Graph potentials, their frequencies and combinatorial reconstructions

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July 10, 2024

Abstract

We prove that a (colored) trivalent graph can be recovered from (the polar dual of) the associated quantum Clebsch–Gordan polytope, and that any isomorphism between such polytopes is induced by a unique isomorphism of the underlying colored graphs. This can be seen as a combinatorial non-abelian Torelli result, because these polytopes arise also from toric degenerations of moduli spaces of rank-2 bundles on a curve. We moreover show how graph potentials introduced by the authors in an earlier work relate to the theory of random walks, and we use our combinatorial Torelli theorem to construct random walks with distinct shapes but equal return probabilities for every number of steps.

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1 Introduction

The question “Can one hear the shape of a drum?” was popularized by Mark Kac in [22] and has its origins in physics. The problem asks whether a sequence of frequencies (eigenvalues of a Laplacian) obtained by a vibrating drum uniquely determines the shape of the drum. Hermann Weyl answered a similar question about frequencies determining the area of the drum positively. However, the question of whether one can hear the shape of the drum has been answered negatively in pioneering works by [27], and subsequent works are surveyed in [18]. Alternatively, one can think of these questions as reconstruction theorems, namely given a sequence of measurable quantitities of phyiscal significance, can we reconstruct the quantity?

Similar questions about "hearing something" using the eigenvalues of a discrete Laplacian have been explored in the context of independent electron models in solid-state physics. The primary objects of interest in that context are the so-called Fermi varieties and the works of Gieseker–Trubowitz–Knörer.

In this paper, we examine a discrete version of the above questions for certain discretization of Laplacian operators in the set-up of random walks, using Laurent polynomials, graphs, and polytopes.

Laurent polynomials in various avatars We first discuss the various roles of a Laurent polynomial. A Laurent polynomial can be used to encode a finite range *random walk* on a d-dimensional lattice $L = \mathbf{Z}^d$. Likewise, it can be used to encode a linear *difference operators*, a character of a representation, or the generating function for a collection of circles on a torus, such as Floer potentials in symplectic topology. It can also be interpreted as a finitely supported function (or measure, or distribution) $p\colon L \to k$ for some value set k, or as an element of the group algebra $k[L]$ if k is a ring.

A distribution p is equivalently encoded by an element $W = \sum_{l\in L} p(l)[l]$, a periodic characteristic function $\phi(\theta) = \sum_{l\in L} p_l \exp(\mathrm{i}\langle l, \theta\rangle)$, an exponential moment-generating function $M(u) = \phi(-iu)$, or more generally as a factorial moment-generating function

$$W(z) = \sum_{l\in L} p_l z^l, \tag{1}$$

that can be evaluated at any d-tuple of non-zero complex numbers. The value set k can be binary $\{0, 1\}$, but it can also be the integers $\mathbf{Z}$, probability values $[0, 1] \subset \mathbf{R}$, or complex numbers $\mathbf{C}$. Sometimes a lattice L comes with extra structure, such as a basis $e_1, \dots, e_d$, or an inner product, but often it has none other than the structure of a (free abelian) group.

We will consider a class of Laurent polynomials, introduced by the authors in [10], named *graph potentials* and establish certain reconstruction results for them. These polynomials are part of a framework for investigating toric degenerations of moduli spaces of vector bundles on curves [9].

Invariants of Laurent polynomials A higher-dimensional analogue of the degree of a polynomial was introduced by Newton and is known as *Newton polytope* $\mathrm{NP}(W) = \mathrm{NP}(p)$ of a Laurent polynomial. It is defined as the convex envelope in $L_{\mathbf{R}} = \mathbf{R}^d$ of the support $\mathrm{Supp}(p)$ of p, i.e., the finite set of vectors $l \in L$ such that $p(l) \neq 0$. Given a linear bijection $b\colon L \to L'$ that maps p to p', the Newton polytope of p is mapped to the Newton polytope of p'.

Another invariant is the *period sequence* (of *probabilities to return to the origin*), or *density state function*. It can be defined as the sequence of numbers $\Pi = (a_m)_{m=0,1,\dots}$, where

$$a_m = \frac{1}{(2\pi)^d} \int_{[-\pi,\pi]^d} \phi(\theta)^m \mathrm{d}\theta \tag{2}$$

is the constant term of the mth power of W, which in random walks can be interpreted as the probability $\mathbb{P}(S_m = S_0)$ to be in the same place after m hops. Here $\mathrm{d}\theta$ is the Lebesgue measure on a d-dimensional torus.

One may also consider more general integral transforms, such as

$$K_p(t) = \int_{u\in\mathbf{R}^d} \exp(-tM(u))\mathrm{d}u \tag{3}$$

or

$$\zeta_p(s) = \int_{u\in\mathbf{R}^d} M(u)^{-s}\mathrm{d}u. \tag{4}$$

Here $\mathrm{d}u$ is the Lebesgue measure on $\mathbf{R}^d$. They are typically convergent for $\Re(t) > 0$ and $\Re(s) > 1$ if a neighbourhood of the origin is contained inside the Newton polytope, which we are going to tacitly assume.

In some applications the latter invariants, such as Π, $K_W(t)$, $\zeta_W(s)$ are either mathematically more accessible or physically more meaningful. Thus one might have a *hope* that the finite distribution p (or equivalently, the Laurent polynomial $W(z)$, or some other incarnation) could be recovered from one of these generating integral functions, hence the analogy with "hearing the shape of a drum".

For example, in symplectic topology the numbers a_m might count holomorphic spheres and are accessible also by algebraic methods, whereas the numbers p_l count holomorphic discs whose formulation so far usually requires analytic tools. In Bloch–Flouquet theory the generating function for a_m is a density state function, which is up to some extent experimentally observable, but the distribution p_l is more about modelling of a discrete Laplacian.

In the one-dimensional case one may use Cauchy's residue theorem, Lagrange's inversion formula, or other tools, and it is well-known that the answer is affirmative up to two minor ambiguities, which also pertains in higher dimensions.

Setting up the stage One problem is that probabilities a_m do not depend on the choice of the lattice $L \subset \mathbf{R}^d$ and stay the same if the lattice L is replaced by an overlattice $L' \supset L$. A simple patch to this problem is to ask that support of p to be a generating set for the lattice L. Random walks with this property are called *aperiodic*; Floer potentials of Fano manifolds are expected to have this property (see Property O and Conjecture O in [13, 14, 15]) in contrast to known examples of periodic potentials for Fano orbifolds in loc. cit.

The second ambiguity is similar, but slightly more subtle: the Lebesgue measures $\mathrm{d}u$ on $\mathbf{R}^d$ and $\mathrm{d}\theta$ on the d-dimensional torus are translation-invariant, so change of coordinates $u \mapsto u + \tau$ for $\tau \in \mathbf{R}^d$ transforms the potential $M(u)$ into the potential $M_\tau(u) = M(u + \tau)$ with equivalent integral invariants, but the transformed distribution has coefficients equal to $p_\tau(l) = \exp(\langle \tau, l\rangle)p(l)$, and thus is clearly distinct from p. One has $M_\tau(0) = M(\tau)$, so unless $M(\tau) = M(0)$, the new distribution stops being probabilistic. One can either rescale it and consider $p_\tau/M(\tau)$, or study in detail the question when $M(\tau) = M(0)$, or more generally $M(\tau_1) = M(\tau_2)$.

