Math808M: Geometric Representation Theory

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Lecture 1: 9/2/14

We recall “basic Lie theory." Take $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$, with generators X, Y, H as usual. We have $[A, B] := AB - BA$ and $(A, B) := \mathrm{tr}(AB)$. This bilinear form is invariant in the sense that

$$([A, B], C) + (B, [A, C]) = 0.$$

We let $\mathfrak{h} := \mathbb{C}H$ and we have $[X, Y] = H$, $[H, X] = 2X$ and $[H, Y] = -2Y$; this is the adjoint representation. If V is a $\mathfrak{g}$-module and $\lambda \in \mathbb{C}$, we define $V^\lambda = \{v : Hv = \lambda v\}$ i.e. the λ-eigenspace. If $w \in V^\lambda$ then we say w has weight λ.

Proposition 1. *1) $V = \bigoplus_{\lambda \in \mathbb{C}} V^\lambda$. 2) If v has weight λ then Xv has weight $\lambda + 2$.*

The proof is easy.

Primitive Weights

V a $\mathfrak{g}$-module. Then $e \in V$ is called primitive of weight λ if $He = \lambda e$ and $Xe = 0$. Not all V have primitive vectors, but the ones we'll be interested in will. They always exist for finite-dimensional V or for quotients of Verma modules.

Structure Theorem

$V \ni e$ as above. Define $e_{-1} = 0$ and $e_n = Y^n e/n!$ for $n \geq 0$. Then

$$He_n = (\lambda - 2n)e_n$$

$$Ye_n = (n + 1)e_{n+1}$$

$$Xe_n = (\lambda - n + 1)e_{n-1}$$

for $n \geq 0$.

As a corollary we see that the e_n are linearly independent (as long as they're nonzero) since they are eigenvectors for H with distinct eigenvalues. Further if $\lambda = m \in \mathbb{Z}_{\geq 0}$ then $e_0, \ldots, e_m$

are independent and $e_{m+1} = 0$. Now define $W_m := \mathbb{C}\{e_1, \ldots, e_m\}$. Then W_m is $\mathfrak{g}$-stable and irreducible. In the case of $\mathfrak{sl}_2$ we in fact have $W_m \cong Sym^m(\mathbb{C}^2)$. There is in this case an isomorphism

$$W_m \otimes W_n \cong \bigoplus_{i=0}^{n} W_{m+n-2i}$$

but in general such a decomposition follows "Littlewood-Richardson" rules, and is more complicated.

Now let's be more general, and let $\mathfrak{g}$ be a (finite-dimensional) Lie algebra over $\mathbb{C}$. We define a "Cartan form" $B : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$, which is invariant in the same sense as above form. One way to define such a B is by

$$B(X, Y) := \mathrm{tr}(\mathrm{ad}\, X \cdot \mathrm{ad}\, Y).$$

Definition/Theorem. We say that $\mathfrak{g}$ is semisimple if the above B is nondegenerate.

Definition. $\mathfrak{g}$ is simple if 1) it is nonabelian, 2) it has no nontrivial proper ideals.

Corollary (of definition). $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$ if simple (or semisimple, because of the next theorem).

Theorem. $\mathfrak{g}$ is semisimple iff it is a product of simple Lie algebras.

Exercise: $\mathfrak{g}$ is semisimple iff the adjoint representation is faithful.

Definition. Let $\mathfrak{g}$ be a semisimple Lie algebra and $X \in \mathfrak{g}$. 1) X is nilpotent if ad_X is nilpotent. 2) X is semisimple if ad_X is semisimple (i.e. diagonalizable).

Theorem (Jordan Decomposition). For any $X \in \mathfrak{g}$ we can uniquely write 1) $x = x_s + x_n$ where x_s, x_n are semisimple, nilpotent, respectively. 2) $[x_s, x_n] = 0$.

Theorem. Jordan decomposition is preserved by any representation.

Theorem. Finite-dimensional representations of semisimple Lie algebras are completely reducible. This can be proved via Weyl's Unitarian Trick, or algebraically with cohomology (since it amounts to showing that an Ext group vanishes). This theorem can also fail when working over positive characteristic fields.

Theorem. If $\mathfrak{g}$ is a semisimple Lie algebra and $\mathfrak{h}$ is a maximal subalgebra consisting of semisimple elements (easy check - this is automatically nonzero), then 1) $\mathfrak{h}$ is abelian, 2) $\mathfrak{h}$ is self-centralizing, 3) Root space decomposition:

$$\mathfrak{g} := \mathfrak{h} \bigoplus_{0 \neq \alpha \in \Phi \subset \mathfrak{h}^*} \mathfrak{g}_\alpha.$$

Now there is no unique maximal such subalgebra, but we can say something. Properties of $\mathfrak{h}$: 1) $B|_{\mathfrak{h}}$ is nondegenerate, 2) All Cartan Subalgebras are conjugate by the group $e^{\mathrm{ad}\, \mathfrak{g}}$ (the group of inner automorphisms), 3) $\mathfrak{g}_\alpha$ are all 1-dimensional.

The Φ above is called a root system. We will prove the 1) above statement that $\mathfrak{h}$ is abelian. To do this, we'll show that the eigenvalues of ad_X are all zero, for given $X \in \mathfrak{h}$. So suppose $\mathrm{ad}_X(y) = ay$

for $a \neq 0$. This would say $\mathrm{ad}_y(X) = -ay$. Now y is an eigenvector for ad_y of zero eigenvalue. Since ad_y is semisimple, we can write $x = \sum a_i v_i$ where v_i are eigenvectors of ad_y. But then get $\mathrm{ad}_y(X)$ is nonzero eigevectors, contradiction.

Root Systems

1) $\Phi \not\ni 0$, and spans $\mathfrak{h}^*$.

2) $\alpha \in \Phi \implies -\alpha \in \Phi$.

3) $s_\alpha(x) := x - 2\frac{(x,\alpha)}{(\alpha,\alpha)}\alpha$, where $s_\alpha : \mathfrak{h}^* \to \mathfrak{h}^*$. Then s_α preserves Φ. (the inner product here is coming from the above B having nondegenerate restriction to $\mathfrak{h}^*$).

4) $\frac{2(\beta,\alpha)}{(\alpha,\alpha)} \in \mathbb{Z}$.

Say that a root system is reduced if $\pm\alpha$ are the only proportional roots in the system.

Bases of Root Systems

1) $S \subset \Phi$ such that S is a basis of $\mathfrak{h}^*$

2) If $\beta \in \Phi$ then $\beta = \sum\limits_{\alpha \in S} m_\alpha \alpha$ where m_α all have same sign.

Theorem. A base S always exists!

Example of $\mathfrak{sl}_n$. Then the Cartan can be taken to be diagonal matrices, and the dual $\mathfrak{h}^*$ is determined by what it does on $n-1$ (independent) elements, which can be chosen nicely. Then $\Phi = \{L_i - L_j : i \neq j\}$ and the base is $S = \{L_i - L_{i+1}\}$.

Lecture 2: 9/4/14

From now on $\mathfrak{g}$ is a semisimple finite-dimensional Lie algebra, with $\mathfrak{h}$ a Cartan subalgebra. Recall last time we had invariant bilinear forms on $\mathfrak{g}$, such as the Killing form, which had nondegenerate restriction to $\mathfrak{h}$. Note $(\alpha,\alpha) \neq 0$ for $\alpha \in \Phi$ so the above definition of s_α makes sense. We will prove the fourth property of root systems above.

Proposition. 1) For $\alpha, \beta \in \Phi$ have $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$. 2) If $x \in \mathfrak{g}_\alpha$ with $\alpha \neq 0$ then $\mathrm{ad}_{\mathfrak{g}} x$ is nilpotent. 3) If $\alpha + \beta \neq 0$ then $\mathfrak{g}_\alpha$ and $\mathfrak{g}_\beta$ are orthogonal with respect to the bilinear form $(\ ,\ )$.

The proof of 1) follows from the definitions and the Jacobi identity. 2) is an immediate corollary of 1). The proof of 3) is also simple computation.

Let $\phi : \mathfrak{h} \to \mathfrak{h}^*$ be the iso coming from the invariant nondegenerate form; write t_α for $\phi^{-1}(\alpha)$.

Proposition ($\mathfrak{sl}_2^\alpha$-triples). 1) $\alpha \in \Phi$, $x \in \mathfrak{g}_\alpha$ and $y \in \mathfrak{g}_{-\alpha}$. Then $[x, y] = (x, y)t_\alpha$. 2) Let $h_\alpha := \frac{2t_\alpha}{(\alpha,\alpha)}$. Given $x_\alpha \in \mathfrak{g}_\alpha$, there exists a unique $y_\alpha \in \mathfrak{g}_{-\alpha}$ such that $[x_\alpha, y_\alpha] = h_\alpha$, and also $h_{-\alpha} = -h_\alpha$.

To prove 1) it suffices to show that both are same after applying $(h, -)$, and using that $\alpha(h) = (t_\alpha, h)$ by definition, then using nondegeneracy. We omit proof of 2). $\square$

$(x_\alpha, y_\alpha, h_\alpha)$ spans the $\mathfrak{sl}_2^\alpha$-triple. Suppose $\beta \neq \pm\alpha$, and let $M = \sum_{i\in\mathbb{Z}} \mathfrak{g}_{\beta+i\alpha}$. Then M is a finite-dimensional module for $\mathfrak{sl}_2^\alpha$. Now $(\beta + i\alpha)(h_\alpha) = \beta(h_\alpha) + 2i$, and this is an integer because it is a weight for an $\mathfrak{sl}_2$-representation. Thus the Cartan integers are integers. Moreover, since in the above we increase by 2, we see that 0 or 1 can't both appear and either can appear only once. It follows that M is actually irreducible as an $\mathfrak{sl}_2^\alpha$-module. (We're implicitly using that this root system is reduced so that $\mathfrak{g}_\alpha$ are 1-dimensional). It follows that we must have an unbroken string $\beta + r\alpha, \dots, \beta - q\alpha$ of roots (and no others of this form). These are the highest and lowest weights, respectively. In fact $\beta + r\alpha = -(\beta - q\alpha)$; applying this to h_α gives $\beta(h_\alpha) = q - r$. Note this is also $\frac{2(\beta,\alpha)}{(\alpha,\alpha)}$.

Let Φ be a root system and S a base. Call elements of S simple [positive] roots, denoted $\alpha_1, \dots, \alpha_\ell$ with $\ell := \operatorname{rank}(\Phi) = \dim \mathfrak{h}$. We remark that $\frac{2(\alpha_i,\alpha_j)}{(\alpha_j,\alpha_j)} =: a_{ij}$ defines the "Cartan matrix." From this matrix one can form the adjacency graph (this is the underlying graph of the Dynkin diagram). Can put an arrow towards the longer root (if applies).

Proposition. If $\beta \in \Phi$ and $\{\alpha_1, \dots, \alpha_\ell\}$ simple roots. Then $\beta = \sum c_i\alpha_i$ with $c_i \in \mathbb{Q}$.

Proof. Apply $(-, \alpha_j)$ then divide by (α_j, α_j) and multiply by 2. Get a system of linear equations with integer entries. The inverse of a matrix with integer entries has rational entries. $\square$

Remark. The $\mathbb{Q}$-span $E_\mathbb{Q}$ of the α_i also has rank ℓ.

If $\lambda, \gamma \in \mathfrak{h}^*$ then $(\lambda, \gamma) = \operatorname{tr}(\operatorname{ad} t_\lambda \operatorname{ad} t_\gamma)$ by definition. What is action of $\operatorname{ad} t_\lambda$ on $\mathfrak{g}$? Well looking at the root space decomposition, it acts by 0 on the $\mathfrak{h}$ summand, and acts by multiplication by $\alpha(t_\lambda)$ by each root space $\mathfrak{g}_\alpha$. Hence

$$(\lambda, \gamma) = \sum_{\alpha\in\Phi} (\lambda, \alpha)(\gamma, \alpha)$$

and in particular

$$(\lambda, \lambda) = \sum_{\alpha\in\Phi} (\lambda, \alpha)^2.$$

Now for any $\beta \in \Phi$ we compute $\frac{1}{(\beta,\beta)} = \sum_{\alpha\in\Phi} \frac{(\beta,\alpha)^2}{(\beta,\beta)^2}$ and deduce that $(\beta, \beta) \in \mathbb{Q}$. Thus the form gives $E_\mathbb{Q} \times E_\mathbb{Q} \to \mathbb{Q}$, and THIS is positive definite (it wouldn't have made sense / been easy to see this over $\mathbb{C}$ where a sum of squares need not be positive).

Proposition. Let Φ be the root system in $\mathfrak{h}^*$, and define $\alpha^\vee = \frac{2t_\alpha}{(\alpha,\alpha)}$ (formerly called h_α). Then $\Phi^* = \{\alpha^\vee : \alpha \in \Phi\}$ is a root system in $\mathfrak{h}$. We call elements of Φ^* coroots. (proof omitted/postponed).

Definition 1. *The root lattice Q is the $\mathbb{Z}$-span of Φ. The coroot lattice $Q^\vee$ is the $\mathbb{Z}$-span of Φ^*. The weight lattice P is the $\lambda \in \mathfrak{h}^*$ such that $\lambda(\alpha^\vee) = 2(\lambda, \alpha)/(\alpha, \alpha) \in \mathbb{Z}$ for all $\alpha \in \Phi$. It is the lattice dual of $Q^\vee$. Similarly the coweight lattice $P^\vee$ is the lattice dual to Q, namely the $h \in \mathfrak{h}$ such that $\alpha(h) \in \mathbb{Z}$ for all $\alpha \in \Phi$.*

We have $(\alpha^\vee, \alpha) = 2$ but in general (α, α) may only be a half-integer. We have inclusions $Q^\vee \subset P^\vee \subset \mathfrak{h}$ and $Q \subset P \subset \mathfrak{h}^*$. Given a choice of base, get "fundamental weights," which are by definition $\omega_1, \ldots, \omega_\ell$ such that $\omega_i(\alpha_j^\vee) = \delta_{ij}$.

Representations of $\mathfrak{g}$

Let $P_+ = \{\lambda \in P : (\lambda, \alpha) \in \mathbb{Z}_{\geq 0}\}$. Then finite-dimensional irreducible representations (up to iso) correspond 1-1 to elements of P_+. This is the theory of primitive weights, quotients of Verma modules.

Lecture 3: 9/9/14

Recommended (French) Reference: Espaces Fibres Algebriques by Serre, 1958.

Last time we defined the root lattice Q as the $\mathbb{Z}$ span of the simple roots $\alpha_1, \ldots, \alpha_\ell$. Similarly we defined the coroot lattice as the span of the simple coroots $\alpha_1^\vee, \ldots, \alpha_\ell$ where $\alpha^\vee = 2t_\alpha/(\alpha, \alpha)$ and t_α is by definition the image of α under the isomorphism $\mathfrak{h} \cong \mathfrak{h}^*$. We also defined the weight lattice $P \subset \mathfrak{h}^*$ to be those λ such that $\lambda(\alpha_i^\vee) \in \mathbb{Z}$ for all i, and similarly the coweight lattice $P^\vee \subset \mathfrak{h}$ to be those H such that $\alpha(H) \in \mathbb{Z}$ for all $\alpha \in \Phi$. In other words, $P^\vee$ is the lattice dual of Q. We have inclusions $Q \subset P \subset \mathfrak{h}^*$ and $Q^\vee \subset P^\vee \subset \mathfrak{h}$. We also defined the set of dominant integral weights to be $P_+ = \{\lambda \in P : 2(\lambda, \alpha)/(\alpha, \alpha) \in \mathbb{Z}_{\geq 0}\}$ i.e. $< \lambda, \alpha > \in \mathbb{Z}_{\geq 0}$.

Now we have a bijection between finite dimensional irreps of $\mathfrak{g}$ and P_+, where V is sent to the weight of its highest weight vector, and in the other direction one takes the quotient of the Verma module. But there's a more direct construction of the representations than as quotients of (infinite dimensional) Verma modules.

Example. Consider $\mathfrak{sl}_n$. Let $H_i = e_{ii}$ and ε_i be the dual of H_i (i.e. in the usual sense of vector spaces). Then $\mathfrak{h}$ is the $\mathbb{C}$-span of $H_i - H_n$, for $i = 1, \ldots, n-1$. Our form can be taken to be $(H_i - H_n, H_j - H_n) = 2\delta_{ij}$ where $(A, B) = \mathrm{tr}(AB)$. In this case we have

$$\Phi = \{\varepsilon_i - \varepsilon_j : 1 \leq i \neq j \leq n\}$$

$$\Phi^+ = \{\varepsilon_i - \varepsilon_j : i < j\}$$

$$S = \{\varepsilon_i - \varepsilon_{i+1} : i = 1, \ldots, n-1\}$$

and we write α_i for $\varepsilon_i - \varepsilon_{i+1}$. Then $\alpha_i^\vee = H_i - H_{i+1}$ for $i = 1, \ldots, n-1$. And also we can take as basis for P the vectors $w_i = (\sum_{j=1}^{i} \varepsilon_j) - \frac{i}{n}(\sum_{j=1}^{n} \varepsilon_j)$, and check that $w_i(\alpha_j^\vee) = \delta_{ij}$ so that the dual of this basis for P is a basis for $Q^\vee$. Also check $P/Q \cong \mathbb{Z}/n$. We'll see this corresponds to automorphisms of the affine dynkin diagram. Then $V_{w_1} = \mathbb{C}^n$ i.e. the standard representation, and more generally $V_{w_i} = \bigwedge^i \mathbb{C}^n$. For a general integral weight $\lambda = \sum_{i=1}^{n} a_i w_i$ with $a_i \geq 0$, what is V_λ? Let v_{w_i} be highest weight vectors in V_{w_i}. Now take $Sym^{a_1}\mathbb{C}^n \otimes \cdots \otimes Sym^{a_{n-1}} \bigwedge^{n-1} \mathbb{C}^n$ and set $v_\lambda = v_1^{\otimes a_1} \otimes \ldots \otimes v_{n-1}^{\otimes a_{n-1}}$. Then we claim that 1) $h \cdot v_\lambda = \lambda(h) v_\lambda$, i.e. is a vector of the correct weight. and 2) v_λ is a primitive vector (so a vector of highest weight, i.e. killed by $\mathfrak{g}_\alpha$ for $\alpha > 0$), and V_λ is the irreducible component of V containing v_λ.

Now we change gears to groups! A motivating question - given a representation of the Lie algebra, how do I get a representation of the group? Let G be a connected complex Lie group (though the following statements will be true of them as algebraic groups).

Definition. G is (complex) semisimple if $\mathfrak{g} = \mathrm{Lie}(G)$ is semisimple.

Given (cartan) subalgebra $\mathfrak{h} \subset \mathfrak{g}$ there exists $H \subset G$ with corresponding Lie algebra $\mathfrak{h}$.

Theorem. H is a closed subgroup of G and H is a product of $\mathbb{C}^*$, so a complex torus.

Recall the character lattice $X(H)$. Recall the exponential map $e : \mathfrak{h} \to H$ mapping $x \mapsto \exp(2\pi i x)$.

Theorem. e is a homomorphism, is surjective, and has discrete kernel $\Gamma(G)$, and we have the property $Q^\vee \subset \Gamma(G) \subset P^\vee$. Also $\pi_1(G) = \Gamma(G)/Q^\vee$.

G is simply connected iff $\Gamma(G) = Q^\vee$ and is adjoint iff $\Gamma(G) = P^\vee$. In particular the connected Lie groups G with a given Lie algebra correspond one to one with subgroups of $P^\vee/Q^\vee$, in particular it's finite.

