Crossed Modular Categories and a Verlinde formula for twisted conformal blocks.

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(joint work with Tanmay Deshpande)

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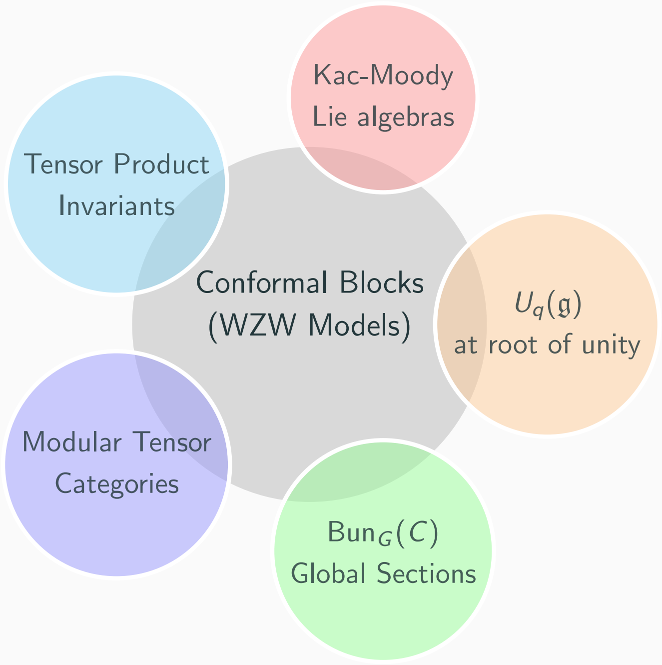
Kac-Moody
Lie algebras
Tensor Product
Invariants
Conformal Blocks
(WZW Models)
$U_q(\mathfrak{g})$
at root of unity
Modular Tensor
Categories
$\mathrm{Bun}_G(C)$
Global Sections

Tsuchiya-Ueno-Yamada construction of WZW models

Consider an affine untwisted Kac-Moody Lie algebra

$$\widehat{\mathfrak{g}} := \mathfrak{g} \otimes \mathbb{C}((t)) \oplus \mathbb{C}c$$

$$[X \otimes f, Y \otimes g] := [X, Y] \otimes fg + (X, Y) \operatorname*{Res}_{t=0} g df .$$

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Remark

We should think of the formal parameter t as a coordinate on the formal disk or a local parameter at a smooth point on a projective curve

Representations

Let $\ell > 0$ and consider the set

$$P_\ell(\mathfrak{g}) := \{\lambda \in P_+(\mathfrak{g}) | (\lambda, \theta) \leq \ell\}.$$

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- Each $\mathcal{H}_\lambda$ is constructed as a quotient of a Verma module.
- $V_\lambda \subset \mathcal{H}_\lambda$.
- $\mathcal{H}_\lambda$ are infinite dimensional.

Conformal Blocks

Let C be a stable curve of genus g and $\vec{p} = (p_1, \ldots, p_n)$ be n-distinct smooth points on C

$$\mathcal{V}_{\vec{\lambda}}(C, \vec{p}, \mathfrak{g}, \ell) := \frac{\mathcal{H}_{\lambda_1} \otimes \cdots \otimes \mathcal{H}_{\lambda_n}}{\left(\mathfrak{g} \otimes \Gamma(C \backslash \vec{p}) \mathcal{H}_{\lambda_1} \otimes \cdots \otimes \mathcal{H}_{\lambda_n}\right)}.$$

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- Moreover as $(C, \vec{p})$ varies in $\overline{\mathcal{M}}_{g,n}$, the spaces $\mathcal{V}_{\vec{\lambda}}(C, \vec{p}, \mathfrak{g}, \ell)$ give a vector bundle $\mathbb{V}_{\vec{\lambda}}(\mathfrak{g}, \ell)$ on $\overline{\mathcal{M}}_{g,n}$.

