Exercise Sheet # 6

1. Stein-Shakarchi: 8, 10, 11, 13, 14.
2. Give an example of a bounded, simply connected domain contained in $\mathbb{C}$ whose boundary is not path connected.
3. Suppose $h_\nu : \mathbb{D} \to \mathbb{D}$ are injective, holomorphic, univalent maps satisfying $h_\nu = 0$, $h'_\nu(0) > 0$, $D_{\rho_\nu}(0) \subset h_\nu(D)$, $\rho_\nu \to 1$. Show that for $z \in \mathbb{D}$, we have $\rho_\nu |z| \leq |h_\nu(z)| \leq |z|$. Further show that $h_\nu(z) \to z$ locally uniformly on D. (Hint: Use a normal family argument)
4. Suppose Ω is a bounded, simply connected domain and $p \in \Omega$. and $f_\nu : \Omega \to \mathbb{D}$ are injective holomorphic maps satisfying $f_\nu(p) = 0$, $f'_\nu(p) > 0$, $f_\nu(\Omega)$ contains a disc of radius ρ_ν such that $\rho_\nu \to 1$. Show that $f_\nu \to f$ locally uniformly on Ω, where $f : \Omega \to \mathbb{D}$ is the function given by the Riemann mapping theroem.
5. Let $\mathbb{D}$ be the unit disc. Let $f : \mathbb{D} \to \mathbb{D}$ is holomorphic. Then show that
$$\frac{|f(0)| - |z|}{1 - |f(0)||z|} \leq |f(z)| \leq \frac{|f(0)| + |z|}{1 + |f(0)||z|}$$
6. (IX.1.5 in Gamelin)Let f be analytic, $|f(0)| \geq r$, then $|f(z)| \geq \frac{(r-|z|)}{(1-r|z|)}$ for $|z| < r$.
7. Let $f : \Omega \to \mathbb{D}$ is a biholomorphism, where $\mathbb{D}$ is the unit disc and
$$\Omega = \{z = re^{i\theta} \in \mathbb{C} \mid r > 0,\ |\theta| < \pi/100\}.$$
Suppose also $f(1) = 0$ and $f'(1) > 0$, compute $f(2)$?
8. Let f be an analytic function on the unit disc $\mathbb{D}$ such that $|f(z)| \leq 1$. Suppose that $z_1, \ldots, z_n$ are the zeroes of f, then show that
$$|f(0)| \leq \prod_{j=1}^{n} |z_i|,$$
where each z_i is repeated according to multiplicities on the right hand side.