Example. $\mathfrak{sl}_2$. Then $P^\vee = \mathbb{Z} < H_1 - \frac{1}{2}(H_1 + H_2) >$ and $Q^\vee = \mathbb{Z} < H_1 - H_2 >$. The quotient is $\mathbb{Z}/2$, and $PGL_2 = SL_2/\{\pm 1\}$. $W_m = Sym^m \mathbb{C}^2$ are all SL_2-modules, but not all are PGL_2-modules. For this need m to be even.

For a more complicated example, consider $0 \to \mathbb{Z}/2 \to Spin(2n+1) \to SO(2n+1) \to 0$. We know $SO(2n+1) \hookrightarrow SL(2n+1)$, so any representation of $SL(2n+1)$ can restrict to $SO(2n+1)$ but the restriction is reducible generally; and will break up into “half-spin" representations. In this sense the representation will have a “square root" which will be a representation of $Spin(2n+1)$ but NOT of $SO(2n+1)$.

Finite-dimensional irreps of $\mathfrak{g}$ biject with finite-dimensional irreps of G where G is simply connected. But what about if G is not simply connected?

G simply connected, $H \subset G$ a Cartan subgroup (i.e. has Lie algebra a Cartan subalgebra) then $e : P = \{\lambda \in \mathfrak{h}^* : \lambda(\alpha_i^\vee) \in \mathbb{Z}\} \to X(H)$ is an iso. We define $X(H)_+$ to be the image of P_+. More generally, if $H \subset G$ is a Cartan in any Lie group, and $\lambda \in P_+$ then V_λ is a (finite dimensional irreducible) representation of G iff $\exp(2\pi i\lambda) \in X(H)$. I.e. the representation V_λ of $\mathfrak{g}$ lifts to a representation of G iff this condition holds.

This ends what we want to say about representations of Lie algebras and Lie groups. Now recall the definition of the positive and negative nilpotent parts $\bigoplus_{\alpha>0} \mathfrak{g}_\alpha =: \mathfrak{n}^+$ and similarly $\mathfrak{n}^-$. Then a Borel subalgebra $\mathfrak{b}$ is defined to be $\mathfrak{h} \bigoplus \mathfrak{n}^+$.

Theorem. 1) $[\mathfrak{b}, \mathfrak{b}] = \mathfrak{n}^+$. 2) $\mathfrak{b}$ is solvable. 3) (Borel-Morozov) Every solvable subalgebra of $\mathfrak{g}$ is conjugate (under $\exp \operatorname{ad} \mathfrak{g}$) to a subalgebra of $\mathfrak{b}$. In particular $\mathfrak{b}$ is a maximal solvable subalgebra.

Next time - suppose $F \subset G$ is closed. Then what is G/F?

Lecture 4: 9/11/14

Today we'll (hopefully) describe G/B as an algebraic variety and state the Borel-Weil-Bott Theorem. Let G be a connected, semisimple, simply connected, complex algebraic group. Maximal solvable subgroups of G are called Borel subgroups. $H \subset B \subset G$ with H a Cartan and B a Borel. We can write $B = H \ltimes [B, B]$. The unipotent radical U of B is $[B, B]$. Or can define as the Lie subgroup whose Lie algebra is $\mathfrak{n}^+$ in yesterday's notation.

Theorem. If $\lambda \in X(H)_+ \cong P_+ = \{\lambda \in P : \lambda(\alpha) \geq 0\}$ (recall the iso was $\exp 2\pi i z$) i.e. a dominant integral weight, then let $V(\lambda)$ be the irreducible G-module with highest weight λ. Then there exists a line in $V(\lambda)$ which is B-stable, and H acts on this line by λ.

(This really just the group-theoretic restatement of the definition of highest weight vector on the Lie algebra side).

Properties of the Group G

Let $N = N_G(H)$ and $W = N/H$. Then W acts by conjugation on H. Recall that we defined for every root (in particular every simple root) Lie subalgebras

$$\mathfrak{sl}_2^{\alpha_i} = \mathfrak{g}_{\alpha_i} \oplus \mathfrak{g}_{-\alpha_i} \oplus \mathbb{C}\alpha_i^\vee \subset \mathfrak{g}.$$

We have a map $\mathfrak{sl}_2^{\alpha_i} \to G$ sending $antidiag(1, -1) \mapsto \bar{s}_i$.

Theorem. 1) $\bar{s}_i \in N_G(H)$. 2) Let s_i be the image in W. Then W is generated by the set $S = \{s_1, \ldots, s_\ell\}$ where $\ell = \operatorname{rank} \mathfrak{g}$. 3) W acts on $\mathfrak{h}^*$ (by differentiating its action on H and identifying the Lie algebra and its dual) by the formula $s_i(\beta) = \beta - \frac{2(\beta,\alpha_i)}{(\alpha_i,\alpha_i)}\alpha_i$.

Theorem (Tits). The 4-tuple (G, B, N, S) forms a Tits system. That is, 1) $H = B \cap N$, 2) S generates W, 3) B, N generate G, 4) $s_i B s_i \not\subset B$ for all i, 5) $Bs_iB \cdot BwB \subset Bs_iwB \bigcup BwB$.

Theorems. 1) Bruhat Decomposition - $G = \coprod_{w \in W} BwB$. 2) $G \supset P \supset B$, and for $I \subset \{1, \ldots, \ell\}$ set $P_I = \coprod_{w \in \{s_i : i \in I\}} BwB$. Then $P = P_I$ for some I.

Also $P \supset B \implies \mathfrak{p} \supset \mathfrak{b} \implies \mathfrak{p} = \mathfrak{b} \bigoplus_{\alpha>0} \mathfrak{g}_\alpha \bigoplus_{i \in I} \mathfrak{g}_{-\alpha_i}$

Theorem. Let F be a closed subgroup of G and V an F-variety. Let $E := G \times_F V = (G \times V)/\sim$ where $(gf, v) \sim (g, fv)$. Then E is a G-variety (action of G is the obvious one on the left). 2) If $V = \{pt\}$ then $G \times_F V = G/F$, set-theoretically. 3) $E \to G/F$ mapping $[(g, v)] \mapsto [g]$ has set-theoretic fibers V. 4) If V an F-module then $E \to G/F$ is a vector bundle.

Recall the definition of vector bundles.

Corollary. $f : G/F \to Y$ is a morphism iff $f \circ \pi : G \to Y$ is a morphism.

To show G/B is a variety then, it suffices to show B is closed in G. Suppose not, and let $\bar{B}$ be the closure (in the Zariski topology). Then $\bar{B}$ is also a group. It suffices now to show that $\bar{B}$ is solvable. It is easy to see (for any group) that $[\bar{B}, \bar{B}] \subset \overline{[B, B]}$. It follows that $D_n(\bar{B}) \subset \overline{D_n(B)}$ for all n, and B is solvable, so done.

Now we want to show that G/B is a projective variety. So let $\lambda \in X(H)_+$ and let $G \to \mathbb{P}(V_\lambda)$ by $g \mapsto [gv]$ where v is (any, since they differ by scalars) highest weight vector. This factors through B by the above, and then gives an injective morphism with closed image. To see this, first let P be the stabilizer of $\{v_\lambda\}$ so $P \supset B$ so $P = P_I$ for some I. Then we argue that I is empty, because if not, then there is an $s_i \in P_I$. But then $s_i(\lambda)$ is given by our previous formula, which (by Lie group theory) is never equal to λ. (essentially because the Weyl group acts on the Weyl chambers simply transitively), and hence s_i do not preserve v_λ, contradiction. So it is indeed injective.

To see that $X = f(G/B)$ is closed, suppose not and let $\bar{X}$ be the closure. Now G acts on $\bar{X}$ and stabilizes $\bar{X} - X$. Therefore this complement is also B-stable. It is a union of closed G-orbits (so a union of projective varieties, so is projective; this is general theory). The Borel Fixed Points Theorem says that if Z is a projective variety and F is a solvable group acting on it, then there's a fixed point. But B having a fixed point in $\bar{X} - X$ is a contradiction since B has a unique preserved line in V_λ. QED.

Also have to prove the inverse is a morphism (will discuss next time, after next week).

This argument fails in positive characteristic.

Lecture 5: 9/23/14

Let's recall some things (it's been a week!)

Let $F \subset G$ closed, then we defined $G \times_F X$ to be $G \times X/\sim$ where $(gf, x) \sim (g, fx)$ where F acts on the left on X. Then $G \times_F X$ is a G-variety and $G \times_F X \to G/F = G \times_F \{pt\}$ is an isotrivial fibration with fiber X.

Corollary. If V is an F-module then $G \times_F V \to G/F$ is a vector bundle (meaning VB in the Zariski topology). Also G/F is an algebraic variety.

Theorem. B is closed.

Theorem. Let $G \to \mathbb{P}(V_\lambda)$ and let v_λ be a highest weight vector; B acts on $[v_\lambda] \in \mathbb{P}(V_\lambda)$ trivially. Hence get $f : G/B \to \mathbb{P}(V_\lambda)$. Then 1) f is injective, and 2) f has closed image X.

Now we must show that in fact the inverse is a morphism, i.e., that G/B is isomorphic to X (hence projective). This follows from the following more general fact, since X is smooth (being the orbit of a G-action).

Theorem. $f : Y \to Z$ a bijective morphism between irreducible algebraic varieties. Suppose Z is normal. Then f is an isomorphism.

Example. $G = \mathrm{SL}_n$ and B the upper triangular matrices. Let $V = \mathbb{C}^n$. Then define $Fl(V)$ to be the set of complete flags in V. This has a transitive G-action, and the stabilizer of the standard flag is B. In particular if $n = 2$ we realize SL_2/B as $\mathbb{P}^1(\mathbb{C})$.

We have an obvious map $Fl(V) \to \prod\limits_{i=1}^{n-1} Gr(i, \mathbb{C}^n)$; it is injective, and called the Plucker Embedding. The map is simply taking each subspace of the flag. This embedding leads to equations that carve out the flag variety; they are very combinatorial, and called Plucker coordinates.

Anyway, let $\lambda \in X(H) = \{\lambda : H \to \mathbb{C}^\times\}$ a character on the torus. Last time showed that $B = H \ltimes [B, B]$. So characters of H are equivalent to characters of B. So regard λ as a map $B \to \mathbb{C}^\times$. From there we defined the line bundle L_λ defined as

$$\pi : G \times_B \mathbb{C}_{-\lambda} \to G/B$$

where we recall the total space is $\{(g, x) \in G \times \mathbb{C}\}/\sim$ where $(gb, x) \sim (g, \lambda(b)^{-1}x)\}$. Now let $H^0(G/B, L_\lambda)$ be the space of global sections. G acts on this by the rule

$$(g \cdot \sigma)(g'B) := g\sigma(g^{-1}g'B)$$

and similarly all the higher H^i have a G-action.

Borel-Weil-Bott Theorem

Let G be simply-connected.

1) $\lambda \in X(H)_+$ then there is a G-module isomorphism $H^0(G/B, L_\lambda) \cong V_\lambda^*$

2) (Bott) Let $w * \lambda := w(\lambda + \rho) - \rho$ for $w \in W$ and $\rho := \frac{1}{2} \sum\limits_{\alpha \in R^+} \alpha$. Then $H^p(G/B, L_{w*\lambda})$ is $V(\lambda)^*$ if $p = \ell(w)$ and is 0 otherwise.

Note that $w * \lambda$ is not dominant anymore in general, i.e. does not lie in the dominant Weyl chamber.

Consider the special case of this theorem when $G = \mathrm{SL}_2$ and $G/B = \mathbb{P}^1$. Then we may can take $\lambda = \omega_1 = \alpha/2$. Then $V(\lambda)$ is the standard representation, and can think of $V(\lambda)^*$ as $\mathbb{C}[x] \oplus \mathbb{C}[y]$. On $\mathbb{P}^1$ this is exactly $H^0(\mathbb{P}^1, \mathcal{O}(1))$. Also $H^i(\mathbb{P}^1, \mathcal{O}(1)) = 0$ for $i \geq 1$, e.g. by Serre duality.

Before we begin the proof proper, let's begin with the following easy construction:

$$G \times_F V \to G/F$$

and using $G \to G/F$ we can get a pullback diagram, hence getting $X_f \xrightarrow{f} G \times_F V$ and X_f is a vector bundle over G. In fact, $X_f \cong G \times V$.

To see this, note first that $X_f = \{([g,x], g_1) \in (G \times_F V) \times G : \pi_1([g,x]) = \pi_2(g_1)\}$. We construct a map $G \times V \to X_f$ by mapping $(g,v) \mapsto ([g,v], g)$. In the other direction we note that $([g,v], g_1) = ([g_1 b, v], g_1) = ([g_1, bv], g_1)$, since $g = g_1 b$ (since the projections agree), and so we map this to (g_1, bv). QED.

Exercise. More generally, if $F \subset G$ is closed and G acts on a vector space M, then $G \times_F M \cong G/F \times M$ over G/F.

We now apply this to our situation of L_λ. Let $\tilde{L}_\lambda$ be the associated (trivial) bundle $\tilde{\pi} : G \times \mathbb{C}_{-\lambda} \to G$. We can take a section of this latter to be $\sigma(g) = (g, f(g))$ where $f : G \to \mathbb{C}_{-\lambda}$. So $H^0(G, \tilde{L}) \cong k[G] \otimes \mathbb{C}_{-\lambda} \cong k[G]$. Get a B-action on these functions in the tensor product by $b \cdot f(g) = \lambda(b)^{-1} f(gb)$. This says, in other words, that our section should satisfy $\sigma(gB) = [g, f(g)] = [gb, f(gb)] = [g, bg(gb)] = [g, \lambda(b)^{-1} f(gb)]$. This can be summarized by saying that

$$H^0(G/B, L_\lambda) \cong H^0(G, \tilde{L}_\lambda)^B.$$

Now $G \times G$ acts on $k[G]$. The Peter-Weyl Theorem and Tannaka-Krein Duality. We have an iso of $G \times G$-modules:

$$k[G] \cong \bigoplus_{\mu \in X(H)_+} V_\mu^* \times V_\mu.$$

The iso can be described by $\Phi_\mu(f \otimes v)(g) = f(gv)$ and taking $\Phi = \sum \Phi_\mu$ to be the iso. Now $k[G] \otimes \mathbb{C}_{-\lambda}$ has a $G \times B$-action on each factor (on the second factor, just have G-factor act trivially). Hence $H^0(G, \tilde{L})^B \cong (k[G] \otimes \mathbb{C}_{-\lambda})^B$; $g \cdot f(x) = f(g^{-1}x)$; the G-action commutes with the B-action. Hence we can take decompositions before or after invariants. Thus as left G-modules we have an iso

$$H^0(G, \tilde{L})^B \cong \bigoplus_{\mu \in X(H)_+} V_\mu^* \otimes (V_\mu \otimes \mathbb{C}_{-\lambda})^B = \bigoplus_{\mu \in X(H)_+} V_\mu^* \otimes (\mathbb{C}_\mu \otimes \mathbb{C}_{-\lambda}) \cong V_\lambda^*.$$

Lecture 6: 9/25/14

Today we'll talk about the Bott part of BWB. But first!

Equivariant Vector Bundles

Suppose G acts on X and V is an algebraic VB over X. V is said to be equivariant with the G action if $G \times V \to V$ over $G \times X \to X$ commutes, and also $g : V_x \to V_{gx}$ is a linear isomorphism. For example, the line bundles $G \times_B \mathbb{C}_{-\lambda} \to G/B$ defined last time.

Equivariant VBs on G/B

The arguments below would actually work for B any closed subgroup.

Let V be a B-module. Then we construct $\tilde{V} := G \times_B V \to G/B$ and this is a G-equivariant VB. Going in the other direction, if we take any $V \to G/B$ any G-equivariant VB, then $V|_B$ is a B-module (V being G-equivariant implies it is B-equivariant, and hence $V|_B$ is B equivariant). These two operations are inverses to each other, and establish a bijection between the set of finite dimensional B-modules (up to iso) and G-equivariant VBs on G/B.

A special case is that equivariant line bundles on G/B correspond 1-1 to characters $B \to \mathbb{C}^\times$ (moduli equivalence on both sides).

Theorem. Suppose G is simply connected. Then the category of equivariant line bundles on G/B is equivalent (via the forgetful functor) to the category of line bundles on G/B.

For proof, see Jacob Lurie's website; search for BWB theorem.

Recall example - $\mathrm{SL}_2/B = \mathbb{P}^1$ and if $\lambda = a\omega_1$ for $a \geq 0$ then $H^0(\mathrm{SL}_2/B, L_\lambda) \cong V_\lambda^* = Sym^a(\mathbb{C}^2)^*$ equals the degree a homogeneous polynomials on $\mathbb{C}^2$ equals $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(a))$.

Let ρ be the half sum of positive roots as usual; it also equals $\sum \omega_i$ (the sum of all the fundamental dominant weights). Then $\rho(\alpha_i^\vee) = 1$ for all i since $\omega_i(\alpha_i^\vee) = \delta_{ij}$. We define $L^\lambda := L_{\lambda-\rho}$.

Let X be a smooth curve of genus g. K_X is the sheaf of differentials; it has degree $2g-2$. If $X = \mathbb{P}^1$ then $g = 0$ so $\deg K_X = -2$. If $g = 1$ then the degree is 0 and this corresponds to elliptic curves not having any holomorphic differentials.

Theorem. Let $E \to S$ be a $\mathbb{P}^1$-bundle, and let $L \to E$ be a line bundle, such that for all s we have $\deg L|_{E_s} = n \geq -1$. Then we have

$$H^i(E, L) \cong H^{i+1}(E, L \otimes K^{n+1}).$$

Here K is the relative canonical bundle (i.e. on every fiber you get K_X). The proof works by applying the Leray-Serre spectral sequence to the fibration $0 \to \mathbb{P}^1 \to E \to S$ and showing $E_2^{p,q} = 0$.

How do we apply this? Let α_i be a simple root and P_i the minimal parabolic containing B. Then have $0 \to P_i/B \to G/B \to G/P_i \to 0$. Here P_i/B is just $\mathrm{SL}_2^{(i)}/(\mathrm{SL}_2^{(i)} \cap B) = \mathbb{P}^1$ (since the Lie algebra of P_i is $\mathfrak{b} + \mathfrak{g}_{-\alpha_i}$ and use the exponential map). We're now in a position to apply the above theorem.

Theorem. Let α_i be a simple positive root and suppose $\lambda(\alpha_i^\vee) \geq 0$, with λ in the weight lattice P. Then $H^p(X, L^\lambda) \cong H^{p+1}(X, L^{s_i(\lambda)})$ where $X = G/B$ and s_i is the Weyl group element corresponding to α_i. (in the original writeup, we incorrectly had $s_i * \lambda$ here).

The theorem follows from the following claim. $L^{s_i * \lambda} \cong L^\lambda \otimes K^{\deg L^\lambda_{|\mathbb{P}^1} + 1}$.

Before we prove this claim, let us use it to complete the proof of BWB. Rewriting this theorem in our old language, it says the following.

Theorem'. Let α_i be a simple root and $\mu(\alpha_i^\vee) \geq -1$ (here thinking of μ as $\lambda - \rho$). If $p \geq 0$ then $H^p(G/B, L_\mu) \cong H^{p+1}(G/B, L_{s_i * \mu})$.

Corollary. $H^p(G/B, L_\mu) \cong H^{p+\ell(w)}(G/B, L_{w*\mu})$. When $w = s_i$ this gives the previous theorem. The proof is basically by induction on length.

At this point the BWB theorem will follow immediately - we have that

$$H^j(G/B, L_\lambda) \cong H^{j+\ell(w_0)}(G/B, L_{w_0 * \lambda})$$

where w_0 is the longest element in W. We observe that $\ell(w_0) = \dim G/B$. But then we're done since cohomology vanishes above the top dimension.

