GEOMETRIC INVARIANT THEORY AND MODULI PROBLEMS

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The main topic of this course is the construction of quotient spaces in algebraic geometry. Indeed, moduli problems reduce to the problem of the existence of good quotients of an algebraic variety acted on by an algebraic group. This is the subject of geometric invariant theory.

§1. Algebraic actions.

Throughout the course we will work over the field of complex numbers $\mathbf{C}$. Let G be an **algebraic group** over $\mathbf{C}$, that is, an algebraic variety provided with regular morphisms $\mu: G \times G \to G$, $e: \{*\} \to G$ and $i: G \to G$ satisfying the usual rules of multiplication, identity element and inverse in a group (the fundamental examples are the general linear group $GL(n, \mathbf{C})$ and the special linear group $SL(n, \mathbf{C})$). A (left) **algebraic action** of G over an algebraic variety X is, by definition, a regular morphism

$$\phi : G \times X \to X$$

such that:

i) $\phi(e, x) = x$ (e denotes the identity element in G) and

ii) $\phi(g_1, \phi(g_2, x)) = \phi(g_1 g_2, x)$ for any $g_1, g_2 \in G$, $x \in X$.

In the following we will suppress ϕ and write simply gx for $\phi(g, x)$.

If $x \in X$, the **orbit** of x under the action of G (for short, the G-orbit of x) is the set defined as

$$O_G(x) = \{gx \mid g \in G\}$$

(we will write just $O(x)$ if no confusion can arise about G, sometimes we will also write Gx for $O_G(x)$).

We would like to take the quotient of X by G. Naively, we may consider the equivalence relation given by:

$$x \sim x' \iff \exists\, g \in G \text{ such that } gx = x'$$

and try to endow the set theoretic quotient — which is just the set of orbits of G in X — with a natural structure of an algebraic variety. In other words, we may look for a variety Y and a surjective morphism $\pi : X \to Y$ such that the fibres of π correspond exactly to the orbits of G in X. Unfortunately, this may not be possible even in the simplest cases, as the following example shows.

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Example 1.1. Let $X = \mathbf{C}^2$ and $G = \mathbf{C}^* = GL(1, \mathbf{C})$, we make G act on X as follows:

$$\mathbf{C}^* \times \mathbf{C}^2 \to \mathbf{C}^2 \qquad (\lambda, (x, y)) \mapsto (\lambda x, \lambda^{-1} y)$$

Thus the orbit of a point in $\mathbf{C}^2$ is:
$O((x,y)) = \{(x', y') \in \mathbf{C}^2 \mid x'y' = xy\}$, if $xy \neq 0$;
$O((x,0)) = \{y = 0\} \setminus \{(0,0)\}$, if $x \neq 0$;
$O((0,y)) = \{x = 0\} \setminus \{(0,0)\}$, if $y \neq 0$ and
$O((0,0)) = \{(0,0)\}$.

Notice that $O((0,y))$ and $O((x,0))$ are not closed and their closures contain the orbit $O((0,0))$. Consider now the map $\pi\colon \mathbf{C}^2 \to \mathbf{C}$ defined by $(x,y) \mapsto xy$. It is clearly a surjective morphism and it is constant on orbits. Indeed its fibre over a point $c \in \mathbf{C} \setminus \{0\}$ is:

$$\pi^{-1}(c) = \{(x,y) \mid xy = c\} = O((1,c)),$$

while the fibre over zero consists of three orbits:

$$\pi^{-1}(0) = O((0,0)) \cup O((0,y)) \cup O((x,0))$$

Thus $\mathbf{C}$ parametrizes only the set of closed orbits.

Remark 1.2. Any morphism $\pi : X \to Y$ which is constant on orbits is also constant — with the same values — on their closures, hence orbits whose closures intersect lie in the same fibre of π. This is one of the reasons why it may not be possible to parametrize all orbits of a given action. We then have to weaken slightly our request and look for a variety Y with a morphism $\pi : X \to Y$ which provide the finest possible parametrization of the orbits of G in X, i.e. such that any other morphism $\psi : X \to Z$ constant on orbits factors through π. This leads to the concept of categorical quotient:

Definition 1.3. Let X be an algebraic variety and G an algebraic group acting on X. A **categorical quotient** for this action is a pair (Y, π), where Y is an algebraic variety and $\pi : X \to Y$ is a G-invariant (i.e. constant on orbits) regular morphism verifying the following universal property: for any G-invariant morphism $\psi : X \to Z$, ψ factors uniquely through π, that is, there exists a unique morphism $\gamma : Y \to Z$ such that $\psi = \gamma \circ \pi$.

Notice that by the universal property a categorical quotient is unique up to isomorphisms.

Still we do not know how we can take the quotient of X by G. To get a more precise idea we should also consider the effect of the group action on the regular functions on the variety X. Let G act algebraically on X and let $A(X)$ be the ring of regular functions $X \to \mathbf{C}$, then G acts in a natural way on $A(X)$ by setting:

$$G \times A(X) \to A(X) \qquad (g, f) \mapsto gf$$

where gf is defined by:

$$gf(x) = f(g^{-1}x)$$

for any $x \in X$ (in this way we define a left action).

Definition 1.4. We say that a function $f \in A(X)$ is **invariant** under the action of G (or, G-**invariant**) if $gf = f$ for any $g \in G$. We put

$$A(X)^G = \{f \in A(X) \mid f = gf \ \forall g \in G\}.$$

This is in fact a subring of $A(X)$.

Remark 1.5. It is easily seen that a function is invariant if and only if it is constant on orbits. On the other hand, regular functions on a quotient space should be functions on X which are constant on orbits. This suggests that a quotient space should be associated with the ring of invariant functions on the variety. To illustrate this, let us look again at Example 1.1:

Example 1.6. Consider the action of the group $\mathbf{C}^*$ on $\mathbf{C}^2$ defined in Example 1.1. The regular functions on $\mathbf{C}^2$ are just the polynomial functions $f\colon \mathbf{C}^2 \to \mathbf{C}$, $f = \sum_{i,j} a_{ij} x^i y^j \in \mathbf{C}[x, y]$. The invariant functions are those polynomials such that

$$\sum_{i,j} a_{ij} x^i y^j = \sum_{i,j} a_{ij} \lambda^{i-j} x^i y^j \qquad \forall \lambda \in \mathbf{C}^*$$

that is, such that $a_{ij} = 0$ for $i \neq j$. These are precisely the polynomials in xy, with which we can naturally associate a variety, the maximal spectrum $\operatorname{Spec} \mathbf{C}[xy]$. Moreover, the inclusion of $\mathbf{C}$-algebras $\mathbf{C}[xy] \to \mathbf{C}[x, y]$ induces a map on the corresponding maximal spectra $\operatorname{Spec} \mathbf{C}[x, y] = \mathbf{C}^2 \to \operatorname{Spec} \mathbf{C}[xy]$, which is in fact a quotient map, as we will see later.

Let us now deal with the general case. First we need some more notation and basic facts about actions of algebraic groups.

Let X be an algebraic variety and G an algebraic group. To denote an (algebraic) action of G on X we will write simply (X, G). With any $x \in X$ we associate the **orbit** $O_G(x) \subset X$ and the **isotropy group of** G **at** x (or the **stabiliser** of x under the action of G), defined as

$$G_x = \{g \in G \mid gx = x\},$$

this is a closed subgroup of G. We also define a map

$$\sigma_x : G \to X \qquad g \mapsto gx,$$

the **orbit map** at x, whose image is just the orbit $O_G(x)$. The fibres of σ_x are cosets of the stabiliser G_x, in particular, the fibre over x is G_x. A subset $X' \subset X$ is G-**invariant** if $\forall x' \in X'$ and $g \in G$, $gx' \in X'$, i.e. X' contains the orbit of any of its points. Notice that the orbit of a point $x \in X$ is by definition the minimal G-invariant subset of X containing x. The closure of an orbit is still G-invariant (indeed, let $y \in \overline{O(x)}$ and consider gy, then for any open neighbourhood I of gy, $g^{-1}I$ is an open neighbourhood of y and $g^{-1}I \cap O(x) \neq \emptyset$; applying g we get $I \cap O(x) \neq \emptyset$, that is $gy \in \overline{O(x)}$). A morphism $\phi : X \to Y$ of algebraic varieties is G-**invariant** if it is constant on orbits (and hence is also constant — with the same values — on the closures of orbits).

Recall that a subset of a topological space is **locally closed** if it is open in its closure or, equivalently, if it is the intersection of an open set with a closed set. A basic result for algebraic actions is the following:

Proposition 1.7. *Let (X, G) be an algebraic action. Then:*

(1) *for any $x \in X$, the orbit $O(x)$ is locally closed (in the Zariski topology) and $\overline{O(x)} - O(x)$ is a union of orbits of strictly lower dimension; in particular, orbits of minimal dimension are closed;*

(2) *for any $x \in X$,*

$$\dim O(x) = \dim G - \dim G_x.$$

(3) *for any integer n, the set*

$$\{x \in X \mid \dim O(x) \geq n\}$$

is open (i.e. $\dim O(x)$ is a lower semicontinuous function of x).

Before proving this Proposition we recall some facts from the general theory (for reference, see Mumford's Red Book, [M]). Let X be a topological space and Y a subset. We say that Y is **constructible** if it is a finite union of locally closed subsets. A topological space is **noetherian** if every open subset is quasi-compact, or, equivalently, if its closed subsets satisfy the descending chain condition, i.e. every strictly descending chain of closed subsets $C_1 \supset C_2 \supset \ldots$ is finite. Notice that the Zariski topology is noetherian.

Proposition. *Let X be a noetherian topological space and Y a constructible subset of X. Then Y contains a dense open subset of $\overline{Y}$.*

Theorem (Chevalley). *Let $f : X_1 \to X_2$ be a morphism of algebraic varieties (with the Zariski topology). Then $f(X_1)$ is a constructible set in Y. More generally, f maps constructible sets in X to constructible sets in Y.* (see Corollary 2 of Theorem 3, Chapter I, in [M])

Corollary. *Let $f : X_1 \to X_2$ be a ~~dominating~~ dominant morphism of algebraic varieties (with the Zariski topology), i.e. $\overline{f(X_1)} = X_2$. Then $f(X_1)$ contains a dense open subset.*

Proof of Proposition 1.7.

(1): Let $x \in X$ and consider the orbit map $\sigma_x : G \to X$. Since $O(x) = \sigma_x(G)$, by Chevalley's theorem the orbit of x is constructible, then it contains a dense open subset U of $\overline{O(x)}$ such that $U \subset O(x) \subset \overline{O(x)}$. Now G acts transitively on $O(x)$ thus we have $O(x) = \bigcup_{g \in G} gU$, i.e. $O(x)$ is union of open subsets of $\overline{O(x)}$. It follows that $\overline{O(x)} - O(x)$ is closed and of strictly lower dimension; moreover, it is clearly G-invariant, hence the thesis.