This is essentially the question about study of level sets of the functions:

$$M(u)\colon \mathbf{R}^d \to \mathbf{R}, \quad W(z)\colon (\mathbf{C}^\times)^d \to \mathbf{C}, \quad \text{and} \quad \tilde{M}(u) := W(\exp(u))\colon \mathbf{C}^d \to \mathbf{C}. \tag{5}$$

They are respectively real algebraic, complex algebraic and complex analytic varieties, known in Bloch–Floquet theory as *Fermi varieties.*

The well-known positive-definiteness of covariance as in [13] means strict convexity of function the M, which in turn by Morse theory implies that its non-empty level sets are $(d-1)$-dimensional spheres with a notable exception of the global minimum at some $M(\tau_0)$, known as *the conifold point* in mirror symmetry ([13, 15]). [1]

So one reasonable way to fix the second ambiguity is to demand $u = 0$ to be the global minimum of the function M, which is equivalent to the vanishing of the derivative $M'(0) = 0$. For random walks this condition corresponds to the *mean zero* property, and is tacitly assumed in the literature. Without this condition the period sequence $(a_m)_m$ has exponentially decaying asymptotics, but with this condition the decay is only polynomial. Further, the condition guarantees that p considered as a difference operator

$$(\Delta_p q)(l) = \sum p(l')(q(l-l') - q(l)) \tag{6}$$

is approximately equal to a continuous Laplacian differential operator, that is, taking Taylor series we obtain the Laplacian as the degree two terms followed up terms of higher order. We refer the reader to Section 6 for more details.

In applications, where W arises as a result of an enumeration, the size of support of p might be known in advance. Sometimes, it is even known to be a subset of an explicit finite range R of candidates with $|R|$ being

[1] SG: According to Y. Benoist passing between two random walks corresponding to two τs at the same level is also known as Doob transform, however I was not able to identify it with what I found about Doob transform in the literature, however it could be another transform of same Doob. FIXME.

of order $O(|\operatorname{Supp}(p)|^3)$, and the requirement for all coefficients of generating function to be in set $\{0, 1\}$ often makes the second ambiguity irrelevant.

A more radical solution to the two obvious ambiguities is not to patch, but to ignore them, and to reformulate the hope in a more robust way:

Question 1.1. Given a period sequence a_m and/or other integral transforms such as $K_p(t)$ or $\zeta_p(s)$, can one recover the Newton polytope $NP(p) \subset \mathbf{R}^d$ at least as a convex polytope up to affine linear transformations?

Both problems can be explained in terms of simple changes of coordinates in the integrals, but with them inventory of obvious such changes exhausts. Moreover, both product essentially the same (biholomorphic) periodic Fermi varieties.

[the above paragraph needs work] [discuss change of coordinate formula, Laurent phenomenon and new hope with cluster algebra discovery]

We now formulate our main result and discuss three interpretations of our results.

Theorem A. *For every $h \geq 3$ there exists $M := M(h) > 1$ subsets S_j, $j = 1, \ldots, M$ of order $8h$ of a sphere of squared radius 3 in a standard Euclidean $3h$-dimensional lattice $L = \mathbf{Z}^{3h} \subset \mathbf{R}^{3h}$, with which we also associate Laurent polynomials $W_j = \sum_{l \in S_j} z^l$ with coefficients in $\{0, 1\}$, difference operators D_j, periodic Schrödinger operators, and lattice random walks with the following properties:*

1. *they have zero mean $-\sum_{l \in S_j} l = 0$;*
2. *the associated difference operators as in equation (6) is approximately equal to a continuous Laplacian differential operator;*
3. *all S_j are convex independent and pairwise non-affine equivalent, in particular all Newton polytopes of W_j are pairwise non-isomorphic;*
4. *all periods, density state functions, and other integral invariants $K_p(t)$, $\zeta_p(t)$ are pairwise-equal;*
5. *for every λ, all Fermi varieties F_λ are not isomorphic as algebraic varieties, but pairwise-birational.*

Another informal way to describe the above theorem is the slogan: *One cannot hear the shape of a random walk.* We refer the reader to Section 5 for interpretations of Theorem A in the context of random walks.

Graph and polytopes The main technical focus of this article that allows us to prove Theorem A is the study of *quantum Clebsch–Gordan polytopes* $P(\gamma) \subset \mathbf{R}^{E(\gamma)}$, a class of convex lattice polytopes associated with graphs γ with edge set E and vertex set V, and the proof of Theorem B, that give an algorithmic recovery of graphs from their polytopes.

The polytopes $P(\gamma)$ and their integer points turn out to be of central interest in mathematics, and already appeared in various contexts at least since 1980s. For example:

- They define the multiplicitive or quantum Horn equalities in the works of Agnihotri–Woodward and Belkale [1, 6]
- They are isomorphic to the images of the Goldman–Jeffrey–Weitsman integrable systems [20, 21] and the Newton–Okounkov bodies arising from conformal blocks in the work of Manon [25, 26].
- Integer points in their k-dilations are in bijection with Bohr–Sommerfeld Lagrangian tori on the moduli space of SU(2)-connections in the symplectic set-up [20, 21].
- The number $n_k(\gamma)$ of such points is a polynomial in k, the so-called Ehrhart polynomial. It coincides with the Hilbert polynomial of the moduli space of rank two bundles in the algebro-geometric setup. It can be computed by the Verlinde formula [31], in particular it depends only on the genus g and is equal to the dimension of the respective space of conformal blocks of level k (or non-abelian theta functions). The location of the roots of these polynomials was studied by the authors in [8].

- They appear in works of Kohno and Tsuchiya–Ueno–Yamada on topological invariants from Knizhnik–Zamolodchikov connection, these might be one of the earliest modern references.

The following theorem is the main combinatorial reconstruction result.

Theorem B. *A graph γ can be algorithmically reconstructed from its quantum Clebsch–Gordan polytope $P(\gamma)$. In particular, the polytopes $P(\gamma)$ and $P(\gamma')$ can be mapped to each other by a pair of (real) affine transformations if and only if the colored graphs γ and γ' are isomorphic.*

This theorem has direct applications to random walks (Section 5), Fermi varieties (Section 6), symplectic topology [9], and an interpretation in algebraic geometry (Section 8).

Earlier we defined *graph potentials* [10], Laurent polynomials whose Newton polytopes are $P(\gamma)$, and proved that their period sequences a_m depend only on the dimension. These results combined prove Theorem A. We also proof here invariance of other integrals, because they are not explicitly discussed in the literature, and for the sake of self-containedness of the article.

Polytopes associated to colored graphs For geometric applications we need a colored variation on the construction of quantum Clebsch–Gordan polytopes. In Section 7, we define colored graphs and isomorphisms between them, which in particular allows us to upgrade Theorem B to a description of all isomorphism groups between the respective polytopes:

Theorem C (Theorem 7.3)**.** *Any affine linear isomorphism between (colored) quantum Clebsch–Gordan polytopes $P(\gamma)$ and $P(\gamma')$ is induced by a uniquely defined isomorphism of colored graphs.*

Analogs of Theorem A also hold for colored graphs. We now discuss three applications of our results in: discrete random walks, Bloch–Floquet theory, symplectic topological, and in algebraic geometry.

Application to discrete random walks In [10] we defined a class of Laurent polynomials using graphs and subsequently discussed their properties. In Section 5 we attach random walks to these Laurent polynomials, and we reinterpret Theorem C in this language.

Application to Schrödinger operators with periodic potentials Section 6 deals with an application of the reconstruction theorem to Bloch–Floquet theory of Schrödinger operators with periodic potentials. We postpone all the relevent definitions for now and refer the reader to Section 6.

Theorem 1.2. *For pairwise distinct graphs γ_i the density state functions of associated periodic Schrödinger operators coincide, but their hoppings (discretizations of Laplacian) are pairwise distinct.*

Application to symplectic topology Our initial motivation to prove these reconstruction theorems was [9, Corollary 3.10] and [16, Example 4.10], where to every graph γ of genus g we associated a monotone Lagrangian torus $L(\gamma)$ in a symplectic manifold $\mathcal{N}_g$, and computed Floer potentials of $L(\gamma)$ to be Laurent polynomials W_γ with Newton polytopes $P(\gamma)$. Since Floer potentials are invariants of monotone Lagrangian tori up to Hamiltonian isotopy, Theorem 7.3 combined with symplectic results of the cited papers proves the following

Theorem D. *Monotone Lagrangian tori $L(\gamma) \subset \mathcal{N}_g$ are pairwise Hamiltonian non-isotopic for distinct graphs γ of genus g.*

The symplectic manifold $\mathcal{N}_g$ of Theorem D is known as an odd SU(2)-character variety. It parametrizes twisted representations of fundamental group of a surface Σ_g of genus g, and its standard/even version parametrizes untwisted representations and is related to even colorings.

