Lecture 26

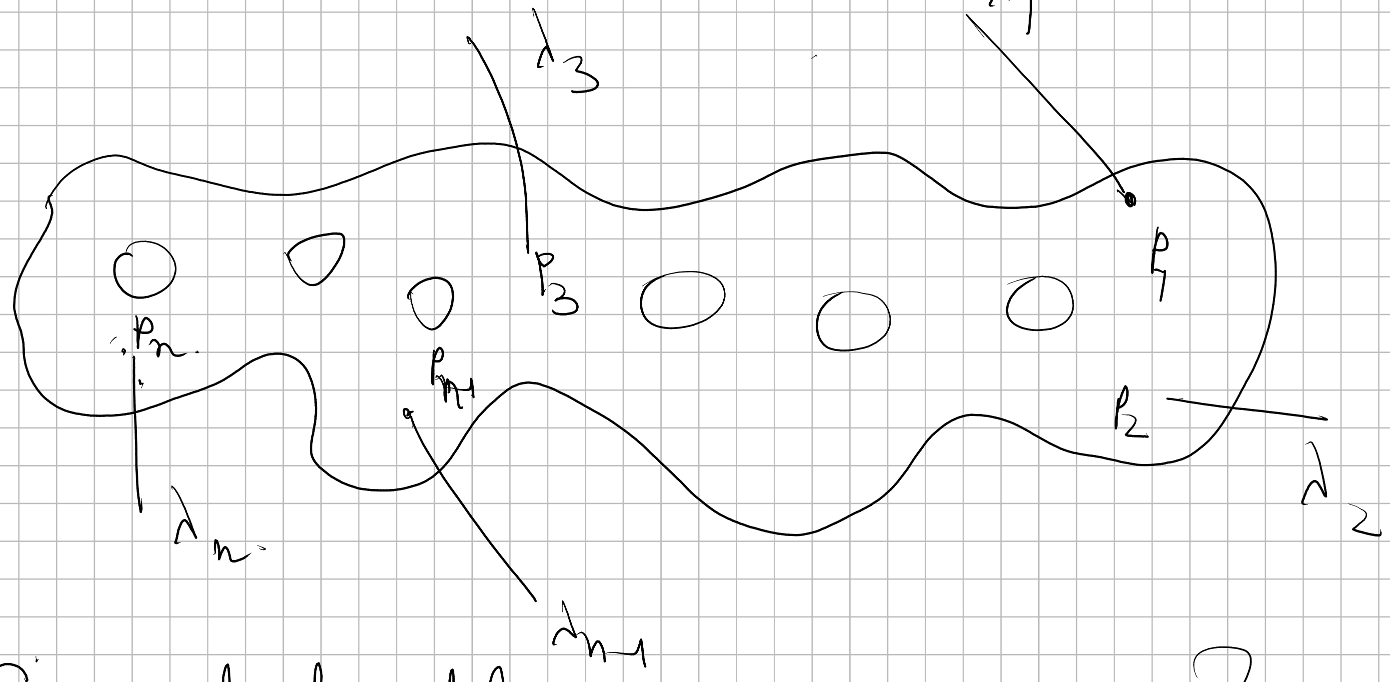
$$P_\ell(\mathfrak{g}) = \{ \lambda \in P_+ \mid (\lambda, \theta) \leq \ell \} \qquad (\theta,\theta) = 2$$

$\lambda \rightsquigarrow \mathcal{H}_\lambda$ – Integrable irreducible $\hat{\mathfrak{g}}$-module with highest weight λ, c = l.

$\cup$

$\rightsquigarrow V_\lambda$ – Irreducible $\mathfrak{g}$-module with highest weight λ

$$\vec{\lambda} = (\lambda_1, \ldots, \lambda_n) \in P_\ell^n(\mathfrak{g})$$



① C is almost nodal.