Recall last time we did the Bruhat decomposition $G = \coprod BwB$. Then $G/B = \coprod BwB/B$ similarly. We define $X_w = BwB/B$ to be the open Schubert cells, which are isomorphic to $\mathbb{C}^{\ell(w)}$. This implies the above dimension observation. This gives a CW complex structure on G/B. A Schubert variety is by definition the closure of X_w in G/B. We can then ask what happens if we restrict L_λ to a Schubert variety, and in particular what the cohomology H^0 of this is. This is more tricky since the Schubert varieties are not smooth in general. But there is a formula for $\dim H^0(\bar{X}_w, L_\lambda|_{\bar{X}_w})$ given by the Demazure Character Formula.

Okay so $L^{s_i * \lambda} = L_{s_i * \lambda - \rho} = L_{-(2\rho - \lambda) - (\alpha_i^\vee, \lambda)\alpha_i} = L^\lambda \otimes (L^{\alpha_i})^{\otimes(\lambda, \alpha_i^\vee)}$.

Lecture 7: 9/30/14

Recall our our $\mathbb{P}^1$-bundle from last time, and the notation there. If L_λ is our line bundle on G/B then $\deg L_\lambda|_{P_i/B} := \lambda(\alpha_i^\vee)$. We have the example for SL_2 in which $L_{a\omega_1} = \mathcal{O}_{\mathbb{P}^1}(a)$ and $a\omega_1(\alpha^\vee) = a \cdot 1$. Also $H^0(\mathrm{SL}_2/B, L_{a\omega_1}) = H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(a))$. And $K = L_{-\alpha_i}$ is the relative ample canonical bundle on G/B. We've defined $L^\lambda = L_{\lambda - \rho}$. Recall the theorem(s) from last time. We must prove the following.

Theorem. If $E \to S$ is a $\mathbb{P}^1$-bundle and $L \to E$ has $\deg L|_{\mathbb{P}^1} = n \geq -1$ then

$$H^p(E, L) \cong H^{p+1}(E, L \otimes K^{n+1})$$

Proof. We NTS that $L^{s(\lambda)} = L^\lambda \otimes K^{\deg L^\lambda + 1}$. Well $L^\lambda = L_{\lambda - \rho}$ and $\deg L^\lambda_| P_i/B = (\lambda - \rho)(\alpha_i^\vee) + 1 = \lambda(\alpha_i^\vee) - \rho(\alpha_i^\vee) + 1 = \lambda(\alpha_i^\vee)$. Next, $L_{s_i(\lambda) - \rho} = L_{\lambda - \rho} \otimes L_{-\alpha_i} = L_{\lambda - \rho} \otimes L_{-\lambda(\alpha_i^\vee)\alpha_i} = L_{\lambda - \rho - \lambda(\alpha_i^\vee)\alpha_i} = L_{\lambda - \lambda(\alpha_i^\vee)\alpha_i - \rho}$ and the subscript here equals $s_i(\lambda) - \rho$. □

Okay now take $\lambda \in P_+$ with $\lambda(\alpha_i^\vee) > 0$ for all i (can think of this as meaning we lie in the interior of the chamber, i.e., not on any wall). We have $G/B \to \mathbb{P}(V_\lambda)$ and the image is the orbit $G \cdot [v_\lambda]$. We know that the stabilizer of $[v_\lambda]$ contains B, and that they must in fact be equal. But this equality used the inequality $\lambda(\alpha_i^\vee) > 0$ to deduce that $s_i(\lambda) \neq \lambda$ since $\lambda(\alpha_i^\vee) \neq 0$ for all i.

Construction. Take $\lambda \in P_+$ and take $G \to \mathbb{P}(V_\lambda)$ as usual. Then we have the stabilizer of $[v_\lambda]$ again containing B but perhaps being bigger. We call this stabilizer P_λ. Then we have

$G/P_\lambda \cong G[v_\lambda]$. Find $i \in \{1,\dots,\ell\}$ such that $s_i \in P_\lambda$: these are precisely the i such that $s_i(\lambda)=\lambda$. Equivalently, $\lambda(\alpha_i^\vee)=0$.

For example, SL_{r+1} and ω_j we must find i such that $\omega_j(\alpha_i^\vee)=0$. The set I in this case is $\{1,2,\dots,j-1,j+1,\dots,\ell\}$.

General Algebraic-Geometry exercise. Suppose we have $X \hookrightarrow \mathbb{P}^(V)$ smooth. Let $L=\mathcal{O}_X(1)$ (i.e. the pullback to X of $\mathcal{O}(1)$). We have $H^0(\mathbb{P}(V),\mathcal{O}_{\mathbb{P}(V)}(1))=V^*$

1) $\phi: V^* \to H^0(X,\mathcal{O}_X(1))$ is onto. (here we're just pulling back global sections, which of course we can do).

2) If X is not contained in a hyperplane, then ϕ is an iso.

3) $f: G/P_\lambda \hookrightarrow \mathbb{P}(V_\lambda)$ is not contained in a hyperplane. Hint - use the G-action (if it lies in one hyperplane argue that it lies in multiple ones, then in all of them via transitivity, and get a contradiction).

Let $L=f^*\mathcal{O}(1)$. We have $H^0(G/P_\lambda,L)\cong V_\lambda^*$. This looks like a BWB theorem!

We have $G/B \to G/P_\lambda$. Whenever we have a map $X \to Y$ with fibers irreducible projective variety, and take the pushforward $f_*\mathcal{O}_X=\mathcal{O}_Y$. This because there are no nonconstant functions on projective varieties.

Projection Formula. Thus we have $f_*(f^*L) - L\otimes f_*\mathcal{O}_{G/B}=L$ for any line bundle L on G/P_λ.

Exercise (alg geo again). $H^0(G/B,f^*L)\cong H^0(G/P_\lambda,L)$.

Applying this observation to our line bundle L_λ and the previous fact, take $G/B \xrightarrow{\pi} G/P_\lambda \hookrightarrow \mathbb{P}(V_\lambda)$ and we get

$$V_\lambda^* = H^0(G/P_\lambda, f^*\mathcal{O}(1)) = H^0(G/B,\pi^*f^*\mathcal{O}(1))$$

and now observe $\pi^*f^*\mathcal{O}(1)=L_\lambda$. This can be shown because G-equivariant line bundles are determined by their characters.

Example. If $G=\mathrm{SL}_r$ write $\lambda=\sum_{i=1}^{r-1} a_i\omega_i$. Then $V_\lambda \subset \bigotimes Sym^{a_i}(\bigwedge^i \mathbb{C}^r)$.

We have a big commutative diagram, in which the top row is $G/B \hookrightarrow \prod_{i=1}^r Gr(i,\mathbb{C}^r)$, (we don't really need this part), the second row is

$$G/P_\lambda \to \prod_{i=1}^r Gr(d_i,\mathbb{C}^r) \to \prod \mathbb{P}(\bigwedge^{d_i},\mathbb{C}^r) \to \prod \mathbb{P}(Sym^{a_i}\bigwedge^{d_i}\mathbb{C}^r) \to \mathbb{P}(\bigotimes_{i=1}^r Sym^{a_i}\bigwedge^{d_i}\mathbb{C}^r)$$

(here the product is over some subset of the indices; we repeat d_i a_i-times). Here we're using the Veronese and Segre embeddings (multiple times). The last row is $\mathbb{P}(V_\lambda)\to\mathbb{P}(\bigotimes_{i=1}^r Sym^{a_i}\bigwedge^{d_i}\mathbb{C}^r)$ One can also find this in Fulton's Young Tableaux (chapter 9). Here d_i is equal to i if $a_i\neq 0$ and is 0 otherwise.

The simple part that we care about is that we get a map $G/B \to \prod Gr(d_i, \mathbb{C}^r)$ over the third row.

Gives example of reading λ from a Young diagram, and thus giving a (partial) flag.

Pulling/pushing the line bundles around this diagram, we can basically write everything in terms of products of $\mathcal{O}(1)$'s. We'll now explain how. G/B is the full flag variety, with line bundle L_λ. There are multiple projections $p_1, \ldots, p_r$ onto the particular subspaces in a given flag. We define $\mathcal{O}(1) \to Gr(r, V)$ to be the dual of the top exterior power of the universal vector bundle (so this will be the one that has sections). We take $\bigotimes_{i=1}^{r} ((p_i^* \mathcal{O}(1))^{\otimes a_i}) = L_\lambda$ by chasing the diagram; we conclude $H^0(G/B, L_\lambda) \cong V_\lambda^*$. All this was for SL_r; one can do it for the other classical groups - see Fulton and Harris.

NEW TOPIC!

Pick positive integers $n \geq r$. Consider young diagrams $\lambda_1, \ldots, \lambda_m$ having at most r rows and at most $n-r$ columns (recall these give weights for SL_r; we'll often think of them this way). $|\lambda_i|$ denotes the number of boxes in the diagram. Further impose that $\sum |\lambda_i| = n(n-r)$. Meanwhile λ_i^T correspond to SL_{n-r}-representations.

Theorem (Strange Duality). $\dim(\bigotimes_{i=1}^{m} V_{\lambda_m})^{\mathrm{SL}_r} = \dim(\bigotimes_{i=1}^{m} V_{\lambda_m^T})^{\mathrm{SL}_{n-r}}$.

In fact the LHS vector space is the dual of the RHS one. It is called strange duality because it does not work (directly) for other groups - one needs to use something other than invariants. This story connects with moduli spaces of vector bundles and of genus g curves with n marked points, Hodge theory, and some accidental isomorphisms.

Lecture 8: 10/2/14

Reference: http://imrn.oxfordjournals.org/content/2004/69/3709.full.pdf (Belkale's article on Invariant Theory of GL_n and intersection theory of Grassmannians)

We begin with the strange duality described last time. There will be some combinatorics. The above isomorphism is not analogously true for other Lie groups, like symplectic ones. For other groups, one needs to use Conformal Blocks; and this leads to interesting information about the geometry of (i.e. vector bundles over) moduli spaces of curves of given genus with n marked points, Nef divisors, Mori Dream conjecture...

Anyway, take λ to be a Young diagram with at most r rows and $n-r$ columns. That is, $n-r \geq \lambda^1 \geq \lambda^2 \geq \ldots \geq \lambda^r$. We then define $i_a = (n-r) + a - \lambda^a$, so that $\underline{i}(\lambda) = \{i_1 < i_2 < \ldots < i_r\} \subset [n] = \{1, \ldots, n\}$. Consider $Gr(r, n)$, λ, and a fixed full flag $F^\bullet : 0 \subset F_1 \subset \ldots \subset F_n = \mathbb{C}^n$. Then we define the open Schubert cell to be

$$\Omega_\lambda^0(F^\bullet) = \{V \in Gr(r, n) : \dim(V \cap F_k) = a \text{ where } i_a \leq k < i_{a+1}\}$$

and the closure of this is (either in the Zariski topology or the ordinary)

$$\Omega_\lambda(F^\bullet) = \{V \in Gr(r,n) : \dim(V \cap F_{i_a}) \geq a\}$$

Reference: Flag Varieties by Lakshmibai and Justin Brown (last chapter).

Now take a vector space V of dimension n, and the affine variety $X = V^{\oplus r}$, thought of as $n \times r$ matrices. Then define $\pi : X \to \bigwedge^r V$ by $(v_1, \ldots, v_r) \mapsto v_1 \wedge \ldots \wedge v_r$. Let $\underline{i} = \{i_1 < \ldots i_r\} \subset [n]$. Then we define $v_{\underline{i}} = (e_{i_1}, \ldots, e_{i_r})$ where we've chosen some basis. We then have

$$\pi : X \backslash D \to \mathbb{P}(\bigwedge^r V).$$

Next define $\Omega^0_{\underline{i}(\lambda)} = \{A \in M_{n\times r} : \operatorname{rank} A = r$ and it has a certain row echelon form $\}$ and the closure of this is then those matrices such that $a_{ij} = 0$ for $i > i_j$. With sufficient staring, one finds that this description is the same as the former.

Facts/theorem: $\Omega_\lambda = \overline{\Omega}^0_\lambda$ and Ω_λ is the disjoint union of the Ω^0_μ such that $\mu \geq \lambda$. Also $\Omega_\lambda \backslash \Omega^0_\lambda$ is the disjoint union of those Ω_μ such that μ is obtained from λ by adding one box.

Theorem (Lesieur). We have $\dim(\bigotimes_{i=1}^r V_{\lambda_i})^{\mathrm{SL}(r)} = \# \bigcap_{i=1}^m \Omega^0_{\lambda_i}$, subject to $\sum |\lambda_i| = r(n-r)$.

To see this, note that if Ω_λ for λ having the restricted number of rows columns as above, then the collection of $[\Omega_\lambda]$ form an additive basis of $H^*(Gr(r,n))$. Hence there are formulas $\Omega_\lambda * \Omega_\mu = \sum c^\nu_{\lambda,\mu} \Omega_{\nu^c}$ where ν^c is the Young diagram corresponding to the complement of ν (in the r by $n-r$ box). On the representation theory side we have $V_\lambda \otimes V_\mu = \sum d^\nu_{\lambda,\mu} V_{\nu^c}$ and the theorem claims that the c's equal the d's. This is in the realm of the "Geometric Horn Problem." There's an algorithm (but no hope for a closed formula) to compute the $d^\nu_{\lambda,\mu}$. A better question is simply: when is it nonzero? By the theorem it's the corresponding question for the c's (so an intersection question) and then one tries an inductive process reducing to smaller Grassmannians. The result is called the Horn Inequalities. On Fulton's webpage there's a survey paper relating to this.

We have the duality $Gr(r,n) \cong Gr(n-r,n)$ sending $V \subset W$ to $\ker[W^* \to V^*]$. The correspondence of their Schubert classes is $\Omega_\lambda \leftrightarrow \Omega_{\lambda^T}$. We deduce from the isomorphism of cohomology rings that we have $[\Omega_{\lambda_1}] * \cdots * [\Omega_{\lambda_r}] = [\Omega_{\lambda_1^T}] * \cdots * [\Omega_{\lambda_m^T}]$ and each side here computes the two sides of the dimension formula of the strange duality theorem, so we're done. Now can we do this story geometrically? We have the whole BWB apparatus, so let's use it. Let $V \in Gr(r,W)$, where $\dim W = n$. Then the tangent space at V is $T_V Gr(r,W) = \mathrm{Hom}(V, W/V)$. Further, $T_V \Omega^0_\lambda(F^\bullet) = \{\phi \in \mathrm{Hom}(V, W/V) : \phi(E_a) \subset G_{i_a - a}\}$. Here if F is a flag on W, then we get induced flags on V and W/V, called $E^\bullet$ and $G^\bullet$. We have (exercise) an iso of complete flag varieties $f : Fl(W) \cong Fl(W^*)$ by $E_\bullet \mapsto E^*_\bullet$

Exercise. For $\Lambda \in Y_{r,n-r}$ i.e. Young diagram as above. Let L_λ be the BWB line bundle. Then show that $f^* L^{E^*_\bullet}_\lambda \cong L^{E_\bullet}_{\lambda^c}$. This map intertwines $\mathrm{SL}(W)$ and $\mathrm{SL}(W^*)$ actions. This duality corresponds to the outer involution on SL_n that we know exists from its Dynkin diagram.

Now let V and Q be vector spaces of dimensions r and $n-r$ (think of $Q = W/V$), and then form $Fl(V)^m \times Fl(Q)^m \supset D$, D a divisor as follows. We have flags $(F^i_\bullet)$ and $(Q^i_\bullet)$, for $1 \leq i \leq m$, and say the corresponding $2m$-tuple $(\mathcal{F}, \mathcal{Q}) \in D$ if there exists nonzero $\phi : V \to Q$ such that $\phi(F^k_a) \subset G_{\underline{i}^a_k - a}$ where we recall $\underline{i}_k \leftrightarrow \lambda_k$.

We'll finish next time.

Lecture 9: 10/7/14

We continue to go through Prakash's paper (reference given last lecture).

Recall we have partitions λ and λ^c (in the $r \times n-r$ box), and $f : Fl(W) \xrightarrow{\cong} Fl(W^*)$ mapping $E_\bullet \mapsto E^*_\bullet$, and $f^* L_\lambda = L_{\lambda^c}$. Define on $Fl(V)$ the line bundle

$$L_i = (\bigwedge^i \tilde{V}_i)^*$$

where $\tilde{V}_i$ is the (rank i) vector bundle on $Fl(V)$ that associates to a flag $0 \subset V_1 \subset \ldots \subset V_n = V$ the vector space V_i. Now if $\lambda = \sum a_i \omega_i$ then define

$$L_\lambda = \bigotimes L_i^{\otimes a_i}.$$

We know $H^0(\mathrm{GL}(n)/B, L_\lambda) \cong V^*_\lambda$. The theorem of Lesieur is that

$$\dim(\bigotimes V_{\lambda_i})^{\mathrm{SL}(r)} = \# \left\{ \bigcap \Omega^0_{\lambda_i} \right\} \in Gr(r,n)$$

where $\sum |\lambda_i| = r(n-r)$. Note here that we're using the Schubert cells and not Schubert varieties (the latter are not smooth so intersection is more subtle, and we can't use tangent spaces as our technique). Weak strange duality says $((\bigotimes V_{\lambda_i})^{\mathrm{SL}(r)})^* \cong (\bigotimes V_{\lambda_i^T})^{\mathrm{SL}(n-r)}$.

Recall our description above of the tangent spaces to open Schubert cells. Let V, Q be vector spaces of dimension r and $s = n-r$. Let $\lambda \in Y_{r,s}$ and $(F_\bullet, G_\bullet) \in Fl(V) \times Fl(Q)$. We define a line bundle fiberwise by

$$P_\lambda|_{(F_\bullet, G_\bullet)} = \{\phi \in \mathrm{Hom}(V,Q) : \phi(F_a) \subset G_{\underline{i}_a - a}\}.$$

This is a subbundle of the trivial vector bundle $\mathrm{Hom}(V,Q)$. Consider the quotient bundle $\mathrm{Hom}(V,Q)/P_\lambda$. If the quotient map is s, then (exercise) we can describe for which pair of flags we have $s = 0$ (on the corresponding fiber).

How can we identify $\mathrm{Hom}(V,Q)/P_\lambda$ as a line bundle? We can think of it as (coming from) a divisor. To do this, let $\lambda_1, \ldots, \lambda_n \in Y_{r,s}$ with $\sum |\lambda_i| = rs$. Let $\mathcal{F}_\bullet$ be an n-tuple of flags and similarly for $\mathcal{G}_\bullet$. Then we define a vector bundle $\tilde{P}_{\lambda_i}(\mathcal{F}_\bullet, \mathcal{G}_\bullet) := P_{\lambda_i}(F^i_\bullet, G^i_\bullet)$. Then we have (at each fiber)

$$\mathrm{Hom}(V,Q) \to \bigoplus_{i=1}^n \mathrm{Hom}(V,Q)/P_{\lambda_i}(\mathcal{F}_\bullet, \mathcal{G}_\bullet)$$

(and we're really getting a vector bundle map). Write this as a map $\mathcal{V} \xrightarrow{S} \mathcal{W}$. Then since the two vector spaces / bundles have the same dimension (as we'll soon show), we can take the determinant of S and view $\det(S)$ as an element of $(\bigwedge \mathcal{V})^* \bigotimes \bigwedge \mathcal{W}$...So we're getting a section of a line bundle, and since it's not the trivial line bundle the section will vanish at some points; the vanishing set is our divisor!

How can we describe this divisor $D \subset Fl(V)^n \times Fl(W)^n$? Well $(F^1_\bullet, \dots, F^n_\bullet, G^1_\bullet, \dots, G^n_\bullet) \in D$ iff there exists $0 \neq \phi \in \mathrm{Hom}(V, Q)$ such that $\phi(F^k_a) \subset G^k_{\underline{i}_k(a)-a}$ for all $k = 1, \dots, n$ where $\lambda_k \leftrightarrow \underline{i}_k$. Now all these line bundles are $\mathrm{GL}(V)$ and $\mathrm{GL}(Q)$-equivariant, and so any point in D has its whole orbit in D.