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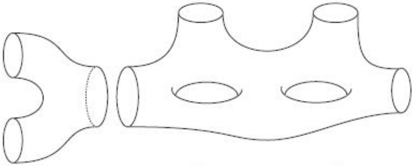
- $\mathbb{V}_{\vec{\lambda}}(\mathfrak{g}, \ell)$ carries a flat projective connection with logarithmic singularities along the boundary of $\overline{\mathcal{M}}_{g,n}$.

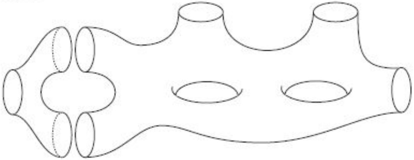
Verlinde formula via 2d-CFT

The spaces of conformal blocks satisfy axioms of Conformal-Field-Theory and motivated by this E. Verlinde conjectured a formula for the dimension of the space of conformal blocks $\mathcal{V}_{\vec{\lambda}}(C, \vec{p}, \mathfrak{g}, \ell)$!!! (1987)

$$(g, n) \Longrightarrow (g, n-1)$$

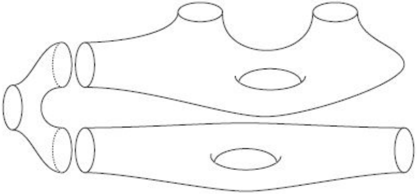


$$(g, n) \Longrightarrow (g-1, n+1)$$



$$(g, n) \Longrightarrow (g_1, n_1)+(g_2, n_2)$$

$$\begin{cases} g = g_1 + g_2 \\ n = n_1 + n_2 - 1 \end{cases}$$



Factorization Theorem (TUY-89)

Consider the natural maps

1. $\overline{\mathcal{M}}_{g-1,n+2} \to \overline{\mathcal{M}}_{g,n}$
2. $\overline{\mathcal{M}}_{g_1,n+1} \times \overline{\mathcal{M}}_{g_2,m+1} \to \overline{\mathcal{M}}_{g_1+g_2,n+m}$

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Globally the vector bundle $\mathbb{V}_{\vec{\lambda}}(\mathfrak{g}, \ell)$ splits into the direct sum,

$$\bigoplus_{\mu \in P_\ell(\mathfrak{g})} \mathbb{V}_{\vec{\lambda},\mu,\mu^*}(\mathfrak{g}, \ell),$$

Fusion ring $\mathcal{R}(\mathfrak{g},\ell)$

Consider the

$$\mathcal{R}(\mathfrak{g},\ell) := \bigoplus_{\lambda \in P_\ell(\mathfrak{g})} \mathbb{Z}[\lambda].$$

$$[\lambda_1] \circ_F [\lambda_2] := \sum_{\mu \in P_\ell(\mathfrak{g})} \dim_{\mathbb{C}} \mathcal{V}_{\lambda_1,\lambda_2,\mu}(\mathbb{P}^1; 0, 1, \infty; \mathfrak{g}; \ell)[\mu^*].$$

The product $\circ_F$ makes $\mathcal{R}(\mathfrak{g},\ell)$ into a commutative associative, ring with identity 0. Moreover $\mathcal{R}(\mathfrak{g},\ell)$ is a Frobenious algebra.

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Remark: There are many other equivalent ways to define fusion rings using tilting modules for quantum groups, cohomology of vector bundles on affine Grassmannians, twisted K-theory (Andersen-Stroppel, Freed-Hopkins-Teleman, Kumar).

Theorem (Faltings-94, Teleman-95)

The characters of the ring $\mathcal{R}(\mathfrak{g}, \ell)$ are of the form

$$[\lambda] \to \mathrm{Tr}_{V_\lambda}\big(\exp \frac{2\pi\sqrt{-1}(\mu+\rho)}{(\ell + h^\vee(\mathfrak{g}))}\big),$$

for $\mu \in P_\ell(\mathfrak{g})$, ρ is the sum of fundamental weights, $h^\vee(\mathfrak{g})$ is the dual-Coxter number.

