TORSORS ON MODULI SPACES OF PRINCIPAL G-BUNDLES ON CURVES

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ABSTRACT. Let G be a semisimple complex algebraic group with a simple Lie algebra $\mathfrak{g}$, and let $\mathcal{M}_G^0$ denote the moduli stack of topologically trivial stable G-bundles on a smooth projective curve C. Fix a theta characteristic κ on C which is even in case $\dim \mathfrak{g}$ is odd. We show that there is a nonempty Zariski open substack $\mathcal{U}_\kappa$ of $\mathcal{M}_G^0$ such that $H^i(C, \operatorname{ad}(E_G) \otimes \kappa) = 0$, $i = 1, 2$, for all $E_G \in \mathcal{U}_\kappa$. It is shown that any such E_G has a canonical connection. It is also shown that the tangent bundle TU_κ has a natural splitting, where U_κ is the restriction of $\mathcal{U}_\kappa$ to the semi-stable locus. We also produce an isomorphism between two naturally occurring $\Omega^1_{M_G^{rs}}$–torsors on the moduli space of regularly stable M_G^{rs}.

1. INTRODUCTION

Let C be a smooth complex projective curve of genus g, with $g \geq 2$, and let G be a complex semisimple affine algebraic group with a simple Lie algebra $\mathfrak{g}$. Denote by $\mathcal{M}_G^0$ the moduli stack of topologically trivial stable G-bundles on C. Fix a theta characteristic κ on C. We assume that κ is even when $\dim \mathfrak{g}$ is odd. We prove that the following (see Corollary 4.2):

Theorem 1.1. *There is a nonempty Zariski open substack $\mathcal{U}_\kappa$ of $\mathcal{M}_G^0$ such that*

$$H^0(C, \operatorname{ad}(E_G) \otimes \kappa) = 0 = H^1(C, \operatorname{ad}(E_G) \otimes \kappa)$$

for all $E_G \in \mathcal{U}_\kappa$.

Using the above result (see Corollary 4.2) the following is proved (see Theorem 5.2). This theorem generalizes earlier results of [BH], [BeBi] for $G = \operatorname{SL}(n, \mathbb{C})$:

Theorem 1.2. *Any principal G–bundle $E_G \in \mathcal{U}_\kappa$. has a natural algebraic connection.*

Let $M_G^{rs} (\subset M_G^{0,ss})$ denote the locus of regularly stable (respectively, semi-stable) principal G-bundles which are topologically trivial. We note that M_G^{rs} is the smooth locus of M_G^{rs} except the only case where $g = 2$ and $G = \operatorname{SL}(2, \mathbb{C})$.

The tangent bundle $T(\mathcal{U}_\kappa \bigcap M_G^{rs})$ of $\mathcal{U}_\kappa \bigcap M_G^{rs}$ (the intersection is happening the moduli stack of regularly stable bundles and then projected to the moduli space) decomposes into a direct sum of vector bundles. This raises the natural question whether distributions on $\mathcal{U}_\kappa \bigcap M_G^{rs}$ given by these direct summands are integrable. (See Question 7.1.)

Now we focus our attention to two natural torsors on $\mathcal{U}_\kappa \bigcap M_G^{rs}$ for its cotangent bundle $T^*(\mathcal{U}_\kappa \bigcap M_G^{rs})$. The first $T^*(\mathcal{U}_\kappa \bigcap M_G^{rs})$–torsor is defined by the moduli space of connections $\mathcal{C}_G$. More precisely, $\mathcal{C}_G$ is the moduli space of pairs of the form (E_G, D), where $E_G \in \mathcal{U}_\kappa \bigcap M_G^{rs}$ and D is an algebraic connection on E_G.

Fix an ample line bundle L on $\mathcal{U}_\kappa \bigcap M_G^{rs}$. The second $T^*(\mathcal{U}_\kappa \bigcap M_G^{rs})$–torsor is given by the sheaf of connections $\text{Conn}(L)$ on L. It is shown that these two $T^*(\mathcal{U}_\kappa \bigcap M_G^{rs})$–torsors are isomorphic (see Section 6). This was proved earlier in [BH] for $G = \text{SL}(n, \mathbb{C})$.

We end the introduction by briefly commenting on the organization of the paper. We start by recalling the notion of conformal embeddings of semisimple Lie algebras and the branching rule of the affine Lie algebras arising from the adjoint representations. Conformal embedding are special as they force finiteness of the branching rules of some infinite dimensional representation of affine Kac-Moody algebras. Next we discuss functorial maps between spaces of conformal blocks induced by conformal embeddings and use the identifications of conformal blocks with global sections of line bundles on $\mathcal{M}_G$ to prove Theorem 1.1. Theorem 5.2 is established in Section 5. Section 6 shows the above mentioned isomorphism of torsors. In Section 7 the decomposition of the tangent bundle is constructed and Question 7.1 is posed.

2. Conformal Embedding and adjoint representations

In this section, we first recall the notion of conformal embedding for affine Lie algebra and consider a particular case of a conformal embedding given by the adjoint representation. We also recall the branching rule for this case.

2.1. **Conformal Embedding.** Let $\mathfrak{g}$ be a simple finite dimensional Lie algebra over $\mathbb{C}$. Consider the corresponding untwisted affine Lie algebra

$$\widehat{\mathfrak{g}} := \mathfrak{g} \otimes \mathbb{C}((\xi)) \oplus \mathbb{C}c,$$

where c is a central element. Fix a Cartan subalgebra of $\mathfrak{g}$ and also a Borel subalgebra containing the Cartan subalgebra. Let $(,)$ be the normalized Killing form on $\mathfrak{g}$ such that $(\theta, \theta) = 2$, where θ is the highest root of $\mathfrak{g}$, and P_+ denotes the set of dominant integral weights. For any positive integer ℓ, consider the set

$$P_\ell(\mathfrak{g}) := \{\lambda \in P_+ \mid (\lambda, \theta) \leq \ell\} \subsetneq P_+. \tag{2.1}$$

The set $P_\ell(\mathfrak{g})$ parameterizes irreducible, integrable representation of the Lie algebra $\widehat{\mathfrak{g}}$. Let $\lambda \in P_\ell(\mathfrak{g})$; then the corresponding integrable representation will be denoted by $\mathcal{H}_{\lambda,\ell}(\mathfrak{g})$. We will often drop ℓ and $\mathfrak{g}$ from the notation of integrable highest weight modules when the context is evident. They satisfy the following properties:

- $\mathcal{H}_\lambda$ is graded.
- The finite dimensional $\mathfrak{g}$ module V_λ is contained in $\mathcal{H}_\lambda$ as the degree zero part.
- The scalar c acts on $\mathcal{H}_\lambda$ as multiplication by the integer ℓ.