(2): Again, consider $O(x)$ as the image of the orbit map σ_x. Since the fibres are cosets of G_x, they all have the same dimension, equal to $\dim G_x$. Now a standard theorem (Thm. 3, Ch. I, [M]) on fibres of a morphism yields the result.

(3): Consider the morphism

$$\Phi : G \times X \to X \times X \qquad (g, x) \mapsto (gx, x)$$

and take the restriction to the diagonal $\Delta \subset X \times X$:

$$\Phi : \Phi^{-1}(\Delta) \to \Delta.$$

Since $\Phi^{-1}(x,x) = G_x \times \{x\}$, $\dim G_x$ is an upper semicontinuous function on $\Phi^{-1}(\Delta)$ (apply Cor. 3 of Thm. 3, Ch. I, [M] and recall that all the components of an algebraic group have the same dimension). This means that for any integer n the set $\{(g,x) \in \Phi^{-1}(\Delta) | \dim G_x \geq n\}$ is closed in $\Phi^{-1}(\Delta)$. Identifying X with the closed subvariety $\{e\} \times X \subset \Phi^{-1}(\Delta)$ and taking the intersection with the above set we get that for any integer n the set $\{x \in X | \dim G_x \geq n\}$ is closed in X. By (2) we obtain the result.

Remark 1.8. The boundary of an orbit may contain infinitely many orbits. Indeed, consider the following example. Let $X = \mathbf{C}^2 \times \mathbf{C}^2$ and let $G = GL(2, \mathbf{C})$ act on $\mathbf{C}^2 \times \mathbf{C}^2$ by matrix multiplication on the left, i.e., if $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, \mathbf{C})$, $v = (v_1, v_2)$ and $w = (w_1, w_2)$ are two vectors in $\mathbf{C}^2$, we set

$$A(v,w) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} v_1 & w_1 \\ v_2 & w_2 \end{pmatrix}$$

Suppose we take v and w to be linearly independent, then the orbit $O((v,w))$ consists only of pairs of linearly independent vectors, but the closure also contains pairs of type $(v', \lambda v')$. Moreover, $O((v', \lambda v')) = O((v'', \mu v''))$ if and only if $\lambda = \mu$, that is, any $\lambda \in \mathbf{C}^*$ determines a different orbit.

We recall that a morphism $f : X \to Y$ of algebraic varieties is **affine** if there exists a cover $\{U_i\}_{i \in I}$ of Y by affine open sets such that $f^{-1}(U_i)$ is affine for any $i \in I$.

Definition 1.9. Let G be an algebraic group acting on an algebraic variety X. A **good quotient** of X by G is a pair (Y, ϕ), where Y is an algebraic variety and $\phi: X \to Y$ is a regular morphism satisfying the following conditions:

(1) ϕ is G-invariant;
(2) ϕ is surjective;
(3) ϕ is affine and for any affine open subset $U \subset Y$ it is $A(U) = A(\phi^{-1}(U))^G$;
(4) the image by ϕ of a G-invariant Zariski-closed subset in X is closed in Y;
(5) any two G-invariant Zariski-closed disjoint subsets have disjoint images in Y.

Sometimes we will refer to the quotient of X by G writing $X//G$ (to distinguish it from the orbit space) and omitting the morphism ϕ.

We now go through some fundamental properties of good quotients.

Proposition 1.10. *Let (Y, ϕ) be a good quotient for (X, G), then two points of X are in the same fibre of ϕ if and only if the closures of their orbits intersect.*

Proof. Suppose $\overline{O(x_1)} \cap \overline{O(x_2)} \neq \emptyset$, then, since ϕ is constant on orbits, it takes the same value on x_1 and x_2, i.e. they lie in the same fibre. To prove the converse, suppose $\overline{O(x_1)} \cap \overline{O(x_2)} = \emptyset$, these are two disjoint closed G-invariant subsets, then they have disjoint images, hence x_1 and x_2 cannot lie in the same fibre.

Proposition 1.11. *Let (Y, ϕ) be a good quotient for (X, G), then each fibre of ϕ contains a unique closed orbit of G in X (thus a good quotient parametrizes closed orbits).*

Proof. Since ϕ is surjective and G-invariant, each fibre of ϕ contains some orbit. Choose an orbit of minimal dimension in the fibre, then, by Prop. 1.7, (1), it is closed. Suppose now there are two closed orbits in the same fibre, then, by Prop. 1.10, they intersect, hence they must coincide.

Definition 1.12. A good quotient is called a **geometric quotient** if each orbit is closed.

Thus, by 1.11, a geometric quotient parametrizes all the orbits of a given action, that is, the underlying set is the orbit space. The concept of good quotient is local, indeed we have:

Proposition 1.13.

(1) *Let (Y, ϕ) be a good quotient for (X, G) and let U be an open subset of Y, then $(U, \phi_{|U})$ is a good quotient for $(\phi^{-1}(U), G)$.*

(2) *Let G act algebraically on X and let $\phi : X \to Y$ be a G-invariant morphism. Suppose there exists an open cover $\{U_i\}_{i \in I}$ of Y such that $(U_i, \phi_|)$ is a good quotient for $(\phi^{-1}(U_i), G)$ for all $i \in I$, then (Y, ϕ) is a good quotient for (X, G).*

Proof. First notice that if $U \subset Y$ is an open subset and $\phi : X \to Y$ is a G-invariant morphism, then $\phi^{-1}(U)$ is an open G-invariant subset of X, hence there is an induced action of G on it.

(1): Properties (1) through (4) of Definition 1.9 are easily seen to hold for $(U, \phi_|)$. To prove (5), consider two disjoint, G-invariant, closed subsets C_1 and C_2 of $\phi^{-1}(U)$ and suppose their images under ϕ are not disjoint. Let $y \in \phi(C_1) \cap \phi(C_2)$, then there exist elements $x_1 \in C_1$ and $x_2 \in C_2$ such that $\phi(x_1) = \phi(x_2) = y$. By Prop. 1.10 the closures of their orbits in X intersect, on the other hand $O(x_i) \subset C_i$ for $i = 1, 2$, thus $\emptyset \neq \overline{O(x_1)} \cap \overline{O(x_2)} \subset \overline{C_1} \cap \overline{C_2} \cap \phi^{-1}(U) = C_1 \cap C_2$, which contradicts the assumption of C_1 and C_2 being disjoint.

(2): $\phi : X \to Y$ is G-invariant by assumption and it is clearly surjective. Property (3) of Def. 1.9 also clearly holds; to prove property (4) consider a closed G-invariant subset $C \subset X$. For any $i \in I$, $C \cap \phi^{-1}(U_i)$ is a closed G-invariant subset of $\phi^{-1}(U_i)$, thus $\phi(C \cap \phi^{-1}(U_i)) = \phi(C) \cap U_i$ is closed in U_i for any $i \in I$, hence $\phi(C)$ is closed. Now we prove property (5). Let C_1 and C_2 be two closed, disjoint, G-invariant subsets of X, then for any $i \in I$ we get two such subsets of $\phi^{-1}(U_i)$. Taking the image under ϕ we still have disjoint subsets, i.e. $U_i \cap \phi(C_1) \cap \phi(C_2) = \emptyset$ for any i, hence $\phi(C_1) \cap \phi(C_2) = \emptyset$.

Proposition 1.14. *A good quotient is a categorical quotient (hence, it is unique up to isomorphisms).*

Proof. Let (Y, ϕ) be a good quotient for (X, G), and suppose $\psi : X \to Z$ is a morphism constant on orbits, then it is also constant on the fibres of ϕ and it factors through a set-theoretic map $\gamma : Y \to Z$. We will show that γ is in fact a regular morphism. We first assume Z is affine. In this case it is enough to prove that for any regular function $f : Z \to \mathbf{C}$ the composite map $f \circ \gamma$ is regular on Y. Consider the map $f \circ \gamma : X \to \mathbf{C}$, this is a G-invariant regular function on X, hence it defines a regular function on the quotient Y which is just $f \circ \gamma$. For the general case, consider an open affine subset $U \subset Z$, then $\gamma^{-1}(U)$ is an open subset of Y – indeed, $\gamma^{-1}(U) = Y - \phi(X - \phi^{-1}\eta^{-1}(U)) = Y - \phi(X - \psi^{-1}(U))$ and $X - \psi^{-1}(U)$ is a closed G-invariant subset of X, hence its image under ϕ is closed. Moreover, by Prop. 1.13, (2), $(\eta^{-1}(U), \phi|)$ is a good quotient for $(\psi^{-1}(U), G)$ and since U is affine we can apply the previous argument.

§2. Affine case.

Let now X be an affine variety and G an algebraic group acting on it. To take the quotient of X by G we proceed as follows: we first show that under a certain assumption on G the $\mathbf{C}$-algebra of G-invariant regular functions on X, $A(X)^G$, is finitely generated — thus its maximal spectrum is an affine algebraic variety — then we prove that the natural map $X = \operatorname{Spec} A(X) \to \operatorname{Spec} A(X)^G$ induced by the inclusion $A(X)^G \hookrightarrow A(X)$ is in fact a good quotient.

We need some preliminaries from representation theory.

Definition 2.1. Let G be a group and V a vector space. A **representation** of G on V is a group homomorphism $\rho: G \to GL(V)$ from G to the group of linear automorphisms of V. If V is finite dimensional, we say that the representation is finite dimensional.

If we are given a representation of G on V, the map ρ makes G act linearly on V, by setting $gv = \rho(g)(v)$, and we also say that V is a **G-module** (by abuse of language we sometimes say that V is a representation of G).

Definition 2.2. Let V_1, V_2 be G-modules. A morphism of G-modules, or a **G-morphism**, is a linear map $\phi: V_1 \to V_2$ such that $\phi(gv) = g\phi(v)\ \forall v \in V_1,\ g \in G$.

Definition 2.3. A subspace V' of a G-module V is called a **G-submodule**, or a subrepresentation of G, if it is (globally) G-invariant, i.e. if $GV' \subseteq V'$.

Remark 2.4. If $\phi: V_1 \to V_2$ is a G-morphism, $\operatorname{Ker}(\phi)$ and $\operatorname{Im}(\phi)$ are G-submodules of V_1, V_2 respectively.

A representation of G on V is **irreducible** if the only G-submodules of V are (0) and the whole space (in this case we also say that V is a **simple G-module**); it is **completely reducible** if it decomposes in a direct sum of irreducible ones (we also say that V is a **semisimple G-module**). Two representations V, W of a group G are **equivalent** or **isomorphic** if there exists a linear isomorphism $f: V \to W$ which is a G-morphism.

