# Exercise Sheet #3

1. Let $\Omega \subseteq \mathbb{C}$ be open and connected and let $f$ be a holomorphic function on $\Omega$. Then $f'(a) \neq 0$ if and only if there is a neighborhood of $a$ on which $f$ is injective.

2. Prove that for each non-empty open subset $\Omega \subseteq \mathbb{C}$ and for all points $a, b \in \mathbb{C}\backslash\Omega$ in the same connected component of $\mathbb{C}\backslash\Omega$, there exists a complex holomorphic function $f : \Omega \to \mathbb{C}$ such that $f(z)^2 = (z-a)(z-b)$.

3. Let $f$ be a meromorphic function on a simply connected open set $\Omega \subset \mathbb{C}$. Prove that $f$ has a primitive that is meromorphic on $\Omega$ if and only if at each pole of $f$, the residue is zero.

4. Consider the meromorphic function $f(z) = \pi^2/\sin^2 \pi z$ and consider the function $H(z) = f(z) - \sum_{n\in\mathbb{Z}} \frac{1}{(z-n)^2}$. Show that $H(z)$ is identically zero. (Exercise from class)

   (I also suggest looking into Problems on Page 312 Exercise 217-219 on Raghavan Narasimhan's book for your practice only.)

5. Let $\Omega = \cup_{i=1}\Omega_i$ where $\Omega_i \subset \mathbb{C}$. For each $i$ consider functions $f_i$ which are meromorphic on $\Omega_i$ such that on $\Omega_i \cap \Omega_j$, the function $f_i - f_j$ is holomorphic. Then show that there exists a meromorphic function $f : \Omega \to \widehat{\mathbb{C}}$ such that $f - f_i$ is holomorphic on $\Omega_i$ for all $i$.

6. Give a proof of Fundamental theorem of algebra from the argument principle.

7. (Conway Page 207)This exercise proves the Weierstrass Factorization theorem from Mittag-Leffler theorem.

   (a) Let $\{a_n\}$ be a sequence of points in the complex plane such that $|a_n| \to \infty$ and $\{b_n\}$ be an arbitrary sequence of complex numbers. Show that if integers $\{k_n\}$ be choosen such that
   $$\sum_{n=1}^{\infty}(\frac{r}{a_n})^{k_n}\frac{b_n}{a_n}$$
   converges absolutely for all $r > 0$ then $\sum_{n=1}^{\infty}(\frac{z}{a_n})^{k_n}\frac{b_n}{z-a_n}$ converges to a meromorphic function on $\mathbb{C}$ with poles along $z = a_n$.

   (b) Show that if $\limsup |b_n| < \infty$, then the above series converges absolutely if $k_n = n$ for all $n$.

   (c) Using the above show that for any sequence $\{a_n\} \in \mathbb{C}$ with $\lim a_n = \infty$ and $a_n \neq 0$, we can find a sequence of integers $k_n$ such that
   $$h(z) = \sum_{n=1}^{\infty}\left(\frac{1}{z-a_n} + \frac{1}{a_n} + \frac{1}{a_n}(\frac{z}{a_n}) + \dots \frac{1}{a_n}(\frac{z}{a_n})^{k_n-1}\right)$$
   is meromorphic with simple poles at $a_1, \dots, a_n, \dots$.

(d) Use the function $f(z) = \exp\left(\int_\gamma h\right)$ where $\gamma$ is any path joining 0 and point $z \in \mathbb{C}\backslash\{a_1, \ldots, ..\}$ and not passing through $\{a_1, \ldots, a_n, \ldots,\}$ as in discussed in class to give a proof of the Weierstrass factorization theorem for the complex plane.