Now consider a orthonormal basis $\{J^1, \cdots, J^{\dim \mathfrak{g}}\}$ of the Lie algebra $\mathfrak{g}$. The Sugawara construction

$$L_{n,\ell}^{\mathfrak{g}} := \frac{1}{2(\ell + h^\vee(\mathfrak{g}))} \sum_{m \in \mathbb{Z}} \sum_{a=1}^{\dim \mathfrak{g}} {}^\circ_\circ J^a(m) J^a(n-m){}^\circ_\circ$$

gives an action of the Virasoro algebra on $\mathcal{H}_\lambda$ at level ℓ. Here $h^\vee(\mathfrak{g})$ denotes the dual Coxeter number of $\mathfrak{g}$ and $X(m) := X \otimes \xi^m$ for any $X \in \mathfrak{g}$. We recall the definition of conformal embedding from Kac-Wakimoto [KW, p. 210, Section 4.2].

Definition 2.1. An embedding $\varphi : \mathfrak{g}_1 \longrightarrow \mathfrak{g}_2$ of simple Lie algebras is conformal if the following equality holds

$$\frac{d_\varphi \dim \mathfrak{g}_1}{d_\varphi + h^\vee(\mathfrak{g}_1)} = \frac{\dim \mathfrak{g}_2}{1 + h^\vee(\mathfrak{g}_2)}, \tag{2.2}$$

where d_φ is the ratio of the normalized Killing form of the embedding φ which is also known as the Dynkin index.

A key feature of conformal embedding is the equality of Virasoro operators $L_{n,d_\varphi}^{\mathfrak{g}_1} = L_{n,1}^{\mathfrak{g}_2}$ as operators on level one integrable representation of $\widehat{\mathfrak{g}}_2$ under the obvious restriction via φ (see [KW, p. 201, Corollary 3.2.1–3.2.2]). Conformal embedding of semi-simple Lie algebras into simple Lie algebras has been classified by Schellekens–Warner [SW] and independently by Bais–Bouwknegt [BaBo]

2.2. **The adjoint representation.** Let $\mathfrak{g}$ be a simple Lie algebra. Consider the adjoint representation

$$\mathrm{ad}_\mathfrak{g} : \mathfrak{g} \hookrightarrow \mathfrak{so}(\mathfrak{g}). \tag{2.3}$$

We will drop in the subscript $\mathfrak{g}$ in the notation of adjoint representation when there is no scope for any confusion. The Dynkin index of the embedding ad is just the dual Coxeter number $h^\vee(\mathfrak{g})$ and it is easy to check that ad satisfied the identity in (2.2). Hence the embedding ad is conformal.

Observe that the Dynkin index of the natural embedding of $\mathfrak{so}(\mathfrak{g}) \longrightarrow \mathfrak{sl}(\mathfrak{g})$ is of index two and the embedding $\mathfrak{g} \longrightarrow \mathfrak{sl}(\mathfrak{g})$ is not conformal since it factors through $\mathfrak{so}(\mathfrak{g})$. Moreover, if G is any connected Lie group with Lie algebra $\mathfrak{g}$, by definition of the adjoint representation the center $Z(G) \subset G$ maps to the identity element in $\mathsf{SL}_{\dim \mathfrak{g}}$. However the natural map $\mathsf{Spin}_{\dim \mathfrak{g}} \longrightarrow \mathsf{SL}_{\dim \mathfrak{g}}$ factors through $\mathsf{SO}_{\dim \mathfrak{g}}$, hence it is not necessarily true that the center of the simply connected group $\widetilde{G}$ maps to identity element under the natural lifting $\widetilde{\mathrm{Ad}} : \widetilde{G} \longrightarrow \mathsf{Spin}_{\dim \mathfrak{g}}$ of the adjoint representation. However the image of $Z(\widetilde{G})$ under the map $\widetilde{\mathrm{Ad}}$ is contained in the following kernel μ_2:

$$1 \longrightarrow \mu_2 \longrightarrow \mathsf{Spin}_{\dim \mathfrak{g}} \longrightarrow \mathsf{SO}_{\dim \mathfrak{g}} \longrightarrow 1. \tag{2.4}$$

The embedding ad in (2.3) gives an embedding of the corresponding untwisted Lie algebra

$$\widehat{\mathrm{ad}} : \widehat{\mathfrak{g}} \longrightarrow \widehat{\mathfrak{so}}(\mathfrak{g})$$
$$X \otimes f \longmapsto \mathrm{ad}(X) \otimes f \quad \text{and} \quad c \longmapsto h^\vee(\mathfrak{g})c.$$

Let us briefly recall that an element of the center $Z(\widetilde{G})$ of the simply connected group can be identified with the subgroup of diagram automorphisms of the affine Dynkin diagram of $\widehat{\mathfrak{g}}$. This induces a bijection of the set $P_\ell(\mathfrak{g})$ of level ℓ weights of $\mathfrak{g}$.

The non-trivial central element $\sigma \in \mu_2$ acts on the level one weights of the affine Lie algebra $\widehat{\mathfrak{so}}(\mathfrak{g})$ by interchanging the zero-th fundamental weight Λ_0 with the first fundamental weight Λ_1. Thus we have the following elementary fact:

Proposition 2.2. *The highest weight integrable moduli $\mathcal{H}_{\omega_0,d_\mathrm{ad}}(\mathfrak{g})$ appears with multiplicity one in the branching of the level one module $\mathcal{H}_{\Lambda_0,1}(\mathfrak{so}(\mathfrak{g}))$. Let Σ be a non-trivial element in $Z(\widetilde{G})$, and let $\widetilde{\mathrm{Ad}}(\Sigma) \in \mu_2$ be its image under the adjoint representation. Then $\mathcal{H}_{\Sigma*\omega_0,d_\mathrm{ad}}(\mathfrak{g})$ appears with multiplicity one in the branching of $\mathcal{H}_{\widetilde{\mathrm{Ad}}(\Sigma)\bullet\Lambda_0,1}(\mathfrak{so}(\mathfrak{g}))$. Here $*$ (respectively, $\bullet$) denotes the action of the center on level d_ad (respectively, 1) weights arising from diagram automorphisms.*

We refer the reader to Kac–Wakimoto [KW, p. 214, Eq. 4.2.13] for a complete description of the branching rule for this conformal embedding.