Its algebro-geometric interpretation, and in particular the association with Torelli theorem is discussed next.

Algebro-geometric interpretations: Reconstruction results as a tropical non-abelian Torelli theorem It is well known that trivalent graph govern degenerations of a smooth curve into the deepest stratum of the Deligne–Mumford–Knudsen compactification of the moduli space of curves. Here, we consider polytopes $P(\gamma)$ associated to trivalent graphs which are polar dual to moment polytopes of toric Fano varieties that can be realized as degenerations of the moduli space $\mathrm{M}_C(2, \mathcal{L})$ of semistable rank-2 bundles over a Riemann surface C of genus $g(C) = g(\gamma)$ with fixed determinant $\mathcal{L}$ [25] obtained via the graph γ. Toric degenerations are often used to tropicalize algebraic varieties.

Analogously to the usual non-abelian Torelli theorem, that recovers the smooth curve C from the moduli space $\mathrm{M}_C(2, \mathcal{L})$, Theorem C recovers the graph γ, a tropicalization of a curve C, from the polytope $P(\gamma)$, a tropicalization of the moduli space $\mathrm{M}_C(2, \mathcal{L})$, so it can be considered as a *tropical/combinatorial non-abelian Torelli theorem*: We refer the reader to Section 8 for precise statements and a more detailed discussion.

Structure of the paper In Sections 2 and 3 we discuss trivalent graphs and its various properties following by a defining the notion of quantum Clebsch-Gordan (qCG) polytopes associated to these graph. Section 3 also contains key lemmas concerning the rays and vertices of these qCG polytopes which are used to prove a key reconstruction theorem in Section 4. Section 5 discusses a reinterpretation of Theorem A in the settings of discrete random walks. In Section 6, we recall the notion of Fermi varieties and interpret our result in that context. Section 7 discusses colored graph and prove Theorem C. This has applications in Section 8 as well. The topics in Section 8 are mostly algebraic geometric in nature, where we prove a discrete/tropical/combinatorial analog of the classical abelian and non-abelian Torelli theorems.

We recommend reading Section 5, for readers with interests in random walks. For readers interested in quantum mechanics or solid state physics, we recommend reading Section 6. We recommend Section 8 for people with interests in algebraic geometry.

Acknowledgements We thank Yves Benoist, Yuri Lima, Rodrigo Matos, Carlos Tomei for interesting discussions.

This collaboration started at the Tata Institute for Fundamental Research (TIFR) during SG's visit in Dec 2019–March 2020 and PB's visit in Feb 2020. We thank TIFR for the very pleasant working conditions.

PB was partially supported by the FWO (Research Foundation—Flanders). SG was partially supported by CNPq grant PQ 315747, and Coordenação de Aperfeiçoamento de Pessoal de Nível Superior – Brasil (CAPES) – Finance Code 001. SM was partially supported by the Department of Atomic Energy, India, under project no. 12-R&D-TFR-5.01-0500 and also by the Science and Engineering Research Board, India (SRG/2019/000513).

2 Graphs

This section serves to set a standard notation and intuition for graphs, we refer reader to [11] for details.

A directed multigraph γ with vertex set $V = V(\gamma)$ and edge set $E = E(\gamma)$ is given by two maps $s, t\colon E \to V$, the source and target of an oriented edge. One can also consider an involution that flips the orientation and exchanges s with t.

The incidence matrix B^γ of size $|E| \times |V|$ is the difference between linearizations of s and t: $B(e, v)$ is $+1$ if v is the source of e ($s(e) = v$), -1 if v is the target of e ($t(e) = v$), and 0 otherwise. In case e being a loop at vertex v one has $B(e, v) = 0$, but we consider graphs without loops.

If B is considered as $|V|$-tuple of vectors in $\mathbf{R}^E$ then its Gram matrix in the standard Euclidean metric equals to $B^{\mathrm{t}}B = A - D$, where D is diagonal matrix with entries $D(v, v)$ equal to valencies of the vertices, and A is an off-diagonal symmetric adjacency matrix with $A(v, v')$ equal to the number of edges incident with both v and

v'. If there are multiple edges between v and v' then $|A(v, v')| > 1$, but we consider graphs without multiple edges.

We say that a graph γ is *trivalent* if D is a scalar matrix with 3 on the diagonal, and will consider only such graphs.

The definitions of qCG-polytopes from Section 3 and graph potentials from [10] make sense for general trivalent multigraphs, with possible leaves, loops, multiple edges, and the theory of graph potentials crucially uses these extra degrees of freedom, however for the applications in this paper it suffices to consider only trivalent graphs

• that are uniquely determined by their adjacency matrix, that would have exactly 3 non-zero entries (equal to 1) in each column.

• for which very connected component has at least 5 vertices.

For the rest of the paper, our trivalent graphs will have the above two additional properties. These properties are chosen to simplify the proof and are sufficient for our purposes.

Free abelian groups of $\mathbf{Z}$-valued functions on V and E are known as the groups of 0-cochains $\mathrm{C}^0(\gamma) = \mathbf{Z}^V$ and 1-cochains $\mathrm{C}^1(\gamma) = \mathbf{Z}^E$, respectively. They come with standard bases $\{v^*\}_{v\in V}$, $\{e^*\}_{e\in E}$ of delta-functions, and with inner products in which these bases are orthonormal.

The random walks of Section 5 can be considered to happen either in the lattice $L = \mathrm{C}^1(\gamma) = \mathbf{Z}^E$ of 1-cochains, or in some particular sublattice.

The incidence matrix B is a matrix of a $\mathbf{Z}$-linear coboundary map $\delta : \mathrm{C}^0(\gamma) \to \mathrm{C}^1(\gamma)$, given by

$$\delta(v^*) = \sum_{e\in E\colon\ t(e)=v} e^* - \sum_{e\in E\colon\ h(e)=v} e^*, \tag{7}$$

i.e., the image of a vertex is a formal sum of outgoing edges minus sum of incoming edges. In our case every column of B has exactly three non-zero entries equal to ± 1.

The map B is known as the coboundary map, and the groups

$$\mathrm{Z}^V := \mathrm{C}^0(\gamma),\ \mathrm{Z}^E := \mathrm{C}^1(\gamma),\ \ker\delta\colon \mathrm{C}^0(\gamma) \to \mathrm{C}^1(\gamma),\ \text{and}\ \operatorname{coker}\delta\colon \mathrm{C}^0(\gamma) \to \mathrm{C}^1(\gamma)\Big) \tag{8}$$

are the groups of 0-cochains, 1-cochains, 0th cohomology $\mathrm{H}^0(\gamma)$ and first cohomology $\mathrm{H}^1(\gamma)$ of the graph, respectively.

The topological interpretation of the transpose matrix B^{t} is of a dual map $\partial\colon \mathrm{C}_1(\gamma) \to \mathrm{C}_0(\gamma)$ from 1-chains to 0-chains, and its kernel and cokernel are known as first homology $\mathrm{H}_1(\gamma)$ and 0th homology $\mathrm{H}_0(\gamma)$, respectively. For these matters one can replace integers $\mathbf{Z}$ by any other abelian group, e.g. $\mathbf{R}$ or $\mathbf{Z}/2 \simeq \{\pm 1\}$, so in this section we consider polytopes in $\mathrm{C}^1(\gamma, \mathbf{R}) = \mathbf{R}^E$ and their polar duals in $\mathrm{C}_1(\gamma, \mathbf{R})$, and in Section 7 we consider coloring functions on vertices, that are essentially 0-chains valued in a group of two elements.

