Exercise Sheet #1

1. Page 65 (Stein and Shakarchi) Problem 7
2. Page 66 (Stein and Shakarchi) Problem 9
3. Page 67 (Stein and Shakarchi) Problem 15
4. Page 68 (Stein and Shakarchi) Problem 3
5. Let $f : \Omega \to \mathbb{C}$ be an analytic function of a open connected domain. Let $z_0 \in \Omega$ be such that $f'(z_0) \neq 0$. Show that
$$\frac{1}{f'(z_0)} = \frac{1}{2\pi\sqrt{-1}} \oint_C \frac{1}{f(z) - f(z_0)} dz,$$
where C is a small circle centered at z_0.
6. If $f : \mathbb{C} \to \mathbb{C}$ is a continuous function such that f is analytic off $[-1, 1]$. Then show that f is entire.
7. If f be an entire function and suppose there is a constant M and $R > 0$ such that $|f(z)| \leq M|z|^n$ for $|z| > R$ and for some positive n. Show that f must be a polynomial of degree $\leq n$.
8. Let $\Omega \subset \mathbb{R}^2$ be an open disk and let $u : \Omega \to \mathbb{R}$ be a harmonic function, i.e $(\partial_x^2 + \partial_y^2)u = 0$. Show that there exists function $v : \Omega \to \mathbb{R}$ such that $f = u + iv$ is a holomorphic function.
9. Recall that a linear transformation $L : \mathbb{R}^2 \to \mathbb{R}^2$ is angle preserving if there exists $\lambda > 0$ such that $L(v) \cdot L(w) = \lambda v \cdot w$ where $\cdot$ denotes the inner product. Show that if $f : \Omega \to \mathbb{C}$ is a real differentiable function, then the following are equivalent:
   - f is holomorphic and f' is zero free in Ω
   - f preserves angle and orientation preserving.