3. Conformal blocks and adjoint representation

In this section, we first recall the notion of conformal blocks and then analyze the functoriality of conformal blocks under the adjoint representation. We use the identification between conformal blocks and the space of non-abelian G theta functions, i.e global section of natural line bundles on $\mathcal{M}_G$ to study the image of $\mathcal{M}_G$ under the adjoint representation $\mathrm{Ad} : \mathcal{M}_G \longrightarrow \mathcal{M}_{\mathsf{SO}(\dim \mathfrak{g})}$.

3.1. **Conformal blocks.** Let $\pi : \mathscr{C} \longrightarrow B$ be a family of stable n-pointed curves satisfying the following conditions:

- There are disjoint sections $s_i : B \longrightarrow \mathscr{C}$, $1 \leq i \leq n$, of the family π, marking smooth points in the fiber $\mathscr{C}_b$.
- $\mathscr{C} \setminus \sqcup_{i=1}^n s_i(B)$ is affine.
- There are isomorphisms $\widehat{\mathcal{O}}_{\mathscr{C}/s_i(B)} \cong \mathcal{O}_B[[\xi_i]]$, where $\xi_1, \cdots \xi_n$ are formal parameters.

We will denote the above family along with a choice of formal parameters by $\mathfrak{X}$. Let $S := \sqcup_{j=1}^n s_i(B)$.

For any choice of n-tuple $\vec{\lambda} = (\lambda_1, \cdots, \lambda_n)$ of level ℓ weights of $\mathfrak{g}$, one can associate the quasi-coherent sheaf of covacua

$$\mathcal{V}_{\vec{\lambda}}(\mathfrak{X}, \mathfrak{g}, \ell) := \frac{\left(\mathcal{H}_{\vec{\lambda}} \otimes_{\mathbb{C}} \mathcal{O}_B\right)}{\mathfrak{g} \otimes \pi_*(\mathcal{O}_{\mathscr{C}}(*S)) \cdot \left(\mathcal{H}_{\vec{\lambda}} \otimes_{\mathbb{C}} \mathcal{O}_B\right)},$$

where $\mathcal{H}_{\vec{\lambda}} := \mathcal{H}_{\lambda_1,\ell}(\mathfrak{g}) \otimes \cdots \otimes \mathcal{H}_{\lambda_n,\ell}(\mathfrak{g})$. Here the action of $\mathfrak{g} \otimes \pi_*(\mathcal{O}_{\mathscr{C}}(*S))$ on $\mathcal{H}_{\vec{\lambda}} \otimes \mathcal{O}_B$ is given by Laurent expansion using the formal parameters ξ_j. This sheaf of covacua was first constructed in the work of Tsuchiya–Ueno–Yamada [TUY]. The dual of the sheaf of covacua is known as the sheaf of conformal blocks and it is denoted by $\mathcal{V}_{\vec{\lambda}}^{\dagger}(\mathfrak{X}, \mathfrak{g}, \ell)$.

We will recall some of it basic properties; the reader is referred to [TUY] for a proof.

Theorem 3.1. *The sheaf of covacua enjoys the following properties:*

- *(Local freeness): The sheaf $\mathcal{V}_{\vec{\lambda}}(\mathfrak{X}, \mathfrak{g}, \ell)$ is locally free of finite rank. Its rank is given by the Verlinde formula.*
- *If $\pi : \mathscr{C} \longrightarrow B$ is a family of smooth curves, then the sheaf $\mathcal{V}_{\vec{\lambda}}(\mathfrak{X}, \mathfrak{g}, \ell)$ carries a flat projective connection constructed out of the Segal–Sugawara action of $L_{n,\ell}^{\mathfrak{g}}$ on integrable modules.*
- *(Propagation of vacua): Let s be a new section for a family of nodal curve $\pi : \mathscr{C} \longrightarrow S$ disjoint from $s_1, \cdots, s_n$, and let $\mathfrak{X}'$ denote the new data. Then there is an isomorphism*

$$\mathcal{V}_{\vec{\lambda}}(\mathfrak{X}, \mathfrak{g}, \ell) \cong \mathcal{V}_{\vec{\lambda},\Lambda_0}(\mathfrak{X}', \mathfrak{g}, \ell),$$

  where Λ_0 is the vacuum representation at level ℓ.
- *(Factorization theorem): Let $\pi : \mathcal{C} \longrightarrow S$ be a family of nodal curves and*

$$s_1, \ldots, s_n, q_1, q_2$$

  are disjoint sections of it marking smooth points, and let $\mathcal{D} \longrightarrow S$ be the family obtained by gluing $\mathcal{C}$ along q_1 and q_2. Then there is an isomorphism

$$\bigoplus \iota_{\mu} : \mathcal{V}_{\vec{\lambda}}(\mathfrak{X}, \mathfrak{g}, \ell) \cong \bigoplus_{\mu \in P_{\ell}(\mathfrak{g})} \mathcal{V}_{\vec{\lambda},\mu,\mu^{\dagger}}(\mathfrak{X}', \mathfrak{g}, \ell),$$

  where $\mathfrak{X}$ (respectively, $\mathfrak{X}'$) is associated to the family $\mathcal{D} \longrightarrow S$ (respectively, $\mathcal{C} \longrightarrow S$).