In Section 8, the graphs γ are discrete/combinatorial counterparts (and tropical limits) of Riemann surfaces C, and the differential $\delta\colon \mathrm{C}^0(\gamma) \to \mathrm{C}^1(\gamma)$ is a counterpart of de Rham differential $\mathrm{d}_{\mathrm{dR}}\colon \Omega^0(C) \to \Omega^1(C)$ from functions to differentials. So analytically the transpose matrix B^{t} can be interpreted as a counterpart of the Hodge-theoretic adjoint map $\mathrm{d}^*_{\mathrm{dR}}\colon \Omega^1(C) \to \Omega^0(C)$; a counterpart of the Hodge metric on C being two inner products on $\mathrm{C}^0(\gamma) = \mathbf{Z}^V$ and $\mathrm{C}^1(\gamma) = \mathbf{Z}^E$ with respect to which the standard bases v^* and e^* are orthonormal. Then B^{t} is intepreted as $\delta^*\colon \mathrm{C}^1(\gamma) \to \mathrm{C}^0(\gamma)$, and the square matrix $B^{\mathrm{t}}B$, i.e., $\delta^*\delta\colon \mathrm{C}^0(\gamma) \to \mathrm{C}^0(\gamma)$ is a counterpart of the Laplace operator acting on functions on a Riemann surface.

We say that a trivalent graph γ has *genus* g if it is connected and has $(2g - 2)$ vertices, which also implies that it has $(3g - 3)$ edges.

3 Quantum Clebsch–Gordan polytopes

We start from a definition and study of some properties of quantum Clebsch–Gordan polytopes (qCG-polytopes for short) attached to graphs, that in the end will help to characterize them and reconstruct the respective graphs.

These polytopes are embedded in the affine space $\mathbf{R}^E$ with coordinates naturally indexed by the edges. The faces of these polytopes $P(\gamma)$ are parameterized by the vertices v: for every vertex $v \in V$ and edges i, j, k incident to v, the corresponding face

For a single vertex $v \in V$ with edges i, j, k incident to it, we consider the *quantum Clebsch–Gordan equations* for the real parameters x, y, z indexed by i, j, k which read

$$\begin{cases} x \le y + z; y \le x + z \text{ and } z \le x + y, & \text{triangle inequality} \\ x + y + z \le 1, & \text{perimeter inequality} \\ \pm x \pm y \pm z \in \mathbf{Z}. \end{cases} \tag{9}$$

These inequalities can be combined as follows. To a graph γ with its set $V(\gamma)$ of vertices associate the vector space $S(\gamma) := \mathbf{R}^E$ generated by a basis $e(i)$ indexed by the edges of γ. The *quantum Clebsch–Gordan polytope* $P(\gamma)$ in $S(\gamma)$ is the convex envelope of the vectors

$$p(v, s) := s(i)e(i) + s(j)e(j) + s(k)e(k) \in S(\gamma), \tag{10}$$

for all $v \in V(\gamma)$ and all sign choices $s(i), s(j), s(k) \in \{\pm 1\}$

$$s(i)s(j)s(k) = 1, \tag{11}$$

where the edges i, j, k are incident to the vertex v.

To avoid confusion between the vertices of the graph and vertices of the polytope, we will refer to the vertices of the polytope as *rays* in what follows, and we are going to consider edges of the graph, but not of the polytope.

For a fixed graph γ we consider a collection of vectors $p(v, s)$ associated to all vertices v and admissible s. Each of them in the basis $\{e(i)\}$ is a permutation of the entries of the vector $(s(1)1, s(2)1, s(3)1, 0, 0, \ldots, 0)$ where $s(1)s(2)s(3) = 1$. Consider the Euclidean norm in which the basis $e(i)$ is orthonormal.

Proposition 3.1. *The scalar product $(p(v, s), p(v', s'))$ takes four values:*

- $+3$ *if and only if $v = v'$ and $s = s'$;*
- $+1$ *if and only if v and v' are adjacent along an edge e and $s(e) = -s'(e)$;*
- -1 *if and only if either of two possibilities hold*
  - *$v = v'$ and $s \neq s'$,*
  - *v and v' are adjacent along an edge e and $s(e) = s'(e)$;*
- 0 *otherwise (i.e., if v and v' are neither coincident nor adjacent).*

Proof. To compute scalar products, note that it equals to

$$(p(v, s), p(v', s') = \sum_e s(e)s'(e) \tag{12}$$

where the sum is over the edges adjacent to both v and v'. If the vertices v, v' are neither coincident, not adjacent, the sum is empty. For coincident vertices there are three terms, all equal to $+1$ for $s' = s$ and one $+1$, two -1s for $s' \neq s$. For adjacent vertices (in the graphs that we consider) there is a single adjacent edge, so a single adjacent term. □

Corollary 3.2. *The vectors $p(v, s)$ are*

1. *pairwise-distinct, i.e., $p(v, s) = p(v', s')$ if and only if $v = v'$ and $s = s'$;*
2. *equidistant from 0;*
3. *and convex independent, i.e., none of them lies in a convex envelope of others.*

Proposition 3.3. *The center of mass of the vectors $p(v, s)$ is the origin, i.e.,*

$$\sum_{v\in V}\sum_{s} p(v, s) = 0. \tag{13}$$

Proof. The set of vectors $p(v, s)$ is partitioned into quadruples $\{p(v, s)\}$ enumerated by the vertices $v \in V(\gamma)$.

If x, y, z, w are four distinct admissible sign assignments then there is a linear relation

$$p(v, x) + p(v, y) + p(v, z) + p(v, w) = 0, \tag{14}$$

so the center of mass of the vectors in any such quadruple is 0, hence the center of mass of all the vectors is 0. □

Corollary 3.4. *The association $(v, s) \mapsto p(v, s)$ is a bijection from the set of pairs (v, s) of a vertex and sign assignment to the set of rays of the polytope $P(\gamma)$. In particular, the polytope $P(\gamma)$ has $4|V(\gamma)|$ rays.*

Let N denote the lattice spanned by the vectors $p(v, s)$. It is a finite-index sublattice in the lattice generated by $e(i)$, which we will denote by $\tilde{N}$.

The discussion above shows that lattice N can be reconstructed from the polytope $P(\gamma)$, however to reconstruct the lattice $\tilde{N}$ and vectors $\pm e(i)$ we need more information: neither Euclidean norm is immediate to reconstruct.

Instead let us study small linear relations between the vectors, since this data will be directly computable from the polytope. Denote the weight of a combination

$$\{w(v, s)\} \tag{15}$$

of integer coefficients $w(v, s)$ to be the sum of absolute values $\sum |w(v, s)|$. Call a combination positive, if all weights are non-negative. Such relations form a cone in the space of combinations, which is a free lattice $\mathbf{Z}^{4V}$. The kernel of the linear evaluation map

$$\{w(v, s)\} \mapsto \sum w(v, s)p(v, s) \in N \tag{16}$$

is the space of linear *relations*, so linear relations are linear combinations such that the respective vector in N is zero.

Since the sum of coordinates of every vector $p(v, s)$ equals to 3 modulo 4 it is clear that $\sum w(v, s)$ is divisible by 4 for any relation, so the smallest weight of a non-zero relation is 4. In this case either $\sum w(v, s) = 0$ and the relation can be rewritten either in form

$$p(v_1, s_1) + p(v_2, s_2) = p(v_3, s_3) + p(v_4, s_4) \tag{17}$$

or in form

$$p(v_1, s_1) - p(v_3, s_3) = p(v_4, s_4) - p(v_2, s_2), \tag{18}$$

or $\sum w(v, s) = \pm 4$ and the relation reads

$$p(v_1, s_1) + p(v_2, s_2) + p(v_3, s_3) + p(v_4, s_4) = 0. \tag{19}$$

So both cases can be rephrased as an equality of sums of pairs of vectors, and additionally in the first case as an equality of differences of pairs of vectors.

Let us compute such sums and differences, divided by two for convenience. It is straightforward to check the next result.

