Exercise Sheet #4

1. Stein and Shakarchi: Page 155, Problems 8,10,11.
2. Let f and g are entire functions such that $h(z) = f(g(z))$ is a polynomial. What can you say about f and g.
3. Find a Laurent series that converges in the annulus $1 < |z| < 2$ to a branch of the function
$$f(z) = \log\left(\frac{(2-z)z}{1-z}\right)$$
4. The following problems involves Bernoulli numbers
   (a) Show that the Laurent series of $(e^z - 1)^{-1}$ at the origin takes the form
   $$\frac{1}{z} - \frac{1}{2} + \sum_{n=1}^{\infty} (-1)^{k-1} \frac{B_k}{(2k)!} z^{2k-1},$$
   where $B_1, \ldots, B_n, \ldots$ are called Bernoulli numbers.
   (b) Show that $\pi z \cot \pi z = 1 - \sum_{n=1}^{\infty} \zeta(2n) z^{2n}$
   (c) Prove that $\zeta(2n) = 2^{2n-1} \frac{B_n}{(2n)!} \pi^{2n}$
5. Compute the following using residues:
   (a) $\int_0^\infty \frac{x^2}{(x^2+a^2)^2} dx$ for $a \in \mathbb{R}$.
   (b) $\int_0^\infty \frac{\log x}{1+x^2} dx$
   (c) $\int_\infty^\infty \frac{\log(x^2+1)}{x^2} dx$
   (d) $\int_0^\infty \frac{\log^2 x}{\sqrt{x}(1-x)^2} dx$
6. Show that
$$\frac{d}{dz} \frac{\Gamma'(z)}{\Gamma(z)} = \sum_{n=0}^{\infty} \frac{1}{(z+n)^2}$$
7. Stein Shakarchi: Page 174: 1, 2, 9(do not submit), 10 (do not submit).