N)/JN$. So it suffices to show that for some $m > n$, the natural map

$$(I^m F \cap N) \to (I^n F \cap N)$$

factors through the submodule $I^n N$. This is an immediate consequence of the Artin-Rees lemma. □

Returning to our situation, we have exact sequences

$$0 \to i_n^* \mathcal{T}or_1^{\mathcal{O}_X}(\mathcal{F}, (i_n)_* \mathcal{O}_{X_n}) \to \mathcal{G}_n \to \mathcal{O}_{X_n}(-a)^{\oplus r} \to \mathcal{F}_n \to 0,$$

where for quasi-coherent $\mathcal{F}$, $\mathcal{G}$, the sheaf $\mathcal{T}or_i^{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ is the quasi-coherent sheaf such that for any affine open $U = \operatorname{Spec} A$, with $M = \Gamma(U, \mathcal{F})$, $N = \Gamma(U, \mathcal{G})$, we have

$$\mathcal{T}or_i^{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \mid_U = \widetilde{\mathrm{Tor}_i^A(M, N)}.$$

We break up the above exact sequence into short exact sequences

$$0 \to i_n^* \mathcal{T}or_1^{\mathcal{O}_X}(\mathcal{F}, (i_n)_* \mathcal{O}_{X_n}) \to \mathcal{G}_n \to \mathcal{H}_n \to 0,$$

$$0 \to \mathcal{H}_n \to \mathcal{O}_{X_n}(-a)^{\oplus r} \to \mathcal{F}_n \to 0.$$

From lemma 7.3, it follows that

$$\lim_{\substack{\longleftarrow \\ n}} H^i(X_n, i_n^* \mathcal{T}or_1^{\mathcal{O}_X}(\mathcal{F}, (i_n)_* \mathcal{O}_{X_n})) = 0 \;\; \forall\, i.$$

Hence, using lemma 7.2 and appropriate long exact sequences in cohomology, we deduce that

$$\lim_{\substack{\longleftarrow \\ n}} H^i(X_n, \mathcal{G}_n) \cong \lim_{\substack{\longleftarrow \\ n}} H^i(X_n, \mathcal{H}_n) \;\; \forall\, i. \tag{7.3}$$

We have an inverse system of exact sequences

$$0 \to H^{N-1}(X_n, \mathcal{F}_n) \to H^N(X_n, \mathcal{H}_n) \to H^N(X_n, \mathcal{O}_{X_n}(-a))^{\oplus r} \to H^N(X_n, \mathcal{F}_n) \to 0,$$

and isomorphisms

$$H^i(X_n, \mathcal{F}_n) \cong H^{i+1}(X_n, \mathcal{H}_n) \ \ \forall\, i < N-1.$$

Taking inverse limits, using lemma 7.2 and (7.3), it follows that (7.1) and (7.2) are valid, as desired. □

We now proceed to give some applications of the Formal Function theorem. An easy consequence is:

Corollary 7.4. *Let $f : X \to Y$ be a proper morphism between Noetherian schemes. For each $y \in Y$, with fiber $X_y = X \times_Y \operatorname{Spec} k(y)$, the stalk $(R^i f_* \mathcal{F})_y$ vanishes for all $i > \dim X_y$.*

Proof. It suffices to prove the completion of this stalk vanishes, which is true from Theorem 7.1 because all the terms in the inverse limit are 0. □

Corollary 7.5. *Let $f : X \to Y$ be a proper morphism between Noetherian schemes which is quasi-finite (i.e., all set-theoretic fibers have finite cardinality). Then f is a finite morphism.*

Proof. Since $f_* \mathcal{O}_X$ is a coherent sheaf on Y, it suffices to prove f is an affine morphism. For this we may assume Y is affine, and have to show X is affine. From Corollary 7.4, we have $R^i f_* \mathcal{F} = 0$ for all $i > 0$, for all coherent $\mathcal{F}$. Since Y is affine, this is the same as saying $H^i(X, \mathcal{F}) = 0$ for all $i > 0$. Now by Serre's criterion (Theorem 1.3), X is affine. □

Theorem 7.6. (Connectedness Theorem*) Let $f : X \to Y$ be a proper morphism with $f_* \mathcal{O}_X = \mathcal{O}_Y$. Then all fibers of f are connected.*

Proof. Let $y \in Y$, and $X_y = X \times_Y \operatorname{Spec} k(y)$ the fiber. Let X_n be the subscheme of Y defined by $X_n = X \times_Y \operatorname{Spec} \mathcal{O}_{Y,y}/\mathbf{m}_y^n$, where $\mathbf{m}_y$ is the maximal ideal of $\mathcal{O}_{Y,y}$. As a topological space, X_n coincides with $X_1 = X_y$. Hence, if X_y is not connected, fix a connected component Z, and let Z_n be the corresponding component of X_n. We can find a unique function $a_n \in \Gamma(X_n, \mathcal{O}_{X_n})$ with $a_n \mid_{Z_n} = 1$ and $a_n \mid_{X_n \setminus Z_n} = 0$; clearly $a_n \mid_{X_{n-1}} = a_{n-1}$. Hence the sequence $\{a_n\}$ determines a well-defined element

$$a \in \varprojlim_n H^0(X_n, \mathcal{O}_{X_n}) \cong (f_* \mathcal{O}_X)_y \otimes \widehat{\mathcal{O}}_{Y,y} \cong \widehat{\mathcal{O}}_{Y,y}$$

which is clearly a non-trivial idempotent element (*i.e.*, $a^2 = a$ with $a \neq 0, 1$). Since $\widehat{\mathcal{O}}_{Y,y}$ is a local ring, this is a contradiction. □

