Generalization to Symmetric KM algebras

K^{wg} (weakly good)

$\cap$

$\mathfrak{h}^*$.

Defin $C_\alpha = \{ \lambda \in \mathfrak{h}^* : \langle \lambda + \rho, \alpha \rangle = \frac{\langle \alpha, \alpha \rangle}{2} \}$

$$C = \bigcup_{\alpha \in \Delta_{im}^+} C_\alpha .$$

$$K^{wg} = \mathfrak{h}^* \setminus \bar{C}$$

Exercise: K^{wg} is preserved under W - Weyl group of $\mathfrak{g}(A)$ under the $*$-action

$W(\lambda) \subset W$ be the group generated by

$$\{ s_\beta \mid \beta \in \Delta_{re}^+ , \langle \lambda + \rho, \beta^\vee \rangle \in \mathbb{Z} \}$$

Prop: Let $\lambda \in K^{wg}$. Then $[M(\lambda) : L(\mu)] > 0$.

iff $\exists$ reflections $s_{\beta_1}, \ldots, s_{\beta_p} \in W(\lambda)$

for $\beta_1, \ldots, \beta_p \in \Delta_{re}^+$, $p \geq 0$ such that

$$\lambda > s_{\beta_1} * \lambda > s_{\beta_2} * s_{\beta_1} \lambda > \cdots\cdots > \left(s_{\beta_p} * s_{\beta_{p-1}} * \cdots s_{\beta_1} * \lambda \right) = \mu .$$

In particular $\mu \in W(\lambda) * \lambda$ and hence in K^{wg}.

Proof: The if part follows from the iff Proposition proved earlier. We only prove the if part.

Since $L(\mu)$ is a subquotient of $M(\lambda)$ of course this implies $\lambda - \mu \in Q^+$

We will induct on $|\lambda - \mu|$

If $|\lambda - \mu| = 0$, then $p = 0$.

This assume $|\lambda - \mu| > 0$.

Since $M(\lambda)/M'(\lambda) \simeq L(\lambda)$ we get from the Janzten formula that $L(\mu)$ is a subquotient of $M(\lambda - n\beta_1)$ for some (β_1, n) appearing in the RHS of Jantzen formula.

But then $\beta_1 \in \Delta_{re}^+$ otherwise $n\beta_1 \in \Delta_{im}^+$ (Since multiple of imaginary is imaginary) contradicting that $\lambda \in K^{wg}$

Moreover $n = \langle \lambda + \rho, \beta_1^\vee \rangle$.

$$\Rightarrow s_{\beta_1} \in W(\lambda)$$

Now $\lambda - n\beta_1 = s_{\beta_1} * \lambda$

Now apply induction

Proof of Characterformula

Need to show

$$\sum_{p \geq 1} \operatorname{ch} M^p(\lambda) = \sum_{(\alpha,n) \in D_\lambda} \operatorname{ch} M(\lambda - n\alpha)$$

where $D_\lambda := \{ (\alpha, n) \in \Delta^+ \times \mathbb{N} \mid \langle \lambda + \rho - \frac{n}{2}\alpha, \alpha \rangle = 0 \}$.

and $M(\lambda) \supset M^1(\lambda) \supset \cdots \supset M^p(\lambda) \supset \cdots$ is the Jantzen filtration

Extend $\hat{S}_\lambda : M(\lambda) \times M(\lambda) \to \mathbb{C}[t]$.

to a bilinear form

$$\hat{S}_\lambda[t] : M(\lambda) \otimes \mathbb{C}[[t]] \times M(\lambda) \otimes \mathbb{C}[[t]] \to \mathbb{C}[[t]].$$

Let $M(\lambda)_{\lambda-\eta}[[t]] := M(\lambda)_{\lambda-\eta} \otimes_{\mathbb{C}} \mathbb{C}[[t]]$ for $\eta \in Q^+$

Since $\exists$ t such $M(\lambda+t\rho)$ is irreducible

$\Rightarrow \hat{S}_\lambda[t]$ is non degenerate restricted to $M(\lambda)_{\lambda-\eta}$.

By Smith normal form for bilinear forms on a PID

we get a basis

$(b_j)_{1 \leq j \leq P(\eta)}$ and $(a_j)_{1 \leq j \leq P(\eta)}$ of

$M(\lambda)_{\lambda-\eta}[[t]]$ over $\mathbb{C}[[t]]$ such that

$\hat{S}_{\lambda,\eta}(t) := \hat{S}_\lambda(t)|_{M(\lambda)_{\lambda-\eta}}$ is of the form

$$\hat{S}_\lambda(t)\left(a_{j_1}, b_{j_2}\right) = t^{n_{j_1}} \delta_{j_1} \delta_{j_2}$$

for some integers $0 \leq n_1 \leq \cdots \leq n_{P(\eta)}$

Now it follows from the definition of Jantzen filtration that

$$M^p(\lambda)_{\lambda-\eta} := \pi\left\{ \sum \mathbb{C}[[t]]\, a_j : n_j \geq p \right\}$$

$$\pi : M(\lambda) \otimes_t \mathbb{C}[[t]] \to M(\lambda)$$

is the evaluat.

Hence

$$\mathrm{ord}_t\left(\det \hat{S}_{\lambda,\eta}(t)\right)$$

$$= \sum_{j=1}^{P(\eta)} n_j$$

$$= \sum_{p \geq 1} \dim M^p(\lambda)_{\lambda-\eta}$$

So

$$\sum ch\, M^p(\lambda) = e^{\lambda} \sum_{\eta \in Q^+} \mathrm{ord}_t\left(\hat{S}_{\lambda,\eta}(t)\right) e^{-\eta}$$

$$= \sum_{\eta \in Q^+} \sum_{(\alpha,n) \in D_\lambda} P(\eta - n\alpha) e^{\lambda-\eta} \quad \text{(By Shapovalov formula)}$$

Rewriting we get

$$= \sum_{(\alpha,n) \in D_\lambda} ch(M(\lambda - n\alpha)) \quad \text{[Using formula for characters of Verma modules]}$$