Character Table

Consider the square matrix $|P_\ell(\mathfrak{g})|\ \Sigma$ of size $|P_\ell(\mathfrak{g})|$ whose (μ,λ)-th entry

$$\Sigma_{\mu,\lambda} := \mathrm{Tr}_{V_\lambda}\big(\exp \frac{2\pi\sqrt{-1}(\mu+\rho)}{(\ell+h^\vee(\mathfrak{g}))}\big),$$

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where Δ is a diagonal matrix with positive entries.

Define the Kac-Moody S matrices

$$S := \Delta^{-1/2}\Sigma$$

Clearly S is unitary and the first column is real and positive. Moreover they are also symmetric.

Rewriting in terms of S-matrices

$$\dim \mathcal{V}_{\lambda,\beta,\gamma}(\mathbb{P}^1; 0, 1, \infty; \mathfrak{g}, \ell) = \sum_{\mu \in P_\ell(\mathfrak{g})} \frac{S_{\lambda,\mu}.S_{\beta,\mu,}.S_{\gamma,\mu}}{S_{0,\mu}}$$

One can write formula for S using the Weyl-Character formula.

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The higher genus version is the following:

$$\dim \mathcal{V}_{\vec{\lambda}}(C, \mathfrak{g}, \ell) = \sum_{\mu \in P_\ell(\mathfrak{g})} \frac{S_{\lambda_1,\mu} \cdots S_{\lambda_n,\mu}}{\left(S_{0,\mu}\right)^{n+2g-2}},$$

Weak Rigidity

A monoidal category $(\mathcal{C}, \otimes, \mathbf{1})$ is said to be an weakly rigid category if:
(i) For each $X \in \mathcal{C}$, the functor $\mathcal{C} \ni Y \mapsto \mathrm{Hom}(\mathbf{1}, X \otimes Y)$ is representable by an object X^*, i.e. we have functorial identifications $\mathrm{Hom}(\mathbf{1}, X \otimes Y) = \mathrm{Hom}(X^*, Y)$.

(ii) The functor $\mathcal{C} \ni X \mapsto X^* \in \mathcal{C}^{\mathrm{op}}$ is an equivalence of categories, with the inverse functor being denoted by $X \mapsto {}^*X$.

(iii) $\mathbf{1}$ is an simple object.

Braided Tensor Category

A Braided-Tensor-Category (BTC) is a monoidal category $(\mathbb{C}, \otimes, \mathbf{1})$ along with functorial isomorphism

$$\beta_{i,j} : [A_i] \otimes [A_j] \cong [A_j] \otimes [A_i]$$

that satisfy Braid group relations.

Rigidity

Let $\mathcal{C}$ be a monoidal category and V be an object. A rigid right dual V^* is a object along with morphism

$$e_A : V^* \otimes V \to \mathbf{1}$$

$$i_A : \mathbf{1} \to V \otimes V^*$$

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such that the compositions are id on V and V^*.

$$V \xrightarrow{i_V \otimes id_V} V \otimes V^* \otimes V \xrightarrow{id_V \otimes e_V} V$$

$$V^* \xrightarrow{id_V^* \otimes i_V} V^* \otimes V \otimes V^* \xrightarrow{e_V \otimes id_V^*} V^*$$

S-matrix

$$tr(f) : \mathbf{1} \xrightarrow{i_V} V \otimes V^* \xrightarrow{f \otimes id} V \otimes V^* \xrightarrow{\delta_V} V^{**} \otimes V^* \xrightarrow{e_{V^*}} \mathbf{1}$$

$$\widetilde{S}_{ij} := tr(A_i \otimes A_j \xrightarrow{\beta_{i,j}} A_j \otimes A_i \xrightarrow{\beta_{j,i}} A_i \otimes A_j)$$

Definition

A Modular-Tensor-Category (MTC) is a semisimple, rigid, BTC such that

- Functorial isomorphism $\delta_V : V \simeq V^{**}$.
- The matrix $\widetilde{S} = (\widetilde{S})_{ij}$ is invertible.

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The Grothendieck Group of $\mathcal{C}$ of a MTC gets a ring structure and the matrix $\widetilde{S}$ gives the character table.