3.2. **Functoriality for conformal blocks.** Let $\varphi : \mathfrak{g}_1 \longrightarrow \mathfrak{g}$ be a homomorphism of simple Lie algebras; this induces a homomorphism $\varphi : \widehat{\mathfrak{g}}_1 \longrightarrow \widehat{\mathfrak{g}}$ between the affine Lie algebras. Let $\vec{\Lambda} := (\Lambda_1, \cdots, \Lambda_n)$ be a choice of level one weights of $\widehat{\mathfrak{g}}$, and let $\vec{\lambda} := (\lambda_1, \cdots, \lambda_n)$ be a choice of level d_{φ} weights of $\mathfrak{g}_1$ such that

$$\mathcal{H}_{\lambda_i,d_{\varphi}}(\mathfrak{g}_1) \hookrightarrow \mathcal{H}_{\Lambda_i,1}(\mathfrak{g}).$$

By functoriality, we get a homomorphism of the corresponding conformal blocks

$$\varphi : \mathcal{V}_{\vec{\lambda}}(\mathfrak{X}, \mathfrak{g}_1, d_{\varphi}) \longrightarrow \mathcal{V}_{\vec{\Lambda}}(\mathfrak{X}, \mathfrak{g}, 1), \tag{3.1}$$

where $\mathfrak{X}$ is the data associated to family of n-pointed nodal curves with a choice of n-formal parameters. Moreover we have the following:

Proposition 3.2. *([Be1, Proposition 5.8]) Assume that the embedding φ is conformal (Definition 2.1), and the family $\mathfrak{X}$ is smooth. Then the functorial map $\varphi : \mathcal{V}_{\vec{\lambda}}(\mathfrak{X}, \mathfrak{g}_1, d_\varphi) \longrightarrow \mathcal{V}_{\vec{\Lambda}}(\mathfrak{X}, \mathfrak{g}, 1)$ in (3.1) is flat with respect to the projective connections. In particular, the rank of φ is constant.*

3.3. **The case of adjoint representations.** We now restrict ourselves to the special case of the adjoint representation. Using the branching rule in Proposition 2.2 together with the functoriality described above, we get a linear map

$$\mathrm{ad} : \mathcal{V}_{\Sigma * \omega_0}(C, \mathfrak{g}, d_{\mathrm{ad}}) \longrightarrow \mathcal{V}_{\widetilde{\mathrm{Ad}(\Sigma)} \bullet \Lambda_0}(C, \mathfrak{so}(\mathfrak{g}), 1), \tag{3.2}$$

where C is a smooth one-pointed curve of genus g. The following question is natural:

Question 3.3. *Is the map ad in (3.2) non-zero?*

3.3.1. *The case where Σ is trivial.* Henceforth, we will restrict to the case where Σ is trivial. Therefore, (3.2) becomes

$$\mathrm{ad} : \mathcal{V}_{\omega_0}(C, \mathfrak{g}, d_{\mathrm{ad}}) \longrightarrow \mathcal{V}_{\Lambda_0}(C, \mathfrak{so}(\mathfrak{g}), 1). \tag{3.3}$$

Theorem 3.4. *The map ad in (3.3) is non-zero for any smooth curve C of genus g.*

Proof. We first note that it is enough to find a curve such that map is non-zero for that curve. Indeed, this follows from the fact that the embedding ad is conformal and the induced map (also denoted by ad) is flat with respect to the projective connections and hence the induced map preserves rank (see Proposition 3.2).

We proceed by induction on the genus of the curve. In the genus zero case $\mathcal{V}_{\omega_0}(\mathbb{P}^1, \mathfrak{g}, d_{\mathrm{ad}}) \cong (V_{\omega_0})^{\mathfrak{g}}$, where $V_{\omega_0} \cong \mathbb{C}$ is the trivial representation of $\mathfrak{g}$. Similarly $\mathcal{V}_{\Lambda_0}(\mathbb{P}^1, \mathfrak{so}(\mathfrak{g}), 1) \cong \mathbb{C}$. Since the trivial $\mathfrak{so}(\mathfrak{g})$-module restricts to the trivial $\mathfrak{g}$ module, the result follows in genus zero by taking invariants of the trivial representation.

Now consider a family $\mathcal{X}$ over $\mathbb{C}[[t]]$ such that for $q \neq 0$, the fibers are all one pointed smooth curve of genus g and X_0 is a nodal curve with exactly one node. By functoriality, we get a map of sheaves over $\mathbb{C}[[t]]$:

$$\mathfrak{ad} : \mathcal{V}^\dagger_{\Lambda_0}(\mathcal{X}, \mathfrak{so}(\mathfrak{g}), 1) \longrightarrow \mathcal{V}^\dagger_{\omega_0}(\mathcal{X}, \mathfrak{g}, d_{\mathrm{ad}}).$$

We have the following diagram:

$$\bigoplus_{\Lambda \in P_1(\mathfrak{g})} \mathcal{V}^\dagger_{\Lambda_0, \Lambda, \Lambda^\dagger}(\widetilde{X}, \mathfrak{so}(\mathfrak{g}), 1) \xrightarrow{\iota_\Lambda} \mathcal{V}^\dagger_{\Lambda_0}(X_0, \mathfrak{so}(\mathfrak{g}), 1)$$
$$\downarrow \mathfrak{ad}_{t=0}$$
$$\mathcal{V}^\dagger_{\omega_0}(X_0, \mathfrak{g}, d_{\mathrm{ad}}) \xrightarrow{\iota_\mu^{-1}} \bigoplus_{\Lambda \in P_{d_{\mathrm{ad}}}(\mathfrak{g})} \mathcal{V}^\dagger_{\omega_0, \mu, \mu^\dagger}(\widetilde{X}, \mathfrak{g}, d_{\mathrm{ad}}). \tag{3.4}$$

Let $\alpha_{\Lambda, \mu}$ be the following map obtained by restricting the composition of maps in (3.4):

$$\alpha_{\Lambda, \mu} : \mathcal{V}^\dagger_{\Lambda_0, \Lambda, \Lambda}(\widetilde{X}, \mathfrak{so}(\mathfrak{g}), 1) \longrightarrow \mathcal{V}^\dagger_{\omega_0, \mu, \mu^\dagger}(\widetilde{X}, \mathfrak{g}, d_{\mathrm{ad}}). \tag{3.5}$$

Observe that the genus of $\widetilde{X}$ is $g - 1$, and taking $\Lambda = \Lambda_0$ and $\mu = \omega_0$ it follows from [BP, Proposition 4.4] (see also [Mu1, Proposition 4.3] and [Mu2]) that the above map $\alpha_{\Lambda_0, \mu}$ is zero if

$\mu \neq \omega_0$. Now we have the following commutative diagram for $\widetilde{X}$:

$$(3.6)$$

The vertical isomorphisms in (3.6) are given by the propagation of vacua. By the induction hypothesis the map $\alpha_{\Lambda_0,\omega_0}$ is non-zero. This applied to (3.4) yields that

$$\mathfrak{ad}_{t=0} \; : \; \mathcal{V}^\dagger_{\Lambda_0}(X_0, \mathfrak{so}(\mathfrak{g}), 1) \; \longrightarrow \; \mathcal{V}^\dagger_{\omega_0}(X_0, \mathfrak{g}, d_{\mathrm{ad}})$$

is non-zero. Consequently, the proof is completed by using semi-continuity. $\qquad\square$

4. Uniformization and Conformal blocks

In this section, we recall the uniformization theorems connecting conformal blocks with global sections of line bundles on moduli stacks. We then have a reinterpretation of Theorem 3.4.