Proposition 3.5. *For all vertices v incident to edges i, j, k and all admissible signs $s \neq s'$ a semi-sum $(p(v,s)+p(v,s'))/2$ takes six values $\pm e(i)$, $\pm e(j)$, $\pm e(k)$, each once. Similarly, a semi-difference $(p(v,s)-p(v,s'))/2$ takes twelve values $\pm e(i) \pm e(j), \pm e(i) \pm e(k), \pm e(j) \pm e(k)$, each once. Note that the semi-sum does not depend on the ordering, but the semi-diference changes its sign.*

Lemma 3.6. *Let $v \in N$ be a point in lattice N. If there are rays p, q and p', q', such that the pairs $\{p, q\}$ and $\{p', q'\}$ are distinct, and $p + q = 2v = p' + q'$, then for any pair of rays p'', q'' such that $2v = p'' + q''$ the pair $\{p'', q''\}$ coincides either with $\{p, q\}$ or with $\{p', q'\}$. In other words, if some lattice point can be represented as semi-sum of two rays in at least two different ways, then it can be represented so in exactly two ways.*

Proof. Suppose that for some vertices $u, u', v, v' \in V(\gamma)$ and signs x, y, z, w there is a relation

$$p(u,x) + p(u',y) = p(v,z) + p(v',w) \tag{20}$$

For each vertex consider the parity of the number of times it appears in a relation. The relation implies that for every edge i the parities of the adjacent vertices are equal. So the assigned parities of vertices are constant in every connected component. Since we have only four terms in a relation, the number of odd vertices is at most four, so in every connected component with at least 5 vertices all vertices are even.

Thus either $u = u' = v = v'$, but then the six semi-sums are all distinct, or there are two vertices and each appears twice in the relation. Hence, either each vertex appears only on one side of the equation, or both vertices appear on both sides.

If each vertex appears on its own side then we have an equation $\pm e(i) = \pm e(j)$ where i is an edge adjacent to u and j is an edge adjacent to v, thus $i = j$ and the vertices u, v are adjacent to each other.

Finally, the equation

$$p(v,x) + p(v',w) = p(v,y) + p(v',z) \tag{21}$$

is equivalent to $p(v,x) - p(v,y) = p(v',z) - p(v',w)$ (where the differences are zero if and only if the summands on both sides are identical). For every vertex v with incident edges i, j, k the 12 semi-differences are equal to

$$\pm e(i) \pm e(j), \pm e(i) \pm e(k), \pm e(j) \pm e(k). \tag{22}$$

So if for two distinct vertices $v, v' \in V(\gamma)$ and some sign choices x, y, z, w the relation (21) holds, then the vertices v and v' would be connected by at least two edges, contradicting our assumption about trivalent graphs. □

4 Reconstruction of graphs from polytopes

The reconstruction is provided by the following procedure, which a priori is defined for any polytope P, but will not work on polytopes not associated with any graph. Recall that we refer to the vertices of a polytope as *rays*, to avoid ambiguity with the vertices of a graph.

[FIXME: what is the tangent space to the polytope in the center of mass?]

1. Let $N_{\mathbf{R}}$ be a tangent space to the polytope in the center of mass of its rays, and $N \subset N_{\mathbf{R}}$ be its abelian subgroup generated by rays. If the polytope is associated with some graph, then N has to be a discrete lattice of full rank.
2. Consider the multi-set $\mathcal{S}$ of semi-sums of pairs of distinct rays which are lattice points, i.e., $\frac{1}{2}(p+q)$ for all $p, q \in P$, provided that the semi-sum is in N. Mark every semi-sum which appears at least twice.
3. For every marked semi-sum in $\mathcal{S}$, consider it as a basis vector $e(i)$ in the real vector space spanned by the semi-sums which appear at least twice, subject to the condition that if a semi-sum and its negative appear you assign them as $e(i)$ and $-e(i)$. The choice of sign will be irrelevant.

4. Choose a basis among the $e(i)$.
5. Let $p(i)$ be the coordinate of the ray $p \in P$.

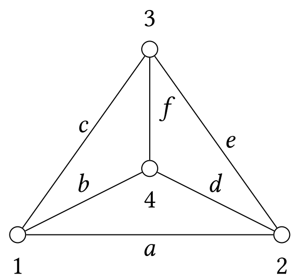
Then define the graph $\gamma' = (V', E')$, where

vertices V' is defined as the rays of the polytope modulo the equivalence relation which says that two rays are equivalent if in the basis given by the $e(i)$ they differ only up to signs;

edges E' is given by the basis indexed by the $e(i)$, and two vertices p and q are connected by an edge $e(i)$ if $p(i) \neq 0$ and $q(i) \neq 0$.

Let us illustrate the procedure in the following toy example of genus 3. In this case the graph contains only 4 vertices, but the procedure works in this case too. Moreover, it is a unique example of a trivalent graph without leaves, loops or multiple edges, that has less than 6 vertices.

Example 4.1. Consider the trivalent graph



(23)

of genus 3, which thus has 4 vertices and 6 edges. For each vertex in the graph we consider the 4 possible sign choices, leading to the following 16 rays in the polytope:

$$\begin{cases} (1,1,1,0,0,0) \\ (1,-1,-1,0,0,0) \\ (-1,1,-1,0,0,0) \\ (-1,-1,1,0,0,0) \end{cases}, \quad \begin{cases} (1,0,0,1,1,0) \\ (1,0,0,-1,-1,0) \\ (-1,0,0,1,-1,0) \\ (-1,0,0,-1,1,0) \end{cases}, \quad \begin{cases} (0,0,1,0,1,1) \\ (0,0,1,0,-1,-1) \\ (0,0,-1,0,1,-1) \\ (0,0,-1,0,-1,1) \end{cases}, \quad \begin{cases} (0,1,0,1,0,1) \\ (0,1,0,-1,0,-1) \\ (0,-1,0,1,0,-1) \\ (0,-1,0,-1,0,1) \end{cases}. \tag{24}$$

Considering the semi-sums of pairs of distinct rays, such that the semi-sum is a lattice point in the standard lattice, we obtain the following list, where we have already indicated a choice of basis.

$$\begin{cases} e_1 := (1,0,0,0,0,0), \\ e_2 := (0,1,0,0,0,0), \\ e_3 := (0,0,1,0,0,0), \\ -e_3 := (0,0,-1,0,0,0), \\ -e_2 := (0,-1,0,0,0,0), \\ -e_1 := (-1,0,0,0,0,0) \end{cases}, \quad \begin{cases} e_1 := (1,0,0,0,0,0), \\ e_4 := (0,0,0,1,0,0), \\ e_5 := (0,0,0,0,1,0), \\ -e_5 := (0,0,0,0,-1,0), \\ -e_4 := (0,0,0,-1,0,0), \\ -e_1 := (-1,0,0,0,0,0) \end{cases}, \quad \begin{cases} e_3 := (0,0,1,0,0,0), \\ e_5 := (0,0,0,0,1,0), \\ e_6 := (0,0,0,0,0,1), \\ -e_6 := (0,0,0,0,0,-1), \\ -e_5 := (0,0,0,0,-1,0), \\ -e_3 := (0,0,-1,0,0,0) \end{cases}, \quad \begin{cases} e_2 := (0,1,0,0,0,0), \\ e_4 := (0,0,0,1,0,0), \\ e_6 := (0,0,0,0,0,1), \\ -e_6 := (0,0,0,0,0,-1), \\ -e_4 := (0,0,0,-1,0,0), \\ -e_2 := (0,-1,0,0,0,0) \end{cases} \tag{25}$$

It is now clear that we can reconstruct the graph using the procedure outlined above, as the coordinates of the points in our chosen basis encode the adjacency relation.

The proof of correctness Let us now explain how the procedure reconstructs the graph γ from the polytope $P(\gamma)$.

If a vertex $v \in V(\gamma)$ is incident to edges i, j, k then the 6 semi-sums of two vectors from a quadruple are equal to $\pm e(i), \pm e(j), \pm e(k)$. So if i is an (internal non-loop) edge connecting vertices v and v' there are at least two distinct ways to write $e(i)$ as a semi-sum of two vectors:

$$2e(i) = p(v, x) + p(v, y) = p(v', z') + p(v', w'), \tag{26}$$

and similarly

$$-2e(i) = p(v, z) + p(v, w) = p(v', x') + p(v', y'). \tag{27}$$

Now Lemma 3.6 shows that no other vector has two representations as a semi-sum, and finishes the proof of correctness of the algorithm.

