Exercise Sheet # 8

For all these problems $\Lambda = \mathbb{Z} \oplus \mathbb{Z}\tau$ where $\operatorname{Im}\tau > 0$.

1. Show that for any $a \in \mathbb{C}$ there exists w such that $\wp(w) = a$. The set of such solutions are $\{w, -w\}$ if $w \notin \{\frac{\omega}{2} | \omega \in \Lambda\}$.
2. Show that the polynomial $4x^2 - g_2(\tau)x - g_3(\tau)$ distinct roots.
3. Consider the cubic $\overline{X} = \{(z, w) \in \mathbb{C} : w^2 = 4z^3 - g_2(\tau)z - g_2(\tau)\} \cup \{\infty\} \subset \mathbb{CP}^2$ and let $\widehat{X}$ be the connected compact Riemann surface $\mathbb{C}/\Lambda$. Show that they are bijective under the map $z \to (\wp(z), \wp'(z), 1)$ if $z \neq 0$ and equals ∞ if $z = 0$.
4. Consider matrix
$$A = \begin{bmatrix} \wp(z_1) & \wp'(z_1) & 1 \\ \wp(z_2) & \wp'(z_2) & 1 \\ \wp(z_3) & \wp'(z_3) & 1 \end{bmatrix}$$
   (a) Show that $\det(A)$ is zero for $z_1 = z_2$, $z_2 = z_3$ and $z_1 = z_3$. (Compare Vandermonde matrix)
   (b) Consider $\det(A)$ as a function $f(z_1)$ and show that $f(z_1)$ has a pole of order 3 at 0.
   (c) Observe that $p_1, \ldots, p_m$ are the zero and $q_1, \ldots, q_m$ are the poles, then $\sum p_i - \sum q_i \in \Lambda$.
   (d) Use $\oint_C z(f'(z)/f(z))dz$ to show that the third zero a of $f(z_1)$ is such that $(a + z_2 + z_3) = 0$ in $\mathbb{C}/\Lambda$
5. If z, w be such that $z + w$ and $z - w$ is not in the lattice. Then one has
$$\wp(z + w) = \frac{1}{4}\left(\frac{\wp'(z) - \wp'(w)}{\wp(z) - \wp(w)}\right)^2 - \wp(z) - \wp(w)$$
6. If $z \in \mathbb{C}$ then show that
$$\wp(2z) = \frac{1}{4}\left(\frac{\wp''(z)}{\wp'(z)}\right)^2 - 2\wp(z)$$
7. Show that $y^n - g(x) = 0$ defines a non-singular plane curve if and only if g has no multiple root.
8. Consider the zero set Z in $\mathbb{CP}^2$ defined by $X^3 = ZY^2 + Z^3$. Is this a Riemann surface? Is so what is the genus? (Use triangulation)
9. Stein and Shakarchi (Chapter 10) Exercise 10,11,12.
10. Consider the smooth cubic curve in $\mathbb{CP}^2$ given by the compactification $\overline{X} = X \cup \{\infty\}$ of the zero set X of the equation $y^2 = 4x^3 - g_2x - g_3$. On the cubic there is a famous group law $(+)$ where ∞ is the identity and for three point, we have $a + b = -c$ where a, b, c are in the same line in $\overline{X}$. Show using the matrix A that from $\mathbb{C}/\Lambda \to \overline{X}$ is a group homomorphism.
11. Reinterpret the identities involving $\wp(z)$ in the previous problems in terms of the group law.