Crossed Modular Tensor Category

Let $\mathcal{C} := \bigoplus_{\gamma \in \Gamma} C_\gamma$ is a Γ graded abelian category with tensor structure

$$\mathcal{C}_{\gamma_1} \otimes \mathcal{C}_{\gamma_2} \to \mathcal{C}_{\gamma_1 \gamma_2}$$

and a monoidal Γ action

$$\gamma : \mathcal{C}_\eta \to \mathcal{C}_{\gamma \eta \gamma^{-1}}$$

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Further there are function crossed brading isomorphism

$$\beta_{M,N} : M \otimes N \xrightarrow{\cong} \gamma(N) \otimes M \quad \text{for } \gamma \in \Gamma,\ M \in \mathcal{C}_\gamma$$

Crossed S-matrix

For each $C \in P_1^\gamma$, let us choose an isomorphism $\psi_C : \gamma(C) \xrightarrow{\cong} C$.
For $M \in P_\gamma, C \in P_1^\gamma$, we set

$$\widetilde{S}^\gamma_{M,C} := \mathrm{tr}(C \otimes M \xrightarrow{\beta_{C,M}} M \otimes C \xrightarrow{\beta_{M,C}} \gamma(C) \otimes M \xrightarrow{\psi_C \otimes \mathrm{id}_M} C \otimes M).$$

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- Define the normalized γ-crossed S-matrix to be

$$S^\gamma := \frac{1}{\sqrt{\dim \mathcal{C}_1}} \cdot \widetilde{S}^\gamma.$$

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- The matrix S^γ is a $P_\gamma \times P_1^\gamma$ unitary matrix.

Categorical Verlinde formula-(Deshpande-M)

Theorem: Deshpande-Mukhopadhyay

Let $\mathcal{C} = \bigoplus_{\gamma \in \Gamma} \mathcal{C}_\gamma$ be a Γ-crossed MTC and consider $A \in \mathcal{C}_{\gamma_1}, B \in \mathcal{C}_{\gamma_2}$ and $C \in \mathcal{C}_{\gamma_1\gamma_2}$ be simple objects.

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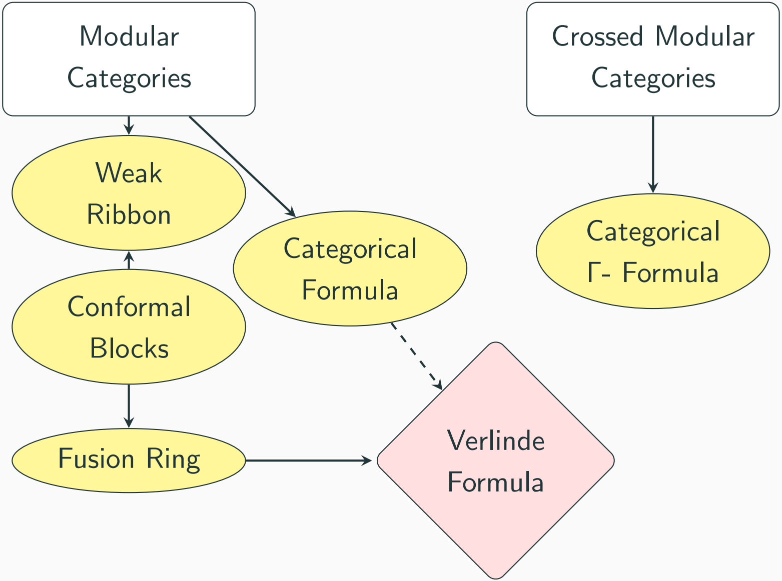
Then the multiplicity $\nu_{A,B}^C$ of C in the tensor product $A \otimes B$ is given by

$$\nu_{A,B}^C = \sum_{D \in P_1^{\langle \gamma_1, \gamma_2 \rangle}} \frac{S_{A,D}^{\gamma_1} \cdot S_{B,D}^{\gamma_2} \cdot \overline{S_{C,D}^{\gamma_1\gamma_2}}}{S_{\mathbf{1},D}},$$

Remark

- Both crossed and uncrossed S-matrices appear in the formula.
- This generalizes the result of T. Deshpande for cyclic groups.
- In general, there are some cocycles in the formula. However in our application, the cocycle does not appear.