Proof of Theorem B. The reconstruction procedure uniquely recovers the graph γ from its polytope $P = P(\gamma)$, thus proving Theorem B. □

5 Random walks on lattices

We now explain how the combinatorial reconstruction is related to the behavior of the period sequence of graph potentials, as considered in [10, 9, 7]. These are typically identified by the name of classical periods in the literature.

For this we translate the setting to that of random walks on lattices, and explain how it yields interesting examples of random walks. The spirit of this example is similar to that of the question "Can one hear the shape of a drum?", where one tries to understand a random walk from its moments, and guess its Newton polytope. We refer the reader to [24, §1.1] for the definition of random walks. Consider a lattice $N \cong \mathbf{Z}^d$ in the d-dimensional real vector space $N_{\mathbf{R}} \cong \mathbf{R}^d$.

Consider the Laurent polynomial

$$W(z_1, \ldots, z_d) = \sum_{v \in \mathbf{Z}^d} p_v z^v, \tag{28}$$

in the ring of Laurent polynomials $\mathbf{R}[N]$ associated to the lattice N.

Assume that

- $\sum_{v\in\mathbf{Z}^d} p_v = 1$ and $p_v \geq 0$ for all $v \in \mathbf{Z}^d$;
- the origin $0 \in N$ is an interior point of P, and the constant term p_0 is zero.

These are the normalised conditions on a Laurent polynomial as considered in [13]. As in the introduction, the convex envelope of the elements $v \in N$ such that $p_v \neq 0$ is a bounded polytope $\mathrm{NP}(W)$ in $N_{\mathbf{R}}$, which we will call the *Newton polytope* of W.

With the assumptions as above, W can be considered as a random walk on the lattice N in the usual way: the vertices of the Newton polytope are the generating set for a random walk, provided there are no other lattice points on the polytope other than the vertices. However observe that we do not insist that the random walk is symmetric, i.e. p_v may not be equal to p_{-v}. Alternatively as in Lawler [24, §2.2], we can also consider the characteristic function $\phi(\theta) = \sum_{v\in\mathbf{Z}^d} p_v \exp(\mathrm{i}\langle v, \theta\rangle)$.

As in the introduction, recall the m-th term a_m of the period sequence written as an integral of the characteristic function in Equation (2). By [24, Corollary 2.2.3], the probability of the walk returning to the origin after m steps

$$a_m := P(S_m = 0 | S_0 = 0), \tag{29}$$

where $S = X_1 + \cdots + X_m$ and $X_1, \ldots, X_m, \ldots$ are identically distributed independent random variables in the lattice $\mathbf{Z}^d$ with characteristic function ϕ.

Recall from [10, §2.3] that a_m is also the constant term $[W^m]_0$ of the mth power of W.

The conifold point and random walks **[FIXME : add discussion of Doob transform]**

The condition that a random walk associated to a Laurent polynomial has mean zero and is truly d-dimensional can be translated to a property of the Laurent polynomial W which was already considered in [13].

We will now explain how this interpretation goes in the notation of op. cit, which will set up a dictionary between random walks and Laurent polynomials as they appear in mirror symmetry.

Let $u_1, \ldots, u_d$ be coordinates on $\mathbf{C}^d$. The exponential map $\mathbf{C}^d \to (\mathbf{C}^\times)^d$ is a topological covering map and identifies $\mathbf{R}^d \subset \mathbf{C}^d$ with $\mathbf{R}^d_+ \subset (\mathbf{C}^\times)^d$. Let $z_i = \exp(u_i)$ and we identify $\mathbf{R}^d$ with $\mathbf{R}^d_+$ via the exponential map. Observe that the partial derivative $\frac{\partial}{\partial u_i}$ coincides with $z_i \frac{\partial}{\partial z_i}$ under this identification.

This gives the following elementary lemma.

Lemma 5.1. *Let W be a Laurent polynomial with associated random walk as above. The condition that it has mean zero, resp. is truly d-dimensional, correspond to*

- *the point $c = (1, 1, \ldots, 1)$ (in exponential notation, i.e., the origin of N) is the unique point of $\mathbf{R}^d_+$ for which all the logarithmic derivatives $\partial W/\partial u_i$ vanish;*
- *the critical point c is Morse point, i.e., its Hessian matrix $(\frac{\partial^2 W}{\partial u_i \partial u_j})_{i,j}$ is non-degenerate.*

The point c is thus the *conifold point* studied in [13].

Proof. The first is a direct translation of the fact that the gradient of W evaluated at c gives $\sum_{v \in N} p_v v$, and thus describes the mean of the random walk. The second is another direct translation, by realising that the Hessian matrix of the Laurent polynomial corresponds to the covariance matrix of the associated random walk, which is invertible by [24, Proposition 1.1.1]. □

The shape of a random walk Let N and N' be two lattices and consider two random walks W and W' in N and N' satisfying the conditions of Lemma 5.1. We define the following:

Definition 5.2. Two random walks W and W'

- have the *same shape* if there exists an injective linear map $\psi \colon N \to N' \otimes_{\mathbf{Z}} \mathbb{Q}$ which maps the Newton polytope of W to the Newton polytope of W';
- have *equal moments* if $a_m(W) = a_m(W')$ for all $m \in \mathbb{N}$.

We ask the following question, where throughout we assume that our random walks are as in Lemma 5.1.

Question 5.3. Can two random walks W and W' have equal moments, but be of different shape?

More generally, can we find a collection of random walks $\{W_1, \ldots, W_k\}$ which all have the same moments, but pairwise they are of different shape?

We will reinterpret our results from Section 4 to answer Question 5.3 affirmatively.

Construction We give a construction of such balanced random walks using the graph potentials we considered in [10]. Let γ be a graph of genus g. Consider the lattice $N_\gamma \cong \mathbf{Z}^{3g-3}$ as in [10, §2.1] and the graph potential $\widetilde{W}_\gamma$ associated to γ as in loc. cit., i.e.,

$$\widetilde{W}_\gamma := \sum_{v \in V} \sum_{\substack{(s_i, s_j, s_k) \in \mathbb{F}_2^{\oplus 3} \\ s_i + s_j + s_k = c(v)}} x_i^{(-1)^{s_i}} x_j^{(-1)^{s_j}} x_k^{(-1)^{s_k}} \tag{30}$$

where x_i, x_j, x_k are variables attached to the edges e_i, e_j, e_k incident to the vertex v. Now consider the normalisation

$$\overline{W}_\gamma := \frac{1}{8(g-1)} \widetilde{W}_\gamma, \tag{31}$$

so that we can consider $\overline{W}_\gamma$ as the defining data for a random walk. By the main result of [13] and Lemma 5.1 we obtain the following.

Proposition 5.4. *For each graph γ, the Laurent polynomial $\overline{W}_\gamma$ defines a random walk in the lattice N_γ with mean zero and which is truly d-dimensional, and it has the conifold property.*

Comparing periods and Newton polytopes Now we investigate the question of comparing periods and Newton polytopes of graph potentials, associated to different graphs (of a fixed genus). We can reinterpret [10, Corollary 2.9] as follows.

Proposition 5.5. *Let γ and γ' be two graphs of the same genus. Then for m, the probabilities of return a_m to the origin after m steps of the associated random walks are the same.*

Remark 5.6. Observe that the number of trivalent graphs of genus g grows very quickly [19]. And each one of them produces random walks $\overline{W}_\gamma$. Thus we are producing many random walks with the same probabilities.

Finally, we can reinterpret Theorem 7.3 as follows, to answer Question 5.3.

Theorem 5.7. *Random walks $\overline{W}_{\gamma_i}$ associated with a collection pairwise-distinct graphs γ_i have pairwise distinct shapes, but pairwise-equal moments.*

6 Interpretation in Fermi varieties

Another reinterpretation of our results appears in the context of algebraic Fermi varieties, for another particular choice of a discrete Laplacian.