The claim above (that the two bundles have the same rank) follows from showing that W_{λ_i} has rank $|\lambda_i|$.

We can view $\det S$ as an element of

$$(\bigotimes_{i=1}^{n} H^0(Fl(V), L_{\lambda_i}))^{\mathrm{SL}(r)} \bigotimes (\bigotimes_{i=1}^{n} H^0(Fl(T), L_{\lambda_i^T}))^{\mathrm{SL}(n-r)}$$

and then using BWB and Leiseur this is a map $V_\Lambda^{\mathrm{SL}(r)} \to V_{\Lambda^T}^{\mathrm{SL}(s)}$. Here we write Λ for the vector of λ_i's.

BREAK!

Now let $P_\lambda(a)(F_\bullet, G_\bullet) = \{\phi \in \mathrm{Hom}(V, Q) : \phi(F_j) \subset G_{i_j} - j \text{ for } j = 1, \dots, a\}$. So that $P_\lambda(0) = \mathrm{Hom}(V, Q)$ and $P_\lambda(r) = P_\lambda$. This again gives us a (bunch of) bundle as before. We have a SES

$$P_\lambda(a+1) \to P_\lambda(a) \to \mathrm{Hom}(F_{a+1}/F_a, Q/G_{i_{a+1}-(a+1)})$$

and the dimension of the RHS term is λ^{a+1} so the claim we made earlier can be proved by induction.

Okay moving on. Note that if we have a SES of VBs (or vector spaces) then the Det of the middle is the product of the dets of the sub and quotient. It follows that $\det P(a+1) = \det P(a) \otimes \det \mathrm{Hom}(F_{a+1}/F_a, Q/G_{i_{a+1}-(a+1)})^*$ where the Hom is the same as before. We need to understand this term; it equals $\det((F_{a+1}/F_a)^* \otimes Q/G_{i_{a+1}-(a+1)})$. Now use the fact that $\det(V \otimes W) \cong (\det V)^{\otimes \operatorname{rank} W} \otimes (\det W)^{\otimes \operatorname{rank} V}$...we then use some induction...get the answer.

To recapitulate, we're trying to show (and the above arguments result in showing) that $\det(\mathcal{V}^* \otimes \mathcal{W}) \cong L_\Lambda \otimes L_{\Lambda^T}$.

Now let's talk some abstract algebraic geometry. Let X, Y be varieties (don't really even need smooth). Suppose we have $L_1 \boxtimes L_2 \to X \times Y$ where by definition the box product of line bundles is $p_1^* L_1 \otimes p_2^* L_2$. Assume that $h^0(X, L_1) = h^0(Y, L_2) = m$ (i.e. same dimensions of global sections). Suppose that $x_1, \dots, x_m \in X$ and $y_1, \dots, y_m \in Y$ such that $s(x_i, y_j) = \delta_{ij}$ for a section $s \in H^0(X \times Y, L_1 \boxtimes L_2)$.

Exercise: THEN 1) $s(x_a, *)$ form a basis of $H^0(Y, L_2)$. Similarly $s(*, y_b)$ form a basis of $H^0(X, L_1)$. (these are well-defined up to constant; just choose to be 1). This sets up a duality isomorphism (i.e. perfect pairing) betwen the H^0 spaces.

Remark - this can be done for X, Y being stacks.

We apply the theorem to $Fl(V)^n/\mathrm{SL}(V) \times Fl(Q)^n/(\mathrm{SL}(Q))$ with line bundles L_Λ and L_{Λ^T}. If we can prove that our element $\det S$ above has the desired property, then we'll be done! We remark that Prekash's paper (listed above) does not talk about strange duality, but the other paper (coauthored by Swarnava) does.

Lecture 10: 10/9/14

Suppose $V \hookrightarrow W \to W/V = Q$ with $r = \dim V$ and $s = \dim W/V$. We were studying $Fl(V) \times Fl(Q)$. We defined, for every point $(F_\bullet, G_\bullet)$ in this product, a vector space $P_\lambda(F_\bullet, G_\bullet) \subset \mathrm{Hom}(V, Q)$, and these pasted together to give a vector bundle. We also studied $X \times Y$ where $X = Fl(V)^n$ and $Y = Fl(Q)^n$, giving a map of vector bundles on $X \times Y$ as

$$\mathrm{Hom}(V,Q) \otimes \mathcal{O}_{X\times Y} \to \bigoplus_{i=1}^{n} (\mathrm{Hom}(V,Q) \otimes \mathcal{O}_{X\times Y})/P_{\lambda_i}$$

We realized the kernel of this as coming from the divisor D described above.
Proposition.

$$\mathcal{O}(D) \cong (L_{\lambda_i} \otimes \cdots L_{\lambda_n}) \boxtimes (L_{\lambda_1^{T,c}} \otimes \ldots \otimes L_{\lambda_n^{T,c}})$$

as vector (line) bundles on $X \times Y$

Call L_Λ and $L_{\Lambda^{T,c}}$ these two factors. Let $s \in H^0(X, L_\Lambda)^{\mathrm{SL}(V)} \otimes H^0(Y, L_{\Lambda^{T,c}})^{SL(Q)}$.
Theorem.

$$a : \bigcap_{i=1}^{n} \Omega^0_{\lambda_i}(E^i_\bullet) \to H^0(X, L_\Lambda)^{\mathrm{SL}(V)}$$

defined by $a(M) = D(M) \subset Fl(V)^n$ (defined below). Then 1) $D(M)$ is not equal to all of $Fl(V)^n$ and 2) a maps distinct Schubert cells to linearly independent sections.

Parts one and two here tell us we have an injection, and then Leiseur's theorem will tell us the dimensions are equal and hence this is actually an iso. Really the codomain here should be the projectivization of the H^0 (since only getting lines, not specific vectors). Or equivalently, we could take the $\mathbb{C}$-span of the domain.

Recall that these Schubert cells being intersected are sitting inside some $G(r, W)$. So a point in such a cell is an r-dimensional vector space, hence (noncanonically) isomorphic to V. So choose an isomorphism of M with V. We know $M \in \Omega^0_{\lambda_i}(E^i_\bullet)$ where $E^i_\bullet$ is a full flag in W. Let $F(E^i_\bullet)$ denote the induced flag on M, hence on V via the iso chosen. So we're getting a point in $Fl(V)^n$. Let $Q = W/M$ and let $G(E^i_\bullet)$ denote the induced flag on W/M.

Using the chosen iso of M with V, we let $D(M)$ be the divisor $D \subset Fl(V)^n$. Recall this works by fixing flags (any, but might as well choose the induced one) $G^i_\bullet \in Fl(Q)$, $i = 1, \ldots, n$ and then using the above construction of a divisor on $Fl(V)^n \times Fl(Q)^n$ to (via restriction) getting one on $Fl(V)^n$.

We claim that the point corresponding to M in $Fl(V)^n$, namely $\tilde{M} := (F(E^1_\bullet), \ldots, F(E^n_\bullet))$ is not in D (this proves part 1 of the theorem). If it were, then we'd have a $\phi \neq 0$, but transversality of the intersection implies the intersection of the tangent spaces is zero, and we're done.

For the 2nd part of the theorem, we argue as follows. Let $M \neq M'$. Then we claim $\tilde{M'} \in D(M)$. This will imply the desired linear independence, because there's a point (really as many points as we need) where all but one of the sections vanishes. To prove the claim, note that we have $F_a(E^j_\bullet)(M') \subset E^j_a(W) \to G_{\underline{i}(\lambda)(a)-a}(E^j)(W/M)$ by definition of induced flags. The composition gives our desired ϕ. In other words, we have a canonical map $0 \to M' \to W \to W/M \to 0$. This is the zero map iff $M = M'$.

This proves strange duality, and a geometrization of Lesieur's theorem. That is, $\# \bigcap \Omega^0_{\lambda_i} \leq \dim(\bigotimes V_{\lambda_i})^{\mathrm{SL}(r)}$. We can say (using the last big from last time) that the two relevant spaces are naturally dual vector spaces.

New topic! (just until the end of this class, in 7 minutes...)

Aside: Generic Strange Duality

Let $SU_C(r)$ be the (coarse) moduli space of semistable vector bundles of rank r on a smooth curve C with trivial determinant. (semistable means it doesn't have a subbundle with greater slope, where slope is something like the ratio of the degree and rank. The reason to care comes from GIT. See the "Hilbert-Mumford Criterion." Stable VBs means that subbundles should always have strictly less slope).

Recall the Euler characteristic of a VB V is $\deg V + \mathrm{rank}(V)(1-g)$. Since $\det V = 1$ we know $\deg V = 0$. Let $L \in J^{g-1}(C)$, a line bundle of degree $g-1$. Then $\chi(V \otimes L) = \deg(V \otimes L) + \mathrm{rank}(V)(1-g) = \mathrm{rank}(V)\deg(L) + \mathrm{rank}(V)(1-g) = 0$. So we can now define a "theta divisor" $\theta_L = \{V \in SU_C(r) : h^0(V \otimes L) \neq 0\}$.

1) θ_L is a divisor in $SU_C(r)$.

2) $\mathcal{L} = \mathcal{O}(\theta_L)$ is independent of $L \in J^{g-1}(C)$.

3) $\mathrm{Pic}(SU_C(r)) = \mathbb{Z}\mathcal{L}$.

We'll soon move on to affine Kac-Moody stuff.

Lecture 11: 10/14/14

Reference for some of this is Mumford's Abelian Varieties, and Tate Lectures on Theta I, II, III by Nori and Ramanujam (not Ramanujan).

Let $g \geq 2$ and C a smooth projective curve of genus g. Then J^{g-1} is the set of degree $g-1$ line bundles on C. We have, for a fixed line bundle L of degree $g-1$ on C, a map $J(C) \xrightarrow{f_L} J^{g-1}(C)$

by $E \mapsto E \otimes L$. Here $J(C)$ is the Jacobian. On Jacobians we have Theta functions. We take a theta divisor $\theta = \{E : h^0(E \otimes L) > 0\} \subset J(C)$. Theta functions are by definition global sections of the line bundles (on the Jacobian) associated to these divisors. We have $\mathcal{O}(k\theta_L) := \mathcal{O}(\theta_L)^{\otimes k}$. Have $h^0(J(C), \mathcal{O}(k\theta_L)) = k^g$.

...For $\mathbb{C}^g/\Gamma$ to be a projective variety we also need a "Riemann Bilinear form..."

We let $SU_C(r)$ be the set of semistable vector bundles $\mathcal{V}$ on C with $\operatorname{rank} \mathcal{V} = r$ and $\det \mathcal{V} = \mathcal{O}_C$. Recall that semistability means that for all subbundles $\mathcal{W} \subset \mathcal{V}$ we have

$$\frac{\deg \mathcal{W}}{\operatorname{rank} \mathcal{W}} := \mu(\mathcal{W}) \leq \mu(\mathcal{V}).$$

References: 1) Newsted "Intro to Moduli and Orbit Spaces" and 2) Seshadri, Le-Poties (French book).

Example. On $\mathbb{P}^1$, the bundle $\mathcal{O}(-1) \oplus \mathcal{O}(1)$ is unstable because the whole thing has slope 0 but the subbundle $\mathcal{O}(1)$ has slope 1.

Since we're assuming $\det \mathcal{V}$ is trivial, the semistability condition just means that subbundles have degree 0. So we now (more generally) let $\theta_L = \{E \in SU_C(r) : h^0(E \otimes L) > 0\}$ where $L \in J^{g-1}(C)$. We let $\mathcal{L} = \mathcal{O}(\theta_L)$.

Theorem. *1) $\mathcal{L}$ is independent of L (i.e. there's a canonical isomorphism).*

2) For generic $L \in J^{g-1}$, θ_L is a (usually nonsmooth...) divisor in $SU_C(r)$ (for some choices it will be the whole space).

3) $\mathcal{L}$ is the "determinant of cohomology."

4) $\operatorname{Pic}(SU_C(r)) = \mathbb{Z}\mathcal{L}$.

5) $\dim SU_C(r) = (\dim \mathrm{SL}_r) * (g-1)$.

We will not prove this.

QUESTION. What is $h^0(SU_C(r), \mathcal{L}^{\otimes k})$? We've seen the answer above for Jacobians - was just k^g. It was conjectured by the Verlinde brothers (by physicists) (and later proved) to be

$$\left(\frac{r}{r+k}\right)^g * \sum_{\mu \in Y_{r,k}} \left(\frac{\sin < \mu + \rho, \alpha >}{(k+r)}\right)^{2-2g}$$

where $Y_{r,k}$ are the Young diagrams with at most r rows and at most k columns. Note this is not even clearly an integer! In RCFT, [TUY], using Kac-Moody theory gave spaces "Conformal Blocks" $\nu_\Lambda^+(\mathfrak{sl}_r, C, k)$ where $\Lambda = (\lambda_1, \ldots, \lambda_n)$ and $\lambda_i \in Y_{r,k}$.

Proposition. *1) $\nu_\Lambda^+(\mathfrak{g}, k) \to \bar{M}_{g,n}$ is a VB (with fiber over C equal to $\nu_\Lambda^+(\mathfrak{g}, C, k)$)*

2) "Factorization." This relates conformal blocks of a curve and its normalization.

3) Have a flat projective connection on $\nu_\Lambda^+(\mathfrak{g}, k) \to M_{0,n}$ (due to KZ, WZW, Hitchin).

Since the dimensions are equal (the Verlinde formula) it's reasonable to ask if the spaces are actually isomorphic. The theorem (due to a number of people) is the following. $H^0(SU_C(r), \mathcal{L}^{\otimes k})$ is canonically isomorphic (up to constants) to $\nu_0^+(\mathfrak{sl}_r, C, k)$.

THE MORAL HERE IS THAT STUDYING INFINITE-DIMENSIONAL LIE ALGERBAS CAN TELL YOU ABOUT THE GEOMTRY OF FINITE-DIMENSIONAL THINGS.

Let $U_C(k)$ be the set of semistable VBs on a smooth curve of rank k and degree $k(g-1)$. We then define a theta divisor $\theta_k = \{E \in U_C(k) : h^0(E) > 0\}$ - note there's no cohice of L here. We define a map

$$(E, F) \mapsto (E \otimes F) : SU_C(r) \times U_C(k) \xrightarrow{\tau} U_C(kr)$$

noting that $\deg(E \otimes F) = \operatorname{rank}(E)\deg(F) = r * k * (g-1)$ as required. We let $M_k = \mathcal{O}(\theta_k)$ be the line bundle associated to this theta divisor.

Theorem. *$\tau^* M_{kr} \cong \mathcal{L}^{\otimes k} \boxtimes M_k^{\otimes r}$ as line bundles on $SU_C(r) \times U_C(k)$. Also*

$$\tau^* S_{\theta_{kr}} \in H^0(SU_C(r), \mathcal{L}^{\otimes k}) \bigotimes H^0(U_C(k), M_k^{\otimes r})$$

where S is the canonical divisor. Then we get the "Strange Duality" map

$$SD : H^0(SU_C(r), \mathcal{L}^{\otimes k})^* \xrightarrow{\tau^* S_{\theta_{kr}}} H^0(U_C(k), M_k^{\otimes r}).$$

This is an iso (proved by Belkale geometrically; Swarnava investigated it via representation theory).

Definition. Let $g = 0$, and $\lambda_i \in Y_{r,k}$ for $1 \leq i \leq n$. Choose n distinct complex numbers z_i, denote the vector of them by z. We define an operator on $V_\Lambda := \bigotimes V_{\lambda_i}$ by

$$T_z(v_1 \otimes \cdots \otimes v_n) = \sum_{i=1}^{n} z_i v_1 \otimes \cdots \otimes x_\theta v_i \otimes \cdots \otimes v_n$$

where $0 \neq x_\theta \in \mathfrak{g}_\theta$ is arbitrary (up to scalars) and θ is the longest root. THEN set

$$\nu_\Lambda(\mathfrak{g}, k, z) := \frac{V_\Lambda}{\mathfrak{g}V_\Lambda + \operatorname{im} T_z^{k+1}}$$

this is a quotient of the coinvariants $V/\mathfrak{g}V$. Note we also have a surjective map from the $k+1$ guys to the k guys.

We have surjective (because it is so on fibers) map of VBs over $X := \bar{M}_{0,n}$ by $\mathcal{O}_X \otimes V_\Lambda/\mathfrak{g}V_\Lambda \to V_\Lambda(\mathfrak{g}, k)$. Here the coinvariants are viewed as a trivial bundle with coinvariants as fibers. The bundle $V_\Lambda(\mathfrak{g}, k)$ (really its determinant) is basepoint-free so we get a map of $\bar{M}_{0,n}$ to the Grassmannian $Grass^{quot}(V_\Lambda/\mathfrak{g}V_\Lambda)$ by $C \mapsto (V_\Lambda/\mathfrak{g}V_\Lambda \mapsto V_\Lambda|C)$. Note that the z_i here are the marked points.

Let $Nef(\bar{M}_{0,n})$ be the (cone of) divisors D such that $D \cdot C \geq 0$ for any 1-dimensional stratum C (i.e. curves, but not necessarily smooth), modulo some equivalence (called "numerically eventually effective"). $Nef(X)$ is actually the closure of the cone of ample divisors $Ample(X)$. Also basepoint-free divisors are Nef (but not conversely).

Conjecture - $Nef(\bar{M_{0,n}})$ is polyhedral - means given by a finite set of linear inequalities. This is false in general, e.g. for Abelian surfaces with Picard rank at most 3. One reason to study conformal blocks is to understand this cone. And to study conformal blocks we need infinite dimensional Lie theory. With this motivation in mind (though we will never speak of it again) we now turn our attention to toward this subject.

Lecture 12: 10/16/14

Let F be a field and V a vector space over F (not necessarily finite-dimensional). Recall the definition of the tensor algebra: $T^0V = F$, $T^1V = V, \ldots, T^nV = V^{\otimes n}$. Then we set $T(V) = \bigoplus T^i(V)$, and this is an associative algebra. $T(V)$ has the following universal property. If X is a set and V_X is vector space generated by X (i.e. with basis set indexed by X), then if $X \to A$ is any vector space map, where A is any associative algebra, then there's a unique map $T(V_X) \to A$ of associative algebras making the triangle commute.