Corollary 7.7. ("Zariski's Main Theorem") *Let $f : X \to Y$ be a birational proper morphism of Noetherian integral schemes, with Y normal. Then all fibers of f are connected.*

Proof. The problem is local on Y, so we may assume $Y = \operatorname{Spec} A$ is affine. By Theorem 7.6, it suffices to prove $f_* \mathcal{O}_X = \mathcal{O}_Y$, which in this case means $\Gamma(X, \mathcal{O}_X) = A$. If $B = \Gamma(X, \mathcal{O}_X)$, then since f is birational, A and B have the same quotient field. On the other hand, Theorem 3.2 implies B is a finite A-module. Hence B is integral over A, and since A is normal, $B = A$. □

Corollary 7.8. (Stein factorization) *Let $f : X \to Y$ be a proper morphism of Noetherian schemes. Then we can uniquely (up to isomorphism) factorize f as a composition $f = g \circ h$, with $h : X \to Z$, $g : Z \to Y$, such that g is finite, and $h_*\mathcal{O}_X = \mathcal{O}_Z$ (in particular, h has connected fibers).*

Proof. Define $Z = \mathbf{Spec} f_*\mathcal{O}_X$, and g, h to be the evident maps. Since $f_*\mathcal{O}_X$ is $\mathcal{O}_Y$-coherent, g is finite; that $h_*\mathcal{O}_X = \mathcal{O}_Z$ follows because $g_*\mathcal{O}_Z = f_*\mathcal{O}_X = g_*h_*\mathcal{O}_X$ by construction, so that the natural map $\mathcal{O}_Z \to h_*\mathcal{O}_X$ becoes an isomorphism on applying g_*, with g finite. □

Proposition 7.9. *Let X be a Noetherian scheme, $Z \subset X$ a closed, local complete intersection subscheme, and $f : Y \to X$ the blow up of Z in X. Then $R^if_*\mathcal{O}_Y = 0$ for all $i > 0$.*

Proof. For simplicity, we assume X, Z (and hence also Y) are regular; the proof in the general case is a little more technical, and is left to the reader.

The question is local on X, so we may assume without loss of generality that $X = \operatorname{Spec} A$ is affine, and $Z = V(I)$ is a complete intersection, where I is generated by a regular sequence $a_1, \dots, a_r \in A$. Since $R^if_*\mathcal{O}_Y$ is a coherent sheaf, it suffices to show that for each closed point $x \in Z$, the stalk $(R^if_*\mathcal{O}_Y)_x$ vanishes.

Now the exceptional divisor $E = f^{-1}Z \cong \mathbb{P}^{r-1}_Z$, as Z-schemes, and the conormal sheaf to E in Y is $\mathcal{O}_{\mathbb{P}^{r-1}_Z}(1)$. This implies that for $x \in Z$, the fiber

$$Y_x = Y \times_X \operatorname{Spec} k(x) \cong E \times_Z \operatorname{Spec} k(x) \cong \mathbb{P}^{r-1}_{k(z)}.$$

Further, the conormal sheaf of Y_x in E is a free $\mathcal{O}_{Y_x}$-module of rank $s = \dim Z$ (this uses that Z is regular, which implies X is regular at points of Z). Hence if $\mathcal{I} = \mathbf{m}_x\mathcal{O}_Y$ is the ideal sheaf of Y_x in Y, where $\mathbf{m}_x$ is the maximal idela of $\mathcal{O}_{X,x}$, then the conormal sheaf $\mathcal{I}/\mathcal{I}^2$ of Y_x in Y fits into an exact sequence

$$0 \to \mathcal{O}_{Y_x}^{\oplus s} \to \mathcal{F} \to \mathcal{O}_{Y_s}(1) \to 0.$$

Since $Y_x \cong \mathbb{P}^{r-1}_{k(x)}$, this sequence must be split exact. Thus, Y_x is a local complete intersection in Y, with conormal sheaf $\mathcal{O}_{Y_x}^{\oplus s} \oplus \mathcal{O}_{Y_x}(1)$.

Let Y_n denote the subscheme defined by the ideal sheaf $\mathcal{I}^n$. Then $Y_1 = Y_x = (Y_n)_{\text{red}}$, and we have exact sequences

$$0 \to S^n(\mathcal{I}/\mathcal{I}^2) \to \mathcal{O}_{Y_{n+1}} \to \mathcal{O}_{Y_n} \to 0,$$

where $S^n(\mathcal{I}/\mathcal{I}^2)$ is the n-th symmetric power, which is a direct sum of invertible sheaves $\oplus_i\mathcal{O}_{\mathbb{P}^{r-1}_{k(x)}}(t_i)$ over a sequence of non-negative integers t_i. Hence $H^i(Y_x, S^n(\mathcal{I}/\mathcal{I}^2)) = 0$ for all $i > 0$. Since $H^i(Y_x, \mathcal{O}_{Y_x}) = 0$ for $i > 0$, we get

$$\varprojlim_n H^i(Y_n, \mathcal{O}_{Y_n}) = 0 \;\; \forall\, n \geq 1, \; \forall\, i > 0.$$

