Shapovalov form

$$\mathcal{U}(\mathfrak{h}) \xrightarrow{pr_2} S(\mathfrak{h})$$

$$S_\lambda(a\otimes 1, b\otimes 1) = P_\lambda\left(pr_2\left(\sigma(a) b\right)\right) \quad \lambda \in \mathfrak{h}^*$$

Properties

(1) S_λ is contravariant

(2) $S_\lambda(M'_\lambda, M_\lambda) = 0$ $\quad M'_\lambda$ is the maximal proper

(3) S_λ descends to a non degenerate form on $L(\lambda) = M(\lambda)/M'(\lambda)$

$$\hat{S}_\lambda : M(\lambda) \times M(\lambda) \longrightarrow \mathbb{C}[t]$$

$$\hat{S}_\lambda(a, b) = P_{\lambda + t\rho}\left(S_\lambda(a, b)\right)$$

$$M^p(\lambda) = \left\{ v_0 \in M(\lambda) \;\middle|\; \exists\, v_1, \dots, v_{p-1} \in M(\lambda) \right.$$

such that

$$\sum S_\lambda(v_i, w)\, t \in (t^p)$$

Clearly $M^1(\lambda) = M'(\lambda)$

Jantzen wanted a filtration such that

(i) $M_i = M^i(\lambda)/M^{i+1}(\lambda)$ have a contravariant form

(ii) $$\sum M^p(\lambda) = \sum_{\substack{\alpha > 0 \\ s_\alpha * \lambda < \lambda}} M(s_\alpha * \lambda) \quad (\dim \mathfrak{g}(A) < \infty)$$

Jantzen's Conjecture: If $\mu < \lambda$ and $M(\mu) \hookrightarrow M(\lambda)$

then $M^i(\mu) = M^i(\lambda) \cap M(\mu)$ upto a shift

(Beilinson - Bernstein using localization of $\mathcal{D}$-module and RH correspondence)

$H_\lambda(v,w)$ is the upto scalar contragradient form

Then $H_\lambda(v,w) = S_\lambda(v, \overline{w})$

Shapovalov determinant formula (Assume symmetrizable KM).

Suppose $\lambda \in \mathfrak{h}^*$ when is $M(\lambda)$ irreducible?

Necessarily: $(\lambda+\rho, \lambda) \neq (\lambda-\beta+\rho, \lambda-\beta)$ for any $\beta \in Q_+$.

$\Leftrightarrow \quad 2\langle \lambda+\rho, \beta\rangle = \langle \beta, \beta\rangle$.

$\eta \in \Delta_-$

Consider $\mathcal{U}(\mathfrak{n}^-)_\eta \subset \mathcal{U}(\mathfrak{g})$

$$S^-_\eta : \mathcal{U}(\mathfrak{n}^-)_\eta \times \mathcal{U}(\mathfrak{n}^-)_\eta \longrightarrow S(\mathfrak{h}) = \mathbb{C}[\mathfrak{h}^*]$$

If we choose a $\mathbb{C}$-basis $F = \{F_I\}$ of $\mathcal{U}(\mathfrak{n}^-)_\eta$

Question

$$\det_F S_\eta^- := \det\left(S_\eta^- (F_I, F_{I'}) \right)_{I, I'}$$ is a basis of $U(n^-)_\eta$

It is well define upto a scalar

Theorem (Shapovalov determinant formula) Shapovalov if $\dim(A) < \infty$. Kac-Kazhdan for symmetrizable KM.

$$\det_F (S_\eta^-)$$

$$= \prod_{\substack{(\alpha, n) \in \Delta^+ \times \mathbb{N} \\ \text{such that } \eta - n\alpha \in Q^+}} \left(v^{-1}(\alpha) + \left(\rho - \frac{n\alpha}{2}, \alpha\right)\right)^{P(\eta - n\alpha)\, \text{mult}\, \alpha}$$

Kostant Partition function

$$\text{ch}\, M(0) = \sum_{\eta \in Q^+} P(\eta) e^{-\eta}$$

$\eta \notin Q^+$ define $P(\eta) = 0$

Corollary: $M(\lambda)$ is irreducible iff

$$2(\lambda + \rho)(v^{-1}(\alpha)) = 2(\lambda + \rho, \alpha) \neq n(\alpha, \alpha)$$

for any $\alpha \in \Delta^+$ and any positive integer n.