② $\vec{p} = (p_1, \ldots, p_n)$ on C $\quad p_i \neq p_j \quad p_i$'s smooth

③ $C \setminus \vec{p}$ is affine

$\Bigg\}\ \mathfrak{X}$

④ $\mathrm{Aut}(C, \vec{p})$ is finite (Deligne–Mumford Stability)

⑤ $t_1, \ldots, t_p$ local parameters around $p_1, \ldots, p_n$

ie $\qquad \mathbb{C}[[t_i]] \simeq \varprojlim_{k} \mathcal{O}_{C,P_i} / m_{P_i}^{k},$

$$\mathfrak{g} \otimes (C - \vec{p}) = \mathfrak{g} \otimes H^0(C, \mathcal{O}_C(*\vec{p}))$$

$$\downarrow$$

$$\mathfrak{g}_{\vec{\lambda}} = \left(\bigoplus \mathfrak{g} \otimes \mathbb{C}((t_i)) \right) \oplus \mathbb{C}_c .$$

$$\mathcal{H}_{\vec{\lambda}} = \mathcal{H}_{\lambda_1} \otimes \cdots \otimes \mathcal{H}_{\lambda_n}$$

$$\mathcal{V}_{\vec{\lambda}}(\mathfrak{X}) = \frac{\mathcal{H}_{\lambda_1} \otimes \cdots \otimes \mathcal{H}_{\lambda_n}}{\mathfrak{g} \otimes (C - \vec{p})(\mathcal{H}_{\lambda_1} \otimes \cdots \otimes \mathcal{H}_{\lambda_n})}$$

(space of covariance)

Co-conformal blocks

$$\mathcal{V}_{\vec{\lambda}}^{\dagger}(\mathfrak{X}) = \left(\mathcal{V}_{\vec{\lambda}}(\mathfrak{X}) \right)^{*}$$

* Gauge Condition

$$= \left\{ \langle \psi | \in \mathcal{H}_{\vec{\lambda}}^{*} \;\middle|\; \langle \psi | x \otimes f = 0 \right\}$$

$$f \in H^0(C, \mathcal{O}_C(*\vec{p})).$$

$$= \left\{ \langle \psi | \in \mathcal{H}_{\vec{\lambda}}^{*} \;\middle|\; \langle \psi | \sum \rho_i (x \otimes f_i) | \Phi \rangle = 0 \quad \forall | \Phi \rangle \in \mathcal{H}_{\vec{\lambda}} , \right\}$$

$$V_{\vec{\lambda}} \hookrightarrow \mathcal{H}_{\vec{\lambda}} \iff \bigotimes_{i=1}^{n} V_{\lambda_i} \hookrightarrow \bigotimes_{i=1}^{n} \mathcal{H}_{\lambda_i}$$

$\Rightarrow$ $\langle \psi \in \mathcal{V}^{+}_{\vec{\lambda}} \hookrightarrow \mathcal{H}_{\vec{\lambda}}$ restrict to $V_{\vec{\lambda}}$.

We get

$$\begin{tikzcd}
\mathcal{V}^{+}_{\vec{\lambda}} \arrow[r] \arrow[dr] & \mathrm{Hom}(V_{\vec{\lambda}}, \mathbb{C}) \\
 & \mathrm{Hom}_{\mathfrak{g}}(V_{\vec{\lambda}}, \mathbb{C}) \arrow[u, hook]
\end{tikzcd}$$

Finally. we show that.

$C = \mathbb{P}^1$, then $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$.

Thm*

$$\mathcal{V}^{+}_{\vec{\lambda}}(\mathbb{P}^1, \vec{z}) \hookrightarrow \mathrm{Hom}_{\mathfrak{g}}(V_{\vec{\lambda}}, \mathbb{C}).$$

Dependence on Parameter $t'_i = h_i(t_i)$ $1 \le i \le n$

Thm where $h_i(0) = 0$ $h_i'(0) \neq 0$.

$h_i(t) \in \mathbb{C}[[t]]$.

Then there is a natural isomorphism between

$$\mathcal{V}^{+}_{\vec{\lambda}}(C, \vec{p}, \vec{t}, \ell) \simeq \mathcal{V}^{+}_{\vec{\lambda}}(C, \vec{p}, \vec{t}', \ell))$$

Exercise $\omega : \mathfrak{g} \to \mathfrak{g}$ that take: $\mathfrak{n}_+ \rightleftarrows \mathfrak{n}_-$

$\lambda \in P_+$

$$\begin{tikzcd} & \mathfrak{g} \arrow[r, "\rho"] & \mathrm{End}(V_\lambda). \\ & \mathfrak{g} \arrow[u, "\omega"] \arrow[ur] & \end{tikzcd}$$

Under $\mathfrak{g} \xrightarrow{\omega\circ\rho} \mathrm{End}(V_\lambda)$ the highest weight is λ^*.