By Theorem 7.1, we have that $(R^if_*\mathcal{O}_Y)_x = 0$ for all $i > 0$. □

8. Base Change and Semicontinuity

Let $f : X \to Y$ be a proper morphism of Noetherian schemes, and $\mathcal{F}$ a coherent sheaf on X. The Formal Function Theorem gives relations between the stalks of $R^i f_*\mathcal{F}$ and the cohomology along the fibers, where we have the take all possible non-reduced structures on these fibers into account. The Base Change and Semicontinuity theorems address the situation where $\mathcal{F}$ is assumed to be *flat* over Y, *i.e.*, , for each $x \in X$, the stalk $\mathcal{F}_x$ is a flat $\mathcal{O}_{Y,f(x)}$-module, where we can (in appropriate situations) restrict attention to the scheme-theoretic fibers themselves.

Since we are interested in comparing the stalks of $R^i f_*\mathcal{F}$ and the fiber cohomology of the restriction of $\mathcal{F}$, we may assume without loss of generality that $Y = \operatorname{Spec} A$ is affine.

We first discuss a criterion for flatness.

Proposition 8.1. *Let T be an integral Noetherian scheme, and $\mathcal{F}$ a coherent sheaf on $\mathbb{P}^N_T$. Let $f : \mathbb{P}^N_T \to T$ be the structure morphism. For any $t \in T$, let $\mathcal{F}_t$ denote the restriction of $\mathcal{F}$ to $\mathbb{P}^N_{k(t)}$, and let $P_t(x) \in \mathbb{Q}[x]$ be the Hilbert polynomial of $\mathcal{F}_t$. Then the following are equivalent.*

(a) *$\mathcal{F}$ is flat over T.*
(b) *For all $n >> 0$, $f_*\mathcal{F}(n)$ is a locally free sheaf on T.*
(c) *The Hilbert polynomial polynomial $P_t(x) \in \mathbb{Q}[x]$ is independent of the choice of $t \in T$.*

Proof. The Theorem is local on T. So we may assume $T = \operatorname{Spec} A$ where A is a Noetherian local domain.

For sufficiently large n, we have that $R^i f_*\mathcal{F}(n) = 0$ (*i.e.*, $H^i(\mathbb{P}^N_T, \mathcal{F}(n)) = 0$) for all $i > 0$. Let $\mathcal{U} = \{U_0, \dots, U_N\}$ be the standard affine open cover of $\mathbb{P}^N_T = \mathbb{P}^N_A$. Then the Čech complex $\check{C}^\bullet(\mathcal{U}, \mathcal{F}(n))$ determines an exact sequence of A-modules

$$0 \to H^0(\mathbb{P}^N_T, \mathcal{F}(n)) \to \check{C}^0(\mathcal{U}, \mathcal{F}(n)) \to \check{C}^1(\mathcal{U}, \mathcal{F}(n)) \to \cdots \to \check{C}^N(\mathcal{U}, \mathcal{F}(n)) \to 0.$$

Now if $\mathcal{F}$ is flat over T, so is $\mathcal{F}(n)$, so that $\check{C}^\bullet(\mathcal{U}, \mathcal{F}(n))$ is a complex of flat A-modules. The above exact sequence then implies $H^0(\mathbb{P}^N_T, \mathcal{F}(n))$ is also a flat A-module; since it is a finitely generated module, it must be a free A-module of finite rank. This means $f_*\mathcal{F}(n)$ is locally free. Conversely, if $f_*\mathcal{F}(n)$ is locally free for all $n \geq n_0$, then $M_n = H^0(\mathbb{P}^N_T, \mathcal{F}(n))$ is a free A-module of finite rank for each $n \geq n_0$. The graded $A[X_0, \dots, X_N]$-module M determined by $M_n = 0$ for $n < n_0$, $M_n = H^0(\mathbb{P}^N_T, \mathcal{F}(n))$ for $n \geq n_0$, has the properties that (i) M is a flat A-module (in fact a freee A-module) (ii) $\mathcal{F} = \widetilde{M}$ is the corresponding sheaf on $\mathbb{P}^N_T = \operatorname{Proj} A[X_0, \dots, X_N]$. In particular, the A-module of sections of $\mathcal{F}$ on U_i is flat over A, since it is a direct summand of M_{X_i}, which is clearly a flat A-module (it is the direct limit of $M \xrightarrow{X_i} M \xrightarrow{X_i} M \xrightarrow{X_i} \cdots$ which is a direct system of free A-modules). Thus, we have shown that (a) and (b) of the Proposition are equivalent.

Let $P_0(x) \in \mathbb{Q}[x]$ denote the Hilbert polynomial of $\mathcal{F}$ on the generic fiber $\mathbb{P}^N_{k(T)}$, where $k(T)$ is the quotient field of A. Let $t \in T$ be the closed point. We show that (b) and (c) are equivalent to:

(c′) the Hilbert polynomial $P_t(x)$ coincides with $P_0(x)$.