Summary so far



Twisted Kac-Moody Lie algebras

Let γ be an automorphism of a finite dimensional simple Lie algebra $\mathfrak{g}$ of order $|\gamma|$. We fix a $|\gamma|$-th root of unity $\epsilon := e^{\frac{2\pi\sqrt{-1}}{|\gamma|}}$ of unity.

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Let γ act on $\mathbb{C}((t))$ by the formula $\gamma.t := \epsilon^{-1}t$.

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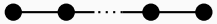
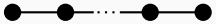
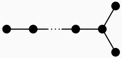
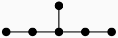
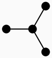
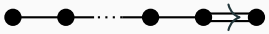
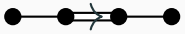
Define the *twisted affine Lie algebra*

$$\widehat{L}(\mathfrak{g},\gamma) := (\mathfrak{g} \otimes \mathbb{C}((t)) \oplus \mathbb{C}c)^{\gamma}.$$

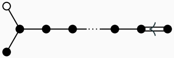
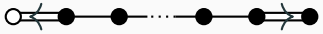
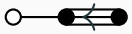
Lie algebra $\mathfrak{g}$

- A_{2n-1}  ,
- A_{2n}  ,
- D_n  ,
- E_6  ,
- D_4 

Fixed point algebra $\mathfrak{g}^\sigma$

- C_n  ,
- B_n  ,
- B_{n-1}  ,
- F_4  ,
- G_2 

Twisted Affine Kac-Moody algebras $X_N^{(m)}$

- $A_{2n-1}^{(2)}$ 
- $A_{2n}^{(2)}$ 
- $D_n^{(2)}$ 
- $E_6^{(2)}$ 
- $D_4^{(3)}$ 

Relation to Diagram automorphism

Let γ be an automorphism of $\mathfrak{g}$ of order $|\gamma|$. Then there exists a Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$ containing a Cartan subalgebra $\mathfrak{h}$ such that

$$\gamma = \sigma \exp(\mathrm{ad}\, \frac{2\pi\sqrt{-1}}{|\gamma|} h),$$

where σ is a diagram automorphism of $\mathfrak{g}$ and $h \in \mathfrak{h}$.

The irreducible, highest weights, integrable modules of $\widehat{L}(\mathfrak{g},\gamma)$ are parametrized by a finite subset of $P_\ell(\mathfrak{g},\gamma)$ and the corresponding modules will be denoted by $\mathcal{H}_\lambda(\mathfrak{g},\gamma)$.

Representations

The irreducible, highest weights, integrable modules of $\widehat{L}(\mathfrak{g}, \gamma)$ are parametrized by a finite subset of $P_\ell(\mathfrak{g}, \gamma)$ and the corresponding modules will be denoted by $\mathcal{H}_\lambda(\mathfrak{g}, \gamma)$.

There is a natural isomorphism:

$$\phi_{\sigma,\gamma} : \widehat{L}(\mathfrak{g}, \sigma) \to \widehat{L}(\mathfrak{g}, \gamma)$$

The map $\phi_{\sigma,\gamma}$ induces a bijection $P_\ell(\mathfrak{g}, \sigma)$ and $P_\ell(\mathfrak{g}, \gamma)$.

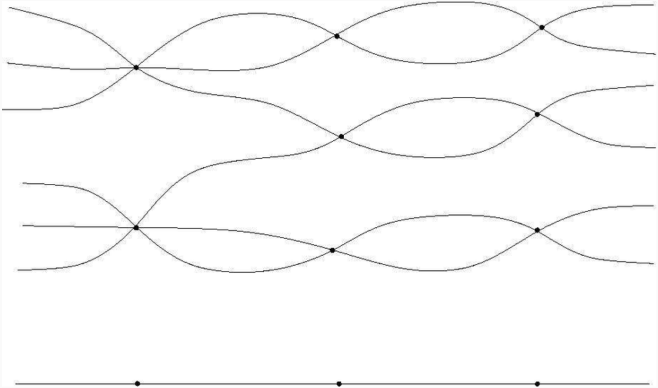
Example

$$\mathfrak{g}_1 \otimes \mathbb{C}((t^2)) \oplus \mathfrak{g}_{-1} \otimes t\mathbb{C}((t^2)) \oplus \mathbb{C}c$$

Ramfied Covering

Remark

Think of "t" as a local coordinate on the top curve at branch points.