In particular we can construct

$$\tilde{\rho} : V_\lambda \to V_\lambda^* \text{ such that}$$

Show that

$$\tilde{\rho}(Xv) = \omega(X)\tilde{\rho}(v).$$

For symplectic kM are constructed this.

$$\mathcal{V}_{\vec{\lambda}}(C, \vec{p}, \vec{t}, \ell)$$

$$\cong$$

$$\mathcal{V}_{\vec{\lambda}^*}(C, \vec{p}, t, \ell).$$

$$\hat{\omega}\ \ \hat{\mathfrak{g}} \to \hat{\mathfrak{g}} \Rightarrow \tilde{\rho} : \mathcal{H}_\lambda \to \mathcal{H}_\lambda$$

$$\tilde{\rho}(Xv) = \hat{\omega}(X)\tilde{\rho}(v)$$

Check that $\oplus\,\hat{\omega}_i : \hat{\mathfrak{g}}_n \to \hat{\mathfrak{g}}_n$ preserves $\mathfrak{g} \otimes (C \setminus \vec{p})$

<u>Question</u>? What is the image?

$$\iota : \mathcal{V}_{\vec{\lambda}}^{\dagger}(\mathbb{P}^1, \vec{z}_i, \ell) \hookrightarrow \mathrm{Hom}_{\mathfrak{g}}(V_{\vec{\lambda}}, \mathbb{C}).$$

$$\langle \psi | \in \iota\left(\mathcal{V}_{\vec{\lambda}}^{\dagger}(\mathfrak{X})\right) \quad \text{where} \quad \mathfrak{X} = (\mathbb{P}^1, \vec{z}_i, \ell)$$

$p_i \neq \infty$.

$$\mathcal{H}_\lambda = \frac{M(V_{\vec{\lambda}}, \ell)}{\left(X_\theta \otimes t^{-1}\right)^{\ell - (\theta, \lambda) + 1}}$$

$t_i = z - z_i$
where z is a global coordinate on $\mathbb{C}$

$$|\Phi\rangle = |\phi_1 \otimes \phi_2 \otimes \cdots \otimes \phi_{k-1} \otimes v_{\lambda_k} \otimes \Phi_{k+1} \otimes \cdots \otimes \Phi_n\rangle$$

v_{λ_k} is the highest weight vector of $\mathcal{H}_{\lambda_k}$.

By definition.

$$\boxed{\langle \psi | \, \rho_k\left(X_\theta \otimes t_k^{-1}\right)^{\ell - (\theta, \lambda_k) + 1} |\Phi\rangle = 0.}$$

$$\ell_k := \ell - (\theta, \lambda_k) + 1.$$

$$\langle \psi | \, \rho_k\left(X_\theta \otimes t_k^{-1}\right)^{\ell_k} |\Phi\rangle = 0.$$

$h = \frac{1}{z - z_k}$. h has simple pole at P_k and holomorphic everywhere.

Now.

Since by assumption $\langle\psi| \in \mathcal{V}_{\lambda}^{*}(\mathfrak{X})$

By Gauge Symmetry we get

$$0 = \langle\psi| \rho_k\left(X_\theta \otimes t_k^{-1}\right)^{l_k} |\phi\rangle$$

$\Rightarrow$

$$= \langle\psi| \rho_k\left(X_\theta \otimes h\right)^{l_k} |\phi\rangle$$

$$= (-1)^{l_k} \sum_{\substack{\vec{m}_k \\ |\vec{m}_k| = l}} \prod \binom{l_k}{m_j} (z_j - z_k)^{-m_k} \langle\psi| \prod_{j \neq k} \rho_j(X_\theta)^{m_j} |\phi\rangle$$

$\Leftrightarrow$ For $\langle\psi| \in i(\langle\psi|)$ $\forall$.

$$(*) \quad (-1)^{l_k} \sum_{\substack{\vec{m}_k \\ |\vec{m}_k| = l}} \prod \binom{l_k}{m_j} (z_j - z_k)^{-m_k} \langle\psi| \prod_{j \neq k} \rho_j(X_\theta)^{m_j} |\phi\rangle = 0.$$

(*) $\langle \psi | \rho_k (X_\theta \otimes t_k^{-1})^{d_k} | \Phi \rangle = 0.$

for $|\Phi\rangle = | \Phi_1 \otimes \cdots \otimes \Phi_{k-1} \otimes V_{\lambda_k} \otimes \Phi_{k+1} \otimes \cdots \otimes \Phi_n \rangle$

Thm: The image is exactly this

NTS $\langle \psi | \in Hom_{\mathfrak{g}}(V_{\vec{\lambda}}, \mathbb{C})$
which satisfy *, then $\langle\psi|$ comes from a conformal block.