We first show (b) $\implies$ (c′). Let $A^m \to A \to k$ be a presentation of the residue field k of A. This determines a presentation

$$\mathcal{F}^{\oplus m} \to \mathcal{F} \to i_*\mathcal{F}_t \to 0$$

where $i : \mathbb{P}^N_k \hookrightarrow \mathbb{P}^N_A$ is the inclusion of the closed fiber. For all large n, this induces (by the Serre vanishing theorem) a presentation

$$H^0(\mathbb{P}^N_A, \mathcal{F}(n))^{\oplus m} \to H^0(\mathbb{P}^N_A, \mathcal{F}(n)) \to H^0(\mathbb{P}^N_k, \mathcal{F}_t) \to 0.$$

On the other hand, there is a similar presentation

$$H^0(\mathbb{P}^N_A, \mathcal{F}(n)) \otimes_A A^m \to H^0(\mathbb{P}^N_A, \mathcal{F}(n)) \to H^0(\mathbb{P}^N_A, \mathcal{F}(n)) \otimes_A k \to 0.$$

Comparing these, we deduce that for large n, we have that

$$H^0(\mathbb{P}^N_A, \mathcal{F}(n)) \otimes_A k \cong H^0(\mathbb{P}^N_k, \mathcal{F}_t(n)).$$

Now by Nakayama's lemma,

$$M_n = H^0(\mathbb{P}^N_A, \mathcal{F}(n)) \text{ is a free } A\text{-module} \iff \operatorname{rank}_A M_n = \dim_k M_n \otimes_A k,$$

since A is a local domain. But $\operatorname{rank}_A M_n = P_0(n)$ for large n, and

$$\dim_k M_n \otimes_A k = \dim_k H^0(\mathbb{P}^N_A, \mathcal{F}_t) = P_t(n)$$

for all large n. Hence (c′) $\iff$ (b). Clearly (c) $\implies$ (c′). So it suffices to show (b) $\implies$ (c). So assume (b), and let $s \in T$ be any point, and $T_s = \operatorname{Spec} A_s$ where $A_s = \mathcal{O}_{T,s}$ is the corresponding local ring. Let $g : \mathbb{P}^N_{A_s} \to \mathbb{P}^N_A$ be the corresponding morphism. We have that $H^i(\mathbb{P}^N_{A_s}, g^*\mathcal{F}(n)) \cong H^i(\mathbb{P}^N_A, \mathcal{F}(n)) \otimes_A A_s$, since the Cech complex computing the cohomology of $\mathcal{F}(n)$ localizes to that computing the cohomology of $g^*\mathcal{F}(n)$. In particular, we reduce to the situation when (b) holds, and s is the closed point of T, in which case we have already proved that (c′) holds. □

Proposition 8.2. (Mumford) *Let $f : X \to \operatorname{Spec} A$ be a proper morphism, where A is Noetherian, and let $\mathcal{F}$ be a coherent sheaf on X which is flat over A. Then there is a bounded complex*

$$0 \to P_0 \to \cdots \to P_n \to 0,$$

where P_i are projective A-modules of finite rank, such that for any Noetherian A-algebra B, if $Y = X \times_{\operatorname{Spec} A} \operatorname{Spec} B$, $g : Y \to X$, then there are isomorphisms

$$H^i(Y, g^*\mathcal{F}) \cong H^i(P_\bullet \otimes_A B),$$

functorial in B.

Proof. Fix an affine open cover $\mathcal{U} = \{U_0, \dots, U_n\}$ os X. Then for any A-algebra B, there is an induced *affine* open cover $\mathcal{U}_B = \{g^{-1}(U_0), \dots, g^{-1}(U_n)\}$, where $g^{-1}(U_i) = U_i \times_{\operatorname{Spec} A} \operatorname{Spec} B$. Hence the Čech complex $\check{C}^\bullet(\mathcal{U}, \mathcal{F})$ is such that there are canonical isomorphisms

$$H^i(Y, g^*\mathcal{F}) \cong H^i(\check{C}^\bullet(\mathcal{U}, \mathcal{F}) \otimes_A B).$$

Since $\mathcal{F}$ is flat over A, all terms in the Čech complex $\check{C}^\bullet(\mathcal{U},\mathcal{F})$ are flat A-modules. If $P_\bullet \hookrightarrow \check{C}^\bullet(\mathcal{U},\mathcal{F})$ is a subcomplex of finitely genrated projective A-modules such that the the induced maops on cohomology modules are all isomorphisms, then the corresponding mapping cone is an exact sequence of flat A-modules, and so remains exact on tensoring with any A-algebra B. Forming the mapping cone commutes with change of rings (in fact it commutes with the functor on complexes associated to any additive functor between abelian categories). Hence $P_\bullet \otimes_A B \to \check{C}^\bullet(\mathcal{U},\mathcal{F}) \otimes_A B$ also induces an isomorphism on cohomology modules. The desired functoriality in B is manifest from the definition.

Hence, using Theorem 5.14, the Proposition follows from Lemma 8.3 below. □

Lemma 8.3. *Let A be a noetherian ring,*

$$0 \to M_0 \to \cdots \to M_n \to 0$$

a bounded complex of flat A-modules, such that the cohomology modules $H^i(M_\bullet)$ are finite A-modules, for all i. Then there exists a complex of finitely generated projective A-modules

$$0 \to P_0 \to \cdots \to P_n \to 0$$

together with an injective map of complexes $P_\bullet \to M_\bullet$ which induces isomorphisms on cohomology modules.

Proof. This is left as an exercise to the reader (see [Ha], III, Lemma 12.3). □

We call a complex $P_\bullet$ as above, associated to a coherent sheaf $\mathcal{F}$ on a proper A-scheme X, a *Mumford complex* (relative to A) for $\mathcal{F}$.