Moduli of Γ-covers

Let $\pi : \widetilde{C} \to C$ be a covering of curves such that

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- π is possibly ramified Galois cover with Galois group Γ.
- $\mathbf{p} = (p_1, \ldots, p_n)$ be n-distinct points on C that includes all ramification points.

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Moduli of Γ-covers

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- π is possibly ramified Galois cover with Galois group Γ.
- $\mathbf{p} = (p_1, \ldots, p_n)$ be n-distinct points on C that includes all ramification points.
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The moduli stack of such data will be denoted by $\overline{\mathcal{M}}^{\Gamma}_{g,n}(\mathbf{m})$ and was studied by Jarvis-Kimura-Kauffmann and a related version by Abramovich-Corti-Vistoli.

Γ-twisted conformal blocks

$$\mathcal{V}_{\vec{\lambda},\Gamma}(\widetilde{C},\widetilde{C},\widetilde{\mathbf{p}},\mathbf{p},\mathfrak{g},\ell) := \frac{\mathcal{H}_{\lambda_1}(\mathfrak{g},m_1)\otimes\cdots\otimes\mathcal{H}_{\lambda_n}(\mathfrak{g},m_n)}{(\mathfrak{g}(\mathcal{A}))^{\Gamma}\Big(\mathcal{H}_{\lambda_1}(\mathfrak{g},m_1)\otimes\cdots\otimes\mathcal{H}_{\lambda_n}(\mathfrak{g},m_n)\Big)},$$

where $\mathcal{A}$ is the space of functions on $\widetilde{C}\backslash\Gamma\cdot\widetilde{\mathbf{p}}$.

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where $\mathcal{A}$ is the space of functions on $\widetilde{C}\backslash\Gamma \cdot \widetilde{\mathbf{p}}$.

Assumption

We will further assume that "Γ preserves a Borel subalgebra of $\mathfrak{g}$".

Properties

- The twisted conformal blocks $\mathcal{V}_{\vec{\lambda},\Gamma}(\widetilde{C}, \widetilde{C}, \widetilde{\mathbf{p}}, \mathbf{p}, \mathfrak{g}, \ell)$ give a vector bundle $\mathbb{V}_{\vec{\lambda},\Gamma}(\mathfrak{g}, \ell)$ on $\overline{\mathcal{M}}^{\Gamma}_{g,n}(\mathbf{m})$.

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- It further carries with a flat projective connection with log singularties along the boundary (Damiolini, Deshpande-M, Kumar-Hong, Sczesny)

Factorization

The twisted conformal blocks decomposition under the map (Damiolini, M, Kumar-Hong)

- $\overline{\mathcal{M}}^{\Gamma}_{g_1,n+1}(\mathbf{m},\gamma)\times\overline{\mathcal{M}}^{\Gamma}_{g_2,m+1}(\mathbf{m}',\gamma^{-1})\to\overline{\mathcal{M}}^{\Gamma}_{g_1+g_2,n+m}(\mathbf{m},\mathbf{m}')$
- $\overline{\mathcal{M}}^{\Gamma}_{g-1,n+2}(\mathbf{m},\gamma,\gamma^{-1})\to\overline{\mathcal{M}}^{\Gamma}_{g,n}(\mathbf{m})$. Moreover the splitting is parametrized by the $P_\ell(\mathfrak{g},\gamma)$.