4.1. **Non-abelian theta functions and functoriality.** Let G be a connected semi-simple group such that the Lie algebra $\mathfrak{g}$ is simple. Let $\mathcal{M}_G$ be the moduli stack of principal G–bundles on a smooth projective curve C. The connected components of the moduli stack $\mathcal{M}_G$ are parametrized by the fundamental group $\pi_1(G)$. We denote by $\mathcal{M}_G^\delta$ the component of $\mathcal{M}_G$ corresponding to $\delta \in \pi_1(G)$. We will mostly be interested the *neutral component* $\mathcal{M}_G^0$ of the moduli stack corresponding to the trivial element in the fundamental group (it is the connected component of $\mathcal{M}_G$ containing the trivial principal G–bundle). Denote by $\widetilde{G}$ the simply connected cover of G. The natural map $\varpi_1 : \widetilde{G} \longrightarrow G$ induces a map

$$\pi_G \; : \; \mathcal{M}_{\widetilde{G}} \; \longrightarrow \; \mathcal{M}_G^0 \qquad\qquad (4.1)$$

The kernel of the above map ϖ_1 will be denoted by A which is just the fundamental group of G. It is known that A is a product of finite cyclic groups; denote $J_A := H^1(C, A)$. Then by Beauville-Laszlo-Sorger [BLS, Proposition 1.5], we have a long exact sequence

$$0 \; \longrightarrow \; J_A \; \longrightarrow \; \mathrm{Pic}(\mathcal{M}_G^\delta) \; \longrightarrow \; \mathrm{Pic}(\mathcal{M}_{\widetilde{G}}^\delta) \; \longrightarrow \; 0.$$

It is known that the Picard group of the moduli stack $\mathcal{M}_{\widetilde{G}}$ is infinitely cyclic [BLS, DN, KNR]. We denote the ample generator of $\mathrm{Pic}(\mathcal{M}_{\widetilde{G}}^\delta)$ by $\mathscr{L}_{\widetilde{G}}$. The space of global sections $H^0(\mathcal{M}_{\widetilde{G}}, \mathscr{L}_{\widetilde{G}}^{\otimes\ell})$ is known as the space of non-abelian theta functions.

Given any nonzero homomorphisms $\varphi' : \mathfrak{g}_1 \longrightarrow \mathfrak{g}$ of simple Lie algebras, consider the corresponding homomorphism $\varphi'' : \widetilde{G}_1 \longrightarrow \widetilde{G}$ between the associated simply connected groups; note that φ' is necessarily injective. This φ'' induces a map of the corresponding moduli stacks via the associated construction

$$\varphi \; : \; \mathcal{M}_{\widetilde{G}_1} \; \longrightarrow \; \mathcal{M}_{\widetilde{G}}. \qquad\qquad (4.2)$$

It follows from [KN, p. 59, Section 5] that $\varphi^*\mathscr{L}_{\widetilde{G}} \cong \mathscr{L}_{\widetilde{G}_1}^{\otimes d_\varphi}$ for the map in (4.2), where d_φ is the Dynkin index of the embedding φ''. This isomorphism induces a map of the global sections

$$\varphi^* \; : \; H^0(\mathcal{M}_{\widetilde{G}}, \mathscr{L}_{\widetilde{G}}) \; \longrightarrow \; H^0(\mathcal{M}_{\widetilde{G}_1}, \mathscr{L}_{\widetilde{G}_1}^{\otimes d_\varphi}) \qquad\qquad (4.3)$$

Now via the uniformization theorems of Beauville-Laszlo [BL], Kumar-Narasimhan-Ramanathan [KNR], Faltings [Fa], Laszlo-Sorger [LS], we get an isomorphism

$$H^0(\mathcal{M}_{\widetilde{G}}, \mathscr{L}_{\widetilde{G}}^{\otimes\ell}) \; \cong \; \mathcal{V}^\dagger_{\Lambda_0}(\mathfrak{X}, \mathfrak{g}, \ell), \qquad\qquad (4.4)$$

where $\mathfrak{X}$ is the data of a smooth curve C with one marked point along with a choice of a formal parameter at the marked point and Λ_0 is the vacuum representation at level ℓ. The following diagram is commutative:

$$
\begin{array}{ccc}
H^0(\mathcal{M}_{\widetilde{G}}, \mathscr{L}_{\widetilde{G}}) & \xrightarrow{\ \varphi\ } & H^0(\mathcal{M}_{\widetilde{G}_1}, \mathscr{L}_{\widetilde{G}_1}^{\otimes d_\varphi}) \\
\Big\downarrow{\cong} & & \Big\downarrow{\cong} \\
\mathcal{V}_{\Lambda_0}^\dagger(\mathfrak{X},\, \mathfrak{g},\, 1) & \xrightarrow{\ \varphi\ } & \mathcal{V}_{\Lambda_0}^\dagger(\mathfrak{X},\, \mathfrak{g}_1,\, d_\varphi)
\end{array}
\tag{4.5}
$$

4.2. **The adjoint representation.** Recall that a square root of the canonical line bundle K_C of C is called a theta characteristic on C. The set of theta characteristics of C is a torsor over the group of two torsion points $J_2(C)$ of the Jacobian. Now identifying $J_2(C)$ with the dual $\widehat{J_2(C)}$ via the Weil pairing, we see that the space of theta characteristics is a torsor for $\widehat{J_2(C)}$.