Theorem 8.4. (Semicontinuity Theorem) *Let $f : X \to Y$ be a proper morhism of Noetherian schemes, and let $\mathcal{F}$ be a coherent sheaf on X which is flat over Y. For $y \in Y$, let $X_y = X \times_Y \operatorname{Spec} k(y)$ be the fiber over y, $\mathcal{F}_y$ the poull-back of $\mathcal{F}$ to a coherent sheaf on X_y. Then for each $i \geq 0$, the function*

$$y \mapsto \dim_{k(y)} H^i(X_y, \mathcal{F}_y)$$

is upper semicontinuous on Y, i.e., the subset of Y on which this function is $\geq m$ is closed, for each $m \geq 0$.

Proof. The theorem is local on Y, so we may assume Y is affine. Then by Proposition 8.2, we are reduced to showing the following:
if $0 \to P_0 \to \cdots \to P_n \to 0$ is a complex of finitely generated projective modules over a Noetherian ring A, then the function on $\operatorname{Spec} A$ defined by

$$\wp \mapsto \dim_{k(\wp)} H^i(P_\bullet \otimes_A k(\wp))$$

is upper semicontinuous, where $k(\wp)$ is the residue field of $A_\wp$. Further localizing, we may assume P_i are free A-modules.

If $\varphi :\to G$ is an A-linear map between free A-modules, then after choosing bases, φ is determined by an $r \times s$ matrix $M \in \mathrm{M}_{r\times s}(A)$, where $r = \operatorname{rank} F$, $s = \operatorname{rank} G$. The rank of $\varphi_\wp = \varphi \otimes_A k(\wp) : F \otimes_A k(\wp) \to G \otimes_A k(\wp)$ is clearly lower semicontinuous on $\operatorname{Spec} A$, *i.e.*, the set

$$\{\wp \in \operatorname{Spec} A \mid \operatorname{rank} \varphi_\wp \leq m\}$$

is closed, for each m, which is true because it is the subset of $\operatorname{Spec} A$ defined by the vanishing of all minors of size $m+1$ of the associated matrix.

Applying this to the segment

$$P_{i-1} \xrightarrow{\varphi_{i-1}} P_i \xrightarrow{\varphi_i} P_{i+1}$$

of our complex $P_\bullet$ of free modules, if $a_i = \operatorname{rank} P_i$, then

$$\{\wp \in \operatorname{Spec} A \mid \dim_{k(\wp)} H^i(P_\bullet \otimes k(\wp)) \geq m$$

$$= \bigcup_{a_i \geq p \geq m} \left(\{\wp \mid \operatorname{rank}(\varphi_i)_{k(\wp)} \leq a_i - p\} \cap \{\wp \mid \operatorname{rank}(\varphi_{i-1})_{k(\wp)} \leq p - m\} \right).$$

This is a closed subset of $SpecA$. □

Corollary 8.5. *With the same hypotheses as Theorem 8.4, assume Y is integral, and suppose that for some $i \geq 0$,*

$$y \mapsto \dim_{k(y)} H^i(X_y, \mathcal{F})_y$$

is a constant function on Y. Then $R^i f_* \mathcal{F}$ is locally free, and the natural maps

$$(R^i f_* \mathcal{F})_y \otimes_{\mathcal{O}_{Y,y}} k(y) \to H^i(X_y, \mathcal{F}_y)$$

are isomorphism for all y.

Proof. We may assume $Y = \operatorname{Spec} A$ is affine, and there is a Mumford complex $(P_\bullet, \varphi_\bullet)$ for $\mathcal{F}$ consisting of free A-modules of finite rank, with $\operatorname{rank} P_i = a_i$.

If m is the constant value of the function in the statement of the Corollary, then we get that

$$m = a_i - \operatorname{rank}(\varphi_i)_{k(\wp)} - \operatorname{rank}(\varphi_{i-1})_{k(\wp)}.$$

Since the functions $\wp \mapsto \operatorname{rank}(\varphi_i)_{k(\wp)}$ and $\wp \mapsto \operatorname{rank}(\varphi_{i-1})_{k(\wp)}$ are lower semicontinuous, but their sum is constant, they must both actually be constant. This means the image and kernel of φ_{i-1} and φ_i must be projective A-modules of finite rank, and hence so is the i-th cohomology module of $P_\bullet$. Thus $R^i f_* \mathcal{F}$ is locally free on $\operatorname{Spec} A$. Further, taking i-th cohomology commutes with tensoring with any A-algebra B, in this situation; in particular letting $B = k(\wp)$ for some prime ideal $\wp$, we get that

$$H^i(X_y, \mathcal{F}_y) \cong H^i(P_\bullet \otimes_A k(\wp)) \cong H^i(P_\bullet) \otimes_A B \cong H^i(X, \mathcal{F}) \otimes_A k(\wp).$$

□

We give an application of this corollary.