Lemma 2.5 (Schur's lemma). *Let G be a group and V and W be simple G-modules. If $\phi: V \to W$ is a G-morphism, then either ϕ is an isomorphism or $\phi = 0$.*

Proof. Consider $\operatorname{Ker}(\phi)$ and $\operatorname{Im}(\phi)$. Since they are G-submodules of simple G-modules they must be either (0) or the whole space. It follows that either $\operatorname{Ker}(\phi) = (0)$ and $\operatorname{Im}(\phi) = W$, i.e. ϕ is an isomorphism, or $\operatorname{Ker}(\phi) = V$ and $Im(\phi) = (0)$, i.e. $\phi = 0$.

Remark 2.6. Note that if one only of the two representations V, W is irreducible, we still get some weaker version of the lemma, namely:

(1) if V is irreducible, then either ϕ is injective or $\phi = 0$;
(2) if W is irreducible, then either ϕ is surjective or $\phi = 0$.

Definition 2.7. Let G be an affine algebraic group. We say that G is **linearly reductive** if for any finite dimensional G-module V and any G-submodule V' there exists a G-submodule V'' such that $V = V' \oplus V''$.

Proposition 2.8. *Let G be an affine algebraic group. G is linearly reductive if and only if every finite dimensional G-module is semisimple.*

Proof. Suppose G is linearly reductive, let V be a G-module and V' a maximal semisimple G-submodule. By assumption, V' has a supplementary V''. If $V'' \neq (0)$, it contains a non-zero simple G-submodule, say W, and $V' \oplus W$ is a semisimple G-submodule contradicting the maximality of V'. It follows that $V'' = (0)$, i.e. V itself is semisimple. Viceversa, suppose there exists a G-module V having some G-submodule with no supplementary. Choose V' maximal among them. Let $V = V_1 \oplus \ldots \oplus V_r$ be a decomposition of V into simple G-submodules. There exists i such that $V_i \not\subset V'$, thus $V' \cap V_i = (0)$. Then $V' \oplus V_i$ is a G-submodule of V with no supplementary, which contradicts the maximality of V'.

Examples. It can be shown that any finite group is linearly reductive and every compact topological group is linearly reductive.

Proposition 2.9. *Let G be an affine algebraic group. Suppose G contains a compact subgroup K (in the usual topology) such that K is Zariski-dense in G. Then G is linearly reductive.*

Proof. Let V be a G-module and V' a G-submodule, then V is also a K-module and V' is K-invariant. Since K is compact, it can be shown to be linearly reductive and hence V' has a K-invariant ~~supplementary~~ complement, say V''. Let $G' := \{g \in G | gV'' \subset V''\}$, G' is a closed subgroup of G and contains K. Since K is Zariski-dense in G, $G' = G$, i.e. V'' is G-invariant.

Example. $GL(n, \mathbf{C})$ is linearly reductive, indeed the unitary group $U(n)$ is a maximal compact subgroup.

Let now G be a linearly reductive group and let Ω be the set of isomorphism classes of irreducible representations of G. If V is a finite dimensional representation of G, and $\gamma \in \Omega$, we define $V_\gamma = \sum_{W \in \gamma, W \subset V} W$ to be the isotypical component of type γ (note that the V_γ's are G-submodules of V), then we can write V as: $V = \bigoplus_{\gamma \in \Omega} V_\gamma$. This is called the **isotypical decomposition** of V (the types and multiplicities of factors are uniquely determined). If γ_0 denotes the class of trivial representations of G, we write $V^G = V_{\gamma_0} = \{v \in V | gv = v \quad \forall g \in G\}$ for the trivial isotypical component.

Lemma 2.10. *Let $f : V_1 \to V_2$ be a G-morphism. Then $f(V_{1,\gamma}) \subset V_{2,\gamma}$ for any $\gamma \in \Omega$.*

Proof. Let $V_2 = \bigoplus_{\alpha_i \in \Omega} V_{\alpha_i}$ be the isotypical decomposition of V_2, then for any α_i we have the projection $p_i : V_2 \to V_{2,\alpha_i}$. Let $W \subset V_1$ be a simple G-submodule of type γ, we obtain for any α_i a G-morphism $p_i \circ f|_W : W \to V_{2,\alpha_i}$. By Schur's Lemma (see Remark 2.6) this morphism is zero if $\gamma \neq \alpha_i$, hence f preserves isotypical components.

Proposition and Definition 2.11. *Let G be a linearly reductive group and V a finite dimensional G-module, then there exists a unique G-morphism $R : V \to V^G$ such that $R|_{V^G} = \mathrm{id}$. It is called* **Reynolds' operator**.

Proof. Consider the isotypical decomposition $V = V^G \oplus \bigoplus_{\gamma \neq \gamma_0} V_\gamma$. The projection on the trivial component V^G is a G-morphism with the required property. This proves the existence of Reynolds' operator. Consider now any simple G-submodule W of V which is not in the component V^G, then by Schur's Lemma $R|_W = 0$. It follows that $\operatorname{Ker} R = \bigoplus_{\gamma \neq \gamma_0} V_\gamma$, hence R is uniquely determined.

Proposition 2.12 (Functoriality of Reynolds' operator). *Let $f : V_1 \to V_2$ be a G-morphism and R_i be the Reynolds' operators on V_i for $i = 1, 2$. Then the following diagram commutes:*

$$\begin{array}{ccc} V_1 & \xrightarrow{f} & V_2 \\ {\scriptstyle R_1}\downarrow & & \downarrow{\scriptstyle R_2} \\ V_1^G & \xrightarrow{f} & V_2^G \end{array}$$

Proof. Since f preserves isotypical components, it induces a morphism $V_1^G \to V_2^G$, which we still denote f. Moreover, since $\operatorname{Ker} R_i = \bigoplus_{\gamma \neq \gamma_0} V_{i,\gamma}$, we have that $f(\operatorname{Ker} R_1) \subset \operatorname{Ker} R_2$. It follows that $R_2 \circ f|_{\operatorname{Ker} R_1} = 0 = f \circ R_1|_{\operatorname{Ker} R_1}$ and $R_2 \circ f|_{V_1^G} = 0 = f \circ R_1|_{V_1^G}$, i.e. $R_2 \circ f$ and $f \circ R_1$ coincide on $V_1 = V_1^G \oplus \bigoplus_{\gamma \neq \gamma_0} V_{1,\gamma}$.

Let us now turn to our problem of constructing a quotient for an algebraic variety acted on by an algebraic group. Let (X, G) be an algebraic action of G on X, we recall that there is an induced action on the algebra of regular functions $A(X)$ given by $gf(x) := f(g^{-1}x)$ and for any $g \in G$ the map $f \mapsto gf$ is a $\mathbf{C}$-algebra automorphism. In particular, $A(X)$ becomes a G-module, in general of infinite dimension.

Definition 2.13. Let A be a commutative $\mathbf{C}$-algebra. We say that A is **finitely generated** (or of **finite type**) if there exists a surjective homomorphism of $\mathbf{C}$-algebras

$$\mathbf{C}[x_1, \dots, x_n] \longrightarrow A \longrightarrow 0$$

for some integer $n \geq 0$, i.e., if there exist elements $f_1, \dots, f_n$ in A such that any $f \in A$ can be written as $f = \sum_{i_j \geq 0} a_{i_1 \dots i_n} f_1^{i_1} \cdots f_n^{i_n}$, with $a_{i_1 \dots i_n} \in \mathbf{C}$.

The central result of the whole theory is the following:

Theorem 2.14. *Let G be a linearly reductive group and V a finite dimensional G-module. Let $S(V^*)$ denote the symmetric algebra over V^*, that is the algebra of polynomial functions over V. Then the subalgebra of G-invariant functions $S(V^*)^G$ is finitely generated.*

Proof. The crucial step is to define Reynolds' operator on $S(V^*)$ (we defined it only for finite dimensional G-modules) and to show that it is $S(V^*)^G$-linear. In order to do this, observe that since the action of G on V is linear, the induced action on $S(V^*) = \bigoplus_{k \geq 0} S^k(V^*)$ preserves the degree. Then we can regard $S(V^*)$ as an increasing union of finite dimensional G-submodules $S_m := \bigoplus_{k \leq m} S^k(V^*)$ with

$m \in \mathbf{N}$. Now on each S_m we have Reynolds' operator and using the natural inclusions $S_m \hookrightarrow S_{m+1}$, by functoriality and uniqueness we obtain an operator R on the whole space $S(V^*)$. In general Reynolds' operator is not an algebra homomorphism, but we do have the identity $R(fh) = fR(h)$ for any $f \in S(V^*)^G$ and $h \in S(V^*)$ (**Reynolds' identity**). Indeed, suppose f is G-invariant and is contained in some S_m and $h \in S_{m'}$, multiplication by f defines a G-morphism $S_{m'} \to S_{m'+m}$, then applying Reynolds' operator we get, by functoriality, the desiderd identity. Consider the ideal I of $S(V^*)$ generated by $S(V^*)^G_+ = \bigoplus_{k>0} S^k(V^*)^G$, by Hilbert basis theorem it admits a finite set of generators, say $f_1, \ldots, f_r$ which we can choose in $S(V^*)^G_+$. We will show that these elements do in fact generate $S(V^*)^G$ as a $\mathbf{C}$-algebra. We use induction on the degree. In degree zero $S^0(V^*)^G = \mathbf{C}$, so there is nothing to prove. Let d be a positive integer and suppose that any element in $S(V^*)^G$ of degree lower than d can be written as a polynomial in $f_1, \ldots, f_r$ with coefficients in $\mathbf{C}$. If $b \in S^d(V^*)^G$ it can be written as $b = \sum_{i=1}^r a_i f_i$ with $a_i \in S(V^*)$ and $\deg(a_i) < d$ for any $i = 1, \ldots, r$, since the f_i have positive degree. Applying Reynolds' operator and Reynolds' identity we get

$$b = R(b) = \sum_{i=1}^{r} R(a_i f_i) = \sum_{i=1}^{r} R(a_i) f_i.$$

Now for all $i = 1, \ldots, r$ we have $\deg(R(a_i)) \le \deg(a_i) < d$. By induction hypothesis $R(a_i)$ can be written as a polynomial in the f_i's with coefficients in $\mathbf{C}$. It follows that $S(V^*)^G$ is a finitely generated $\mathbf{C}$-algebra.