For every theta characteristic κ of C, Laszlo-Sorger [LS, p. 517, Section 7.8] constructed a natural square-root $\mathcal{P}_\kappa$ of the determinant of cohomology on $\mathcal{M}_{\mathsf{SO}_r}$ — which is known as the Pfaffian line bundle — along with a canonical Pfaffian section s_κ. The divisor corresponding to the Pfaffian section consists of the following associated bundles:

$$
\Xi_\kappa \;:=\; \{E \in \mathcal{M}_{\mathsf{SO}_r} \mid h^0(E \otimes \kappa) \neq 0\} \subseteq \mathcal{M}_{\mathsf{SO}_r}.
\tag{4.6}
$$

Let us recall the following results [Be2] (for $\mathcal{M}_{\mathsf{SO}_r}^0$) and [MW, Proposition 3.5] (for $\mathcal{M}_{\mathsf{SO}_r}^-$) about these Pfaffian sections s_κ being non-zero. Consider the decomposition

$$
\mathcal{M}_{\mathsf{SO}_r} = \mathcal{M}_{\mathsf{SO}_r}^0 \sqcup \mathcal{M}_{\mathsf{SO}_r}^-,
$$

where $\mathcal{M}_{\mathsf{SO}_r}^-$ parametrizes bundles with non-trivial Stiefel-Whitney class. Then, we get:

(1) If r is even and κ is any theta characteristic, then $H^0(\mathcal{M}_{\mathsf{SO}_r}^0, \mathcal{P}_\kappa)$ is one dimensional.
(2) If r is even and κ is any theta characteristic, then $H^0(\mathcal{M}_{\mathsf{SO}_r}^-, \mathcal{P}_\kappa)$ is zero dimensional.
(3) If r is odd, then $H^0(\mathcal{M}_{\mathsf{SO}_r}^0, \mathcal{P}_\kappa)$ is one dimensional if and only if κ is even.
(4) If r is odd, then $H^0(\mathcal{M}_{\mathsf{SO}_r}^-, \mathcal{P}_\kappa)$ is one dimensional if and only if κ is odd.
(5) Each Pfaffian section s_κ is projectively flat with respect to the Hitchin connection.
(6) The Pfaffian sections $\{s_\kappa\}$ are linearly independent and hence
  - if r is even, then $\dim_\mathbb{C} H^0(\mathcal{M}_{\mathsf{Spin}_r}, \mathscr{L}_{\mathsf{Spin}_r}) = 2^{2g}$, and
  - if r is odd, then $\dim_\mathbb{C} H^0(\mathcal{M}_{\mathsf{Spin}_r}, \mathscr{L}_{\mathsf{Spin}_r}) = 2^{g-1}(2^g + 1)$.

A choice of a theta characteristic induces an action of $J_2(C)$ on $H^0(\mathcal{M}_{\mathsf{Spin}_r}, \mathscr{L}_{\mathsf{Spin}_r})$ and consequently there is a decomposition (see [Be2, p. 2], [MW, p. 14, Proposition 3.7])

$$
H^0(\mathcal{M}_{\mathsf{Spin}_r}, \mathscr{L}_{\mathsf{Spin}_r}) = \bigoplus_{\chi \in \widehat{J_2(C)}} H^0(\mathcal{M}_{\mathsf{SO}_r}^0, \mathcal{P}_\kappa \otimes \mathcal{L}_\chi),
\tag{4.7}
$$

where $\mathcal{L}_\chi$ is the line bundle associated to the character χ. Moreover by [BLS, Proposition 5.2], [Be2, Proposition 2.2], [MW, Proposition 3.9] we have $\mathcal{P}_\kappa \otimes \mathcal{L}_\chi \cong \mathcal{P}_{\kappa'}$, where κ' is just the image of κ under the action of $\chi \in \widehat{J_2(C)}$.

The adjoint representation of any connected semisimple group G gives a homomorphism

$$
\mathrm{Ad}_G \,:\, G \longrightarrow \mathsf{SO}_{\dim \mathfrak{g}}
$$

whose kernel is finite, and we have the following commutative diagram

$$\begin{array}{ccc}
\widetilde{G} & \xrightarrow{\ \widetilde{\mathrm{Ad}}\ } & \mathsf{Spin}_{\dim\mathfrak{g}} \\
{\scriptstyle \pi_G}\downarrow & {\scriptstyle \mathrm{Ad}_{\widetilde{G}}}\searrow & \downarrow{\scriptstyle \pi} \\
G & \xrightarrow[\ \mathrm{Ad}_G\ ]{} & \mathsf{SO}_{\dim\mathfrak{g}},
\end{array} \tag{4.8}$$

where $\widetilde{G}$ is the simply connected cover of G. In particular, when $G = \widetilde{G}$, we have the following commutative diagram:

$$\begin{array}{ccc}
& & \mathcal{M}_{\mathsf{Spin}_{\dim\mathfrak{g}}} \\
& {\scriptstyle \widetilde{\mathrm{Ad}}}\nearrow & \downarrow{\scriptstyle \pi} \\
\mathcal{M}_{\widetilde{G}} & \xrightarrow{\ \mathrm{Ad}_{\widetilde{G}}\ } & \mathcal{M}^0_{\mathsf{SO}_{\dim\mathfrak{g}}} \\
{\scriptstyle \pi_G}\downarrow & \nearrow{\scriptstyle \mathrm{Ad}_G} & \\
\mathcal{M}^0_G & &
\end{array} \tag{4.9}$$

The following is a consequence of Theorem 3.4.