Given any associative algebra A, it is naturally a Lie algebra, denoted $\mathcal{L}(A)$ under $[x, y] = xy - yx$. In the other direction, we can associate to a Lie algebra its universal enveloping algebra $\mathcal{U}(L)$, which is just the quotient of $T(L)$ by the ideal generated by $[x, y] - (x \otimes y - y \otimes x)$. If we think of L as left-invariant vector fields, then $\mathcal{U}(L)$ is the invariant differential operators. From this perspective it's clear that we should have an injection $L \hookrightarrow \mathcal{U}(L)$, though proving this is nontrivial and uses the PBW theorem.
Theorem. *If $L \to \mathcal{L}(A)$ is a Lie algebra map, then there is a unique map $\mathcal{U}(L) \to \mathcal{L}(A)$ of associative algebras making the triangle commute.*
Corollary 2. *Suppose $\mathcal{W}$ is an L-module. Then we get a unique map $\mathcal{U}(L) \to \mathrm{End}(\mathcal{W})$.*

Consider the ideal $\tilde{I} \subset T(L)$ generated by tensors of the form $\{a \otimes b - b \otimes a\}$. The corresponding quotient is the symmetric tensors, $Sym(L) = T(L)/\tilde{I}$. Note $\tilde{I}$ is graded/homogeneous. That is, $\tilde{I} = \bigoplus(\tilde{I} \bigcap T^i(L))$. There's a filtration on $\mathcal{U}(L)$, viewed as differential operators, defined by $\mathcal{U}_m(L)$ being those operators of degree $\leq m$. Then $\mathcal{U}_m/\mathcal{U}_{m-1}$ is isomorphic to degree m homogeneous polynomials. There's an obvious filtration $T_m = T^0 \oplus \cdots \oplus T^m$, and we define $\mathcal{U}_m = \pi(T_m)$ (this coincides with the one in the previous paragraph). Multiplication induces $\mathcal{U}_m \times \mathcal{U}_n \to \mathcal{U}_{m+n}$. We define $G^m = \mathcal{U}_m/\mathcal{U}_{m-1}$. Then we have $G^m \times G^n \to G^{m+n}$ for the same reason. And finally, we let $\mathcal{G} := \bigoplus G^m$ be the associated graded algebra. We have $\phi_m : T^m \hookrightarrow T_m \xrightarrow{\pi} \mathcal{U}_m \to G^m$. Then ϕ_m is onto, because $\pi(T_m \backslash T_{m-1}) = \mathcal{U}_m \backslash \mathcal{U}_{m-1}$. Taking the direct sum of all these gives a map $T(L) \to \bigoplus G^m$. This is trivial on $\tilde{I}$ and we get a map $\phi : Sym(L) \to \mathcal{G}$. The PBW (Poincare-Birkoff-Witt) theorem says that ϕ is an iso of algebras.
Corollary 3. *1) If $W \subset T^m(L) \to Sym^m(L)$. Suppose $W \cong Sym^m(L)$. Then $\pi(W)$ is a complement to $\mathcal{U}_{m-1}$ in $\mathcal{U}_m$.*

2) $L = T^1(L) = Sym^1(L)$ and $L \to \mathcal{U}(L)$ is 1-1.

3) Suppose $H \subset L$ is a Lie subalgebra. Then we can extend this to a map $\mathcal{U}(H) \to \mathcal{U}(L)$ and $\mathcal{U}(L)$ is a free $\mathcal{U}(H)$-module. Suppose that $\{h_s\}$ is an ordered basis of H and that $\{h_s, \ell_t\}$ is an ordered *basis of L extended from this. Then a basis for $\mathcal{U}(L)$ over $\mathcal{U}(H)$ can be expressed as monomials in these.*

Exercise. 1) $\mathcal{U}(L) = Sym(L)$ if L is abelian. 2) $\dim L < \infty$ implies $\mathcal{U}(L)$ is a domain. 3) Let $\mathfrak{g}$ be the only nonabelian 2-dimensional Lie algebra. Then show directly that it embeds in $\mathcal{U}(\mathfrak{g})$.

Free Lie Algebras

Let X be a set, and have a map $X \to L$ where L is a Lie algebra. Then there is a universal object $L(X)$ (coming with an inclusion $X \hookrightarrow L(X)$) such that there exists a unique map $L(X) \to L$ making the triangle commute. To see that this object exists, let A_X be the free algebra on X; note this is not associative. Then let $I = \{aa, (ab)c + (bc)a + (ca)b\}$ for $a, b, c \in A_X$. We define $L(X) = A_X/I$.
Proposition. *$\mathcal{U}(L(X)) \cong T(V_X)$ as associative algebras.*

The proof is by universality. That is, by taking $X \to L(X) \to \mathcal{U}(L(X))$ and $X \to T(V_X)$ we get the two maps needed for the proposition.

We can also define $L(X)$ as the Lie subalgera of $T(V_X)$ generated by X.

Lecture 13: 10/21/14

Today we recall the structure theory / classification of complex simple Lie algebras. Let Φ be a root system of $(\mathfrak{g}, \mathfrak{h})$ where $\mathfrak{g}$ is a fin-dim simple $\mathbb{C}$-Lie algebra. Let $S = \{\alpha_1, \ldots, \alpha_\ell\} \subset \Phi$ be the simple positive roots. We have $\Phi^* \subset \mathfrak{h}$ and $S^* = \{\alpha_1^\vee, \ldots, \alpha_\ell^\vee\}$. We have $(\alpha, \beta) = |\alpha||\beta|\cos(\phi)$. We define $n(\alpha, \beta) = \alpha(\beta^\vee) = (\alpha, 2\beta/(\beta, \beta)) = 2(\alpha, \beta)/(\beta, \beta)$. This is an integer by the definition of a root system. We see that $n(\alpha, \beta)n(\beta, \alpha) = 4\cos^2(\phi)$ and this has to be an integer, so this integer can only be 0, 1, 2, 3, or 4.
Proposition. *$(\alpha, \beta) \leq 0$ if $\alpha, \beta \in S$ and $\alpha \neq \beta$. In particular $n(\alpha, \beta) \leq 0$.*
Corollary 4. *$n(\alpha, \beta) \in \{0, -1, -2, -3\}$ for $\alpha, \beta \in S$.*

We define the Cartan matrix $(n(\alpha, \beta))_{\alpha, \beta \in S}$.

Ex. the Cartan matrix for G_2 has rows $(2, -1)$ and $(-3, 2)$, and for A_3 (i.e. $\mathfrak{sl}_4$) has rows $(2, -1, 0; -1, 2, -1; 0, -1, 2)$.

Recall how to use the Cartan matrix to construct the Dynkin diagram.

Write $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$.
Theorem. *We can choose generators $e_i \in \mathfrak{g}_{\alpha_i}$, $f_i \in \mathfrak{g}_{-\alpha_i}$ such that $[e_i, f_i] = \alpha_i^\vee$, and $[\alpha_i^\vee, e_i] = 2e_i$ and $[\alpha_i^\vee, f_i] = -2f_i$, where the α_i are the simple positive roots.*

Let $n(ij) = n(\alpha_i, \alpha_j)$. Then we have $\mathrm{ad}(e_i)^{-n(ji)+1} e_j = 0$ and $\mathrm{ad}(f_i)^{-n(ji)+1} f_j = 0$.

Choose a set of symbols $\alpha_1^\vee, \ldots, \alpha_\ell^\vee$ and $e_1, \ldots, e_\ell$ and $f_1, \ldots, f_\ell$. Let F be the free Lie algebra on these symbols. We define $\overline{\mathfrak{g}}$ to be $F/\sim$ where $\sim$ corresponds to the above relations (the 3 "obvious" (Weyl) ones and the 2 Serre relations). Also assume the $\alpha_i^\vee$ commute with each other.
Theorem. *$\overline{\mathfrak{g}} \cong \mathfrak{g}$ as Lie algebras.*

Verma Modules

Let $\lambda \in \mathfrak{h}^*$ and $\mathbb{C}_\lambda$ the 1-dimensional $\mathfrak{h}$-module where $\mathfrak{h}$ acts by λ. Write $\mathfrak{g} = \bigoplus_{\alpha\in\Phi^-} \mathfrak{g}_\alpha \oplus \mathfrak{h} \oplus \bigoplus_{\alpha\in\Phi^+} \mathfrak{g}_\alpha$. If $X \in \mathfrak{g}_\alpha$ for $\alpha \in \Phi^+$ then define an extension to the Borel subalgebra $\mathfrak{b} \subset \mathfrak{g}$ by $Xv = 0$. Now we have $\mathcal{U}(\mathfrak{b}) \to \mathcal{U}(\mathfrak{g})$ making $\mathcal{U}(\mathfrak{g})$ into a free $\mathcal{U}(\mathfrak{b})$-module, so we simply define

$$M_\lambda := \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{b})} \mathbb{C}_\lambda = \mathrm{Ind}_{\mathfrak{b}}^{\mathfrak{g}} \mathbb{C}_\lambda$$

so $M_\lambda \cong \mathcal{U}(\mathfrak{n}^-) \otimes_{\mathbb{C}} \mathbb{C}_\lambda$, because we can write $\mathcal{U}(\mathfrak{g}) = \mathcal{U}(\mathfrak{n}^-) \otimes_{\mathbb{C}} \mathcal{U}(\mathfrak{b})$. This module is always infinite-dimensional.

Now require that λ be a dominant integral weight, i.e., an element of $P_+ := \{\lambda \in \mathfrak{h}^* : \lambda(\alpha_i^\vee) \in \mathbb{Z}_{\geq 0}\ \forall i\}$. Then there exists a (unique) maximal proper $\mathfrak{g}$-submodule J_λ such that $V_\lambda = M_\lambda/J_\lambda$ is finite-dimensional irreducible, and all finite-dimensional $\mathfrak{g}$-modules so arise.

Aside - can realize J_λ as kernel of a bilinear form on M_λ, called the Shapovalov form.

Aside - let $\tilde{g}$ be $F/\sim$ where $\sim$ is just the Weyl relations (no Serre relations). Then the result is infinite-dimensional, but we can construct the Verma module $\tilde{M}_\lambda$ just the same, though now $\mathcal{U}(\mathfrak{n}^-)$ is now a free Lie algebra. We have the obvious surjective map $\tilde{\mathfrak{g}} \to \overline{\mathfrak{g}}$ and this induces a map $\tilde{M}_\lambda \to M_\lambda$.

Take $\lambda_1, \ldots, \lambda_n \in P_+(\mathfrak{g})$. Now write $\sum \lambda_i = \sum n_\alpha \alpha$, summing over the simple positive α and $n_\alpha \in \mathbb{Z}_{\geq 0}$. Now if $(V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_n})^{\mathfrak{g}} \neq 0$ then $(\bigotimes V_{\lambda_i})_0 \neq 0$ so $(\lambda_1 - \sum n_\alpha \alpha) + \ldots + \lambda_n - \sum n_\alpha \alpha$ implies $\sum \lambda_i \in Q$ (the root lattice). Let $M = \sum n_\alpha$. Let $z_1, \ldots, z_n \in \mathbb{C}$ disinct. Then define

$$X_M = \{(t_1, \ldots, t_M) \in \mathbb{C}^M : t_i \neq t_j\ ,\ t_i \neq z_i\}.$$

This is called a "hyperplane arrangement." These are quite interesting and have a whole theory. Let $f_{ij} = t_i - t_j$ and $a_{ij} = t_i - z_j$. Consider the (differential graded) algebra A_{X_M} generated by df_{ij}/f_{ij} and da_{ij}/a_{ij} (i.e. log forms). These are holomorphic forms on the hyperplane arragement. On the boundary these have simple poles. Check that d of a log form is zero. The algebra multiplication here is $\wedge$; the grading is just the number of d's in the expression. Now consider the top degree (i.e. degree M) forms $\log \Omega_{X_M}^M$ in A_{X_M}. Since log forms have d equal to zero, we get a map $\log \Omega_{X_M}^M \to H^M(X, \mathbb{C})$.

The following is badass.
Theorem (Schectman, Varchenko). *$(\log \Omega_{X_M}^M)^\Sigma \cong (\tilde{M}_{\lambda_1} \otimes \cdots \otimes \tilde{M}_{\lambda_n})_0^*$ where Σ is a subgroup of S_M, namely $S_{n_{\alpha_1}} \times \cdots \times S_{n_{\alpha_\ell}}$.*

End Aside!

Let $A = (a_{ij})$ be an $\ell \times \ell$ matrix. We say A is a generalized Cartan matrix (GCM) if $a_{ii} = 2$ and $a_{ij} \leq 0$ if $i \neq j$ and $a_{ij} = 0$ iff $a_{ji} = 0$.
Definition 5. *A realization of A is a triple $(\mathfrak{h}, \Pi, \Pi^\vee)$ such that $\mathfrak{h}$ is a vector space of dimension $\ell + (\ell - \operatorname{rank} A)$, and $\Pi = \{\alpha_1, \dots, \alpha_\ell\} \subset \mathfrak{h}^*$, and $\Pi^\vee$ is similar. These are subject to the conditions:*

i) α_i are linearly independent.

ii) $\alpha_i^\vee$ linearly independent.

iii) $< \alpha_i^\vee, \alpha_j >= a_{ij}$.

Note that in the finite-dimensional case the rank equals ℓ and $\mathfrak{h}$ always has dimension ℓ. The first obvious questions are of existence and uniqueness of realizations. We say 2 realizations are isomorphic if there exists an iso $\phi : \mathfrak{h} \to \mathfrak{h}'$ such that $\phi : \Pi^\vee \to \Pi^{\vee\prime}$ and $\phi^* : \Pi' \to \Pi$.
Theorem. *Realizations exist and are unique up to iso.*

The idea is to pick an arbitrary vector space V and choose elements of V and V^* satisfying $i - iii$; this is easily done. Then it is shown that $\dim V \geq \ell + (\ell - \operatorname{rank} A)$. Then let $\mathfrak{h}$ be the (span of the) $\alpha_i^\vee$ together with the inverse image of the cokernel of A. (?)

We let $\mathfrak{g}(A)$ be the quotient of the free Lie algebra generated by $\mathfrak{h}$, and e_i, f_i for $1 \leq i \leq \ell$ modulo the relations:

1) $[\mathfrak{h}, \mathfrak{h}] = 0$,

2) $[h, e_i] = \alpha_i(h) e_i$ and $[h, f_i] = -\alpha_i(h) f_i$,

3) $[e_i, f_j] = \delta_{ij} \alpha_i^\vee$,

4) $(\operatorname{ad} e_i)^{1-a_{ij}}(e_j) = 0$ and

5) $(\operatorname{ad} f_i)^{1-a_{ij}}(f_j) = 0$.

Then $\mathfrak{g}(A)$ is called the Kac-Moody algebra associated to A.

Lecture 14: 10/23/14

Last time we defined a GCM, so let $(a_{ij}) = A$ a GCM. Recall this means $a_{ii} = 2$ and $a_{ij} = 0$ iff $a_{ji} = 0$ and $a_{ij} \leq 0$ for $i \neq j$. We said a realization of A was a vector space $\mathfrak{h}$ together with ℓ linearly indepdendent vectors in $\mathfrak{h}$ and in $\mathfrak{h}^*$, with some properties. We ended by defining the Kac-Moody algebra $\mathfrak{g}(A)$. This was invented in the 1960s; has found uses is physics etc. $\mathfrak{g}(A)$ is just a free Lie algebra on 2ℓ generators, modulo the 3 Weyl and 2 Serre relations.

Let $\mathfrak{n} = \mathfrak{n}^+$ be the Lie subalgebra of $\mathfrak{g}(A)$ generated by the e_i and $\mathfrak{n}^-$ the Lie subalgebra generated by the f_i. We define the standard Borel $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}$. We also write $\tilde{\mathfrak{g}}(A)$ to be the quotient of the same free Lie algebra by just the Weyl relations (omit the Serre relations). So we have the obvious quotient $\tilde{\mathfrak{g}}(A) \to \mathfrak{g}(A)$.

Consider the Serre relation $(\operatorname{ad} e_i)^{(1-a_{ij})}(e_j) = 0$ for $i \neq j$. If $A = (2)$, which is the Cartan matrix for $\mathfrak{sl}_2$, then we're looking at $\{e, f, \alpha^\vee\}$ with the relations $[e, f] = \alpha^\vee$ and $[\alpha^\vee, e] = 2e$ and $[\alpha^\vee, f] = -2f$ and actually the Serre relations are automatic so $\tilde{\mathfrak{g}}(A) = \mathfrak{g}(A)$ in this case.

For $1 \leq i \leq \ell$, we let $\mathfrak{g}(i)$ be the Lie subalgebra of $\mathfrak{g}(A)$ generated by $\{e_i, f_i, \alpha_i^\vee\}$. We do the same thing for everything with tildes. The previous paragraph though shows that $\mathfrak{g}(i) \cong \tilde{\mathfrak{g}}(i) \cong \mathfrak{sl}_2$.

Define a map

$$\omega : F(A) \to F(A)$$

by $e_i \mapsto -f_i$ and $f_i \mapsto -e_i$ and $h \mapsto -h$ for any $h \in \mathfrak{h}$. One can check that this is a Lie algebra map and that ω preserves all 5 relations, hence descends to $\mathfrak{g}(A) \to \mathfrak{g}(A)$. This ω is called the Cartan Involution (note it obviously squares to 1). It is easy to see that ω preserves $\mathfrak{h}$ and switches $\mathfrak{n}^\pm$.

Take two disjoint subsets $Y_1 \cup Y_2 = \{1, \ldots, \ell\}$, such that $a_{ij} = 0$ if $i \in Y_1$ and $j \in Y_2$. That is, up to permutation we can write A as a block diagonal matrix. Then $\mathfrak{g}(A) \cong \mathfrak{g}(A_1) \oplus \mathfrak{g}(A_2)$ as Lie algebras.

We define the root lattice $Q = \bigoplus_{i=1}^{\ell} \mathbb{Z}\alpha_i \subset \mathfrak{h}^*$ and Q_+ to be the same with $\mathbb{Z}_{\geq 0}$. Note the root lattice may not be full rank. We similarly define, for each $\alpha \in \mathfrak{h}^*$ the spaces

$$\mathfrak{g}_\alpha := \{x \in \mathfrak{g}(A) : [h, x] = \alpha(h)x \ \forall h \in \mathfrak{h}\}.$$

These can now be more than 1-dimensional (a priori they could even be infinite dimensional).
Theorem. *(Kac-Serre). Assume A is indecomposable.*

a) $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}$.

b) $\mathfrak{n}^\pm = \bigoplus_{\alpha \in Q_+ \backslash \{0\}} \mathfrak{g}_{\pm\alpha}$.

c) $\dim \mathfrak{g}_\alpha < \infty$

d) $\mathfrak{n}^+$ is generated by the e_i with the relation R4 (the Serre relation) (i.e. it's a free Lie algebra mod this one relation). Similarly for $\mathfrak{n}^-$. (this part is used to prove a).

We omit the proof of this.

Let Δ be the $0 \neq \alpha \in Q$ such that $\mathfrak{g}_\alpha \neq 0$. I.e. the set of roots. Similarly $\Delta^+ = \Delta \cap Q_+$ and $\Delta^- = -\Delta^+$ and then Δ is the disjoint union of $\Delta^\pm$. We say the α_i, $1 \leq i \leq \ell$, are the simple positive roots. Now for $\alpha \in \Delta \cup \{0\}$ we can write $\alpha = \sum n_{\alpha_i}\alpha_i$ with $n_{\alpha_i} \in \mathbb{Z}_{\geq 0}$. We define the height of α to be $|\alpha| = \sum n_{\alpha_i}$. We define the multiplicity of α to be $mult\alpha = \dim \mathfrak{g}_\alpha$.

Exercise. (hint - use part d of theorem). If α_i is simple then 1) $mult\alpha_i = 1$ and 2) $n\alpha_i \in \Delta$ iff $n = \pm 1$.

Let $Y \subset \{1, \ldots, \ell\}$, possibly empty. Then let $\Delta_Y := \Delta \cap (\bigoplus_{i \in Y} \mathbb{Z}\alpha_i)$ and similarly $\Delta_Y^{\pm}$. We then define

$$\mathfrak{g}_Y = \mathfrak{h} \oplus (\bigoplus_{\alpha \in \Delta_Y} \mathfrak{g}_\alpha)$$

and

$$U_Y = \bigoplus_{\alpha \in \Delta \setminus \Delta_Y^+} \mathfrak{g}_\alpha$$

and similarly U_Y^-. We also define standard parabolics

$$P_Y = \mathfrak{g}_Y \oplus U_Y$$

and the opposite parabolic

$$P_Y^- = \mathfrak{g}_Y \oplus U_Y^-$$

and these two are permuted by the Cartan involution. We call $\mathfrak{g}_Y$ the Levi subalgebras and U_Y the nilradical. If $Y = \{i\}$ then the P_Y is called a minimal parabolic.

Last definition for today. We define $s_i \in \mathrm{Aut}(\mathfrak{h}^*)$ by $s_i(\chi) = \chi - \chi(\alpha_i^\vee)\alpha_i$. We have $s_i^2 = 1$ and we let the Weyl group of $\mathfrak{g}(A)$ be the group generated by the s_i for $1 \le i \le \ell$. We'll see later this is still a Coxeter group, though it may be infinite.