We shall now deal with the general case. Let X be an affine variety acted on by an algebraic group G, as in the case just treated in Theorem 2.14, we wish to define Reynolds' operator on $A(X)$, although this is in general an infinite dimensional G-module. We have:

Lemma 2.15. *Let (X, G) be an algebraic action of G on X and let W be a finite-dimensional subspace of $A(X)$ (as a $\mathbf{C}$-vector space). Then W is contained in a finite-dimensional, G-invariant subspace of $A(X)$ on which the action of G is algebraic.*

Proof. The action $\phi : G \times X \to X$ induces a map of $\mathbf{C}$-algebras: $\phi^* : A(X) \to A(G \times X)$ such that $\phi^* f(g, x) = f(\phi(g, x)) = f(gx)$. On the other hand, $A(G \times X) = A(G) \otimes A(X)$, thus $\phi^* f$ is a finite sum of the form $\sum H_k \otimes F_k$, with $H_k \in A(G)$ and $F_k \in A(X)$. Let $f_1, \ldots, f_N$ be a basis for $W \subset A(X)$, then $\forall g \in G$ we have:

$$gf_i(x) = f_i(g^{-1}x) = \phi^* f_i(g^{-1}, x) = \sum_{k=1}^{r_i} H_{ik}(g^{-1}) F_{ik}(x) = (\sum_{k=1}^{r_i} H_{ik}(g^{-1}) F_{ik})(x)$$

That is, GW is contained in the subspace generated by $\{F_{i1}, \ldots, F_{ir_i} | i = 1, \ldots, N\}$, hence the subspace spanned by GW is finite-dimensional (and clearly it is G-invariant). That the action is algebraic is clear by the above formula.

Remark 2.16. The lemma says, in particular, that any orbit of G in $A(X)$ generates a finite-dimensional subspace of $A(X)$.

Suppose now the group G is linearly reductive. Any $f \in A(X)$ is contained in some finite dimensional G-invariant subspace, on which Reynolds' operator is defined. By uniqueness and functoriality, it is independent of the subspace chosen and in this way we can define Reynolds' operator on the whole space $A(X)$.

Theorem 2.17. *Let X be an affine algebraic variety and G a linearly reductive group acting on X. Let $A(X)$ be the $\mathbf{C}$-algebra of regular functions on X and $A(X)^G$ the $\mathbf{C}$-algebra of invariant functions. Then $A(X)^G$ is finitely generated.*

Proof. We will use the special case treated in Theorem 2.14. Since $A(X)$ is a finitely generated $\mathbf{C}$-algebra, let $f_1, \ldots, f_m$ be a set of generators and let W be the G-module spanned by them (i.e., the minimal G-invariant subspace containing the linear span of $f_1, \ldots, f_m$). By 2.15, W is a finite dimensional subspace of $A(X)$, then letting $S(W)$ denotes the symmetric algebra over W we have a surjective homomorphism of algebras $S(W) \to A(X) \to 0$ which is also a G-morphism, since the action of G on functions is linear. Applying Reynolds' operator we get a commutative diagram:

$$\begin{array}{ccccc} S(W) & \xrightarrow{\alpha} & A(X) & \longrightarrow & 0 \\ {\scriptstyle R}\downarrow & & {\scriptstyle R}\downarrow & & \\ S(W)^G & \xrightarrow{\alpha} & A(X)^G & & \end{array}$$

which shows that the map $S(W)^G \to A(X)^G$ is also surjective (indeed, take $f \in A(X)^G \subset A(X)$ and lift it to some element $h \in S(W)$, by commutativity $\alpha R(h) = R(\alpha(h)) = R(f) = f$). By Theorem 2.14 we know that $S(W)^G$ is finitely generated, hence $A(X)^G$ is too.

Remark 2.18. Note that the surjection $S(W) \to A(X) \to 0$ yields an embedding of the variety X in the G-module $W^* = \operatorname{Spec} S(W)$, in such a way that X is G-invariant and the induced action on X is the original one.

We shall now prove that the maximal spectrum $\operatorname{Spec} A(X)^G$ with the map $\phi : X \to \operatorname{Spec} A(X)^G$ induced by the inclusion $A(X)^G \hookrightarrow A(X)$ is a good quotient for the given action of G on X. Let us first state two results we will need in the proof:

Lemma 2.19. *With notation as above, let $\mathcal{B}$ be an ideal of $A(X)^G$ and $\mathcal{B}'$ the ideal in $A(X)$ generated by $\mathcal{B}$, then $\mathcal{B}' \cap A(X)^G = \mathcal{B}$.*

Proof. Clearly $\mathcal{B} \subset \mathcal{B}' \cap A(X)^G$. Let $f \in \mathcal{B}' \cap A(X)^G$, then $f = \sum_i a_i h_i$, with $h_i \in \mathcal{B}$, $a_i \in A(X)$. Applying Reynolds' operator we get: $f = R(f) = \sum_i R(a_i) h_i$, hence $f \in \mathcal{B}$.

Lemma 2.20. *Let (X, G) be an algebraic action of a linearly reductive group G on an affine variety X. Let W_1, W_2 be two disjoint G-invariant closed subsets of X, then there exists $f \in A(X)^G$ such that $f(W_1) = 0$ and $f(W_2) = 1$.*

Proof. The algebraic version of Urysohn lemma states that there exists a function $\phi \in A(X)$ such that $\phi(W_1) = 0$ and $\phi(W_2) = 1$. Then $f = R(\phi)$ will do, since by functoriality it still takes the value 0 on W_1 and 1 on W_2.

Theorem 2.21. *Let X be an affine algebraic variety and G a linearly reductive group acting on it. Then the pair (Y, ϕ), where $Y := \operatorname{Spec} A(X)^G$ and $\phi : X \to Y$ is the map induced by the inclusion $A(X)^G \hookrightarrow A(X)$, is a good quotient for the given action of G on X.*

Proof. We have to check properties (1) through (5) of Definition 1.9:

(1): Suppose that ϕ is not G-invariant, that is $\phi(x) \neq \phi(gx)$ for some $x \in X$ and some $g \in G$. Since Y is affine, there exists a regular function $h \in A(Y)$ taking different values at $\phi(x)$ and $\phi(gx)$. On the other hand, $A(Y) = A(X)^G$, that is, h is constant on orbits, which yields a contradiction.

(2): Let $\mathcal{M}$ be a maximal ideal in $A(X)^G$ and $\mathcal{M}A(X)$ the ideal generated by $\mathcal{M}$ in $A(X)$, then $\mathcal{M}A(X)$ is a proper ideal (otherwise $\mathcal{M}$ would be the whole ring $A(X)^G$, see Lemma 2.19 above). Let $\mathcal{M}'$ be a maximal ideal of $A(X)$ containing it. Clearly $\mathcal{M} \subset \mathcal{M}' \cap A(X)^G$ and since they are both maximal they coincide. So that ϕ is surjective.

(3): To prove (3) it is enough to check it on the special affine open subsets $Y_f := \{y \in Y | f(y) \neq 0\}$, $f \in A(Y)$, which form a basis for the Zariski topology. Since functions on Y are invariant functions on X, we clearly have $\phi^{-1}(Y_f) = X_f$, hence ϕ is affine. Let us now show that $A(Y_f) = A(X_f)^G$. But $A(Y_f)$ is just the localization

$$A(Y)_f = \{h/f^n, h \in A(Y), n \in \mathbf{N}\} = \{h/f^n, h \in A(X)^G, n \in \mathbf{N}\} = (A(X)^G)_f$$

and $(A(X)^G)_f \subset (A(X)_f)^G = A(X_f)^G$. On the other hand, let $h/f^n \in A(X_f)^G$, with $h \in A(X)$, then $g(h/f^n) = h/f^n$ for any $g \in G$, but since f is G-invariant (and we may assume $f \neq 0$) this implies $gh = h$ for any $g \in G$, that is $h \in A(X)^G$.

(4) and (5): By Lemma 2.20 above, we know that for any pair of closed disjoint G-invariant subsets W_1, W_2 of X there exists a G-invariant function f such that $f(W_1) = 0$ and $f(W_2) = 1$. Since f can be regarded as a function on Y, we have $f(\phi(W_1)) = 0$, $f(\phi(W_2)) = 1$, hence $\overline{\phi(W_1)} \cap \overline{\phi(W_2)} = \emptyset$. This proves (4). Let now W be a closed invariant subset of X, suppose its image under ϕ is not closed and let $y \in \overline{\phi(W)} - \phi(W)$. Applying the previous argument with $W_1 = W$ and $W_2 = \phi^{-1}(y)$ we get $\overline{\phi(W)} \cap y = \emptyset$, hence a contradiction. This proves (5).

§3. Projective case.

Suppose G is a linearly reductive group which acts linearly on a vector space V, then this action commutes with that of $\mathbf{C}^*$ (acting by multiplication) and hence G also operates on the projective space $\mathbf{P}(V)$. Does there exists a quotient of $\mathbf{P}(V)$ under this action ? We will show that there is a G-invariant open subset of $\mathbf{P}(V)$ which admits a good quotient and such a quotient is projective. The same holds for any projective variety acted on by a linearly reductive group in such a way that the action extends to a linear action on some ambient projective space (Thm. 3.9).

Exercise. Show that $\mathbf{C}^n - \{0\}$ with the action of $\mathbf{C}^*$ defined by multiplication admits a geometric quotient (which is, of course, projective space $\mathbf{P}^{n-1}$).

To start with we need a general result. Consider a finitely generated graded $\mathbf{C}$-algebra $B = \bigoplus_{m \geq 0} B_m$ with $B_0 = \mathbf{C}$ and let B_+ be the maximal ideal $\bigoplus_{m>0} B_m$.

The grading allows to define a natural action of $\mathbf{C}^*$ on B setting, for any graded component B_m:

$$\mathbf{C}^* \times B_m \to B_m \qquad (\lambda, b) \mapsto \lambda^{-m} b.$$

We thus obtain an action of $\mathbf{C}^*$ on $\operatorname{Spec} B$ which leaves 0 invariant (0 is the point corresponding to B_+), hence it induces an action on $\operatorname{Spec} B - \{0\}$. We have:

Proposition and Definition 3.1. *With the above notation, ($\operatorname{Spec} B - \{0\}$, $\mathbf{C}^*$) admits a geometric quotient which is a projective variety called* $\operatorname{Proj} B$.

To prove this Proposition we need a few more preliminaries.

Recall that a morphism $\phi : X \to Y$ of algebraic varieties is **quasi finite** if all of its fibres are finite. It is **finite** if there exists an affine open cover $\{U_i\}$ of Y such that $\phi^{-1}(U_i)$ is affine for every i and the restricted morphism $\phi^{-1}(U_i) \to U_i$ corresponds to a ring homomorphism $B \to A$ with A a finitely generated B-module.