Corollary 4.1. *If* $\dim\mathfrak{g}$ *is odd (respectively, even), then a choice of an even (respectively, any) theta characteristic* κ *gives a non-zero map*

$$\mathrm{Ad}_{\widetilde{G}}\,:\,H^0(\mathcal{M}_{\mathsf{SO}_{\dim\mathfrak{g}}},\,\mathcal{P}_\kappa)\,\longrightarrow\,H^0(\mathcal{M}_{\widetilde{G}},\,\mathscr{L}^{d_{\mathrm{ad}}}_{\widetilde{G}}),$$

where d_{ad} *is the Dynkin index of the adjoint representation of* $\mathfrak{g}\hookrightarrow\mathfrak{so}(\mathfrak{g})$. *Moreover, this map factors through* $H^0(\mathcal{M}^0_G,\,\mathscr{P}_\chi)$, *where* $\mathscr{P}_\chi$ *is a line bundle on* $\mathcal{M}^0_G$ *which is pulled back from* $\mathcal{P}_\kappa$; *in other words, there is a commutative diagram*

$$\begin{array}{ccc}
H^0(\mathcal{M}_{\mathsf{SO}_{\dim\mathfrak{g}}},\,\mathcal{P}_\kappa) & \xrightarrow{\ \mathrm{Ad}_{\widetilde{G}}\ } & H^0(\mathcal{M}_{\widetilde{G}},\,\mathscr{L}^{d_{\mathrm{ad}}}_{\widetilde{G}}) \\
& \searrow & \uparrow \\
& & H^0(\mathcal{M}^0_G,\,\mathscr{P}_\chi).
\end{array}$$

We note an immediate consequence of Corollary 4.1.

Corollary 4.2. *If* $\dim\mathfrak{g}$ *is odd (respectively, even), let* κ *be an even (respectively, any) theta characteristic. Then the image of the morphism*

$$\mathcal{M}^0_G\,\longrightarrow\,\mathcal{M}_{\mathsf{SO}_{\dim\mathfrak{g}}},$$

given by Ad_G *in* (4.8), *is not contained in the divisor* Ξ_κ *(see* (4.6)).

Proof. Under the above assumption we know that $\dim H^0(\mathcal{M}_{\mathsf{SO}_{\dim\mathfrak{g}}},\,\mathcal{P}_\kappa) = 1$ [Be2, MW], and the divisor for any nonzero section of $\mathcal{P}_\kappa$ is Ξ_κ. Therefore, if the image of the above morphism $\mathcal{M}^0_G\,\longrightarrow\,\mathcal{M}_{\mathsf{SO}_{\dim\mathfrak{g}}}$ is contained in Ξ_κ, then the homomorphism

$$H^0(\mathcal{M}_{\mathsf{SO}_{\dim\mathfrak{g}}},\,\mathcal{P}_\kappa)\,\longrightarrow\,H^0(\mathcal{M}^0_G,\,\mathscr{P}_\chi)$$

in the statement of Corollary 4.1 becomes the zero map. But Corollary 4.1 says that this map is nonzero. In view of this contradiction we conclude that the image of the morphism

$$\mathcal{M}^0_G\,\longrightarrow\,\mathcal{M}_{\mathsf{SO}_{\dim\mathfrak{g}}}$$

is not contained in the divisor Ξ_κ. $\square$

5. A NATURAL CONNECTION

The main goal of this section is consider the non-empty Zariski open substack in $\mathcal{M}_G^0$ given by Corollary 4.2 as the set of algebraic principal G–bundles on C such that $\mathrm{Ad}(E_G)$ is not in the divisor Ξ_κ, and show that every element of this set admits a natural algebraic connection.

Let E_G be an algebraic principal G–bundle on C. An algebraic connection on E_G produces an algebraic connection on any algebraic fiber bundle associated to E_G, in particular, an algebraic connection is induced on the adjoint vector bundle $\mathrm{ad}(E_G)$. Let $\mathcal{C}(E_G)$ and $\mathcal{C}(\mathrm{ad}(E_G))$ be the spaces of algebraic connections on E_G and $\mathrm{ad}(E_G)$ respectively. Let

$$\Phi_0 \,:\, \mathcal{C}(E_G) \,\longrightarrow\, \mathcal{C}(\mathrm{ad}(E_G))$$

be the above map.

Lemma 5.1. *There is a natural map*

$$\Phi \,:\, \mathcal{C}(\mathrm{ad}(E_G)) \,\longrightarrow\, \mathcal{C}(E_G)$$

such that $\Phi \circ \Phi_0 \,=\, \mathrm{Id}_{\mathcal{C}(E_G)}$.

Proof. Consider the adjoint homomorphism $\mathfrak{g} \,\hookrightarrow\, \mathfrak{g} \otimes \mathfrak{g}^* \,=\, \mathfrak{gl}(\mathfrak{g})$. We have the symmetric bilinear form on $\mathfrak{gl}(\mathfrak{g})$ defined by $A \otimes B \,\longmapsto\, \mathrm{trace}(AB)$. Its restriction to $\mathfrak{g}$ is a constant scalar multiple of the Killing form of $\mathfrak{g}$. Consider the corresponding orthogonal decomposition

$$\mathfrak{gl}(\mathfrak{g}) \,=\, \mathfrak{g} \oplus \mathfrak{g}^\perp.$$

Let

$$P \,:\, \mathfrak{gl}(\mathfrak{g}) \,\longrightarrow\, \mathfrak{g} \tag{5.1}$$

be the projection constructed using the above decomposition of $\mathfrak{gl}(\mathfrak{g})$. The adjoint action of G on $\mathfrak{g}$ produces an action of G on $\mathfrak{g} \otimes \mathfrak{g}^* \,=\, \mathfrak{gl}(\mathfrak{g})$. The projection P in (5.1) is a homomorphism of G–modules.

Let $E_{\mathrm{GL}(\mathfrak{g})}$ be the principal $\mathrm{GL}(\mathfrak{g})$–bundle C corresponding to the vector bundle $\mathrm{ad}(E_G)$. We note that $E_{\mathrm{GL}(\mathfrak{g})}$ is the quotient of $E_G \times \mathrm{GL}(\mathfrak{g})$ where $(z,\, B)$ is identified with $(zg,\, \mathrm{Ad}(g^{-1})B\mathrm{Ad}(g))$ for all $g \in G$. We have a natural map

$$\Psi \,:\, E_G \,\longrightarrow\, E_{\mathrm{GL}(\mathfrak{g})}$$

that sends any $z \in E_G$ to the equivalence class of $(z,\, \mathrm{Id}_\mathfrak{g})$.