EXAMPLE

Let $R = \mathbb{C}[t, t^{-1}]$. Then let $\mathfrak{g}$ be a finite-dimensional Lie algebra. Let $L(\mathfrak{g}) := \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$. We give this the obvious bracket: $[x \otimes f, y \otimes g] = [x, y] \otimes fg$. We now extend further and let $\bar{L}(\mathfrak{g}) := (\mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]) \oplus \mathbb{C}(t\frac{d}{dt})$. We let $d = td/dt$ and define $[x \otimes t^m, d] = mx \otimes t^m$.

We next define $\hat{L}(\mathfrak{g})$ as (what happens to be the unique nontrivial one, called the universal central extension) the nontrivial central extension

$$0 \to \mathbb{C}c \to \hat{L}(\mathfrak{g}) \to \bar{L}(\mathfrak{g}) \to 0.$$

Write (as vector spaces) $\hat{L} = \bar{L} \oplus \mathbb{C}c$ and define the bracket by 1) $[\bar{L}, c] = 0$ and 2) $[x \otimes f, y \otimes g] = [x, y] \otimes fg + ((x, y)\,\mathrm{res}_{t=0}\, gdf/dt)c$ where (x, y) is the Cartan-Killing form. Then $\hat{L}(\mathfrak{g})$ is called the affine Kac-Moody algebra (Swarnava also calls it the untwisted affine KM algebra). $L(\mathfrak{g})$ is called the Loop algebra (it is the Lie algebra of a loop group...?). Summing with $\mathbb{C}c$ gives the affine Lie algebra.
Theorem 6. *$\hat{L}$ IS a Kac-Moody algebra, that is, it is isomorphic to $\mathfrak{g}(A)$ for some GCM A.*

Idea of proof. We know that the $\mathfrak{g}$ we started with is a Kac-Moody algebra associated to some matrix A', of size $\ell \times \ell$. The A we want will be $\ell + 1 \times \ell + 1$ with A as the lower-right chunk. Well $a_{00} = 2$ is forced. For the rest, set $a_{0j} = -\alpha_j(\theta^\vee)$ where $\theta^\vee$ is the coroot corresponding to θ, the highest root of $\mathfrak{g}$ (N.B. we don't have a notion of highest root for general KM algebras). And lastly set $a_{j0} = -\theta(\alpha_j^\vee)$.

We define $\Pi^\vee$ to be $\{c-\theta^\vee, \alpha_1^\vee, \ldots, \alpha_\ell^\vee\}$ and $\mathfrak{h}(A) = \mathfrak{h} \oplus \mathbb{C}c \oplus \mathbb{C}d$. We define $\delta \in \mathfrak{h}(A)^*$ by having it be zero on the first 2 summands and $\delta(d) = 1$. That is, the dual element of d. With this we let $\Pi = \{\alpha_0 := \delta - \theta, \alpha_1, \ldots, \alpha_\ell\}$.

It is then not hard to show that A has corank 1.

We can write

$$\hat{L} = \mathfrak{h}(A) \bigoplus_{0\neq j\in\mathbb{Z}, k\in\mathbb{Z}} t^j \otimes \mathfrak{h} \bigoplus_{\beta\in\Phi} t^k \otimes \mathfrak{g}_\beta$$

where Φ is the root system for the original $\mathfrak{g}$. The roots Δ here are $j\delta$ for $j \neq 0$, and $k\delta + \beta$ for $k \in \mathbb{Z}$ and $\beta \in \Phi$ (the above is its root space decomposition). One can show $mult(j\delta) > 1$ for all $j \neq 0$ and $mult(j\delta + \beta) = 1$. We call roots with multiplicity >1 imaginary, and with multiplicity $=1$ real. This phenomena does not happen in the finite-dimensional case.

Last thing: $\Delta^+ = \{j\delta : j > 0\} \cup \{k\delta + \beta : k > 0, \beta \in \Phi\} \cup \{\beta \in \Phi^+\}$.

Lecture 15: 10/28/14

Recall last time we defined, for a GCM A, realizations $(\mathfrak{h}, \Pi, \Pi^\vee)$, the Kac-Moody algebra $\mathfrak{g}(A)$, and $\tilde{\mathfrak{g}}(A)$ the Kac-Moody algebra without Serre relations.
Definition 7. *Let A be a GCM. We say A is symmetrizable if there exists a diagonal matrix $D = \mathrm{diag}(\varepsilon_1, \ldots, \varepsilon_\ell)$ with $\varepsilon_i \in \mathbb{Q}$ such that $D^{-1}A$ is symmetric.*

Writing out what this means in entries of the matrices gives $\varepsilon_i^{-1} a_{ij} = \varepsilon_j^{-1} a_{ji}$ or $\frac{a_{ij}}{a_{ji}} = \frac{\varepsilon_i}{\varepsilon_j} > 0$ so we can choose all $\varepsilon_i > 0$. Note we can multiply such a D by a scalar and it will still be such a matrix. If given two such matrices D^1, D^2, with entries in $\mathbb{Z}$, we write $D^1 \leq D^2$ if $\varepsilon_i^1 \leq \varepsilon_i^2$ for all i. This is a partial ordering and it has minimal elements.

For any symmetrizable matrix A, there exists a minimal diagonal matrix D with integer entries such that $D^{-1}A$ is symmetric. By this we mean that $\varepsilon_i \in \mathbb{Z}_{>0}$ for all i, and D is minimal wrt the ordering. We will care about symmetrizable KMAs because only for them will we have a W-invariant symmetric bilinear form (akin to the Killing form).
Proposition. *Let W be the Weyl group of $\mathfrak{g}(A)$, and assume A is symmetrizable. Then $\mathfrak{h}$ carries a W-invariant form, which is symmetric nondegenerate. This form extends to the whole Lie algebra.*

Proof. Let $\mathfrak{h}' = \bigoplus \mathbb{C}\alpha_i^\vee$. Choose a vector space complement $\mathfrak{h} = \mathfrak{h}' \oplus \mathfrak{h}''$. Now choose D minimal so that $D^{-1}A$ is symmetric. Now set $\langle \mathfrak{h}'', \mathfrak{h}'' \rangle = 0$ and $\langle h, \alpha_i^\vee \rangle = \langle \alpha_i^\vee, h \rangle := \alpha_i(h)\varepsilon_i$. We remark that this is well-defined. To see that this is W-invariant, compute $\langle s_j h_1, s_j h_2 \rangle = \langle h_1 - \alpha_j(h_1)\alpha_j^\vee, h_2 - \alpha_j(h_2)\alpha_j^\vee \rangle = \langle h_1, h_2 \rangle - \alpha_j(h_2)\alpha_j(h_1)\varepsilon_j - \alpha_j(h_1)\alpha_j(h_2)\varepsilon_j + 2\alpha_j(h_1)\alpha_j(h_2)\varepsilon_j = \langle h_1, h_2 \rangle$ and done. Note we're using here that $\alpha_j(\alpha_j^\vee) = 2$. $\square$

N.B. that $\mathfrak{h}''$ is not unique, and our form depended on its choice.

Exercise 1. A is symmetrizable iff for all k and all subsets $\{i_1, \ldots, i_k\} \subset \{1, \ldots, \ell\}$, we have equality of products $a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_k i_1} = a_{i_2 i_1} a_{i_3 i_2} \cdots a_{i_1 i_k}$

Exercise 2. A a GCM and $\mathfrak{h}'$ as in the proposition. Let $(\ , \ ) : \mathfrak{h}' \times \mathfrak{h}' \to \mathbb{C}$ be a W-invariant bilinear form (maybe not symmetric or nondegenerate, but NOT identically zero).

i) Then A is symmetrizable. Hint: let $\varepsilon_i = (\alpha_i^\vee, \alpha_i^\vee)/2$ and compare $(s_j \alpha_i^\vee, s_j \alpha_i^\vee) = (\alpha_i^\vee, \alpha_i^\vee)$ to get $a_{ij}\varepsilon_j = a_{ji}\varepsilon_i$.

ii) The form $\langle \ , \ \rangle$ from the proposition is equal, up to a nonzero multiple, of $(\ , \ )$. In particular, $(\ , \ )$ is automatically nondegenerate, and the restriction of the proposition's form to $\mathfrak{h}'$ is essentially unique.

Theorem 8. *Let $\mathfrak{g}(A)$ be a symmetrizable KM algebra. Then there exists a form $<,>$ on $\mathfrak{g}(A)$ such that*

1) $< [X, Y], Z > + < Y, [X, Z] >= 0$

2) $<,>$ restricted to $\mathfrak{h}$ is the same as the one defined in the previous proposition.

Proposition. *We have*

1) $< \mathfrak{g}_\alpha, \mathfrak{g}_\beta >= 0$ unless $\alpha + \beta = 0$ for $\alpha, \beta \in \Delta \cup \{0\}$.

2) $[X, Y] =< X, Y > v^{-1}(\alpha)$ where $X \in \mathfrak{g}_\alpha$ and $Y \in \mathfrak{g}_{-\alpha}$.

In 2) we're using $v : \mathfrak{h} \cong \mathfrak{h}^*$ defined by $v(h)(h_1) :=< h, h_1 >$

Proof. Now we define, for $k \in \mathbb{Z}$,

$$\mathfrak{g}_k = \bigoplus \mathfrak{g}_\alpha$$

where we sum over all $\alpha \in \Delta \cup \{0\}$ with $|\alpha| = k$. We also set

$$\mathfrak{g}(N) = \bigoplus_{k=-N}^{N} \mathfrak{g}_k$$

for $N \geq 0$. For example $g_0 = \mathfrak{h}$ and $\mathfrak{g}(0) = \mathfrak{g}_0$. We stated last time that $\mathfrak{g}_\alpha$ is a line if α is simple, and so we can define $< f_j, e_i >=< e_i, f_j >= \delta_{ij}\varepsilon_i$ on $\mathfrak{g}(1)$, and $< \mathfrak{g}_0, \mathfrak{g}_{\pm 1} >= 0$. We note that this pairing restricts to $\mathfrak{g}_0$ to give the previous pairing, as we desire.

So now we try to extend this inductively. We claim that for $N \geq 1$, there exists a form $<,>$ on $\mathfrak{g}(N)$ such that $< \mathfrak{g}_{k_1}, \mathfrak{g}_{k_2} >= 0$ unless $k_1 + k_2 = 0$ and $|k_i| \leq N$, and satisfies condition 1) whenever $x, y, z, [x, y], [x, z]$ lie in $\mathfrak{g}(N)$. Inductively, suppose we've done it for $\mathfrak{g}(N-1)$. For $x \in \mathfrak{g}_{\pm N}$ and $y \in \mathfrak{g}_{\mp N}$, we write $y = \sum [u_i, v_i]$ where $u_i \in \mathfrak{g}(N-1)$ and $v_i \in \mathfrak{g}(N-1)$. Now note that $[x, u_i] \in \mathfrak{g}(N-1)$. Now define

$$< x, y >:= \sum < [x, u_i], y > .$$

We cite without (the horrible) proof the fact/theorem that this is well-defined. □

In the case of affine KM algebras we can give a more intrinsic and transparent description of $<,>$. So following last class's example, let $\hat{L}(\mathfrak{g}) := \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c \oplus \mathbb{C}(td/dt)$ and $d = td/dt$. We defined the $[,]$ structure last time. This is a symmetrizable Lie algebra. The way this works is, if D' was the diagonal matrix symmetrizing the Cartan matrix for $\mathfrak{g}$ (know exists since fin-diml), then $D = \mathrm{diag}(1, D')$ will symmetrize the GCM for $\hat{L}$. The $<,>$ on $\hat{L}$ can be described directly:

$$< x \otimes p, y \otimes q >=< x, y > \mathrm{res}(t^{-1}pq)$$

$$< \mathbb{C}c + \mathbb{C}d, \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] >= 0$$

$$< c, c >=< d, d >= 0$$

$$< c, d >= 1$$

and note that the restriction of $<,>$ to $\mathfrak{g}$ inside $\hat{L}(\mathfrak{g})$ is $\mathfrak{g}$'s $<,>$. Now $\mathfrak{h}(A)^* = \mathfrak{h}^* \perp (\mathbb{C}\delta \oplus \mathbb{C}w_0)$ where δ is the dual element of d and w_0 is defined by being zero on $\mathfrak{h} + \mathbb{C}d$ and $w_0(c) = 1$.

End example.

Let $\mathfrak{g}(A)$ be any KM algebra and S any ideal of $\mathfrak{g}(A)$. We can write $S = \bigoplus_{\alpha\in\Delta}(S \cap \mathfrak{g}_\alpha) \bigoplus (S \cap \mathfrak{h})$. Let S_1 be an ideal of $\mathfrak{g}(A)$ such that $S_1 \cap \mathfrak{h} = 0.$, and S_2 another such ideal. Then $(S_1 + S_2)] \cap \mathfrak{h} = 0$ still. So let R be the sum of all such ideals, so $R \cap \mathfrak{h} = 0$. In some books the KM algebra is defined to be $\mathfrak{g}(A)/R$. Everything we've done still goes through for this guy; for symmetrizable KM algebras $R = 0$ so no issue.

Next time we'll talk about the affine Weyl group.

Lecture 16: 10/30/14

Consider $\mathrm{SL}_2(\mathbb{C}[t, t^{-1}])$. So an element is a matrix $p_{ij} \in \mathbb{C}[t, t^{-1}]$ with determinant 1. This is an example of an "ind-group." Anyway, today we want to discuss affine Weyl groups. Consider the Weyl group element r_α for the finite guy (i.e. antidiagonal $1, 1$). Note the way we're writing it, it is not normalizer mod centralizer (since it's not in SL_2). Consider also the matrices $\mathrm{diag}(t^i, t^{-i})$. Let $T(i)$ be conjugation by this, viewed as an automorphism of $\mathfrak{sl}_2(\mathbb{C}[t, t^{-1}])$.

For example, $T(i) : (0, t^a; 0, 0) \mapsto (0, t^{a+2i}; 0, 0)$. Now let $S(i) = r_\alpha T(i)$ (multiplication of matrices). Then $S(i)$ also acts on the Lie algebra.

Definition 9. *The Affine Weyl Group* $Aff(\mathfrak{sl}_2) \subset \mathrm{Aut}(\mathfrak{sl}_2(\mathbb{C}[t, t^{-1}]))$ *is the group generated by the* r_α *and* $T(i)$.

In this case the AWG is isomorphic to $\mathbb{Z} \rtimes \mathbb{Z}/2$.

Single Shift Automorphisms

Let $\mu \in P^\vee(\mathfrak{g})$ (an element of the coroot lattice). We define an automorphism $\tilde{\mu}$ on

$$\hat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c$$

where $\mathfrak{g}$ corresponds to a finite dimensional semisimple Lie algebra, by setting

$$\tilde{\mu}(X_\alpha(n)) = X_\alpha(\alpha(\mu) + n)$$

$$\tilde{\mu}(h(n)) = h(n) + (< \mu, h > \cdot c)\delta_{0,n}$$

where $X_\alpha(n) = X_\alpha \otimes t^n$, with $X_\alpha \in \mathfrak{g}_\alpha$, and fixing c, and similarly for $h(n)$ for $\mathfrak{h}$. Part of the reason for this correction at $n = 0$ is coming from the fact that we're dealing with a nontrivial central extension. The image of the group map $\mu \mapsto \tilde{\mu} : P^\vee \to \mathrm{Aut}(\mathfrak{g})$ gives the (abelian) lattice part of the AWG.

Now recall $H_{ii} = e_{ii}$ (the matrix with 1 at ii and 0 elsewhere) and $X_1 := H_1 - \frac{1}{n}\sum(H_{ii})$. Let $\tau = \tau_{X_1} = t^{X_1}$. Conjugation by τ is well-defined (though choosing different branches of logarithm can change τ itself, since we're using $t^{1-1/n}$ and $t^{1/n}$). So we get a map c_τ is an automorphism of $\mathfrak{sl}_2(\mathbb{C}[t, t^{-1}])$ (assuming $n = 2$), and if we choose $\mu = X_1$ then we're basically getting a single shift automorphism. The SSA's are a somewhat nonstandard way of getting at AWGs, but it's a useful perspective for physics.

Note that the X_1 above is in $P^\vee$ but not $Q^\vee$ and so does not contradict the next theorem (noting that τ isn't in the group since its t's has fractional exponents).
Theorem. *Let $\mu \in Q^\vee(\mathfrak{g})$ where $\mathfrak{g}$ is of classical type. Then $\tau_\mu \in G(\mathbb{C}[t, t^{-1}])$ where G is the simply connected group with Lie algebra $\mathfrak{g}$.*

Let $X_i = \sum^i H_{kk} - \frac{i}{n}\sum^n H_{kk}$, and these are a basis for $P^\vee$. Then $\tau_{X_i} := t^{X_i}$; though this doesn't quite make sense, conjugation by τ_{X_i} on $\mathfrak{sl}_n \otimes \mathbb{C}[t, t^{-1}]$is well-defined. It needs to be checked that this conjugation will not accidentally introduce fractional powers of t.
Definition 10. *The (general) AWG is $Aff(W) = W \ltimes \mathbb{Z}Q^\vee \subset \mathrm{Aut}(h^*)$. Here $\mathfrak{h}$ is the Cartan of the affine KM Lie algebra defined previously (so $\mathfrak{h} = \mathfrak{h}^\circ \oplus \mathbb{C}c \oplus \mathbb{C}d$). For h in the finite Cartan $\mathfrak{h}^\circ$, we let $T_h \in \mathrm{Aut}(h^*)$ by $T_h(\lambda) = \lambda + \lambda(c)v(h) - [\lambda(h) + 1/2 * < h, h > \lambda(c)]\delta$. Here $v : \mathfrak{h} \cong \mathfrak{h}^*$ and T gives an embedding of the finite Cartan into $\mathrm{Aut}(\mathfrak{h}^*)$. We define $T_{wh} = wT_hw^{-1}$. This defines the semidirect product and thus the AWG. Note we're letting w act on $\mathfrak{h}^*$ by fixing c and d.*

Lecture 17: 11/4/14

Extra long class today; none Thursday.

We're working towards Category O - this will be the category of *reasonable* modules. Reference - S. Kumar's book.

Let $T : V \to V$ an operator on a vector space. We say T is locally finite if $\forall v \in V$, there exists a finite dimensional subspace $W \subset V$ such that $v \in W$ and $T(W) \subset W$. In other words, the orbit of any vector under T spans a finite-dimensional space. We say that T is locally nilpotent if, in addition, $T|_W$ is nilpotent.

Let $\mathfrak{g}(A)$ be a KM algebra and V a $\mathfrak{g}(A)$-module. We say V is a weight module if

$$V \cong \bigoplus_{\lambda \in \mathfrak{h}^*} V_\lambda$$

with $\dim V_\lambda < \infty$. V is called integrable if, in addition, all e_i and f_i (for $1 \leq i \leq \ell$) act locally nilpotently.

Example/Theorem. $\mathfrak{g}(A)$ is an integrable $\mathfrak{g}(A)$-module (by the adjoint action). This requires a proof (which we maybe give later).

Suppose $\pi : \mathfrak{g} \to \mathrm{End}(V)$ with V integrable. Then we can define an automorphism

$$s_i(\pi) : V \to V$$

$$s_i(\pi) := \exp(\pi f_i) \circ \exp(-\pi e_i) \circ \exp(\pi f_i).$$

In this case one has

$$s_i(\pi)(V_\lambda) = V_{s_i(\lambda)}$$

for $\lambda \in \mathfrak{h}^*$. Here the RHS s_i is from the Weyl group. As a corollary we see that for $\alpha \in \Delta$ we have $\dim \mathfrak{g}_\alpha = \dim \mathfrak{g}_{-\alpha}$, since $s_\alpha(\alpha) = -\alpha$. The same holds for arbitrary roots.