Theorem: Deshpande-Mukhopadhyay

The twisted conformal blocks associated to a finite group Γ, a simple Lie algeba $\mathfrak{g}$ at level ℓ for define a Γ-crossed MTC

$$\mathcal{C} = \bigoplus_{\gamma \in \Gamma} \mathcal{C}_\gamma$$

such that

- The simple object of Γ are parametrized by the set $P_\ell(\mathfrak{g}, \ell)$.
- For $m_1, m_2, m_3 \in \Gamma^3$ such that $m_1.m_2.m_3 = 1$

$$\mathrm{Hom}(\mathbf{1}, [\lambda_1] \otimes [\lambda_2] \otimes [\lambda_3]) := \mathcal{V}_{\vec{\lambda},\Gamma}(\widetilde{C}, \mathbb{P}^1, \mu_n, \mathfrak{g}, \ell)^*,$$

where $\lambda_i \in P_\ell(\mathfrak{g}, m_i)$.

Rigidity of Abstract Crossed Modular Tensor Category

How does one check for rigidity for abstract crossed modular categories? In general there are no known examples of weakly rigid but not rigid. The following theorem answer a question communicated by V. Drinfeld in this direction.

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Theorem: Deshpande-Mukhopadhyay

Let $\mathcal{C}$ be any weak Γ crossed ribbon category. Assume that $\mathcal{C}_1$ is rigid, then weak rigidity of $\mathcal{C}$ implies that $\mathcal{C}$ is rigid.

Remarks about rigidity issues

- Ridigity of weakly rigid braided tensor categories arising out of conformal blocks in the untwisted case is a very important and hard result. This was proved by Yi-Zhi Huang. Huang's works in the general set up of vertex algebras. (Related work of M. Finkelberg in the quantum group set up)

Remarks about rigidity issues

- Ridigity of weakly rigid braided tensor categories arising out of conformal blocks in the untwisted case is a very important and hard result. This was proved by Yi-Zhi Huang. Huang's works in the general set up of vertex algebras. (Related work of M. Finkelberg in the quantum group set up)

- The proof of the Verlinde formula for twisted Kac-Moody Lie algebras, uses the rigidity result of Deshpande-M to reduce rigidity issues to the untwisted case.

Verlinde Formula for twisted conformal blocks

Since the conformal blocks define a Γ crossed modular tensor category. We have a formula to compute the dimension:

Theorem: Deshpande-Mukhopadhyay

Assume that "Γ preserves a Borel subalgebra of $\mathfrak{g}$".

$$\operatorname{rank} \mathbb{V}_{\vec{\lambda},\Gamma}(\widetilde{C}, C, \widetilde{\mathbf{p}}, \mathbf{p}) = \sum_{\mu \in P_\ell(\mathfrak{g})^\Gamma} \frac{S^{m_1}_{\lambda_1,\mu} \cdots S^{m_n}_{\lambda_n,\mu}}{\left(S_{0,\mu}\right)^{n+2g-2}},$$

Let γ be an automorphism of $\mathfrak{g}$ and let $\lambda_1, \lambda_2, \lambda_3$ are fixed by γ.

Twisted Fusion ring

Let γ be an automorphism of $\mathfrak{g}$ and let $\lambda_1, \lambda_2, \lambda_3$ are fixed by γ.

$$\gamma : \mathcal{V}_{\lambda_1,\lambda_2,\lambda_3}(\mathbb{P}^1; 0, 1, \infty; \mathfrak{g}, \ell) \to \mathcal{V}_{\lambda_1,\lambda_2,\lambda_3}(\mathbb{P}^1; 0, 1, \infty; \mathfrak{g}, \ell).$$

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The twisted fusion ring

$$\mathcal{R}_\gamma(\mathfrak{g}, \ell) := \sum_{\lambda \in P_\ell(\mathfrak{g})^\gamma} \mathbb{C}[\lambda]$$

$$[\lambda_1] \circ_\gamma [\lambda_2] = \sum_{\lambda_3 \in P_\ell(\mathfrak{g})^\gamma} \mathrm{tr}(\gamma)[\lambda_3^*]$$

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Remark

The crossed S-matrix S^γ is the normalized character table of $R_\gamma(\mathfrak{g}, \ell)$.