Step 1: $M(\lambda)$ is irreducible if $2\langle \lambda + \rho, \beta \rangle \neq \langle \beta, \beta \rangle$ for every $\beta \in Q_+ \setminus \{0\}$.

Step 2: $\det S_\eta^-$ breaks up as a product of linear factors of the form $v^{-1}(\beta) + \langle \rho - \beta, \beta \rangle$.

$Z_\eta \subset \mathfrak{h}^*$ be the hypersurface defined by $\det S_\eta^-$

$$Z_\eta \subseteq \{ \lambda \in \mathfrak{h}^* : M(\lambda) \text{ is not irreducible} \}$$

$$\subseteq \bigcup_{\beta \in Q^+ \setminus 0} \{ \lambda \in \mathfrak{h}^* \mid \underset{**}{\langle \lambda + \rho - \tfrac{\beta}{2}, \beta \rangle = 0} \},$$

$\Rightarrow$ $\det S_\eta^- \sim$ power of linear factor for the set $\beta \in Q^+ \setminus 0$ satisfy $**$.

Step 3: For any $\eta \in Q_+$ $\det(S_\eta^-)$ has leading term

$$\prod \left(v^{-1}(\alpha) \right)^{P(\eta - n\alpha)\, \text{mult}\, \alpha}$$

where $(\alpha, n) \in \Delta^+ \times \mathbb{N}$ such that

$$\eta - n\alpha \in Q^+$$

Jantzen filtration (General Case)

$\mathbb{K}$ be a field and R be a $\mathbb{K}$-algebra which is also a p.i.d.

Let (t) be a prime ideal in R.

$$k = R/tR \xleftarrow{\phi} R \quad (\text{valuation at } 0)$$

If $\tilde{M}$ be a R-module then (free R-module)

$$\phi : \begin{pmatrix}\text{Free} \\ R\text{-modules}\end{pmatrix} \to (k\text{-vector space})$$

$$\tilde{M} \longmapsto \tilde{M} \otimes_R k \quad (\text{reduction functor})$$

$(\, , \,)_{\tilde{M}} : \tilde{M} \times \tilde{M} \to R$. and let $M = \phi(\tilde{M}) = \tilde{M} \otimes_R k$.

We get a bilinear form on M.

$$\left(\phi(v), \phi(w)\right)_M = \phi\left(\left(v, w\right)_{\tilde{M}}\right)$$

where $v, w \in \tilde{M}$

Def (Jantzen) For $m \in \mathbb{Z}_{\geq 0}$

$$\tilde{M}^m := \left\{ v \in \tilde{M} \;\middle|\; (v, \tilde{M})_{\tilde{M}} \subset (t^m) \right\} \subset \tilde{M}$$

$\iota_m : \tilde{M}^m \hookrightarrow \tilde{M}$ be the obvious map.

Then define a filtration on M by the formula

$$M^m := \operatorname{Im} \phi\left(\iota_m(\tilde{M}^m)\right).$$

Basic Properties (Jantzen)

① $\bigcap_{m \in \mathbb{Z}_{\geq 0}} M^m = \{0\}$

② $M^1 = \{ v \in M \mid (v, M) = 0 \} = \ker (\,,\,)_M$.