We now wish to define the Casimir operator. What is it and why is it important? Consider $\mathfrak{sl}_2(\mathbb{C}) = \mathbb{C}f \oplus \mathbb{C}h \oplus \mathbb{C}e$ (identified with traceless 2×2 matrices). Let ϕ be the standard 2-dimensional representation. Define a bilinear form $\beta : \mathfrak{sl}_2 \times \mathfrak{sl}_2 \to \mathbb{C}$ by $\beta(x, y) = \mathrm{tr}(\phi(x)\phi(y)) = \mathrm{tr}(xy)$. We get $\beta(h, h) = 2$ and $\beta(e, f) = 1$ so β is nondegenerate. We then define $c_\phi = h^2/2 + ef + fe$ inside the universal enveloping algebra. This element is in fact central: $[c_\phi, e] = [c_\phi, f] = [c_\phi, h] = 0$ so $c_\phi =\in Z(\mathcal{U}(\mathfrak{sl}_2))$ [to be careful should say $Z(\phi(\mathcal{U}(\mathfrak{sl}_2)))$]

More generally, let $\phi : \mathfrak{g} \to \mathrm{End}(V)$ be a faithful representation of a simple finite-dimensional $\mathfrak{g}$ (e.g. the adjoint representation). Then we define a bilinear form β as above to be $\mathrm{tr}(\phi(x)\phi(y))$. This form is nondegenerate (exercise to see why). Let $x_1, \ldots, x_n$ be a basis of $\mathfrak{g}$ and y_i be the dual basis wrt β. Let $c_\phi = \sum_{i=1}^{n} \phi(x_i)\phi(y_i) \in \mathrm{End}(V)$. Remark: c_ϕ is independent of the basis chosen (basically because it acts by the identity), and $\mathrm{tr}(c_\phi) = \dim \mathfrak{g} = n$. The Casimir operator is a key ingredient in proving the complete reducibility of finite-dimensional modules for finite-dimensional semisimple Lie algebras.

To do this for our KM algebras we'll need an infinite sum, so we'll work in a completion. Recall now lecture 15. Assume $\mathfrak{g}(A)$ is symmetrizable and define $\bar{\mathfrak{g}} = \mathfrak{g}(A)/R$. Then $\mathcal{U}(\bar{\mathfrak{g}}) = \bigoplus_{d \geq 0} \mathcal{U}(\bar{\mathfrak{b}}^-) \otimes \mathcal{U}_d(\bar{\mathfrak{n}})$ where $\bar{\mathfrak{g}} = \overline{\mathfrak{b}^-} \oplus \bar{\mathfrak{n}}$. We define $\overline{\mathcal{U}}(\bar{\mathfrak{g}})$ to be the same thing with $\bigoplus_{d \geq 0}$ replaced by $\prod_{d \geq 0}$. In here $(\sum x_d)(\sum y_m) = \sum_{k \geq 0} \sum_{d,m} (x_d y_m)_k$. Remark - given k there are only finitely many d, m

such that $(x_d y_m)_k \neq 0$, so this product on the completion of the universal enveloping algebra makes sense.

We again have $\dim \overline{\mathfrak{g}_\alpha} = \dim \overline{\mathfrak{g}_{-\alpha}}$. Let $\{e^1_\alpha, \ldots, e^{p_\alpha}_\alpha\}$ be a basis of $\overline{\mathfrak{g}_\alpha}$ and $e^i_{-\alpha}$ the dual basis of $\overline{\mathfrak{g}_{-\alpha}}$ wrt the normalized invariant bilinear form on $\overline{\mathfrak{g}}$ (see lecture 15 - THIS is where we're using the assumption of symmetrizability). We similarly let $u_1, \ldots, u_k$ be a basis of $\mathfrak{h}$ and u^i be the dual basis of $\mathfrak{h}$. Define

$$\Omega_0 := \sum u^k_k$$

$$\Omega_\alpha := \sum_{i=1}^{p_\alpha} e^i_{-\alpha} e^i_\alpha$$

And we let the Kac-Casimir operator be

$$\Omega_\rho := 2v^{-1}(\rho) + \Omega_0 + 2 \sum_{\alpha \in \overline{\Delta^+}} \Omega_\alpha \in \overline{\mathcal{U}}(\overline{\mathfrak{g}})$$

Here $\rho \in h^*$ is such that $< \rho, \alpha_i^\vee >= 1$ for all i. In the KM case ρ is not unique (not just the half sum of positive roots). Problem is that the $\alpha_i^\vee$ may no longer span $\mathfrak{h}$. Again $v : \mathfrak{h} \cong \mathfrak{h}^*$ is our iso induced by $<>$.

CLAIM. Ω_α is independent of the choice of basis $x_\alpha := \sum_i e^i_{-\alpha} \otimes e^i_\alpha \in \bar{\mathfrak{g}}_{-\alpha} \otimes \bar{\mathfrak{g}_\alpha} \cong \mathrm{End}(\bar{\mathfrak{g}}_\alpha)$ since $\bar{g}_{-\alpha} \cong \bar{\mathfrak{g}_\alpha}^*$. The definition then implies $x_\alpha = \mathrm{Id}$. The following theorem now follows.
Theorem 11. Ω *only depends on the choice of* ρ.

Example. If $\dim \mathfrak{g} < \infty$ and $\mathfrak{g}$ simple, and $\phi : \mathfrak{g} \to \mathrm{End}(V)$ a faithful represenation, we've assigned objects c_ϕ and $\Omega \in \mathcal{U}(\mathfrak{g})$. These two differ by a scalar multiple (exercise). Basically because invariant bilinear forms are unique up to scalars.
Theorem 12. $\Omega \in Z(\overline{\mathcal{U}}(\overline{\mathfrak{g}}))$

Proof. We must show Ω commutes with the generators. It is easy to see that it commutes with $\mathfrak{h}$ (the first two terms are obvious since they come from the Cartan; the last term is because $[h, e_{-\alpha} \otimes e_\alpha] = 0$). We NTS $[\Omega, e_\alpha] = 0$.

Lemma 13. *If* $\alpha \neq \beta$. *We know* $\mathfrak{g}_\alpha$ *is dual to* $\mathfrak{g}_{-\alpha}$ *(similarly for* β*). Let* $v \in \mathfrak{g}_{-\alpha} \otimes \mathfrak{g}_\beta$. *Suppose* $< v, e \otimes f >= 0$ *for all* $e \otimes f \in \mathfrak{g}_\alpha \otimes \mathfrak{g}_{-\beta}$. *Then* $v = 0$ *(note that such* $e \otimes f$ *were the only ones that *could* have paired nontrivially with* v*).*

For proof of the lemma see Kumar's book.

Moving on, let $\alpha, \beta \in \overline{\Delta}$ and let $z \in \overline{\mathfrak{g}}_{\beta-\alpha}$. Then

$$\sum_{s=1}^{\dim \mathfrak{g}_\alpha} e^s_{-\alpha} \otimes [z, e^s_\alpha] = \sum_{t=1}^{\dim \mathfrak{g}_\beta} [e^t_{-\beta}, z] \otimes e^t_\beta$$

This is obvious for $\alpha = \beta$ so assume otherwise. CLAIM: let L and R be the LHS and RHS. Then

$$< L, e \otimes f >=< [z, e], f >$$

$$< R, e \otimes f >=< e, [f, z] >$$

where $e \otimes f \in g_\alpha \otimes \mathfrak{g}_{-\beta}$ as before. Then the Weyl-group invariance plus the lemma implies $L = R$.

Proof of claim: $\sum_s < e^s_{-\alpha} \otimes [z, e^s_\alpha], e \otimes f >= \sum_s < e^s_{-\alpha}, e >< [z, e^s_\alpha], f >= \sum_s < e^s_{-\alpha}, e >< e^s_\alpha, [f, z] >=< e, [f, z] >$. The other is similar.

Corollary now is that

$$\sum_s [e^s_{-\alpha}, [z, e^s_\alpha]] = -\sum_t [[z, e^t_{-\beta}], e^t_\beta] \in \mathfrak{g}$$

$$\sum_s e^s_{-\alpha}[z, e^s_\alpha] = -\sum_t [z, e^t_{-\beta}] e^t_\beta \in \mathcal{U}(\mathfrak{g})$$

Now let $\bar{\Omega} = 2\sum \Omega_\alpha$ the last summand of Ω. Then $[\bar{\Omega}, e_{\alpha_i}]$ is, after unwinding the definition, $[e_{-\alpha_i}, e_{\alpha_i}]e_{\alpha_i} + 2 \sum_{\alpha_i \neq \alpha \in \Delta_+} (\sum_s [e^s_{-\alpha}, e_{\alpha_i}]e^s_\alpha + \sum_t [e_{\alpha_i}, e^t_{-\alpha-\alpha_i}]e^t_{\alpha+\alpha_i}) = -2v^{-1}(\alpha_i)e_{\alpha_i} = -2 < \alpha_i, \alpha_i > e_{\alpha_i} - 2e_{\alpha_i}v^{-1}(\alpha_i)$. Here just using that $[v^{-1}(\alpha_i), e_{\alpha_i}] =< \alpha_i, \alpha_i >$.

Next, compute, for $x \in \bar{\mathfrak{g}}_\alpha$, $[\Omega_0, x] = 2xv^{-1}(\alpha) + < \alpha, \alpha > x$. But $v^{-1}(\alpha) = \alpha_i^\vee / \varepsilon_i$ by definition of the Cartan-Killing form. This then implies (compute) that $2 < \rho, \alpha_i >=< \alpha_i, \alpha_i >$. Thus we have that $[2v^{-1}(\rho) + \Omega_0, e_{\alpha_i}] = 2 < \alpha_i, \alpha_i > e_{\alpha_i} + 2e_{\alpha_i}v^{-1}(\alpha)$.

Putting all these together implies $[\Omega, e_{\alpha_i}]$. $\square$

Lecture 18: 11/11/14

Plan for rest of semester is (among other things) going to hopefully hit affine Grassmannians. But today we want to play with the Kac-Casimir operators that we constructed last time, and define category $\mathcal{O}$. Reference is chapters 1 and 2 of S. Kumar's book.

Let $\mathfrak{g} = \mathfrak{g}(A)$ be any KM algebra (we won't assume symmetrizable today). We introduce a partial order $\leq$ in $\mathfrak{h}^*$ by $\mu \leq \lambda$ if $\lambda - \mu \in Q^+ := \bigoplus \mathbb{Z}_{\geq 0}\alpha_i$. This is the usual Bruhat order. Then we can define the subsets $\mathfrak{h}^*_{\leq \lambda} \subset \mathfrak{h}^*$, whose meaning is obvious. For $\lambda \in \mathfrak{h}^*$ we let $\mathbb{C}_\lambda$ be the 1-dimensional $\mathfrak{h}$-representation where $\mathfrak{h}$ acts by λ. Now $\mathfrak{n}$ acts trivially on any 1-dimensional representation, so we extend $\mathbb{C}_\lambda$ trivially to a representation of $\mathfrak{b}$. We now define the Verma Module

$$M(\lambda) := \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{b})} \mathbb{C}_\lambda = \mathcal{U}(\mathfrak{n}^-) \otimes_{\mathbb{C}} \mathbb{C}_\lambda$$

where the second equality is via the PBW theorem $[\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{b}$ then $\mathcal{U}(\mathfrak{g}) = \mathcal{U}(\mathfrak{n}^-) \otimes_{\mathbb{C}} \mathcal{U}(\mathfrak{b})]$.

Properties. 1) $M(\lambda)$ is an $\mathfrak{h}$-weight module, i.e.,

$$M(\lambda) = \bigoplus_{\mu \in \mathfrak{h}^*} (M(\lambda))_\mu$$

such that $\dim M(\lambda)_\mu < \infty$. 2) If μ is a weight in $M(\lambda)$ then $\mu \leq \lambda$.

For the latter, note that $h(f_i v_\lambda) = (f_i h + [h, f_i])v_\lambda = (\lambda - \alpha_i)(h)(f_i v_\lambda)$.

Definition 14. *A Highest Weight Module is any quotient of a Verma Module.*

Exercise. If L is a highest weight module, compute $\mathrm{End}_{\mathfrak{g}}(L)$. Hint - the action of an element is determined by what it does to a highest weight vector (so the answer is: scalars).

Exercise. For $\mathfrak{sl}_2$, if $\lambda \in \mathbb{Z}^+$, then the kernel of $M(\lambda) \to V_\lambda \to 0$ (V_λ being the unique irreducible quotient) is of the form M_μ for some μ (exercise - $\mu = -\lambda - 2$).

Remark. The category of highest weight modules is not semisimple.

We shall show that $M(\lambda)$ has a unique irreducible quotient, and that this is the unique irreducible highest weight module of weight λ. But first we define Category $\mathcal{O}$, the category of "representations that are not insane."

Let $\mathcal{O}$ (for fixed $\mathfrak{g}$) be the category of $\mathfrak{g}$-modules M such that 1) M is an $\mathfrak{h}$-weight module: $M = \bigoplus_{\mu \in P(M)} M_\mu$ with $0 < \dim M_\mu < \infty$. 2) There exists $\lambda_1, \ldots, \lambda_k \in \mathfrak{h}^*$ (depending on M) such that $P(M) \subset \bigcup_{j=1}^k \mathfrak{h}^*_{\leq \lambda_j}$.

Easy remark. $\mathcal{O}$ is closed under submodules and quotient modules, as well as $\oplus$ and $\otimes$. But it is not semisimple (since Verma modules don't split).

The category $\mathcal{O}$ has an involution. Recall we defined the Cartan involution $\omega : \mathfrak{g}(A) \to \mathfrak{g}(A)$, by $e_i \mapsto -f_i$, $f_i \mapsto -e_i$, and $h \mapsto -h$. We want to promote this to $\mathcal{O}$. So if $M \in \mathcal{O}$ write $M = \oplus M_\mu$, and then define (note this is a restricted dual) $M^* := \oplus_{\mu \in P(M)} M_\mu^*$ (note the summands are finite-dimensional). We let $\mathfrak{g}$ act on this by $(x \cdot f)(v) = -f(\omega(x)v)$. Then the map $M \mapsto M^*$ gives our involution of $\mathcal{O}$. We remark that M_μ^* is of weight μ for M^*, though it would be $-\mu$ for the contragredient.

For $\lambda \in \mathfrak{h}^*$, we now introduce e^λ. Let $A = A_{\mathfrak{h}}$ be the set of formal sums $\sum a_\lambda e^\lambda$ with $a_\lambda \in \mathbb{Z}$, such that if $a \in A$ then there exist $\lambda_1, \ldots, \lambda_k$ (depending on a) such that $a_\lambda = 0$ for $\lambda \notin \bigcup_{i=1}^k \mathfrak{h}^*_{\leq \lambda_i}$. We see that addition in $A_{\mathfrak{h}}$ makes sense, and in fact there is a ring structure. By definition, the multiplication is

$$\left(\sum_\lambda a_\lambda e^\lambda\right) \cdot \left(\sum_\mu a_\mu e^\mu\right) = \sum_{\theta \in \mathfrak{h}^*} \left(\sum_{\lambda+\mu=\theta} a_\lambda b_\mu\right) e^\theta.$$

These sum and product operations are designed to capture the weights of the sum and tensor product of two modules.

Definition 15. *$M \in \mathcal{O}$ define*

$$ch(M) := \sum_{\lambda \in P(M)} (\dim M_\lambda) e^\lambda \in A$$

Easy remarks. $ch(M \otimes N) = ch(M)ch(N)$ and $ch(M \oplus N) = ch(M) + ch(N)$ and $ch(M^*) = ch(M)$. (Note this last is NOT true for contragredient representations!) Also more generally $ch(M/N) = ch(M) - ch(N)$.

Proposition. *If $M(\lambda)$ has a proper (i.e. not 0 and not the whole thing) $\mathfrak{g}$-submodule, then there is a unique maximal proper $\mathfrak{g}$-submodule $M' \subset M(\lambda)$.*

Corollary 16. *$M(\lambda)$ admits a unique irreducible quotient $L(\lambda) = M(\lambda)/M'$.*

Proof of Proposition. Suppose $0 \subset M_1 \subsetneq M(\lambda)$. Let v_λ be the unique highest weight vector (up to scalars) of weight λ. Then $v_\lambda \notin M'$. Now if $0 \subsetneq M', N' \subsetneq M$ then $M' + N'$ is also proper (it cannot contain v_λ by considering the weight decompositions of the sum). So just let M' be the submodule generated by all proper submodules. □

Proposition. *For any irreducible module $L \in \mathcal{O}$, there exists a unique $\lambda \in \mathfrak{h}^*$ such that $L \cong L(\lambda)$.*

Proof. Choose a maximal weight λ of L. This must be unique because L is irreducible. So have a $v_\lambda \in L$ a unique (up to scalars) vector of weight λ. Now we can define a $\mathfrak{b}$-module map $\mathbb{C}_\lambda \to L$ by $1 \mapsto v_\lambda$. This then gives a (surjective) $\mathfrak{g}$-module map $M(\lambda) \to L$. The previous corollary does it. Uniqueness of λ is obvious. □

Proposition. *Let $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$ and $\lambda \in \mathfrak{h}^*$. Then $L(\lambda)$ is irreducible as a $\mathfrak{g}'$-module.*

Proof. Suppose $V \subset L(\lambda)$ is a $\mathfrak{g}'$-submodule. It suffices to prove that $v_\lambda \in V$. Write $0 \neq v = \sum v_k \in V$ where $v_k \in L(\lambda)_{\lambda_k}$ (and in V), and assume the λ_k's are distinct. (in other words using that V is a weight module). Choose v such that the quantity $\sum_k |\lambda - \lambda_k|$ is minimal (for your favorite choice of norm on the finite-dimensional vector space $\mathfrak{h}^*$). Now if $\lambda_k \neq \lambda$ for some k then there exists j such that $e_j v_k \neq 0$, and $e_j \in \mathfrak{g}'$, so that $e_j v_k \in V$, and this will contradict minimality of v, since e_j pushes up the weight. □

Next time we talk about the category of integrable representations (note the Verma modules are not integrable in general).

Exercise. If $\mathfrak{g}$ is simple finite-dimensional, and $\lambda \in P_+(\mathfrak{g})$ (a dominant integral weight), then the Casimir operator $\Omega : V_\lambda \to V_\lambda$ acts by a scalar (by Schur's Lemma). Compute this scalar. In particular, do the case where λ is adjoint. (if you normalize the bilinear form properly, you'll get the "dual Coxeter number").

We'll see you get the same thing for integrable representations.

Lecture 19: 11/13/14

Last time we defined category $\mathcal{O}$ for KM algebras $\mathfrak{g}(A)$. We also defined Verma modules $M(\lambda) \in \mathcal{O}$, and a ring $A := A_\mathfrak{h}$ of characters. We defined, for $M \in \mathcal{O}$, a character $ch(M) \in A$. We classified the irreducible objects in $\mathcal{O}$ as the $L(\lambda) = M(\lambda)/M'$, and we showed that $L(\lambda)$ remain irreducible as $[\mathfrak{g}, \mathfrak{g}]$-modules.

We wish now to contrast the behavior of $\mathfrak{g}$ a fin-dim ss Lie algebra and $\mathfrak{g}(A)$ a KM algebra. In both cases we have $M(\lambda)$ for λ in the respective Cartans. In the former case, when $\lambda \in P_+(\mathfrak{g})$, we got a finite-dimensional $V(\lambda)$ as a quotient of the Verma module $M(\lambda)$. We need corresponding objects for $\mathfrak{g}(A)$. At issue is that $\mathcal{O}$ is not semisimple, but this smaller category of *integrable modules* will be.

