Exercise Sheet # 5

1. Consider the Bessel equation
$$\frac{d^2u}{dz} + \frac{1}{z}\frac{du}{dz} + (1 - \frac{\nu^2}{z^2})u = 0$$
Show the following:
    (a) The power series $J_\nu(t) := \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(k+\nu+1)} \left(\frac{t}{2}\right)^{2k+\nu}$ valied for $t \in \mathbb{C}\backslash(-\infty, 0]$ solves the Bessel equation
    (b) Show that $J_{-\nu}(t)$ also solves the Bessel equation and show that if $\nu \notin \mathbb{Z}$ the solutions $J_\nu(t)$ and $J_{-\nu}(t)$ are linearly independent.
    (c) Show that if $n \in \mathbb{Z}$, then $J_n(t) = (-1)^n J_{-n}(t)$.

2. Define $(a)_0 := 1$ and $(a)_k := a(a+1)\cdots(a+k-1)$. For $b \notin \{-1, -2, -3, \cdots\}$, consider the two power series
    - ${}_1F_1(a, b, z) := \sum_{k=0}^{\infty} \frac{(a)_k}{(b)_k} \frac{z^k}{k!}$
    - ${}_2F_1(a_1, a_2, b, z) := \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k}{(b)_k} \frac{z^k}{k!}$

    Show the following:
    (a) Find the radius of convergence of ${}_1F_1$ and ${}_2F_1$.
    (b) Show that $u = {}_1F_1(z)$ satisfies $zu''(z) + (b-z)u'(z) - au(z) = 0$.
    (c) Show that $u = {}_2F_1(z)$ satisfies $z(1-z)u''(z) + (b - (a_1 + a_2 + 1)z)u'(z) - a_1 a_2 u(z) = 0$.
    (d) Show that any solution to the above differential equation in part(c) has an analytic continuation to any simply connected domain in $\mathbb{C}\backslash\{0, 1\}$. (no need to submit this part)
    (e) Show that ${}_1F_1(a, b, z) = \lim_{c\to\infty} {}_2F_1(a, c, b, c^{-1}z)$.

3. Consider the Airy equation: $u''(z) = zu(z)$, $u(0) = u_0$ and $u'(0) = u_1$.
    (a) Show that by taking $v = (u, u')^T$ gives linear differential equation of the form $\frac{dv}{dz} = (A_0 + A_1 z)v$, where A_0 and A_1 are 2×2 matrices.
    (b) If $v(z) = \sum_{k=0}^{\infty} v_k z^k$ is a solution, find a recusion involving v_k's.
    (c) Use this to directly show that $v(z)$ converges for all $z \in \mathbb{C}$. (no need to submit this part)

4. Recall that we know how to solve differential equations of the form $\frac{d^2u}{dx} + k^2 u = 0$ in terms of $\sin kx$ and $\cos kx$. We now consider a two variable analog of the above equation:

    Let Δ be the two variable Laplacian and u is a function of two variable. Consider the equation $\Delta u + k^2 u = 0$.
    (a) Show that if $v = ku$, then the above equation is just $(\Delta + 1)v = 0$.
    (b) If $x = r\cos\theta$ and $y = r\sin\theta$ and let $v = w(r)e^{i\nu\theta}$. Show that v satisfies $(\Delta + 1)v = 0$ if $w(r)$ satisfies the Bessel equation.