Consider the lattice $\mathbf{Z}^d$ and a sublattice $\Gamma = \bigoplus_{i=1}^d \mathbf{Z}a_i e_i$ where the e_i denote the standard basis vectors. Consider the $a_1 \cdots a_d$-dimensional complex Hilbert space $\mathrm{L}^2(\mathbf{Z}^2/\Gamma)$ of complex-valued Γ-periodic functions on $\mathbf{Z}^d$ with the inner product of two functions ϕ and ψ defined as

$$\langle \phi, \psi \rangle := \sum_{x \in \mathbf{Z}^d/\Gamma} \phi(x)\overline{\psi}(x). \tag{32}$$

The potential function $q(x)$ in the setup of Gieseker–Knörrer–Trubowitz [30] (see also Peters [30]) lies in the Hilbert space $\mathrm{L}^2(\mathbf{Z}^d/\Gamma)$ and they define the following discrete Laplacian

$$\Delta = \sum_{j=1}^d S_j + S_j^{-1}, \tag{33}$$

where the S_j's are the shift operators acting on functions by the formula

$$S_j\psi(x_1, \ldots, x_d) := \psi(x_1, \ldots, x_{j-1}, x_j + 1, x_{j+1}, \ldots, x_d). \tag{34}$$

Gieseker–Knörrer–Trubowitz introduce the following variety which they call the *Bloch variety*

$$B(q) := \{(t_1, \ldots, t_d, \lambda) \in (\mathbf{C}^\times)^d \times \mathbf{C} \mid \exists\psi \neq 0 \text{ a solution to the discrete model (36)}\} \tag{35}$$

for the system

$$\begin{cases} (-\Delta + q - \lambda)\psi = 0), & \text{where } \lambda \in \mathbf{C} \\ S^{a_j}\psi = t_j\psi, & \text{where } t_j \in \mathbf{C}^\times \text{ and } 1 \leq j \leq d \end{cases} \tag{36}$$

This discrete model is an analog of the model from solid-state physics called the independent state model. The role of $q(x)$ is played by a field potential, Γ by ions and the ψ are wave functions for gas of electrons moving in $\mathbf{R}^d/\Gamma$ under the influence of $q(x)$. (36) is the Schrödinger equation with periodic boundary conditions. The

Fermi variety F_λ can be considered inside the real part of the Bloch variety as the fiber over $\lambda \in \mathbf{R}$ under the second projection from $(\mathrm{S}^1)^d \times \mathbf{R}$. When $d = 2$, they are referred to as *Fermi curves.*

The following is one of the main theorems in [17]

Theorem 6.1. *If $d = 2$ and a_1, a_2 are distinct odd primes, then generic potentials determine the Bloch variety. More precisely, there exists a Zariski-dense subset $L \subset \mathrm{L}^2(\mathbf{Z}^2/\Gamma)$ such that, if $B(q) = B(q')$ for $q \in L$ and $q' \in \mathrm{L}^2(\mathbf{Z}^2/\Gamma)$ then there exists $(x_0, y_0) \in \mathbf{Z}^2/\Gamma$ for which*

$$q'(x, y) = q(\pm x + x_0, \pm y + y_0). \tag{37}$$

Analogy with graph potentials The analogy also includes the *shape* of the objects involved. The definition of the Bloch variety and Fermi varieties starts from a discrete Laplacian. Likewise, one can show that the graph potentials from [10] (see also Section 5) can be interpreted in terms of a discretization of the standard Laplacian in $\mathbf{R}^{3g-3}$.

For this we consider the normalised graph potential in (31), and consider its series expansion around the point $(1, \ldots, 1)$ in logarithmic coordinates over the real numbers, i.e., after having applied $\exp\colon \underline{N_{\mathbf{R}}} \to \mathbf{R}_+^{3g-3}$ in the setup of (30). The special properties of this point are discussed in Lemma 5.1. The expansion $\overline{W}_{\gamma,c}(\exp(\hbar\partial))$ has constant term $(1, \ldots, 1)$, no linear term, and the coefficient for $\hbar$ is precisely the standard Laplacian. This can be checked from the definition in (30), where the sum over the different sign choices allows us to cancel and rewrite accordingly. Thus the translation of $\overline{W}_{\gamma,c}(\exp(\hbar\partial))$ to cancel the constant term is truly a discretization of the (rescaled) standard Laplacian in $\mathbf{R}^{3g-3}$. We leave the details to the interested reader, as they play no role in what follows.

7 Groupoids of colored graphs and their polytopes

In this section we will upgrade the context to include colorings, and prove Theorem C as Theorem 7.3.

Colored graphs and their isomorphisms A *colored graph* is a pair (γ, c) consisting of a graph γ and a coloring function $c\colon V(\gamma) \to \{\pm 1\}$. We will consider $\{\pm 1\}$ as a group under the multiplication. Whenever needed we will reinterpret $\{\pm 1\}$ as $\mathbf{Z}/2\mathbf{Z}$. We define the following.

Definition 7.1. An *isomorphism* of colored graphs (γ, c) and (γ', c') consists of a pair (f, g), where

- f is a graph isomorphism $f\colon \gamma \xrightarrow{\sim} \gamma'$,
- g is a function $g\colon E(\gamma') \to \{\pm 1\}$ such that for any vertex $v \in V(\gamma)$ with edges i', j', k' incident to $f(v) \in V(\gamma')$ the relation

  $$c'(fv)/c(v) = g(i')g(j')g(k') \tag{38}$$

  holds.

In particular if f is the identity we can regard the colorings c, c' as 0-chains of the graph with values in $\mathbf{Z}/2\mathbf{Z}$, which we will consider as a ring. Then the set of all g's such that (id, g) is an isomorphism is the set of 1-chains whose boundary is the ratio of c' and c in $\{\pm 1\}$, so the set of automorphisms is a torsor under the first relative homology group $\mathrm{H}_1(\gamma, \partial\gamma; \mathbf{Z}/2\mathbf{Z})$.

Quantum Clebsch-Gordan polytope of a colored graph Next we generalise the definition $P(\gamma)$ to incorporate the coloring function.

Definition 7.2. Define the *colored quantum Clebsch–Gordan polytope* $P(\gamma, c)$ as the convex envelope in $S(\gamma)$ of the rays $p(v, s)$ in (10) where we now ask

$$s(i)s(j)s(k) = c(v). \tag{39}$$

Note that $P(\gamma) = P(\gamma, c_0)$, where c_0 is the constant coloring taking $c_0(v) = +1$ for all $v \in V(\gamma)$, as (39) reduces to (11).

Groupoid of colored graphs The composition of two colored isomorphisms $(f, g)\colon (\gamma, c) \to (\gamma', c')$ and $(f', g')\colon (\gamma', c') \to (\gamma'', c'')$ is $(f' \circ f, g'')$ is given by

$$g''(i'') := g'(i'') \cdot \prod_{f'(i')=i''} g'(i'). \tag{40}$$

Generators for the isomorphism groupoid of colored graphs can be chosen to be

$$(f, g)\colon (\gamma, c) \to (\gamma', c') \tag{41}$$

with either $c = c' \circ f, g = 1$ or f being the identity and g is an assignment of ± 1 to a single edge.

Functor of quantum Clebsch-Gordan polytopes Any isomorphism

$$(f, g)\colon (\gamma, c) \to (\gamma', c') \tag{42}$$

of colored graphs naturally induces an isomorphism $P(f, g)\colon P(\gamma, c) \to P(\gamma', c')$ of the associated colored quantum Clebsch–Gordan polytopes. The generators of the first kind correspond to "permutations" and the generators of the second kind correspond to linear transformations that send $e(i)$ to $-e(i)$ for the chosen edge i and preserve all other basis vectors.

Colored combinatorial non-abelian Torelli theorem We can upgrade the reconstruction result of Section 4 as follows, which is the more precise version of Theorem C.