(1) Need to produce elements $\mathcal{H}_{\vec{\lambda}}^*$

(2) We need to check Gauge Condition.

Step 1: Denote $M_\lambda = M(V_\lambda, \ell)$ (New Notation).

$$\mathcal{H}_\lambda \twoheadleftarrow M_\lambda = M(V_\lambda, \ell) \leftarrow \mathcal{U}(\hat{\mathfrak{g}})\left((X_\theta \otimes t^{-1})^{\ell - (\theta, \lambda)+1}\right)$$

Let F_p be the degree filtration on M_λ.

$$F_p M_{\vec{\lambda}} = \bigoplus_{p_1 + \cdots + p_n = p} F_{p_1} M_{\lambda_1} \otimes \cdots \otimes F_{p_n} M_{\lambda_n}$$

We will show that $\langle\psi| \in Hom_{\mathfrak{g}}(V_{\vec{\lambda}}, \mathbb{C}) + *$

can be extended to an element in

$$\langle \psi^p | \in \mathrm{Hom}_{\mathbb{C}}\left(F_p M_{\vec{\lambda}}, \mathbb{C} \right),$$

such that if $f \in H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(*\vec{z}))$, $|v\rangle \in F_p M_{\vec{\lambda}}$

then

$$\sum_{i=1}^{n} \langle \psi^p | \, \rho_i (X \otimes f) \, |v\rangle = 0.$$

$\underline{p=0}$ $\quad F_0(M_{\vec{\lambda}}) = V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_n}$

$\langle \psi^0 | = \langle \psi |$ and f is a constant funct.

then $\langle \psi | X \otimes f = 0$ since $\langle \psi |$ is $\mathfrak{g}$-invariant

If $|\phi\rangle \in F_{p+1}(M_{\vec{\lambda}})$

$$|\phi\rangle = \rho_j(-n)\, v \qquad v \in F_q(M_{\vec{\lambda}})$$

$$h(z) = \frac{1}{(z - z_j)^n}$$

Define

$$\langle \psi^{p+1} | \Phi \rangle := - \sum_{i \neq j} \langle \psi^p | \, \rho_i(X \otimes h) \, v \rangle$$

$\underbrace{}_{\in F_p(M_{\vec{\lambda}})}$

Step 2: $\langle \psi^{p+1} | \in \mathrm{Hom}_{\mathbb{C}}(I_{p+1} M_{\vec{\lambda}}, \mathbb{C})$

satisfy Gauge condition for Verma Module

$\Rightarrow \langle \psi | \in \mathrm{Hom}_{\mathbb{C}}(M_{\vec{\lambda}}, \mathbb{C})$ that satisfy Gauge condition for all Verma Module

$\langle \psi |$ descends to a function on $\mathcal{H}_{\vec{\lambda}}$.

i.e an element of $\mathrm{Hom}_{\mathbb{C}}(\mathcal{H}_{\vec{\lambda}}, \mathbb{C})$.

Let

$$| J_{\lambda_k} \rangle = \left(X_\theta \otimes t_k^{-1} \right)^{l_k} | v_{\lambda_k} \rangle$$

and.

$$J_{\lambda_k} = U(\hat{g}_-) | J_k \rangle.$$

$$\hat{g} = \hat{g}_- \oplus \underset{\sim}{\hat{g}_+} \oplus g \oplus \mathbb{C}c \qquad \mathcal{H}_{\lambda_k} = \frac{M(V_\lambda)}{J_{\lambda_k}} = \frac{M_{\lambda_k}}{J_{\lambda_k}}$$

Need to show that

$$\langle \psi | M_{\lambda_1} \otimes \cdots \otimes M_{\lambda_{k-1}} \otimes J_{\lambda_k} \otimes \cdots \otimes M_{\lambda_n} \rangle = 0$$

Proof we show that

$$\ast\ast \quad \langle \psi | V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_{k-1}} \otimes J_{\lambda_k} \otimes V_{\lambda_{k+1}} \cdots \otimes V_{\lambda_n} \rangle = 0$$

$F_p J_{\lambda_k}$ is a filtration given by filtration on $\mathcal{U}(\hat{\mathfrak{g}}_-)$.