③ There exists a symmetric bilinear form $(\,,\,)_m$ on M^m such that

$$M^{m+1} = \{ v \in M^m \mid (v, M^m)_m = 0 \}.$$

$\Rightarrow$ on the graded pieces we have a symmetric non degenerate bilinear form

Prop (Jantzen)

$$\sum_{m \geq 1} \dim_k M^m = \operatorname{val}_t \left(\det (\,,\,)_M \right)$$

with respect to a basis

Corollary (Jantzen) (corollary of Shapovalov determinant) *

For $\lambda, \mu \in \mathfrak{h}^*$ with $\mu < \lambda$, we get

$M^p(\lambda)_\mu = 0$ for large enough p

Moreover

$$\sum_{p \geq 1} \operatorname{ch} M^p(\lambda) = \sum_{(\alpha, n) \in D_\lambda} \operatorname{ch} M(\lambda - n\alpha)$$

$$D_\lambda = \left\{ (\alpha, n) \in \Delta^+ \times \mathbb{N} \;\middle|\; \left(\lambda + \rho - \tfrac{n}{2}\alpha, \alpha\right) = 0 \right\}.$$

Corollary of Jantzen character formula

$\mu \leq \lambda \in \mathfrak{h}^*$, $L(\mu)$ is a component of $M(\mu)$

iff $\exists$ a sequence of positive roots $\beta_1, \dots, \beta_p$ $p \geq 0$.

and positive integers $k_1, \dots, k_p$. such that

a) $\lambda - \mu = \sum_{i=1}^{p} k_i \beta_i$

b) for all $1 \leq j \leq p$ we have

$$2\left\langle \lambda + \rho - \sum_{i=1}^{j-1} k_i \beta_i, \beta_j \right\rangle = k_j \langle \beta_j, \beta_j \rangle$$

Proof: We can induct of $|\lambda - \mu|$. If $\lambda - \mu = 0$ then nothing to prove.

So assume $|\lambda - \mu| > 0$.

By the Jantzen Character formula:

$L(\mu)$ is a component of $M(\lambda)$ iff

$L(\mu)$ is a component of $M(\lambda - k_1\beta_1)$ for some $(k_1, \beta_1) \in D_\lambda$

By $\lambda - k_1\beta_1$ has strictly less weight
Hence apply induction

Strong Linkage Principle (If $\mathcal{Y}(A) < \infty$), the following is known a BGG-reciprocity (Bernstein-Gelfand-Gelfand)

Theorem If $\lambda, \mu \in \mathfrak{h}^*$, $\mu \leq \lambda$ ($\mathcal{Y}(A)$ is of finite)

(Verma) If μ is strong related to λ, then $M(\mu) \hookrightarrow M(\lambda)$
in particular $[M(\lambda) : L(\mu)] \neq 0$.

(BGG) If $[M(\lambda) : L(\mu)] \neq 0$ then μ is strongly linked to λ.

Generalization to Symmetric KM algebras

K^{wg} (weakly good)

$\subset \mathfrak{h}^*$.

Define $C_\alpha = \{ \lambda \in \mathfrak{h}^* : \langle \lambda + \rho, \alpha \rangle = \frac{\langle \alpha, \alpha \rangle}{2} \}$

$$C = \bigcup_{\alpha \in \Delta_{im}^+} C_\alpha .$$

$$K^{wg} = \mathfrak{h}^* \setminus C$$

Exercise: K^{wg} is preserved under W - Weyl group of $\mathfrak{g}(A)$ under the $*$-action

$W(\lambda) \subset W$ be the group generated by

$$\{ s_\beta \mid \beta \in \Delta_{re}^+ , \langle \lambda + \rho, \beta^\vee \rangle \in \mathbb{Z} \}$$

Prop: Let $\lambda \in K^{wg}$. Then $[M(\lambda) : L(\mu)] > 0$ iff $\exists$ reflections $s_{\beta_1}, \dots, s_{\beta_p} \in W(\lambda)$ for $\beta_1, \dots, \beta_p \in \Delta_{re}^+$, $p \geq 0$ such that

$$\lambda > s_{\beta_1} * \lambda > s_{\beta_2} * s_{\beta_1} \lambda > \cdots > (s_{\beta_p} * s_{\beta_{p-1}} * \cdots s_{\beta_1} * \lambda) = \mu .$$

In particular $\mu \in W(\lambda) * \lambda$ and hence in K^{wg}.