So let (V, π) be an integrable $\mathfrak{g}(A)$-module. That is, e_i and f_i act locally nilpotently, and π is an $\mathfrak{h}$-weight module (may not be in $\mathcal{O}$ because may not have the finitely-many λ_i's bounding the possible weights).

Definition 17. *$D := \{\lambda \in \mathfrak{h}^* :< \lambda, \alpha_i^\vee >\in \mathbb{Z}_{\geq 0}\}$ (for all $\alpha_i^\vee$). I.e. this is $P_+(\mathfrak{g}(A))$.*

For $\lambda \in D \backslash \{0\}$ we have $M_1(\lambda) \subset M(\lambda)$ defined as

$$M_1(\lambda) := \{f_i^{\lambda(\alpha_i^\vee)+1} \otimes 1_\lambda : 1 \leq i \leq \ell\}$$

where we recall that $M(\lambda) = \mathcal{U}(\mathfrak{n}^-) \otimes \mathbb{C}_\lambda$.

Lemma 18. *For $1 \leq i, j \leq \ell$ we have $e_j \cdot f_i^{\lambda(\alpha_i^\vee)+1} \otimes 1_\lambda = 0$.*

Note that $M_1(\lambda)$ does not contain $1 \otimes 1_\lambda$, so is a proper subset. The lemma implies that it is a (proper) submodule. The lemma also says that $M_1(\lambda)$ has many highest-weight vectors.

Define $L^{max}(\lambda) = M(\lambda)/M_1$. In general this may be reducible (i.e. $M' \supsetneq M_1$), but it is named the **maximal integrable highest-weight $\mathfrak{g}(A)$-module with highest weight λ.**

Proof of Lemma. If $i \neq j$ then e_j and f_i commute, and e_j kills 1_λ (since it is a highest weight vector in $\mathbb{C}_\lambda$) so that case is easy. So suppose $i = j = 1$ (WLOG). By induction, one routinely proves that $e_1 f_1^n = f_1^n e_1 + n f_1^{n-1}(\alpha_1^\vee - n + 1)$. Now set $n = \lambda(\alpha_1^\vee) + 1$ and just use that $\alpha_i^\vee$ acts on 1_λ by $\lambda(\alpha_i^\vee)$. $\square$

So now let's prove that $L^{max}(\lambda)$ is actually integrable. We remark that the Verma modules themselves are not integrable since the f_i need not act locally nilpotently (though the e_i will since they raise the weight). First we need the following.

Lemma 19. *Let $\mathfrak{s}$ be any Lie algebra and $x \in \mathfrak{s}$.*

1) $\mathfrak{s}_x := \{y \in \mathfrak{s} : \mathrm{ad}(x)^{n_y} y = 0,\ n_y \geq 0\}$ is a Lie subalgebra of $\mathfrak{s}$.

2) (V, π) an $\mathfrak{s}$-representation, then $V_x := \{v \in V : \pi(x)^{n_v} v = 0\} \subset V$ is a representation of $\mathfrak{s}_x$.

As an application, take $x = f_i \in \mathfrak{g}(A)$. Then $\mathfrak{s}_x = \mathfrak{g}(A)$ since the adjoint representation is integrable. Let $L_i := \{v \in L^{max}(\lambda) : f_i^v = 0,\ m = m(v)\}$. We have that $1_\lambda \in L_i$ and so $L_i = L^{max}(\lambda)$. This implies that f_i acts locally nilpotently on $L^{max}(\lambda)$ (for all i). We also know the e_i's act locally nilpotently (because they already did on the Verma module).

Proposition. *Any integrable highest weight module is a quotient of $L^{max}(\lambda)$.*

Proof. Let $L = M(\lambda)/\!\sim$ be our integrable highest weight module. Recall the $\mathfrak{sl}_2$-triple $\mathfrak{g}(i) \hookrightarrow \mathfrak{g}(A)$ generated by $e_i, f_i, \alpha_i^\vee$. We know L is (also) integrable as an $\mathfrak{g}(i)$-module. Let L_λ be the (1-dimensional) weight space of L of weight λ. By integrability of the $\mathfrak{g}(i)$-module, the $\mathfrak{g}(i)$-module generated by L_λ is finite-dimensional. This then requires that $\lambda(\alpha_i^\vee) \in \mathbb{Z}_{\geq 0}$. That is, $\lambda \in D$. We can conclude that $f^{\lambda(\alpha_i^\vee)+1} v_\lambda = 0$ in L. . $\square$

Corollary 20. *Integrable, irreducible, highest weight $\mathfrak{g}(A)$-modules $L(\lambda)$ correspond 1-1 to $\lambda \in D$.*

We end today with the following proposition, which we don't prove.
Proposition. *$M \in \mathcal{O}$. Then there exists a (possibly infinite) chain $0 = M_0 \subset M_1 \subset \cdots \subset M$ of $\mathfrak{g}$-modules such that $M = \bigcup M_i$ and M_i/M_{i-1} is a highest weight module (that is, a quotient of a Verma module) of weight λ_i. Further, if $\lambda_i > \lambda_j$ then $i < j$. Finally, if $\lambda \in P(M)$ then there exists an r such that $(M/M_r)_\lambda = 0$.*

Plan now is to prove the Kac-Weyl character formula, and prove the semisimplicity of the subcategory of integrable modules in $\mathcal{O}$.

Lecture 20: 11/18/14

Proposition. *$V \in \mathcal{O}$ and $\lambda \in \mathfrak{h}^*$. Then there is a filtration (which will not be unique) $0 = V_0 \subset \ldots \subset V_p = V$ such that, if we set $W_i := V_{i+1}/V_i$, at least one of the following holds (for each i):*

1) W_i is irreducible of highest weight $\lambda_i \geq \lambda$.

2) $(W_i)_\mu = 0$ for any $\mu \geq \lambda$.

Proof. Consider $a(V, \lambda) := \sum_{\mu \geq \lambda} \dim V_\mu$ where V_μ is the weight μ space of V. Suppose $a(V, \lambda) = 0$. Then $0 = V_0 \subset V_1 = V$ works. So assume $a(V, \lambda) > 0$. Then choose a μ maximal in the Bruhat order such that $\mu \geq \lambda$. Let W be the $\mathfrak{g}$-module spanned by v_μ (a weight vector of weight μ). So this is a Verma module. Now we have $0 \subset W' \subset W \subset V$ where W' is the maximal proper submodule of W. Now observe that

$$a(W', \lambda) < a(V, \lambda)$$

$$a(V/W, \lambda) < a(V, \lambda)$$

and apply induction to the number $a(V, \lambda)$, using that $\mathcal{O}$ is closed under subs and quotients. $\square$

Let $\mu \in \mathfrak{h}^*$ and $V \in \mathcal{O}$. Choose $\lambda \leq \mu$ and let F_λ be a filtration as in the proposition. Note that since our category is not semisimple, multiplicity doesn't really make sense, so we need a substitute.
Definition 21. $[V : L(\mu)]_{F_\lambda} := \#\{i : W_i \cong L(\mu)\}$
Proposition. *For $V \in \mathcal{O}$ we have*

$$ch(V) = \sum_{\nu \geq \lambda} [V : L(\nu)]_{F_\lambda} chL(\nu) + R_\lambda$$

where $R_\lambda = \sum a_\theta(\lambda) e^\theta \in A_\mathfrak{h}$ such that $a_\theta(\lambda) = 0$ if $\theta \geq \lambda$.

The two terms above are coming from the 2 possibilities in the previous proposition.

Remark. Consider another filtration $F'_{\lambda'}$ where $\lambda' \leq \mu$ or $\lambda = \lambda'$. Then for all $\nu \geq \lambda$ and $\nu \geq \lambda'$ we have $[V : L(\nu)]_{F_\lambda} = [V : L(\nu)]_{F'_{\lambda'}}$. To see this, argue by contradiction and suppose ν_0 is maximal such that $[V : L(\nu_0)]_{F_\lambda} \neq [V : L(\nu_0)]_{F'_{\lambda'}}$. By the definition of the filtration (equivalently, the previous proposition), $[V : L(\nu_0)]_{F_\lambda}$ is the coefficient of e^{ν_0} in $ch(V)$. This provides the contradiction.

Definition 22. *$[V : L(\mu)] \neq 0$ then we say that $L(\mu)$ is a component of V; and the $L(\mu)$'s are subquotients of V.*

Lemma 23. $ch(V) = \sum_{\mu \in \mathfrak{h}^*} [V : L(\mu)] ch(L(\mu))$

Before proving this, we need to clarify a point. Suppose we have a family $a_i; = \sum_\lambda a_i(\lambda) e^\lambda$ of elements of $A_\mathfrak{h}$. When does the sum $\sum a_i$ make sense?

Definition 24. *We say $\{a_i\}$ is a locally finite family if for any $\lambda \in \mathfrak{h}^*$ the set*

$$I_\lambda := \{i \in I : a_i(\lambda) \neq 0\}$$

is finite

We claim that $\{[V : L(\mu)] ch(L(\mu))\}_{\mu \in \mathfrak{h}^*}$ is a locally finite family. This is necessary for the lemma to make sense. So choose $\lambda \in \mathfrak{h}^*$. There are only finitely-many $\nu \geq \lambda$ such that $[V : L(\nu)]_{F_\lambda} \neq 0$. And by definition $[V : L(\mu)]$ can only be nonzero for $\mu \geq \lambda$.

The proof of the lemma (now that we know it makes sense) is actually pretty obvious now. Observe that $ch(V) = \sum_{\mu \geq \lambda} [V : L(\mu)]_{F_\lambda} ch(L(\mu)) + R_\lambda$; meanwhile the RHS is $\sum_{\mu \geq \lambda} + \sum_{\mu > \lambda}$ of the same expression; but we can do this for any λ! So this is ultimately a tautology.

Exercise (see p.152 in Kac's book). $ch(M(\lambda)) = e^\lambda \cdot \prod_{\alpha \in \Delta^+} (1 - e^{-\alpha})^{-mult\alpha}$ where $(1 - e^{-\alpha})^{-mult\alpha} := (1 + e^{-\alpha} + e^{-2\alpha} + \ldots)^{mult\alpha}$.

Proposition. *$M \in \mathcal{O}$ is integrable iff all its components are integrable.*

Proof. For the nonobvious direction, suppose towards a contradiction that M is not integrable. Then there is a $v \in M$ such that for some fixed $i \in [1, \ell]$, we have $f_i^m \neq 0$ for any $m \in \mathbb{Z}_{\geq 0}$. We can assume WLOG that v is a highest weight vector (otherwise just push it up; it must have been gotten by applying f's to a highest weight vector anyway) of weight λ_0. Fix $q \in \mathbb{Z}_+$ large enough such that $\lambda_0 + q'\alpha_i$ is not a weight of M for $q' \geq q$.

Claim. $\lambda_0 - k\alpha_i$ is not a weight of any component, for $k \geq q + < \lambda_0, \alpha_i^\vee >$.

This claim implies that $M_{\lambda_0 - k\alpha_i} = 0$. The weight of $f_i^k v$ will be $\lambda_0 - k\alpha_i$, but then $f_i^k v = 0$ and this is a contradiction.

So we must prove this claim. Note that we haven't used integrability of components yet. Since exp of f_i and e_i thus make sense, we get an action of the Weyl group (as described previously). Let L be a component. Now (by applying s_i) we get $mult_{\lambda_0 - k\alpha_i} L = mult_{\lambda_0 + [k - <\lambda_0, \alpha_i^\vee>]\alpha_i} L$. Now use the definition of q and the assumption on k to get the claim. $\square$

Exercise. If M is integrable, then the restricted dual M^* is also integrable (this immediate from the proposition).

We can now state the Kac-Weyl character formula (in the symmetrizable setting).

Kac-Weyl Character Formula

(below equivalent to assume $\mathfrak{g}(A)$ is symmetrizable.) Let L be an integrable $\overline{\mathfrak{g}}$-module with highest weight λ. Then

$$ch(L) := \left[\sum_{w\in W} \varepsilon(w)e^{w*\lambda}\right] \cdot \left(\prod_{\alpha\in\overline{\Delta^+}} (1-e^{-\alpha})^{-\overline{mult_\alpha}}\right) \in A$$

where $*$ is the shifted action: $w * \lambda = w(\lambda+\rho) - \rho$ and ε is the sign representation of W. Recall also that $\overline{mult}(\alpha) := \dim \mathfrak{g}_\alpha$. We remark that the reciprocal of the product appearing here equals the $\sum_{w\in W} \varepsilon(w)e^{w*0}$.

NOTE we aren't assuming L is irreducible here! But this means that we get the following awesome corollary.
Corollary 25. *Any integrable highest weight $\overline{\mathfrak{g}}$-module is irreducible. In particular $L(\lambda)^{max}$ is irreducible.*

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Let $D := \left(\prod_{\alpha\in\overline{\Delta^+}} (1-e^{-\alpha})^{\overline{mult_\alpha}}\right)$, and let L be an integrable $\overline{\mathfrak{g}}$-module of highest weight λ. Then today we shall prove

$$ch(L) = \frac{\left[\sum_{w\in W} \varepsilon(w)e^{w*\lambda}\right]}{D}$$

Proposition 2. *$\overline{\mathfrak{g}} := \mathfrak{g}(A)/R$, A is symmetrizable. Then Ω acts on $\overline{M}(\lambda)$ as multiplication by the scalar $< \lambda, \lambda + 2\rho >= |\lambda+\rho|^2 - |\rho|^2$.*

Proof. Recall the definition of $\Omega := 2v^{-1}(\rho) + \sum u_k u^k + 2\sum_{\alpha\in\Delta^+} e_\alpha^{-i} e_\alpha^i$ but this last term kills the highest weight vecot. So we just need to compute $\Omega \cdot 1_\lambda$. This is a direct computation. $\square$

Lemma 26. *Let V be a highest weight $\overline{\mathfrak{g}}$-module with highest weight λ. Then $ch(V) = \sum c_\mu \overline{M}(\mu)$, where the $c_\mu \in \mathbb{Z}$ are some constants, and we sum over $\mu \leq \lambda$ with $< \mu+\rho, \mu+\rho >=< \lambda+\rho, \lambda+\rho >$.*

Proof. We omit bars on $L(\mu)$. Recall we know from last time that $ch(V) = \sum_{\mu\ni\mathfrak{h}^*} [V : L(\mu)]ch(L(\mu))$. It is enough to prove the above for $V = L(\mu)$. Let $s(\lambda) = \{\mu \in \mathfrak{h}^* : \mu \leq \lambda,\ |\mu+\rho|^2 = |\lambda+\rho|^2\}$. Since $\overline{M}(\mu) \in \mathcal{O}$ again, we can write $ch(\overline{M}(\mu)) = \sum a_{\mu,\theta} ch(\overline{L}(\theta))$ where we sum over $\theta \leq \mu$ with $|\theta+\rho|^2 = |\mu+\theta|^2$. We know $a_{\mu,\mu} = 1$ and that $a_{\mu,\theta} = 0$ if $\theta > \mu$. So we have a lower triangular unipotent matrix, so we can invert it and deduce

$$ch(\overline{L}(\theta)) = \sum d_\mu ch\overline{M}(\mu)$$

where we sum over the μ subject to the same conditions. $\square$

Lemma 27. *Let $\lambda \in \mathfrak{h}^*$ be such that $< \lambda, \alpha_i^\vee > \geq 0$ for all $1 \leq i \leq \ell$. Then for any $\nu \in \mathfrak{h}^*$ such that*

1) $\nu \leq \lambda + \rho$,

2) $< \nu, \nu > = < \lambda + \rho, \lambda + \rho >$, and

3) $< \nu, \alpha_i^\vee > \geq 0$ for all i,

we have $\nu = \lambda + \rho$.

Proof. From 1) we can write $\nu = \lambda + \rho - \sum n_i \alpha_i$. Then plug this into 2) to deduce that $0 = - < \sum n_i \alpha_i, \nu > - < \lambda + \rho, \sum n_i \alpha_i >$, and then this implies all n_i are zero, using 3) and the assumption on λ. $\square$

We now define an action of W on $A_{\mathfrak{h}}$. First note, for example, that if L is an integrable highest weight module, then W permutes $P(L)$, the set of weights appearing. We should have $w(ch(L)) := w\left(\sum_{\mu \in P(M)} \dim V_\mu e^\mu\right) = \sum_{\mu \in P(\mu)} \dim V_\mu e^{w(\mu)} = ch(L)$. So the action on the characters of highest weight integrable modules is trivial. The action in general is defined by $w(\sum a_\lambda e^\lambda) := \sum a_\lambda e^{w(\lambda)}$.

Exercise. $w(a_1 \cdot a_2) = w(a_1)w(a_2)$. I.e. W acts by ring automorphisms on $A_{\mathfrak{h}}$.

Remark. It is nontrivial to extend Harish-Chandra's theory to this "completed" ring $A_{\mathfrak{h}}$ to study the W-invariants.

We claim that $w(e^\rho \cdot D) = \varepsilon(w) e^\rho D$. It is clearly enough to show that $s_i(e^\rho D) = -e^\rho D$. The only fact needed is that s_i preserves $\Delta_+ \backslash \{\alpha_i\}$. Also note that $\overline{mult}\alpha_i = 1$ since α_i is a simple root. The rest is some fun simple algebra from the definitions.

Now we proceed with the proof of the main result.

Proof. This is equivalent to showing $e^\rho Dch(L) = \sum_{w \in W} \varepsilon(w) e^{w(\lambda)}$. By a simple exercise we know $ch(M(\mu)) = e^\lambda D^{-1}$. Plugging this, we get the LHS is equal to to

$$e^\rho D \sum_{\mu \leq \lambda,\ |\mu+\rho|^2 = |\lambda+\rho|^2} c_\mu ch\overline{M}(\mu) = \sum_{\mu \leq \lambda,\ |\mu+\rho|^2 = |\lambda+\rho|^2} c_\mu e^{\mu+\rho}.$$

Now we compute both sides of the equality $w(e^\rho Dch(L)) = \varepsilon(w) e^\rho Dch(L)$ independently. We can rewrite $\varepsilon(w) e^\rho Dch(L) = \sum_\nu \varepsilon(w) c_v e^{\nu+\rho}$ and $w(e^\rho Dch(L)) = \sum c_\mu e^{w(\mu+\rho)}$ (in both we're summing over guys $\leq \lambda$ with the same length condition as we've had). We see now that $\nu = w(\mu+\rho) - \rho (= w * \mu) \leq \lambda$. From the above, by comparing coefficients we have the following:

$$c_\mu = \varepsilon(w) c_\nu \text{ if } \nu = w * \mu$$

But this is same as claiming that $c_\mu = \varepsilon(w) c_{w*\mu}$ for all $w \in W$. So now choose $w_0 = w(\mu)$ such that the height $|\lambda - w_0 * \mu|$ is minimal. By the following exercise and the previous Lemma, we have $w_0 * \mu + \rho = \lambda + \rho$. Thus $\lambda = w_0 * \mu = w_0(\mu + \rho) - \rho$ so $\mu = w_0^{-1}(\lambda + \rho) - \rho = w_0^{-1} * (\lambda + \rho)$. Since w and w^{-1} have the same, combined with the fact that $\lambda = w_0 * \mu$ and $c_\mu = \varepsilon(w_0)$ gives us the Kac-Weyl character formula. □

Exercise. Let $\nu = w_0 * \mu + \rho$. Then ν satisfies the 3 conditions for the previous lemma.

Corollary 28. *Let M be an integrable module in $\overline{\mathcal{O}}$ for $\overline{\mathfrak{g}}$, then*

$$M \simeq \oplus_{\lambda \in D} L(\lambda)^{[M:L(\lambda)]}$$

Proof. Copy from the homework solutions □