Corollary 8.6. *Let Y be an integral scheme of finite type over an algebraically closed field, $f : X \to Y$ be a proper, flat morphism with integral fibers. Let $\mathcal{L}$, $\mathcal{M}$ be any invertible sheaves on X such that their restrictions to each fiber X_y are isomorphic. Then $\mathcal{M} = \mathcal{L} \otimes f^* \mathcal{N}$ for some invertible sheaf $\mathcal{N}$ on Y.*

Proof. Replacing $\mathcal{L}$ by $\mathcal{L}\otimes_{\mathcal{O}_X}\mathcal{M}^{-1}$, we reduce to the statement that if an invertible sheaf $\mathcal{L}$ on X has trivial restriction to all fibers of f, then $\mathcal{L} \cong f^*\mathcal{N}$ for some invertible sheaf $\mathcal{N}$ on Y. In fact, we will show this is true with $\mathcal{N} = f_*\mathcal{L}$. Since $\dim_{k(y)} H^0(X_y, \mathcal{L}_y) = \dim_{k(y)} H^0(X_y, \mathcal{O}_{X_y}) = 1$ for all closed points, by Corollary 3.3, the semicontinuity theorem implies that $\dim_{k(y)} H^0(X_y, \mathcal{L}_y) = 1$ for all y, and by Corollary 8.5, $f_*\mathcal{L}$ is a locally free sheaf, and hence an invertible sheaf on Y (since the generic stalk of $f_*\mathcal{L}$ is a 1-dimensional vector space over the function field of Y). Further, the map $\psi_0(y) : (f_*\mathcal{L})_y \to H^0(X_y, \mathcal{L}_y)$ is an isomorphism for all y. Since $\mathcal{L}_y \cong \mathcal{O}_{X_y}$ for all y, this means that the natural map $f^*f_*\mathcal{L} \to \mathcal{L}$ is surjective, hence an isomorphism. □

We now give the statement of the technically important Base Change Theorem. We do not give the proof here; see [Ha] III, Theorem 12.11.

Theorem 8.7. (Base Change Theorem) *Let $f : X \to Y$ be a proper morphism of Noetherian schemes, and $\mathcal{F}$ a coherent sheaf on X, such that $\mathcal{F}$ is flat over Y. Let $y \in Y$; let X_y be the fiber, and $\mathcal{F}_y$ the pullback of $\mathcal{F}$ to X_y.*

(a) *If the natural map*

$$\psi_i(y) : (R^i f_*\mathcal{F})_y \otimes k(y) \to H^i(X_y, \mathcal{F}_y)$$

is surjective, then it is an isomorphism, and the same is true for all y' in an open neighbourhood of y.

(b) *Suppose $\psi_i(y)$ is surjective. Then:*

$$(R^i f_*\mathcal{F} \text{ is locally free in a neighbourhood of } y) \iff \psi_{i-1}(y) \text{ is also surjective.}$$

□

We give an application to illustrate how the Base Change Theorem is used.

Corollary 8.8. *Let $f : X \to Y$ and $\mathcal{F}$ be as above. Suppose $H^i(X_y, \mathcal{F}_y) = 0$ for some y. Then $R^i f_*\mathcal{F}$ vanishes in an open neighbourhood of y, and the map $\psi_{i-1}(y) : (R^{i-1} f_*\mathcal{F})_y \otimes k(y) \to H^{i-1}(X_y, \mathcal{F}_y)$ is an isomorphism.*

9. Vanishing theorems, formal duality and applications

We first state some important vanishing theorems for the cohomology of certain sheaves on non-singular projective varieties over an algebraically closed field k of characteristic 0. Though these theorems now admit algebraic proofs using characteristic p techniques, based on the results of Deligne and Illusie, the original proofs are by reducing to the case $k = \mathbb{C}$ and using analytic methods. For a more systematic discussion of this important topic, see the book [EV].

Theorem 9.1. (Kodaira Vanishing Theorem) *Let $\mathcal{L}$ be an ample invertible sheaf on a non-singular irreducible projective variety over a field k of characteristic 0. Then $H^i(X, \mathcal{L}^{-1}) = 0$ for all $i < \dim X$, or equivalently $H^i(X, \omega_X \otimes \mathcal{L}) = 0$ for all $i > 0$.*

This can be generalized in two different directions, as follows.

Theorem 9.2. (Kodaira-Akizuki-Nakano Vanishing) *With the same hypotheses as Theorem 9.1, we have* $H^i(X, \Omega^j_{X/k} \otimes \mathcal{L}) = 0$*, provided* $i + j > \dim X$*. Equivalently,* $H^i(X, \Omega^j_{X/k} \otimes \mathcal{L}^{-1}) = 0$ *for all* $i + j < \dim X$*.*

Thus, we have a vanishing result for sheaves more complicated than invertible sheaves. The equivalence of the tweo forms of the theorem follows from Serre duality, since

$$\mathcal{H}om_{\mathcal{O}_X}(\Omega^i_{X/k}, \omega_X) \cong \Omega^{n-i}_{X/k}.$$

Definition 9.3. Let $\mathcal{L}$ be an invertible sheaf on a proper variety X over a field k.

(i) If for each morphism $f : C \to X$ from an irreducible curve C, the degree of the invertible sheaf $f^*\mathcal{L}$ on C is non-negative, we say $\mathcal{L}$ is *nef.*

(ii) If for some constant $A > 0$, we have

$$\dim_k H^0(X, \mathcal{L}^{\otimes n}) \geq A n^{\dim X}$$

for a sequence of integers n tending to infinity, we say $\mathcal{L}$ is *big.*

Theorem 9.4. (Kawamata-Viehweg) *Let* X *be a non-singular irreducible projective variety over an algebraically closed field* k*, and* $\mathcal{L}$ *an invertible sheaf on* X*. Assume* $\mathcal{L}$ *is nef and big. Then* $H^i(X, \mathcal{L}^{-1}) = 0$ *for* $i < \dim X$*.*

Thus we get the conclusion of Kodaira Vanishing with a weaker hypothesis on $\mathcal{L}$ than ampleness. The case $\dim X = 2$ of this result is due to C. P. Ramanujam.