Lemma 3.2 (A. Ramanathan, [R]). *Let G be a linearly reductive group acting on two algebraic varieties X and Y and suppose $f : X \to Y$ is a G-morphism (i. e. $f(gx) = gf(x)$ for any $x \in X$ and $g \in G$).*

(1) *If f is affine and Y has a good quotient, then X also has a good quotient and the induced morphism $\bar{f} : X//G \to Y//G$ is still affine.*
(2) *If f is finite and Y has a good quotient then X also has a good quotient and the induced morphism $\bar{f} : X//G \to Y//G$ is finite; moreover, if $Y//G$ is a geometric quotient then so is $X//G$.*

Theorem 3.3 (R. Hartshorne, [H], Ch. 1). *Let $f : X \to Y$ be a finite morphism of algebraic varieties. If Y is projective then so is X and if L is an ample line bundle on Y then $f^*(L)$ is ample on X.*

Proof of Proposition 3.1. Suppose B is generated as a $\mathbf{C}$-algebra by homogeneous elements $f_1, \ldots, f_k$ of degrees $m_1, \ldots, m_k$ respectively. The natural surjection of algebras $\mathbf{C}[x_1, \ldots, x_k] \to B$ induces an embedding $\operatorname{Spec} B \hookrightarrow \operatorname{Spec} \mathbf{C}[x_1, \ldots, x_k] \cong \mathbf{C}^k$ which maps the point 0 to 0, hence we get an inclusion

$$\phi : \operatorname{Spec} B - \{0\} \hookrightarrow \mathbf{C}^k - \{0\}.$$

Consider now on $\mathbf{C}^k - \{0\}$ the following $\mathbf{C}^*$ action, which makes ϕ a $\mathbf{C}^*$-morphism:

$$(*) \qquad \mathbf{C}^* \times (\mathbf{C}^k - \{0\}) \to \mathbf{C}^k - \{0\} \qquad (\lambda, (z_1, \ldots, z_k)) \mapsto (\lambda^{m_1} z_1, \ldots, \lambda^{m_k} z_k)$$

We claim that this action admits a geometric quotient. To show this we will apply Lemma 3.2, so we define a morphism $f : \mathbf{C}^k - \{0\} \to \mathbf{C}^k - \{0\}$ such that $(z_1, \ldots, z_k) \mapsto (z_1^{m/m_1}, \ldots, z_k^{m/m_k})$ where $m = l.\ c.\ m.\{m_1, \ldots, m_k\}$ and make $\mathbf{C}^*$ operate on the second $\mathbf{C}^k - \{0\}$ in such a way to make f invariant, that is by $(\lambda, (z_1, \ldots, z_k)) \mapsto (\lambda^m z_1, \ldots, \lambda^m z_k)$. Now f is finite because $\mathbf{C}[x_1, \ldots, x_k]$ is finitely generated over $\mathbf{C}[x_1^{m/m_1}, \ldots, x_k^{m/m_k}]$. Since a geometric quotient of the second $\mathbf{C}^k - \{0\}$ exists (it is projective space $\mathbf{P}^{k-1}$), by Lemma 3.2 the first $\mathbf{C}^k - \{0\}$ with the action (*) also has a geometric quotient and, by Thm. 3.3, it is projective. It is called **weighted projective space**. Now applying again Lemma 3.2 and Theorem 3.3 to ϕ, we conclude that $\operatorname{Spec} B - \{0\}$ admits a geometric quotient which is projective.

Remark 3.4. Suppose A and B are graded algebras and $f : B \to A$ a homomorphism of degree zero. One could expect there is a corresponding map from $\operatorname{Proj} A$ to $\operatorname{Proj} B$, as in the case of spectra, but this is not true. Instead we can do the following: consider the map $f_* : \operatorname{Spec} A \to \operatorname{Spec} B$, the action of $\mathbf{C}^*$ on the algebras determined by the grading also gives an action on the spectra and f_* is a $\mathbf{C}^*$-morphism. Now the inverse image under f_* of $0 \in \operatorname{Spec} B$ is a closed, $\mathbf{C}^*$-invariant subset of $\operatorname{Spec} A$, which we call $\widetilde{N}$, so we get a map

$$\operatorname{Spec} A - \widetilde{N} \to \operatorname{Spec} B - \{0\}.$$

Besides, $\widetilde{N} - \{0\}$ is a closed $\mathbf{C}^*$-invariant subset of $\operatorname{Spec} A - \{0\}$, so that its image in the quotient $\operatorname{Proj} A$ is a closed subset, say N, and we finally obtain a map

$$\operatorname{Proj} A - N \to \operatorname{Proj} B.$$

For example, if A is the polynomial ring $\mathbf{C}[x_0, \dots, x_n]$ and B is the subring of invariants for some action of a linearly reductive group G, then the subset N of $\operatorname{Proj} A = \mathbf{P}^n$ consists of points on which every homogeneous G-invariant polynomial of positive degree vanishes and, as we shall see, $\mathbf{P}^n - N \to \operatorname{Proj} B$ is a good quotient. So we cannot get a quotient of the whole projective space but just of an open subset, which we are going to define precisely:

Definition 3.5. Let G be a linearly reductive group acting linearly on a finite dimensional vector space V. A point $x \in \mathbf{P}(V)$ is **semistable** for the induced action of G on $\mathbf{P}(V)$ if there exists a homogeneous G-invariant polynomial f of positive degree such that $f(x) \neq 0$. The subset of semistable points is denoted $\mathbf{P}(V)^{ss}$.

Remark 3.6. Clearly $\mathbf{P}(V)^{ss}$ is an open subset and it is G-invariant.

More generally, if $Y \subset \mathbf{P}(V)$ is a G-invariant subvariety we can say that $y \in Y$ is semistable if it is such as a point of $\mathbf{P}(V)$. This definition however is not very satisfactory because it depends on the embedding of Y in $\mathbf{P}(V)$. Instead, we can give an intrinsic definition in a different setting. Let Y be a complete variety acted on by some algebraic group G and L an ample line bundle on Y. A G-**linearization** of L is a lift of the action of G on Y to L as a line bundle automorphism. Notice that in this case G also operates on any tensor power $L^{\otimes r}$ of L and on the space of global sections $H^0(Y, L^{\otimes r})$ for any $r \in \mathbf{N}$. In this setting we put:

Definition 3.7. Let Y be a complete variety acted on by a linearly reductive group G and L an ample line bundle on Y with a G-linearization. A point $y \in Y$ is **semistable** with respect to this G-linearization if there exists an invariant section $s \in H^0(Y, L^{\otimes n})$ for some $n \geq 1$ such that $s(y) \neq 0$. The subset of semistable points is denoted Y^{ss}.

Remark 3.8. Again, Y^{ss} is an open and G-invariant subset of Y.

It is not very hard to check that for a complete variety the two definitions of semistability give in fact the same notion: suppose Y and L and G are as in Definition 3.7, then a suitable tensor power $L^{\otimes d}$ gives an embedding of Y in a projective space $j : Y \hookrightarrow \mathbf{P}(H^0(Y, L^{\otimes d}))$ in such a way that the action on $H^0(Y, L^{\otimes d})$ induces the original action on Y. With respect to this embedding, a point $y \in Y$ is semistable if there exists a homogenous G-invariant polynomial of positive degree k on $\mathbf{P}(H^0(Y, L^{\otimes d}))$ which does not vanish at y, but this gives, by restriction to Y, a G-invariant section in $H^0(Y, L^{\otimes dk})$. Viceversa, suppose G is a linearly reductive group acting linearly on some finite dimensional vector space V and $Y \subset \mathbf{P}(V)$ is a G-invariant subvariety. Let $L = \mathcal{O}_{\mathbf{P}(V)}(1)_{|Y}$ then of course L is an ample line bundle on Y (in fact, very ample) and the action of G on V induces a G-linearization of L. Now consider the exact sequence of sheaves on $\mathbf{P}(V)$:

$$0 \to \mathcal{I}_Y \to \mathcal{O}_{\mathbf{P}(V)} \to \mathcal{O}_Y \to 0.$$

Tensoring by $\mathcal{O}_{\mathbf{P}(V)}(d)$ and taking cohomology we get a homomorphism of finite dimensional vector spaces $\psi : H^0(\mathcal{O}_{\mathbf{P}(V)}(d)) \to H^0(L^{\otimes d}))$ which is surjective for d large enough (because $H^1(\mathcal{I}_Y(d)) = 0$ for $d \gg 0$ by Serre Theorem B). Since both the vector spaces are G-modules we can apply Reynolds' operator and by functoriality the induced homomorphism $H^0(\mathcal{O}_{\mathbf{P}(V)}(d))^G \to H^0(L^{\otimes d}))^G$ is still surjective. Thus a G-invariant section of $L^{\otimes d}$ lifts to a G-invariant homogeneous polynomial of degree d for $d \gg 0$.

Suppose now again that we have a complete variety Y with an ample line bundle L and a linearly reductive group G acting on Y with a G-linearization of L. Let $S(Y)$ be the homogeneous coordinate ring of Y with respect to the embedding $Y \hookrightarrow \mathbf{P}(H^0(Y, L^{\otimes n}))$ defined by a suitable tensor power $L^{\otimes n}$ of L and let $S(Y)^G$ be the subring of invariants. Then $Y = \operatorname{Proj} S(Y)$ and by Remark 3.4 we have a morphism $\operatorname{Proj} S(Y) - N \to \operatorname{Proj} S(Y)^G$, where N is the set of zeroes of the ideal of $S(Y)$ generated by $S(Y)^G_+$, that is N is the set of non semistable points of Y, hence $\operatorname{Proj} S(Y) - N = Y^{ss}$.

Theorem 3.9. *With the above notation, the morphism $Y^{ss} \to \operatorname{Proj} S(Y)^G$ induced by the inclusion $S(Y)^G \subset S(Y)$ is a good quotient for the given action of G on Y and the given linearization.*

Proof. The inclusion $S(Y)^G \subset S(Y)$ induces a morphism $\operatorname{Spec} S(Y) \to \operatorname{Spec} S(Y)^G$ which is a good quotient (Thm. 2.21) and the restricted morphism (Remark 3.4) $\operatorname{Spec} S(Y) - \widetilde{N} \to \operatorname{Spec} S(Y)^G - \{0\}$ is still a good quotient. Now we can make $\mathbf{C}^*$ act on both sides to get the corresponding Proj varieties, so we obtain a diagram where the vertical arrows are also good quotients (Prop. 3.1):

$$\begin{array}{ccc} \operatorname{Spec} S(Y) - \widetilde{N} & \longrightarrow & \operatorname{Spec} S(Y)^G - \{0\} \\ \downarrow & & \downarrow \\ \operatorname{Proj} S(Y) - N & \longrightarrow & \operatorname{Proj} S(Y)^G \end{array}$$

To conclude that $\operatorname{Proj} S(Y) - N \to \operatorname{Proj} S(Y)^G$ is a good quotient we just apply the following Lemma on transitivity of good actions to the product group $G \times \mathbf{C}^*$.

Lemma 3.10. *Let H be an affine algebraic group and K a closed normal subgroup. Suppose that H acts on an algebraic variety X and that there exists a good quotient $X//K$. Then there is a good quotient for the action of H/K on $X//K$ if and only if there exists a good quotient for the action of H on X (and the quotient space is the same).*

§4. Hilbert-Mumford semistability criterion.

