Let us discuss how to get a hyper cohomology class from a pair (L, ∇)

Theorem

The hypercohomology group $H^1(M, \underline{\mathbb{C}}^*_M \xrightarrow{d\log} \underline{A}^1_M)$ -

classifies pair (L, ∇) - line bundles upto isomorphisms

$$\begin{tikzcd} \underline{\mathbb{C}}^*_M \arrow[r] \arrow[d] & \underline{A}^1_M \arrow[d, "d"'] \\ 0 \arrow[r] & \underline{A}^2_M \end{tikzcd}$$

This is a map of complexes

$$H^1(M, \underline{\mathbb{C}}^*_M \to \underline{A}^1_M)$$
$$\downarrow$$
$$H^0(M, \underline{A}^2_M)$$

$(\underline{g}, -\underline{\alpha})$ is a cocycle class

$$\downarrow$$

$$-d\underline{\alpha}$$

$\underline{\alpha} = (\alpha_i)$ where $\alpha_i \in \Gamma(U_i, \underline{A}^1_M)$

$$\nabla_i s_i = \alpha_i s_i$$

$$\nabla_j s_j = \alpha_j s_j$$

Now on $U_{ij} = U_i \cap U_j$ $\quad s_i = g_{ij} s_j$

$$\nabla_j s_i = \nabla_i (g_{ij} s_j) = dg_{ij}\, s_j + g_{ij} \nabla s_j$$

$$\nabla_{j} s_i = d g_{ij} s_j + g_{ij} \alpha_j s_j$$

$$\frac{\nabla_{j} s_i}{s_i} = \frac{d g_{ij} s_j}{s_i} + g_{ij} \alpha_j \frac{s_j}{s_i}$$

$$\alpha_i = \frac{d g_{ij}}{g_{ij}} + \frac{g_{ij} s_j}{s_i} \alpha_j$$

$$\Rightarrow \alpha_i = d\log g_{ij} + \alpha_j \Rightarrow \alpha_i - \alpha_j = d\log g_{ij}$$

$\star_1$ $\Rightarrow$ $d\alpha_i = d\alpha_j$ on $\mathcal{U}_{ij}$

This gives the global two form M, which is the curvature K $\star_2$

$$\underline{\mathbb{C}}^*_X \to \check{\mathcal{C}}^0(\mathcal{U}, \underline{\mathbb{C}}^*_X) \to \check{\mathcal{C}}^1(\mathcal{U}, \underline{\mathbb{C}}^*_X) \to \check{\mathcal{C}}^2(\mathcal{U}, \underline{\mathbb{C}}^*_X) \to \cdots$$

$$\check{\mathcal{C}}^1(\mathcal{U}, \underline{\mathbb{C}}^*_X) = \prod_{\{i_0 < i_1\}} \underline{\mathbb{C}}^*_X(\mathcal{U}_{i_0 i_1})$$

$$\check{\mathcal{C}}^2(\mathcal{U}, \underline{\mathbb{C}}^*_X) = \prod_{\{i_0 < i_1 < i_2\}} \underline{\mathbb{C}}^*_X(\mathcal{U}_{i_0 i_1 i_2})$$

$$\delta(\underline{g})_{i_0 i_1 i_2} = g_{i_1 i_2} - g_{i_0 i_2} + g_{i_0 i_1} = 0$$

$$\delta(\underline{\alpha})_{i_0 i_1} = \alpha_{i_1} - \alpha_{i_0}$$

$$\begin{tikzcd}
0 & \underline{\mathbb{C}}^*_M \arrow[r, "d\log"] \arrow[d] & \underline{A}^1_M \arrow[d] & \\
(0,0) & \check{C}^0(\underline{\mathbb{C}}^*_M) \arrow[r, "d\log"] \arrow[d] & \check{C}^0(\underline{A}^1_M) \arrow[d] & (1,0) \\
(0,1) & \check{C}^1(\underline{\mathbb{C}}^*_M) \arrow[r, "d\log"] \arrow[d] & \check{C}^1(\underline{A}^1_M) \arrow[d] & (1,1) \\
(0,2) & \check{C}^2(\underline{\mathbb{C}}^*_M) \arrow[r, "d\log"] \arrow[d] & \check{C}^2(\underline{A}^1_M) \arrow[d] & (1,2) \\
(0,3) & \check{C}^3(\underline{\mathbb{C}}^*_M) \arrow[r, "d\log"] \arrow[d] & \check{C}^3(\underline{A}^1_M) \arrow[d] & (1,3) \\
 & \vdots & \vdots &
\end{tikzcd}$$