Theorem 7.3. *The association $P(-)$ is a* full *functor from the isomorphism groupoid of colored positive graphs to the groupoid of convex polytopes and affine isomorphisms. That is, any affine linear isomorphism between colored quantum Clebsch–Gordan polytopes $P(\gamma, c)$ and $P(\gamma', c')$ is induced by a uniquely defined isomorphism of colored graphs.*

Proof. The polytopes $P(\gamma, c)$ and $P(\gamma', c')$ are considered as in Proposition 3.3, thus affine linear isomorphisms do not take translations into account. The steps of the algorithm above can be applied to the colored set-up without modifications. Then for any of the $2^{\dim S(\gamma)}$ bases of $S(\gamma)$ with given coordinate axes $\pm e(i)$ the coloring of the vertices of the graph is well-defined and uniquely determined by the parity of the number of minus signs in the coordinates of the respective rays in the given basis. A unique choice of signs corresponds to the original coloring, all other choices produce homologically equivalent colorings provided the total parity (even or odd) of vertices in every leafless connected component remains unchanged.

It suffices to notice that the change of the sign of a single vector in the basis corresponds to simultaneous change of the color of the two vertices adjacent to this edge. □

8 Algebro-geometric interpretation

Abelian and non-abelian Torelli The usual Torelli theorem recovers a curve C from the Jacobian $\operatorname{Jac} C$ together with its polarisation given by the Theta divisor, i.e., from the pair $(\operatorname{Jac} C, \Theta)$ [33]. See [2, §VI.3] for a modern discussion. More generally, the "Torelli package" also provides a description of the polarised automorphism group in terms of automorphisms of C, and the full package also classifies all isomorphisms between the pairs $(\operatorname{Jac} C, \Theta)$ and $(\operatorname{Jac} C', \Theta')$.

A similar *non-abelian* Torelli packages exist for moduli spaces of bundles with non-abelian structure groups, such as moduli spaces $\mathrm{M}_C(r,d)$ and $\mathrm{M}_C(r,\mathcal{L})$ of semistable vector bundles of higher rank $r > 1$ and degree d (resp. fixed determinant $\mathcal{L} \in \operatorname{Pic} C$).

To state the non-abelian Torelli package, observe that

- an isomorphism of curves $f\colon C \xrightarrow{\sim} C'$ induces an isomorphism $\mathrm{M}_C(r,\mathcal{L}) \xrightarrow{\sim} \mathrm{M}_{C'}(r,f_*\mathcal{L})$ given by $\mathcal{E} \mapsto f_*\mathcal{E}$,
- a line bundle $\mathcal{N} \in \operatorname{Pic} C$ induces an isomorphism $\mathrm{M}_C(r,\mathcal{L}) \xrightarrow{\sim} \mathrm{M}_C(r,\mathcal{L}\otimes\mathcal{N}^{\otimes r})$ given by $\mathcal{E} \mapsto \mathcal{E}\otimes\mathcal{N}$.

These isomorphisms generate a groupoid, with objects being pairs $(C,\mathcal{L})$ of a smooth projective curve C of genus $g \geq 2$ and a line bundle $\mathcal{L} \in \operatorname{Pic} C$ and morphisms from $(C,\mathcal{L})$ to $(C',\mathcal{L}')$ given by triples $(f,\mathcal{N}',\iota)$ of

- an isomorphism of curves $f\colon C \xrightarrow{\sim} C'$,
- a line bundle $\mathcal{N}' \in \operatorname{Pic} C'$,
- an isomorphism of line bundles $\iota\colon f_*\mathcal{L} \xrightarrow{\sim} \mathcal{L}'\otimes\mathcal{N}'^{\otimes r}$.

Hence for fixed $r \geq 2$ the association $(C,\mathcal{L}) \mapsto \mathrm{M}_C(r,\mathcal{L})$ is promoted to a functor $\mathcal{M}$ from this groupoid to the groupoid of algebraic varieties and isomorphisms.

Unlike Jacobians the spaces $\mathrm{M}_C(r,\mathcal{L})$ have anticanonical polarizations, and a non-abelian Torelli theorem for smooth curves was formulated by in rank 2 independently by Mumford–Newstead [28, Corollary] and Tyurin [32, Theorem 1], and in arbitrary rank by Narasimhan–Ramanan [29, Theorem 3]. The Torelli package was completed by a description of the automorphism group by Kouvidakis–Pantev [23, Theorems B and D], leading to the following statement. Let us phrase it for $r = 2$, the general statement is similar but requires more notation;

Theorem 8.1 (Non-abelian Torelli)**.** *If* $\mathrm{M}_C(2,\mathcal{L}) \cong \mathrm{M}_{C'}(2,\mathcal{L}')$ *with* $\deg\mathcal{L} = \deg\mathcal{L}' = d$*, then* $C \cong C'$*. Moreover, any automorphism of* $\mathrm{M}_C(2,\mathcal{L})$ *is induced uniquely by an isomorphism of pairs* $(C,\mathcal{L})$*, i.e., the functor* $\mathcal{M}$ *is full.*

The automorphism group of the moduli space is an extension of the automorphisms of the curve and the r-torsion subgroup in Jacobian.

From geometry to combinatorics (and back) In the combinatorial versions of these theorems trivalent graphs will correspond to curves, colorings to line bundles, polytopes to moduli spaces, and the functor P from Section 7 is a version of the functor $\mathcal{M}$.

The analogy between curves and trivalent graphs is standard and can be explained as follows. With a graph one can associate a *graph curve* $C(\gamma)$ as studied by Bayer–Eisenbud [5], the union of projective lines enumerated by the vertices of the graph with the intersections corresponding to the edges of the graph.

Algebro-geometric property of very ampleness of the canonical bundle of graph curves curves corresponds to three-connectedness of the respective trivalent graphs, and forbids loops and multiple edges.

The leaves (if there are any) correspond to additional marked smooth points. These curves index the finite set of zero-dimensional strata in the usual stratification of the Deligne–Mumford–Knudsen moduli space of stable marked curves, with the graph γ being the dual graph of the marked curve $C(\gamma)$.

A path between such a point and the interior of the moduli space corresponds to a degeneration of a smooth algebraic curve to the graph curve. Topologically, the respective vanishing cycles give the Thurston cut system of the corresponding Riemann surface into trinions (i.e., spheres with three holes, or pairs-of-pants) encoded by the same graph.

Moduli spaces such as Jacobians and $\mathrm{M}_C(r,\mathcal{L})$ vary together with the smooth curve C, and we can try to construct limits or degenerations as the curve tends to a graph curve $C(\gamma)$. It is natural to ask whether there are Torelli-type theorem for these limiting varieties.

Combinatorial Torelli theorems For stable curves of genus g with rational components Artamkin [4] constructed a compactification $A(\gamma)$ of the generalized Jacobian which is a toric variety whose polytope is built from the combinatorial data of cycles in γ. Artamkin [3] furthermore proves that $A(\gamma)$ recovers the graph γ for graphs which are k-connected for $k \geq 3$. Similar Torelli-type theorems for the Albanese torus for 3-connected graph curves have been proved by Caporaso–Viviani [12].

Manon [25] constructed a toric degeneration of the moduli space of *rank two* bundles with fixed determinant for any trivalent colored graph (γ, c), with the translation between op. cit. and our setup using colored graphs being explained in [9, §2.2]. A modular interpretation of these toric varieties is not known. But in the spirit of results of Artamkin [3] and Caporaso–Viviani [12], it is interesting to ask a Torelli-type question: are these toric varieties uniquely determined by the maximally degenerate curves?

The polytopes $P(\gamma, c)$ defined in Definition 7.2 are the moment polytopes of the toric varieties constructed by Manon [25, 26], see also [9, §2.2]. Since a projective toric Fano variety with its anticanonical polarization is uniquely determined by its moment polytope, Theorem 7.3 can be considered as a non-abelian analog of a Torelli-type theorem for the toric varieties constructed by Manon.

Corollary 8.2 (Combinatorial non-abelian Torelli)**.** *Let (γ, c) and (γ', c') be colored trivalent graphs of genus g, such that the toric degenerations are isomorphic. Then $\gamma \cong \gamma'$ and $c = c'$ in $\mathrm{H}_1(\gamma, \mathbb{Z}/2\mathbb{Z})$. Moreover, any polarised automorphism of the toric degeneration is uniquely determined by a colored graph isomorphism, i.e., the functor is full.*

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