The problem raised by Theorem 3.9 is how to get hands on semistable points? Hilbert-Mumford criterion answers the question in an effective way.

Proposition 4.1. *Let G be a linearly reductive group acting linearly on the vector space $\mathbf{C}^N$. Consider a point $x \in \mathbf{P}^{N-1}$ and let $\hat{x} \in \mathbf{C}^N - 0$ be a vector which represents it. Then x is semistable if and only if $0 \notin \overline{G\hat{x}}$.*

Proof. Suppose $0 \in \overline{G\hat{x}}$ and let f be a homogeneous G-invariant polynomial of positive degree. Then f is constant on $\overline{G\hat{x}}$ and $f(0) = 0$, therefore f vanishes on $\overline{G\hat{x}}$ and hence on x. Viceversa, assume $\{0\} \cap \overline{G\hat{x}} = \emptyset$, then by Lemma 2.20 there exists an invariant polynomial f such that $f(0) = 0$ and $f(\overline{G\hat{x}}) = 1$. Write $f = f_1 + \cdots + f_m$ where f_i is the homogeneous component of degree i (notice that we have no constant term f_0 since $f(0) = 0$). The f_i's are also G-invariant and there exists at least one of them which does not vanish at $\overline{G\hat{x}}$ and hence at x.

Definition 4.2. A **1-parameter subgroup** of a group G is a nontrivial group homomorphism $\lambda : \mathbf{C}^* \to G$ of the multiplicative group $\mathbf{C}^*$ into G.

If G acts on a projective variety X and $\lambda : \mathbf{C}^* \to G$ is a 1-parameter subgroup then $\mathbf{C}^*$ acts on X via λ. If we think of $\mathbf{C}^* \subset \mathbf{C}$ then we can make $s \in \mathbf{C}^*$ tend to 0, meaning the zero of $\mathbf{C}$.

Now we come to Hilbert-Mumford criterion:

Theorem 4.3 (Hilbert-Mumford). *Let G be a linearly reductive group acting linearly on $\mathbf{C}^N$. Then a point $x \in \mathbf{P}^{N-1}$ is semistable if and only if for any 1-parameter subgroup λ of G the point x is semistable for the action of $\mathbf{C}^*$ given by λ.*

Proof. The "only if" part is clear. We shall prove that if x is not semistable there exists a 1-parameter subgroup $\lambda : \mathbf{C}^* \to G$ such that $\lim_{s\to 0} \lambda(s)\hat{x} = 0$, where as usual $\hat{x}$ denotes a point in $\mathbf{C}^N - \{0\}$ representing x. The proof consists of two steps. First we reduce to consider the case of a torus showing that if x is not semistable there exists a maximal torus $T \subset G$ such that $0 \in \overline{T\hat{x}}$. The crucial result we use here is that the group G decomposes as KTK, with K a maximal compact subgroup and T a maximal torus (i. e. every element of G can be written as hdk with $h,k \in K$ and $d \in T$, roughly we say that "the noncompactness of G is contained in a maximal torus") (reference?). The second step consists in proving that for a given torus T we can find a 1-parameter subgroup $\lambda : \mathbf{C}^* \to T$ such that $\lim_{s\to 0} \lambda(s)\hat{x} = 0$.

Step 1. Recall that a torus is a product $(\mathbf{C}^*)^m$ of copies of the multiplicative group $\mathbf{C}^*$. Suppose by contradiction that for any maximal torus T of G we have

$0 \notin \overline{T\hat{x}}$. Then for any point $z \in G\hat{x}$ we still have $0 \notin \overline{Tz}$, indeed, writing $z = g\hat{x}$ we get

$$\{0\} \cap \overline{Tg\hat{x}} = \{0\} \cap \overline{gg^{-1}Tg\hat{x}} = g(\{0\} \cap \overline{T'\hat{x}}) = \emptyset$$

since $g(0) = 0$ (G acts linearly on $\mathbf{C}^N$) and $T' = g^{-1}Tg$ is still a maximal torus. Thus, by Lemma 2.20, for any $z \in G\hat{x}$ we can find a T-invariant regular function $f_z : \mathbf{C}^N \to \mathbf{C}$ such that $f_z(0) = 0$ and $f_z(\overline{Tz}) = 1$. Let U_z be the open subset $\{u \mid f_z(u) \neq 0\}$ and K a maximal compact subgroup of G such that $G = KTK$, then $K\hat{x}$ is compact and we can find a finite open covering $U_{z_1} \cup \cdots \cup U_{z_r}$ of $K\hat{x}$. Define

$$\phi : \mathbf{C}^N \to \mathbf{R} \qquad u \mapsto |f_{z_1}(u)| + \cdots + |f_{z_r}(u)|.$$

By construction the function ϕ is strictly positive on $K\hat{x}$ and also on $TK\hat{x}$ since the f_{z_i}'s are T-invariant, while it vanishes on zero since the f_{z_i}'s do. In conclusion we get that $\{0\} \cap cl(TK\hat{x}) = \emptyset$, where $cl(-)$ denotes the closure in the usual topology. It follows that $\{0\} \cap Kcl(TK\hat{x}) = K(\{0\} \cap cl(TK\hat{x})) = \emptyset$. Now consider the inclusions

$$G\hat{x} = KTK\hat{x} \subset Kcl(TK\hat{x}) \subset Kcl(G\hat{x}) \subset cl(G\hat{x})$$

Observe that the last two inclusions are in fact equalities because $Kcl(TK\hat{x})$ is closed. On the other hand, recall that for a locally closed subset (in the Zariski topology) of an algebraic variety over $\mathbf{C}$ the closures in the usual topology and in the Zariski topology coincide (see [GAGA]). Since the orbit $G\hat{x}$ is locally closed (Prop. 1.7) we conclude that $0 \notin Kcl(TK\hat{x}) = cl(G\hat{x}) = \overline{G\hat{x}}$ which contradicts the hypothesis.

Step 2. Consider a torus T acting linearly on $\mathbf{C}^N$ and let $\mathrm{Hom}(T, \mathbf{C}^*)$ be its group of characters and $X_*(T) = \mathrm{Hom}(\mathbf{C}^*, T)$ be the group consisting of 1-parameter subgroups and the trivial homomorphism. Then there is a pairing

$$X_*(T) \times X(T) \to \mathbf{Z} \qquad (\lambda, \chi) \mapsto <\lambda, \chi>$$

where $<\lambda, \chi>$ is the integer defined by the endomorphism $\chi \circ \lambda : \mathbf{C}^* \to \mathbf{C}^*$ such that $\chi(\lambda(s)) = s^{<\lambda,\chi>}$. Recall that a linear action of a torus on $\mathbf{C}^N$ can be diagonalized, that is there exists a basis $e_1, \ldots, e_N$ of $\mathbf{C}^N$ such that $te_i = \chi_i(t)e_i$ for some character χ_i, so that to any $t \in T$ we can associate an invertible diagonal matrix $diag(\chi_1(t), \ldots, \chi_N(t))$. This defines a map $\rho : T \to D^* \subset D$, where D denotes the set of $N \times N$ diagonal matrices and D^* the subset of the invertible ones. Let $\hat{x} = \sum \hat{x}_i e_i$ and consider the characters χ_i corresponding to the nonzero coefficients $\hat{x}_i$ of $\hat{x}$; up to renumbering assume they are $\chi_1, \ldots, \chi_l$. Now suppose $0 \in \overline{T\hat{x}}$, we claim that the trivial character — i. e. the constant function $T \to \mathbf{C}^*$ equal to 1 — cannot be written as $\chi_1^{m_1} \cdot \ldots \cdot \chi_l^{m_l}$ with the m_i's nonnegative integers and at least one of them positive. Indeed, if it was the case, the morphism $D \to \mathbf{C}$ defined by $diag(a_1, \ldots, a_N) \mapsto a_1^{m_1} \cdot \ldots \cdot a_l^{m_l}$ would be constant equal to 1 on $\rho(T)$ and hence also on its closure $\overline{\rho(T)}$ in D. On the other hand, $0 \in \overline{T\hat{x}}$ implies that the zero matrix belongs to $\overline{\rho(T)}$ and of course its image under the above morphism is zero, which yields a contradiction. Consider now the lattice $X(T) \otimes \mathbf{Q}$. The condition that the trivial character cannot be written as $\chi_1^{m_1} \cdot \ldots \cdot \chi_l^{m_l}$ with the m_i's nonnegative integers and at least one of them positive means that 0 does not belong to the convex closure of the set $\{\chi_1, \dots, \chi_l\}$ in $X(T) \otimes \mathbf{Q}$. This in turn says that there exists a hyperplane in $X(T) \otimes \mathbf{Q}$ such that the χ_i's are all on one side of it, in other words, there exists a linear function f with rational coefficients on $X(T) \otimes \mathbf{Q}$ such that $f(\chi_i) > 0$ for all $i = 1, \dots, l$, or else a 1-parameter subgroup $\lambda \in X_*(T) \otimes \mathbf{Q}$ such that $< \lambda, \chi_i >$ is positive for any $i = 1, \dots, l$. Now we are done: the action of $\mathbf{C}^*$ on $\hat{x}$ via λ is given by

$$\lambda(s)\hat{x} = \sum \hat{x}_i \lambda(s) e_i = \sum \hat{x}_i \chi_i(\lambda(s)) e_i = \sum \hat{x}_i s^{<\lambda, \chi_i>} e_i$$

since by construction $< \lambda, \chi_i >$ is positive for any i such that $\hat{x}_i \neq 0$ we see that $\lim_{s \to 0} \lambda(s)\hat{x} = 0$.

This theorem translates easily into a numerical criterion for semistability. If $\lambda : \mathbf{C}^* \to G$ is a 1-parameter subgroup of G and G acts linearly on $\mathbf{C}^N$, then $\mathbf{C}^*$ also acts on $\mathbf{C}^N$ via λ and the action can be diagonalized. Let $e_1, \dots, e_N$ be a basis of $\mathbf{C}^N$ which diagonalizes this action, then for any $t \in \mathbf{C}^*$ we have $\lambda(s)e_1 = s^{r_i} e_i$ for some integer r_i. Let $x \in \mathbf{P}^{N-1}$ and $\hat{x} = \sum \hat{x}_i e_i \in \mathbf{C}^N - 0$ a vector representing it. We set:

Definition 4.4. With the above notation:

$$\mu(\lambda, x) := max\{-r_i \mid \hat{x}_i \neq 0\}$$

It is clear that the definition does not depend on the vector $\hat{x}$ representing x.

