Exercise Sheet # 7

1. Page 9 of Forster's book: 1.4, 1.5
2. Page 39 of Forster's book: 5.2, 5.4
3. Page 59 of Forster's book: 8.1, 8.2
4. Consider the upper half plane $\mathbb{H}$ and its group of holomorphic automorphisms $\mathrm{Aut}(\mathbb{H})$. We know that $\mathrm{Aut}(\mathbb{H}) = \mathrm{PSL}_2(\mathbb{R})$. Let $\gamma \in \mathrm{Aut}(\mathbb{H})$ and let $\gamma(z) = \frac{az+d}{cz+d}$. Let $c \neq 0$ and consider the discrimimant (of the quadratic equation $\gamma(z) = z$) $\Delta = (d-a)^2 + 4bc$.
   - If $\Delta < 0$, then show that upto conjugates γ is a rotation. Such a γ is called elliptic. In particular γ may or may not have finite order.
   - If $\Delta > 0$, show that there are two distinct fixed points and that's in $\mathbb{R} \cup \{\infty\}$. Use this to show that such a γ is conjugate to a μ in $\mathrm{PSL}_2(\mathbb{R})$ that fixed ∞ and 0. Hence $\mu(z) = \lambda(z)$ for $\lambda > 0$ and $\lambda \neq 1$ and γ has infinite order. Such elements are called hyperbolic.
   - If $\Delta = 0$, then show that there is one fixed point and that's in $\mathbb{R} \cup \{infty\}$. Use this to show that γ is conjugate to an element μ in $\mathrm{PSL}_2(\mathbb{R})$ sending 0 and ± 1 and fixing ∞. Such elements are called parabolic and also have infinite order.
5. Let p be a prime number and consider matrices of the form

$$\widetilde{\Gamma}_p = \{\pm \begin{bmatrix} ap+1 & * \\ * & bp+1 \end{bmatrix}\} \cap \mathrm{SL}_2(\mathbb{Z}).$$

   Let Γ_p be the image of $\widetilde{\Gamma}_p$ in $\mathrm{PSL}_2(\mathbb{R})$. Show that if $p \geq 5$, then all element of Γ_p are either elliptic or parabolic. In particular there have no fixed points in $\mathbb{H}$. Thus $\mathbb{H}/\Gamma_p$ is a Riemann surface.