Home Work #6

1. Read the proof of vanishing of $H^1(X, \mathcal{M}_X)$ (Corollary 17.17) and also the proof vanishing of $H^1(X, \mathcal{M}_X^*)$ for a compact Riemann surface from Forsters's book.

2. Forster's book, Page 16.4.

3. Forster's book, Page 145-17: 17.1, 17.6, 17.7,

4. Let X be a compact Riemann surface. Suppose $\mathcal{O}(D)$ is a degree $2g-2$ invertible sheaf. Show that it has $g-1$ or g sections, and it has g sections if and only if $\mathcal{O}(D)$ is the canonical sheaf K_X

5. Let X be a compact Riemann surface, then X has genus 0 if and only if all degree 0 line bundles (or rank one locally free sheaves of $\mathcal{O}_X$ modules) are trivial.

6. Let X be a compact Riemann surface of genus 2, show that K_X is base point free. Hence construct a branch cover to $\mathbb{P}^1$. Use the above to show that every genus two curves is a double cover of $\mathbb{P}^1$.

7. A compact Riemann surface X of genus g is called hyper-elliptic if it admits a double cover (may be ramified) to $\mathbb{P}^1$ i.e $\phi : X \to \mathbb{P}^1$. Show that $\phi^*\mathcal{O}_{\mathbb{P}^1}(1)^{\otimes 2g-2} = K_X$. Use this to show that if X is a compact Riemann surface of genus at least 2, then it admits at most one double cover of $\mathbb{P}^1$.

8. Use the above to show that there are infinite numbers of pairwise not isomorphic compact Riemann surfaces of genus 2.

9. Observe that if X is of genus 3, then K_X is base point free and admits a map $\phi : X \to \mathbb{P}^2$. Show if the map ϕ is not an embedding then there exists points p and q such that $h^0(X, \mathcal{O}_X(p+q)) = 2$. Use it to show that X is hyperelliptic. Conclude that if X is not hyperelliptic, then K_X is very ample.

10. Show that there are non-hyperelliptic compact Riemann surfaces of genus 3.

11. Show that if X is not a hyperelliptic compact Riemann surface of genus g, then K_X is very ample.

12. If $\mathcal{V}$ and $\mathcal{W}$ are vector bundles of rank r and s compute $\det(\mathcal{V} \otimes \mathcal{W})$ in term of $\det \mathcal{V}$ and $\det \mathcal{W}$. Same problem for $\mathcal{V} \oplus \mathcal{W}$.

13. Hartshorne Page 321 (assume C is a compact Riemann surface): 2.6 (a,b), 2.6 (c)-just read the problem statement and think about it, 2.5 (you can also read about it from a text book).

14. Consider the degree d Vernonese embedding from $v_d : \mathbb{P}^m \to \mathbb{P}^N$. Compute $v^*\mathcal{O}_{\mathbb{P}^N}(1)$.

15. (Hard Problem) Let X be a compact Riemann surface and $p_1, \ldots, p_n$ are marked points with coordinates $\xi_1, \ldots, \xi_n$. Consider the pairing

$$\langle\ ,\ \rangle : \bigoplus_{i=1}^{n} \mathbb{C}((\xi)) \times \bigoplus_{i=1}^{n} \mathbb{C}((\xi))d\xi \longrightarrow \mathbb{C},$$
$$\langle \{f_i\}, \{g_i d\xi_i\} \rangle := \sum_{i=1}^{N} \operatorname{Res}_{\xi_i=0} f_i g_i d\xi_i.$$

Now consider the Laurent expansion along the point $p_1, \ldots, p_n$ to get maps

$$H^0(X, \Omega_X(*\vec{p})) \longrightarrow \bigoplus_{i=1}^{n} \mathbb{C}((\xi))d\xi, \text{ and } H^0(X, \mathcal{O}_X(*\vec{p})) \longrightarrow \bigoplus_{i=1}^{n} \mathbb{C}((\xi)).$$

Here $*\vec{p}$ indicates poles of arbitary order and can be defined as a limit of finite order poles at the points $\vec{p} = (p_1, \ldots, p_n)$ Show the following:

(a) The above maps given by Laurent expansions are injective.

(b) Under the pairing $\langle\ ,\ \rangle$, the vector spaces $H^0(X, \Omega_X(*\vec{p}))$ and $H^0(X, \mathcal{O}_X(*\vec{p}))$ annihilate each other.

This is a very useful tool to detect global meromorphic function or forms from local expansions.