Character of twisted Fusion ring

Analgous to the result of Faltings and Teleman for the untwisted case:

Theorem: J. Hong

The characters of the twisted fusion rings are given are parameterized by the set $P_\ell(\mathfrak{g}, \gamma)$ and are given by "traces of representations."

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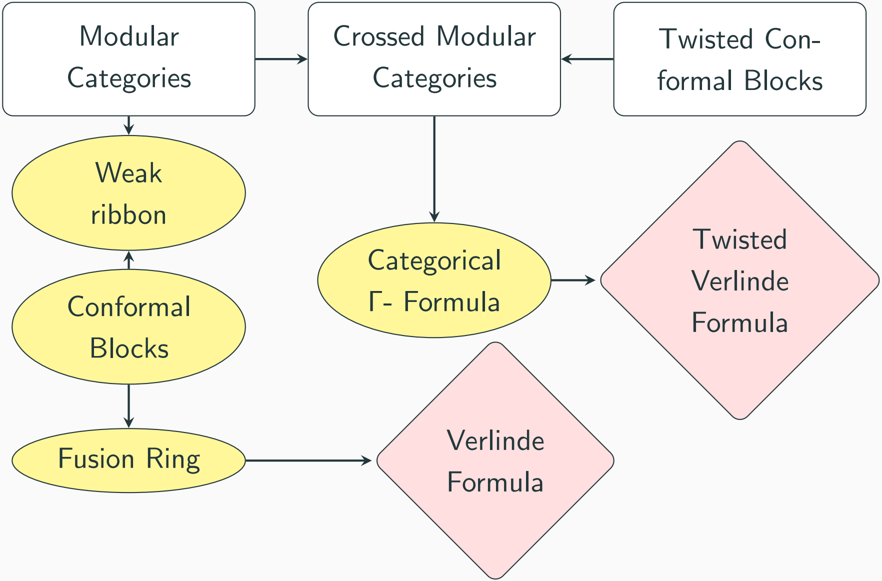
Remark

- $\mathcal{R}_\gamma(\mathfrak{g}, \ell)$ is defined over $\mathbb{Z}$.
- The coefficients can be negative (Countering Hong's claim).
- We can write crossed S^γ matrices in terms of the Weyl-Character formula.

Summary of the main steps



Example:Étale case

- Let C be smooth curve of genus g, then

$$\dim \mathcal{V}_0(C, \mathfrak{sl}(r), 1) = r^g,$$

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- $\widetilde{C} \to C$ be a cyclic étale-Galois cover of order m and $\Gamma = \langle \gamma \rangle$, then

$$\dim \mathcal{V}_{0,\Gamma}(\widetilde{C}, C, \mathfrak{sl}(r), 1) = |P_1(\mathfrak{sl}(r))^\gamma| r^{g-1},$$

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Remark

If $\Gamma = \mathbb{Z}/2$, then the above result can be crossed verified independent by H. Zelaci's thesis where the result of Beauville-Narasimhan-Ramanan is generalized for Prym varieties.

Examples: Ramified Cases

- Consider $A^{(2)}_{2r-1}$ and a double cover $\widetilde{C} \to C$ ramified at $2n$ points $\widetilde{p}$ and let g be the genus of C.

$$\dim_{\mathbb{C}} \mathbb{V}^{\dagger}_{\vec{0};\mathbb{Z}/2}(\widetilde{C}, C, \widetilde{\mathbf{p}}, \mathbf{p}) = 2^{g} r^{g+n-1}.$$

- Let $E \to \mathbb{P}^1$ be a ramified Galois cover of order three with three ramification points, For the Lie algebra $D_4^{(3)}$, we get

$$\dim_{\mathbb{C}} \mathbb{V}^{\dagger}_{0,0,0;\mathbb{Z}/3}(E, \mathbb{P}^1, \widetilde{\mathbf{p}}, \mathbf{p}) = 2.$$

Thank You !!!!