Theorem 9.5. (Grauert-Riemenschneider Vanishing) *Let* $f : X \to Y$ *be a proper birational morphism of irreducible varieties over a field* k *of characteristic 0, where* X *is non-singular. Then* $R^i f_* \omega_X = 0$ *for all* $i > 0$*.*

Proof. The question is local on Y, so we may assume first that Y is affine, then may replace Y by a normal projective compatification $\bar{Y}$. By Hironaka's theorem on resolution of singularities, we can find a non-singular $\bar{X}$ with a proper morphism to $\bar{Y}$ which extends the given map f. In other words, it suffices to consider the case when Y is projective.

First assume X is also projective. Let $\mathcal{L}$ be a very ample invertible sheaf on Y. Then $f^*\mathcal{L}$ is a nef and big (see Definition 9.3) invertible sheaf on X. Hence by Theorem 9.4, $H^i(X, f^*\mathcal{L}^{-n}) = 0$ for all $i < n$, or equivalently $H^i(X, \omega_X \otimes f^*\mathcal{L}^n) = 0$ for all $n > 0$.

Consider the Leray spectral sequence

$$E_2^{p,q} = H^p(Y, R^q f_* \omega_X \otimes f^*\mathcal{L}^n) \implies H^{p+q}(X, \omega_X \otimes \mathcal{L}^n).$$

By the projection formula (see 5.10), we have isomorphisms $R^i f_*(\omega_X \otimes f^*\mathcal{L}^n) \cong (R^i f_* \omega_X) \otimes \mathcal{L}^n$, for all $i \geq 0$. Since $\mathcal{L}$ is ample on Y, Serre vanishing implies that we have $E_2^{p,q} = 0$ for $p > 0$, for all q, for all large n. Hence the spectral sequence degenerates at E_2 and we have isomorphisms

$$H^0(Y, R^i f_*(\omega_X) \otimes \mathcal{L}^n) \cong H^i(X, \omega_X \otimes f^*\mathcal{L}^n) = 0 \text{ for } i > 0..$$

But $(R^i f_* \omega_X) \otimes \mathcal{L}^n$ is generated by its global sections, for all i, for large enough n. Hence we must have $R^i f_* \omega_X = 0$ for $i > 0$.

If now X is merely proper over k, we may find a projective birational morphism $g : Z \to X$ such that $f \circ g : Z \to Y$ is projective (and birational of course), and Z is non-singular. In fact, if $X' \to X$ is a birational morphism with X' projective, which exists by Chow's Lemma, we may (by Hironaka's theorem) take Z to be a projective resolution of singularities of X'. The morphism $g : Z \to X$ is a projective birational morphism with Z non-singular and X normal (in fact non-singular), so by the case already considered, we have $R^i g_* \omega_Z = 0$ for all $i > 0$. Also, we have a natural map $g^* \omega_X \to \omega_Z$ which is an isomorphism over the open subset where $Z \to X$ is an isomorphism; now applying g_* to this, we obtain an inclusion $\omega_X \hookrightarrow g_* \omega_Z$ which is an isomorphism outside a codimension 2 subset of X. Since ω_X is invertible and $g_* \omega_Z$ is torsion-free, we must have $\omega_X = g_* \omega_Z$.

Now consider the Grothendieck spectral sequence for the hihgher direct images of a composition,

$$E_2^{p,q} = R^p f_* \circ R^q g_* \omega_Z \implies R_*^{p+q} (f \circ g)_* \omega_Z.$$

We have $E_2^{p,q} = 0$ for $q > 0$ and all p, so that the spectral sequence degenerates, and we have isomorphisms

$$E_2^{i,0} = R^i (f \circ g)_* \omega_Z = 0 \; \forall \; i > 0,$$

where we have used the Theorem for $f \circ g$ (which is a morphism between projective varieties). But $E_2^{i,0} = R^i f_* (g_* \omega_Z) = R^i f_* \omega_X$. □

We now state a duality theorem for cohomology with support, which will allow us to get an equivalent form of Theorem 9.5 which is useful for applications.

Theorem 9.6. (Formal Duality) *Let X be an irreducible non-singular proper variety of dimension n over a field k, and let Y be a closed subset. Let $\mathcal{I}_Y$ be the ideal sheaf for the corresponding reduced subscheme of X. Let $\mathcal{F}$ be a locally free sheaf of finite rank on X. Then we have a functorial isomorphism of (not necessarily finite dimensional) k-vector spaces*

$$\varprojlim_m H^i(X, \mathcal{F} \otimes \mathcal{O}_X / \mathcal{I}_Y^m) \cong \mathrm{Hom}_k(H_Y^{n-i}(X, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \omega_X)), k).$$

Proof. We have natural isomorphisms

$$H_Y^j(X, \mathcal{G}) = \varinjlim_m \mathrm{Ext}^j_{\mathcal{O}_X}(\mathcal{O}_X / \mathcal{I}_Y^m, \mathcal{G})$$

for any $\mathcal{O}_X$-module $\mathcal{G}$, for all j. Indeed, both sides define δ-functors (in $\mathcal{G}$) on $\mathcal{M}od(X)$, which vanish if $\mathcal{G}$ is injective (the cohomology with support vanishes because $\mathcal{G}$ is flasque), and are both hence universal δ-functors. Since they coincide when $j = 0$ (why?), they must coincide for all j.