Theorem 4.5 (Numerical criterion for semistability). *Let G be a linearly reductive group acting linearly on $\mathbf{C}^N$. A point $x \in \mathbf{P}^{N-1}$ is semistable if and only if for any 1-parameter subgroup λ of G we have $\mu(\lambda, x) \geq 0$.*

Proof. The action of $\mathbf{C}^*$ via λ on $\hat{x}$ is given by:

$$\lambda(s)\hat{x} = \sum \hat{x}_i \lambda(s) e_i = \sum \hat{x}_i s^{r_i} e_i$$

then by definition of $\mu(\lambda, x)$ we see that $\mu(\lambda, x) < 0$ if and only if $\lim_{s \to 0} \lambda(s)\hat{x} = 0$ (note that if $\mu(\lambda, x) \geq 0$ then either the limit does not exists in $\mathbf{C}^N$ or it is different from zero). Thus, by Theorem 4.3, $\mu(\lambda, x) < 0$ for some λ if and only if x is not semistable.

For a complete variety we have formulated an intrinsic definition of semistability (Def. 3.7) which does not depend on a particular projective embedding, so in this case we wish to interpret $\mu(\lambda, x)$ without referring to the ambient space. Suppose X is a complete variety acted on by a linearly reductive group G and L is an ample line bundle on X with a G-linearization. Let $\lambda : \mathbf{C}^* \to G$ be a 1-parameter subgroup of G, so that $\mathbf{C}^*$ acts on X via λ. For a point $x \in X$ consider the orbit map of the $\mathbf{C}^*$-action $\sigma_x : \mathbf{C}^* \to X$ defined by $\sigma_x(s) = \lambda(s)x$. Since X is complete, by the valuative criterion σ_x can be extended to $\mathbf{P}^1_{\mathbf{C}}$ (think of it as $\mathbf{C}^* \cup \{0\} \cup \{\infty\}$). Let x_0 be the image of 0 under the extended map. Then the $\mathbf{C}^*$-action leaves x_0 fixed hence $\mathbf{C}^*$ operates on the fibre L_{x_0} which is a 1-dimensional vector space. This means that for any $v \in L_{x_0}$ and any $s \in \mathbf{C}^*$ the action is $\lambda(s)v = s^{\mu_0} v$ for some integer μ_0 and we take $\mu(\lambda, x) = \mu_0$.

§5. Stable points.

The set of semistable points of a projective variety acted on by a linearly reductive group contains a subset on which the action is closed (i. e. all the orbits of these points are closed) so that the quotient of this subset is a geometric quotient.

As in the case of semistability we can give a definition depending on a projective embedding or formulate an intrinsic definition and the two notions coincide. Here we will continue to use the former approach:

Assume G is a linearly reductive group acting linearly on a vector space $\mathbf{C}^N$ and let $X \subset \mathbf{P}^{N-1}$ be a G-invariant subvariety of the corresponding projective space.

Definition 5.1. A point $x \in X$ is **stable** if there exists an invariant homogeneous polynomial of positive degree f such that $f(x) \neq 0$ (i. e. x is semistable) and moreover the isotropy group G_x is finite (i. e. $\dim O(x) = \dim G$, by Prop. 1.7) and the action of G on the affine open subset $X_f = \{y \in X \mid f(y) \neq 0\}$ is closed.

The subset of stable points is denoted X^s.

Remark 5.3. It is not clear from the definition that X^s is an open subset, but this is in fact true. It follows from the fact that the set of points where the orbits have maximal dimension is open by semicontinuity (Prop. 1.7).

There are several equivalent characterizations of stable points, we list them here leaving the proof to the reader:

Proposition 5.4. *A point $x \in X$ is stable if and only if there exists an invariant homogeneous polynomial f of positive degree such that $f(x) \neq 0$ and any of the following condition is satisfied:*

1. $\dim O(x) = \dim G$ *and the action of G on X_f is closed (this is just the definition);*
2. $\dim O(x) = \dim G$ *and the orbit $O(x)$ is closed in X_f;*
3. $\dim O(y) = \dim G$ *for every $y \in X_f$;*
4. *the orbit map $(\sigma_x)_f : G \to X_f$ is proper;*
5. *the orbit map $\sigma_x : G \to X^{ss}$ is proper;*
6. $\dim O(x) = \dim G$ *and $O(x)$ is closed in X^{ss}.*

To prove the equivalence of these conditions one needs the following result:

Lemma 5.5. *: Let G be an algebraic group acting on X. Then the orbit map $\sigma_x : G \to X$ is proper if and only if the orbit $O(x)$ is closed and the stabiliser G_x is finite.*

Proof. If σ_x is proper then $O(x)$ is closed because it is the image of σ_x and the stabiliser $G_x = \sigma_x^{-1}(x)$ is affine and complete, hence finite. Conversely, if $O(x)$ is closed and G_x is finite then the morphism $G \to O(x)$ induced by σ_x has finite fibres (i. e. it is quasi-finite). Then there exists a nonempty open subset U of $O(x)$ such that $\sigma_x^{-1}(U) \to U$ is finite. Now we can cover $O(x)$ with the open sets gU for $g \in G$ so that σ_x is in fact finite, hence proper.

As announced, the orbits on stable points are closed:

Theorem 5.6. *With the above notation, there exists a geometric quotient for the action of G on X^s. More precisely, if $\phi : X^{ss} \to Y$ is a good quotient for the action* *of G on X, there exists an open subset U of Y such that $\phi^{-1}(U) = X^s$ and U is a geometric quotient of X^s.*

Remark 5.7. Since Y is projective we can think of Y as a compactification of the geometric quotient $X^s//G$.

Again we would like to have a criterion for determining stable points. It turns out that applying the numerical criterion for semistability one can distinguish at once stable points among the semistable ones, indeed we have:

Theorem 5.8 (Hilbert-Mumford). *A point $x \in X$ is stable with respect to a G-action if and only if $\mu(\lambda, x) > 0$ for every 1-parameter subgroup $\lambda : \mathbf{C}^* \to G$.*

REFERENCES

[D] J. Dieudonné, *Précis de géométrie algébrique élémentaire, Tome 2*, Presses Universitaires de France, 1974.

[D-C] J. Dieudonné, J. B. Carrell, *Invariant Theory, old and new*, Academic Press, 1971.

[F] J. Fogarty, *Invariant Theory*, W. A. Benjamin, 1969.

[H] R. Hartshorne, *Ample Subvarieties of Algebraic Varieties*, Springer-Verlag, Lecture Notes in Mathematics 156, 1970.

[K] H. Kraft, *Geometrische Methoden in der Invariantentheorie*, Fried. Vieweg & Sihn, 1984.

[K-S-Sp] H. Kraft, P. Slodowy, T. A. Springer, *Algebraic transformation groups and Invariant theory*, Birkhaüser, D. M. V. Seminar Band 13, 1989.

[M] D. Mumford, *The Red Book of Varieties and Schemes*, Springer-Verlag, Lecture Notes in Mathematics 1358, 1988.

[M-F] D. Mumford, J. Fogarty, *Geometric Invariant Theory*, Springer-Verlag, Ergebnisse 34, 1982.

[N] P. Newstead, *Introduction to Moduli Problems and Orbit Spaces*, Springer-Verlag, Tata Institute Lecture Notes, 1978.

[R] A. Ramanathan, *Stable principal bundles on a complete Riemann Surface - Construction of Moduli Space*, Ph. D. Thesis, University of Bombay (1976).

[S] T. A. Springer, *Invariant Theory*, Springer-Verlag Lecture Notes in Mathematics 585, 1977.

HILBERT-MUMFORD CRITERION FOR TORAL ACTIONS

1. Preliminary definitions For keeping the language simple, we will work over an algebraically closed field k, though the results suitably generalize to all fields. Let $GL_1 = k^\times$ be the multiplicative group of k, and let $T = (GL_1)^n$ be the n-dimensional torus over k, regarded as a linear algebraic group. Let V be a finite dimensional k-vector space, and let there be given a *rational representation* ρ of T on V. In concrete terms, this means the following. There will exist some basis $v_1, \ldots, v_d$ of V where d is the dimension of V, and an $n \times d$ integral matrix A such that for any $\tau = (t_1, \ldots, t_n) \in T$ the matrix of $\rho(\tau) \in GL(V)$ is the diagonal matrix with entries

$$\prod_{i=1}^{n} t_i^{A_{i,j}}$$

where $j = 1, \ldots, d$. For $\tau \in T$ and $v \in V$, we will denote the vector $\rho(\tau)v$ simply by τv.

Given such a ρ, we get an induced right action of T on the ring $k[X_1, \ldots, X_d]$ of all polynomial functions $f : V \to k$, defined by $f^\tau(v) = f(\tau^{-1}v)$. From the above matrix description of ρ, it can be seen that a monomial $X_1^{q_1} \cdots X_d^{q_d}$ is invariant under the action if and only if the vector $(q_1, \ldots, q_d) \in \mathbf{Z}^d$ lies in the kernel of the homomorphism $A : \mathbf{Z}^d \to \mathbf{Z}^n$ defined by the $n \times d$ integral matrix A.

By definition, a 1-parameter subgroup λ of T is a homomorphism $\lambda : GL_1 \to T$ given by the formula

$$\lambda(t) = (t_1^{a_1}, \ldots, t_n^{a_n})$$

where the a_i are integers. Note that such a λ need not be injective.

Theorem 1 *Let $\rho : T \to GL(V)$ be a rational representation of the torus T. Let $v \in V$ such that the Zariski closure of the orbit Tv of v contains the origin $0 \in V$. Then there exists a 1-parameter subgroup λ of T such that*

$$\lim_{t \to 0} \lambda(t)v = 0$$

2. Basic Lemma

The proof will make use of the following lemma.

Lemma 2.1 *Let S be a finite subset of the n-dimensional real vector space $\mathbf{R}^n$, such that its conves hull $Con(S)$ does not contain the origin $0 \in \mathbf{R}^n$. Let $Y_1, \ldots, Y_n$ be the cartesian coordinates on $\mathbf{R}^n$. Then there exists a linear functional $\varphi = a_1Y_1 + \ldots + a_nY_n = 0$ on $\mathbf{R}^n$, where the a_i are integers, such that $\varphi(s) > 0$ for each $s \in S$.*

Proof By compactness of $Con(S)$, there exists some point $u \in Con(S)$ such that its euclidean norm $\|u\|$ is minimum, moreover, $u \neq 0$ as $Con(S)$ does not contain the origin of $\mathbf{R}^n$. By convexity of $Con(S)$, the linear functional $\varphi_u : \mathbf{R}^n \to \mathbf{R} : w \mapsto u \cdot w$ (inner product with u) takes a positive value at each point of $Con(S)$. By finiteness of S there exists some $\alpha > 0$ such that $\varphi_u(w) > \alpha$ for each $w \in S$. This last condition defines an open neighbourhood U of u in $\mathbf{R}^n$. Choose some $v \in U$, all whose coordinates are rational; such a v exists as $\mathbf{Q}$ is dense in $\mathbf{R}$. Then the desired linear functional φ can be taken to be a positive integral multiple of φ_v (where it is enough to multiply by the l.c.m of tl denominators of the coordinates of v).