$F_0 J_{\lambda_k} = |J_{\lambda_k}\rangle$.

Here the $\ast\ast$ follows by the equation in the assumption.

For the general case of $F_p J_{\lambda_k}$ induction and use Gauge Condition

Then we need to show that

$$\langle \psi | X(-n) V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_{k-1}} \otimes J_{\lambda_k} \otimes V_{\lambda_{k+1}} \otimes \cdots \otimes V_{\lambda_n} \rangle = 0.$$

(By applying the Gauge trick)

Finally use the filtration on $\left(F_{p_1} M_{\lambda_1} \otimes \cdots \otimes J_{\lambda_k} \otimes F_{p_{k+1}} M_{\lambda_{k+1}} \otimes \cdots \right)$

to complete

Corollary: If $n=2$, $\lambda \neq \mu^*$

$$\mathcal{V}_{\lambda,\mu}(\mathbb{P}^1, P_1, P_2, 0, 1, \ell)$$

$$\int$$

$$\mathrm{Hom}_{\mathfrak{g}}(V_\lambda \otimes V_\mu, \mathbb{C}).$$

But

$$\mathrm{Hom}_{\mathfrak{g}}(V_\lambda \otimes V_\mu, \mathbb{C})$$

$$\simeq \mathrm{Hom}_{\mathfrak{g}}(V_\lambda, V_\mu^*)$$

$$\simeq \mathrm{Hom}_{\mathfrak{g}}(V_\lambda, V_{\mu^*})$$

$$= 0 \text{ unless } \{\ \lambda = \mu^*.$$

From the Equation or directly we can show

If $\lambda = \mu^*$ then

$$\left\{ \mathcal{V}_{\lambda,\mu}(\mathbb{P}^1, P_1, P_2, \ell) = \mathbb{C} \right\}$$

Again if $\lambda \neq 0$.

then $\vartheta_\lambda^{\mathbb{P}^1} \hookrightarrow \mathrm{Hom}_{\mathfrak{g}}(V_\lambda, \mathbb{C})$.

$$= 0 \quad \text{since } \lambda \neq 0.$$

$\Rightarrow \dim \vartheta_\lambda^{\mathbb{P}^1} = 0$ unless $\lambda = 0$.

Recall that.

$$\mathfrak{sl}(2) \simeq \mathfrak{sl}_{2,\theta} = \mathbb{C}X_\theta \oplus \mathbb{C}X_{-\theta} \oplus \mathbb{C}[X_\theta, X_{-\theta}].$$

$$\mathfrak{g}.$$

$$V_\lambda \simeq \bigoplus_{i=0}^{\ell/2} W_{\lambda,i} \qquad W_{\lambda,i} \simeq \mathrm{Sym}^{2i}\mathbb{C}^2.$$

Then we proved that

If $\psi \in \mathrm{Hom}_{\mathfrak{g}}(V_{\lambda_1} \otimes V_{\lambda_2} \otimes V_{\lambda_3}, \mathbb{C})$.

satisfy:

$$(\ast\ast\ast) \qquad \psi\big|_{W_{\lambda_1,i} \otimes W_{\lambda_2,j} \otimes W_{\lambda_3,k}} = 0 \quad \text{if } i+j+k > \ell.$$

Then $\psi \in \vartheta_{\vec{\lambda}}^{\dagger}(\mathbb{P}^1)$ ← in the sense of (Beauville, TSV)

Show that if $\psi \in \mathrm{Hom}_{\mathfrak{g}}(V_{\lambda_1} \otimes V_{\lambda_2} \otimes V_{\lambda_3}, \mathbb{C})$

satisfy $(* * *)$, then

$\langle \psi \rangle \in \mathcal{V}_{\lambda}^{+}(\mathbb{P}^1)$ in the sense of TUY