Hence if $V^\vee = \mathrm{Hom}_k(V, k)$ for any k-vector space V, we have

$$H_Y^j(X, \mathcal{G})^\vee = \varprojlim_m \mathrm{Ext}^j_{\mathcal{O}_X}(\mathcal{O}_X / \mathcal{I}_Y^m, \mathcal{G})^\vee.$$

Apply this to $\mathcal{G} = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \omega_X)$ and $j = n - i$. Since $\mathcal{F}$ is locally free, we have

$$\begin{aligned} \mathrm{Ext}^{n-i}_{\mathcal{O}_X}(\mathcal{O}_X / \mathcal{I}_Y^m, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \omega_X)^\vee &\cong \mathrm{Ext}^{n-i}_{\mathcal{O}_X}(\mathcal{F} \otimes \mathcal{O}_X / \mathcal{I}_Y^m, \omega_X)^\vee \\ &\cong H^i(X, \mathcal{F} \otimes \mathcal{O}_X / \mathcal{I}_Y^m) \end{aligned}$$

by Serre Duality on X (see Theorem 4.14 and 4.17). Substituting this in the inverse limit formula for $H^{n-i}_Y(X,\mathcal{G})^\vee$, the result follows. □

Corollary 9.7. *Let $f : X \to Y$ be a surjective morphism, with X non-singular projective. Let $y \in Y$ be a closed point, with fiber X_y. Then for any locally free $\mathcal{O}_X$-module $\mathcal{F}$ of finite rank, we have an isomorphism*

$$H^i_{X_y}(X,\mathcal{F})^\vee \cong (R^{n-i}f_*\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F},\omega_X))_y \otimes_{\mathcal{O}_{Y,y}} \widehat{\mathcal{O}_{Y,y}}.$$

Proof. This is a consequence of formal duality for the subscheme X_y of X, and the Formal Function Theorem (Theorem 7.1. □

Corollary 9.8. *Let $f : X \to Y$ be a birational proper morphism, where X is non-singular and proper over a field k of characteristic 0. Then for any $y \in Y$, we have that*

$$H^i_{X_y}(X,\mathcal{O}_X) = 0 \ \ \forall\, i < n.$$

Proof. This follows from Theorem 9.5 and Corollary 9.8. □

We give two applications of this.

Example 9.9. Let $(A,\mathbf{m})$ be the local ring of a closed point on a normal variety over a field of characterstic 0. Suppose that $f : X \to \operatorname{Spec} A$ is a resolution of singularities, such that $H^i(X,\mathcal{O}_X) = 0$ for all $i > 0$ (we then say $\operatorname{Spec} A$ has *rational singularities*). Then A is Cohen-Macaulay.

Indeed, let $Y = \operatorname{Spec} A$, $X_y \subset X$ the fiber over the closed point, $U = Y \setminus \{\mathbf{m}\}$, $V = X \setminus X_y = f^{-1}(U)$. Since $H^i(X,\mathcal{O}_X) = 0$ for $i > 0$ and Y is affine, we have $R^if_*\mathcal{O}_X = 0$ for $i > 0$. From the Leray spectral sequence, we deduce that $H^i(U,\mathcal{O}_U) = H^i(U,f_*\mathcal{O}_V) \cong H^i(V,\mathcal{O}_V)$ for all i. From the exact sequence

$$H^i(X,\mathcal{O}_X) \to H^i(V,\mathcal{O}_V) \to H^{i+1}_{X_y}(X,\mathcal{O}_X)$$

and Corollary 9.8, we deduce that

$$H^i(U,\mathcal{O}_U) \cong H^i(V,\mathcal{O}_V) = 0$$

for $1 \le i \le n-2$. From Corollary 6.2, we deduce that $H^i_{\mathbf{m}}(A) = 0$ for $2 \le i \le n-1$. Since A is normal, we deduce that A has depth n, and is hence Cohen-Macaulay.

Example 9.10. Let $(A,\mathbf{m})$ be the local ring of a closed point on a normal variety of dimension n over a field k of characteristic 0. Assume the punctured spectrum of A is regular. Let $f : X \to \operatorname{Spec} A$ be a resolution of singularities. Then we have isomorphisms

$$H^i(X,\mathcal{O}_X) \cong H^{i+1}_{\mathbf{m}}(A) \ \ \forall\, i \le n-2.$$

In particular A is Cohen-Macaulay $\iff H^i(X,\mathcal{O}_X) = 0$ for $1 \le i \le n-2$ $\iff R^if_*\mathcal{O}_X = 0$ for $1 \le i \le n-2$.

Indeed, in the notation of the previous example, we have $V \cong U$, so the same argument applies, giving isomorphisms

$$H^i(X,\mathcal{O}_X) \cong H^i(V,\mathcal{O}_V) \cong H^i(U,\mathcal{O}_U) \cong H^{i+1}_{\mathbf{m}}(A)$$

for $1 \le i \le n-2$.

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