In pictorial terms, the above lemma is obvious: as $C = Con(S)$ is convex and does n contain the origin, there will exist a hyperplane $H \subset \mathbf{R}^n$ passing through the origi such that C lies on one side of H. As C is compact, we can perturb H slightly withou intersecting C, and thereby choose H to be defined over $\mathbf{Q}$.

It is possible to eliminate all references to real numbers and convexity in this article b using systematically the following remark but the resulting treatment would be furthe from 'pictorial intution'.

Remark 2.2 Let $S \in \mathbf{Z}^j \subset \mathbf{R}^j$ be a finite subset and let $Con(S) \subset \mathbf{R}^j$ be its conve hull in $\mathbf{R}^j$. Then $0 \in Con(S)$ if and only if 0 can be written as a linear combination c elements of S with coefficients in $\mathbf{N} = \{0, 1, 2, \dots\}$ with atleast one coefficient nonzerc This is because rational points are dense in any linear subspace of $\mathbf{R}^j$ which is defined ove $\mathbf{Q}$.

3. Proof of Theorem 1

Let $v = (x_1, \dots, x_d)$ in terms of the basis (v_i). By permuting the indices if necessary we can assume that there exists an integer m with $1 \leq m \leq d$ such that $x_i \neq 0$ for al $1 \leq i \leq m$ and $x_i = 0$ for all $m+1 \leq i \leq d$. Let $W \subset V$ be the vector subspace spanne by $v_1, \dots, v_m$. Then W is Zariski closed and T-invariant, and $v \in W$, so we can replac V by W in the proposition, so that we can assume without loss of generality that eacl coordinate x_i of v is non-zero. Hence every monomial $f = X_1^{q_1} \cdots X_d^{q_d}$ in $k[X_1, \dots, X_d$ takes a nonzero value at v. As $0 \in V$ lies in the Zariski closure of Tv, we have $f(v) = f(0$ for any invariant function f, which shows that there is no invariant monomial f of positive degree. As remarked before, a monomial $f = X_1^{q_1} \cdots X_d^{q_d}$ is invariant if and only if the vector $(q_1, \dots, q_d) \in \mathbf{Z}^d$ lies in the kernel of $A : \mathbf{Z}^d \to \mathbf{Z}^n$. Let $e_1, \dots, e_d$ be the standard basis of $\mathbf{Z}^d$. As a consequence of remark 2.2, the non-existence of an invariant monomial o positive degree implies that the convex hull of the subset $S = \{Ae_1, \dots, Ae_d\} \subset \mathbf{Z}^n \subset \mathbf{R}^n$ does not contain the origin $0 \in \mathbf{R}^n$. Hence by lemma 2.1, there exists a homomorphism $\varphi : \mathbf{Z}^n \to \mathbf{Z}$ such that the composite $\varphi \circ A : \mathbf{Z}^d \to \mathbf{Z}$ takes a value $p_i > 0$ on each e_i for $1 \leq i \leq d$. Let $\varphi(Y_1, \dots, Y_n) = a_1 Y_1 + \dots + a_n Y_n$ where the a_i are integers. Now if $\lambda : GL_1 \to T$ is defined by $t \mapsto (t^{a_1}, \dots, t^{a_n})$, then it follows that

$$\lambda(t)v = (t^{p_1}x_1, \dots, t^{p_d}x_d)$$

From this it follows that $\lim_{t \to 0} \lambda(t)v = 0$, which proves the theorem 1.

4. Limit of orbits

The following theorem is more general than Theorem 1.

Theorem 4.1 *Let $\rho : T \to GL(V)$ be a rational representation as before, and let $v \in V$ such that the orbit $Tv \subset V$ is not closed. Then there exists a 1-parameter subgroup λ of T and a point $u \in V - Tv$ of such that*

$$\lim_{t \to 0} \lambda(t)v = u$$

For the proof, we need the following more general version of the combination of lemma 2.1. and remark 2.2, which immediately imply the special case $m = d$ in what follows.

Lemma 4.2 *Let $A : \mathbf{Z}^d \to \mathbf{Z}^n$ be a homomorphism, and let m be an integer with $1 \leq m \leq d$, such that the following two conditions hold.*

(1) If $r = (r_1, \ldots, r_d) \in \mathbf{Z}^d$ lies in the intersection $\mathbf{N}^d \cap ker(A)$, then $r_i = 0$ for $i \leq m$.

(2) There exists some $r = (0, \ldots, 0, r_{m+1}, \ldots, r_d) \in ker(A)$, such that $r_{m+1}, \ldots, r_n$ are nonzero positive integers.

Then there exists a homomorphism $\psi : \mathbf{Z}^n \to \mathbf{Z}$ such that the composite map $\psi \circ A : \mathbf{Z}^d \to \mathbf{Z}$ takes a positive value on each basis vector $e_i \in \mathbf{Z}^d$ for $i \leq m$ and takes the value zero on each e_i for $i \geq m + 1$.

Proof of lemma 4.2 Let $W \subset \mathbf{Z}^n$ be the $\mathbf{Z}$-submodule spanned by the Ae_i for $i \geq m+1$, and consider the quotient $q : \mathbf{Z}^n \to \mathbf{Z}^n/W$. We will first show that the convex hull of the finite subset

$$S = \{\overline{Ae_1}, \ldots, \overline{Ae_m}\} \subset \mathbf{R} \otimes (\mathbf{Z}^n/W)$$

does not contain the origin $0 \in \mathbf{R} \otimes (\mathbf{Z}^n/W)$. Equivalently (see remark 2.2), we have to show that a linear combinition of the elements of S with non-negative integer entries cannot equal 0 unless all coefficients are zero.

Suppose the contrary holds, so that $\sum_{1 \leq i \leq m} r_i \overline{Ae_i} = 0 \in \mathbf{Z}^n/W$, where each $r_i \in \mathbf{N}$ and atleast one r_i is nonzero. Then by definition of the quotient $\mathbf{Z}^n/W$, we will have integers $s_{m+1}, \ldots, s_d$ such that the element

$$r_1 e_1 + \ldots + r_m e_m + s_{m+1} e_{m+1} + \ldots + s_d e_d$$

of $\mathbf{Z}^d$ lies in the kernel of $A : \mathbf{Z}^d \to \mathbf{Z}^n$. Now using the hypothesis (2) of the lemma, we can if necessary change the s_i so that $s_i \geq 0$ for each $i \geq m + 1$. But this contradicts the hypothesis (1). Hence we have shown that $0 \in \mathbf{R} \otimes (\mathbf{Z}^n/W)$ does not lie in the convex hull of S.

Now it follows by lemma 2.1 that there exists a linear functional $\varphi : \mathbf{Z}^n/W \to \mathbf{Z}$, which takes positive values at all elements of S. Therefore we can take the desired homomorphism ψ to be the composite

$$\psi = \varphi \circ q : \mathbf{Z}^n \to \mathbf{Z}^n/W \to \mathbf{Z}$$

which proves the lemma.

5. Proof of theorem 4.1

As in section 1, there will exist some basis $v_1, \ldots, v_d$ of V where d is the dimension of V, and an $n \times d$ integral matrix A such that for any $\tau = (t_1, \ldots, t_n) \in T$ the matrix of $\rho(\tau) \in GL(V)$ is the diagonal matrix with entries $\prod_{i=1}^{n} t_i^{A_{i,j}}$ where $j = 1, \ldots, d$.

Let $v = (x_1, \ldots, x_d)$. As in the proof of theorem 1, we can reduce to the case where all the x_i are nonzero. Hence v lies in the Zariski open subset V_0 of V consisting of all points all whose coordiantes are nonzero. Let $D \subset GL(V)$ be the subgroup of diagonal matrices. Now, the variety V_0 is isomorphic to D under the morphism $\eta : D \to V_0 : \sigma \mapsto \sigma(v)$. Under this isomorphism, the image $\rho(T) \subset D \subset GL(V)$ maps isomorphically to the orbit Tv. As $\rho(T)$ is closed in T, it follows that Tv is closed in V_0. Hence if $u \in V - Tv$ is a limit point of Tv then atleast some coordinate of u must be zero. As by hypothesis such a limit point u exists, it follows that there exists some index j with $1 \leq j \leq d$ such that for each T-invariant monomial $X_1^{q_1} \cdots X_d^{q_d}$, we must have $q_j = 0$.

Hence by permuting the basis of V, we can assume that there is an integer $1 \leq m \leq d$ such that the variables $X_1, \ldots, X_m$ do not occur in any T-invariant monomial $X_1^{q_1} \cdots X_d^{q_d}$ (that is, each T-invariant monimial lies in $k[X_{m+1}, \ldots, X_d]$), while each of the remaining variables X_i for $i \geq m+1$ occurs with a positive exponent in atleast one invariant monomial, say f_i. Let f be the product $f = \prod_{i=m+1}^{d} f_i \in k[X_{m+1}, \ldots, X_d]$. This shows the existence of an invariant polynomial

$$f = X_{m+1}^{r_{m+1}} \cdots X_d^{r_d}$$

where $r_i > 0$ for each $m+1 \leq i \leq d$. Note that by definition $f = 1$, in the case where $m = d$.

Let the vector $(q_1, \ldots, q_d) \in \mathbf{N}^d \subset \mathbf{Z}^d$ be associated to the monomial $X_1^{q_1} \cdots X_d^{q_d}$. As T-invariant monomials correspond to the intersection of $\mathbf{N}^d$ with the kernel of $A : \mathbf{Z}^d \to Z^n$, it now follows that the hypothesis of lemma 4.2 is satisfied. Let $\psi : Z^n \to \mathbf{Z}$ be the homomorphism given by the lemma, so that the integers $p_i = \psi(Ae_i)$ satisfy the following: $p_i > 0$ for $1 \leq i \leq m$ and $p_i = 0$ for $m+1 \leq i \leq d$. Let $\psi(Y_1, \ldots, Y_n) = a_1Y_1 + \ldots + a_nY_n$ where the a_i are integers. Now if $\lambda : GL_1 \to T$ is defined by $t \mapsto (t^{a_1}, \ldots, t^{a_n})$, it follows that

$$\lambda(t)v = (t^{p_1}x_1, \ldots, t^{p_d}x_d)$$

From this it follows that

$$\lim_{t \to 0} \lambda(t)v = (0, \ldots, 0, x_{m+1}, \ldots, x_d)$$

As $m \geq 1$, the above limit point does not lie in V_0, hence lies outside Tv, which proves the theorem.

These results are important in geometric invariant theory. The above purely expository note. prepared by Nitin Nitsure, is based on a paper of **David Birkes** *in Annals of Math. vol 93 (1971). The only possibly novel feature of the exposition is the use of monomials.*

September 1997