$$D = d^{hor} + (-1)^p d^{ver}$$

on (p,q)-th component

$$\check{\mathcal{C}}^0 = \check{\mathcal{C}}^0(\underline{\mathbb{C}}^*_M)$$

$$\check{\mathcal{C}}^1 = \check{\mathcal{C}}^1(\underline{\mathbb{C}}^*_M) \oplus \check{\mathcal{C}}^0(\underline{A}^1_M)$$

$$\check{\mathcal{C}}^2 = \check{\mathcal{C}}^2(\underline{\mathbb{C}}^*_M) \oplus \check{\mathcal{C}}^1(\underline{A}^1_M)$$

$$\check{C}^0 \xrightarrow{D^2} \check{C}^1 \xrightarrow{D^1} \check{C}^2 \xrightarrow{D^2} \check{C}^3 \to \cdots$$

$$H^1\left(M, \underline{\mathbb{C}}^*_M \xrightarrow{d\log} \underline{A}^1_M\right) = \frac{\ker D^1}{\text{Image } D^2}$$

$$\begin{array}{ccc} (0,1) & \check{C}^1(\underline{\mathbb{C}}^*_M) \oplus \check{C}^0(\underline{A}^1_M) & (1,0) \\ & \big\downarrow D & \\ (0,2) & \check{C}^2(\underline{\mathbb{C}}^*_M) \oplus \check{C}^1(\underline{A}^1_M) & (1,1) \end{array}$$

$(0,1)$

$D = d^{hor} + d^{ver}$

$$\begin{tikzcd}
\underline{(g_{ij})} \in \check{C}^1(\underline{\mathbb{C}}^*_M) \arrow[d, "\delta^1_{\underline{\mathbb{C}}^*_M}"] \arrow[dr, "d\log"] & \\
\underline{(g_{ij} + g_{jk} - g_{ik})} \in \check{C}^2(\underline{\mathbb{C}}^*_M) & \underline{\frac{dg_{ij}}{g_{ij}}} \in \check{C}^1(\underline{A}^1_M) \quad -(\alpha_i - \alpha_j) \\
& \underline{(\alpha_i)} \in \check{C}^0(\underline{A}^1_M) \arrow[u, "\delta^0_{\underline{A}^1_M}"']
\end{tikzcd}$$

So for this $(\underline{g_{ij}})$, $(\underline{\alpha_i})$ to be a cocycle

$\Rightarrow$ $(\alpha_i - \alpha_j) = d\log g_{ij}$. This we get

$$(\underline{g}, \underline{\alpha}) \in \check{C}^1(\underline{\mathbb{C}}^*_M) \oplus \check{C}^0(\underline{A}^1_M)$$

is a 1-cocycle iff

$$\boxed{\begin{array}{ll} * \quad g_{ij} + g_{jk} = g_{ik} & \\ & \text{for all } i, j, k. \\ * \quad (\alpha_i - \alpha_j) = d\log g_{ij} & \end{array}}$$

Now observe that

degree 0 degree 1

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z}(1) \arrow[r] & \underline{\mathbb{C}}_M \arrow[r] \arrow[d] & \underline{A}^1_M \arrow[d, equal, "Id"] & \\
 & & \underline{\mathbb{C}}^*_M \arrow[r] \arrow[d] & \underline{A}^1_M \arrow[d, "d"] & (\underline{\alpha}) \arrow[d] \\
 & & 0 \arrow[r] & \underline{A}^2_M & d\underline{\alpha}
\end{tikzcd}$$

This are quasi isomorphic

$*_1$ and $*_2$

Taking hypercohomology we get

$$H^1(M, \mathbb{Z}(1) \to \underline{\mathbb{C}}_M \to \underline{A}^1_M)$$

$$\simeq H^1(M, \underline{\mathbb{C}}^*_M \to \underline{A}^1_M) \longrightarrow H^0(M, \underline{A}^2_M)$$