Let ∇ be an algebraic connection on $\mathrm{ad}(E_G)$. So ∇ is an algebraic 1–form on $E_{\mathrm{GL}(\mathfrak{g})}$, with values in $\mathfrak{gl}(\mathfrak{g})$, satisfying certain conditions. Therefore, $P \circ (\Psi^*\nabla)$ is a $\mathfrak{g}$–valued algebraic 1–form on E_G, where P is the projection in (5.1). Using the fact that P in (5.1) is a homomorphism of G–modules it is straightforward to check that $P \circ (\Psi^*\nabla)$ satisfies the two conditions needed to define a connection on E_G. It is also evident that the map

$$\Phi \,:\, \mathcal{C}(\mathrm{ad}(E_G)) \,\longrightarrow\, \mathcal{C}(E_G)$$

constructed this way satisfies the condition $\Phi \circ \Phi_0 \,=\, \mathrm{Id}_{\mathcal{C}(E_G)}$. $\square$

Take a theta characteristic κ on C. We assume that if $\dim \mathfrak{g}$ is odd, then κ is an even theta characteristic. There is no condition on κ when $\dim \mathfrak{g}$ is even.

Consider the map

$$F \,:\, \mathcal{M}_G^0 \,\longrightarrow\, \mathcal{M}_{\mathrm{SO}_{\dim \mathfrak{g}}}$$

in Corollary 4.2 that sends any E_G to $\mathrm{ad}(E_G)$ equipped with the Killing form on the fibers. Take any principal G–bundle $E_G \in \mathcal{M}_G^0$ such that

$$F(E_G) \,\notin\, \Xi_\kappa \tag{5.2}$$

(see (4.6)); from Corollary 4.2 we know that the locus, in $\mathcal{M}_G^0$, of all such E_G is a nonempty Zariski open substack. From (5.2) we know that

$$H^0(C, \text{ad}(E_G) \otimes \kappa) = 0 = H^0(C, \text{ad}(E_G) \otimes \kappa), \qquad (5.3)$$

which implies that $\text{ad}(E_G)$ is semistable. This in turn implies that the principal G-bundle E_G is semistable.

Since $H^0(C, \text{ad}(E_G) \otimes \kappa) = 0 = H^1(C, \text{ad}(E_G) \otimes \kappa)$, from [BeBi], [BH] we know that $\text{ad}(E_G)$ has a canonical algebraic connection. We will now briefly recall the construction of this connection on $\text{ad}(E_G)$.

For $i = 1, 2$, let $p_i : C \times C \longrightarrow C$ be the natural projections. Let

$$\Delta := \{(x, x) \in C \times C \,|\, x \in C\} \subset C \times C$$

be the reduced diagonal divisor. We will identify Δ with C using the map $x \longmapsto (x, x)$, where $x \in C$. The restriction of $(p_1^* \kappa) \otimes (p_2^* \kappa)$ to Δ is evidently identified with K_C. Also, the restriction of $\mathcal{O}_{C \times C}(\Delta)$ to Δ is identified with the tangent bundle TC using the Poincaré adjunction formula (see [GH, p. 146]). Therefore, we have the following short exact sequence of coherent sheaves on $C \times C$:

$$0 \to (p_1^*(\text{ad}(E_G) \otimes \kappa) \otimes (p_2^*(\text{ad}(E_G)))) \otimes \kappa)) \to (p_1^*(\text{ad}(E_G) \otimes \kappa)) \otimes (p_2^*(\text{ad}(E_G) \otimes \kappa)) \otimes \mathcal{O}_{C \times C}(\Delta) \to$$

$$\to \text{ad}(E_G)^{\otimes 2} \longrightarrow 0$$

where $\text{ad}(E_G)^{\otimes 2}$ is supported on $\Delta = C$. Let

$$0 \longrightarrow H^0(C \times C, (p_1^*(\text{ad}(E_G) \otimes \kappa)) \otimes (p_2^*(\text{ad}(E_G) \otimes \kappa)))$$

$$\downarrow$$

$$H^0(C \times C, (p_1^*(\text{ad}(E_G) \otimes \kappa)) \otimes (p_2^*(\text{ad}(E_G) \otimes \kappa)) \otimes \mathcal{O}_{C \times C}(\Delta)) \to H^0(C, \text{ad}(E_G)^{\otimes 2}) \to$$

$$\to H^1(C \times C, (p_1^*(\text{ad}(E_G) \otimes \kappa)) \otimes (p_2^*(\text{ad}(E_G) \otimes \kappa)))$$

$$(5.4)$$

Since

$$H^m(C \times C, (p_1^*(\text{ad}(E_G) \otimes \kappa)) \otimes (p_2^*(\text{ad}(E_G) \otimes \kappa)))$$
$$= \bigoplus_{i=0}^{m} H^i(C, \text{ad}(E_G) \otimes \kappa) \otimes H^{m-i}(C, \text{ad}(E_G) \otimes \kappa)$$

from (5.3) we conclude that

$$H^m(C \times C, (p_1^*(\text{ad}(E_G) \otimes \kappa)) \otimes (p_2^*(\text{ad}(E_G) \otimes \kappa))) = 0$$

for all $m \geq 0$. Consequently, from (5.4) it is deduced that

$$H^0(C \times C, (p_1^*(\text{ad}(E_G) \otimes \kappa)) \otimes (p_2^*(\text{ad}(E_G) \otimes \kappa)) \otimes \mathcal{O}_{C \times C}(\Delta)) = H^0(C, \text{ad}(E_G)^{\otimes 2}). \quad (5.5)$$

The isomorphism

$$\text{ad}(E_G) \xrightarrow{\sim} \text{ad}(E_G)^* \qquad (5.6)$$

given by the fiberwise Killing form on $\text{ad}(E_G)$ produces a section

$$\gamma \in H^0(C, \text{ad}(E_G)^{\otimes 2}). \qquad (5.7)$$

From the construction of γ it is evident that the composition of homomorphisms

$$\mathrm{ad}(E_G) \xrightarrow{\sim} \mathrm{ad}(E_G)^* \xrightarrow{\gamma} \mathrm{ad}(E_G) \qquad (5.8)$$

coincides with the identity map of $\mathrm{ad}(E_G)$, where the first isomorphism is the one in (5.6)

Let

$$\widetilde{\Gamma} \in H^0(C \times C, (p_1^*(\mathrm{ad}(E_G) \otimes \kappa)) \otimes (p_2^*(\mathrm{ad}(E_G) \otimes \kappa)) \otimes \mathcal{O}_{C \times C}(\Delta)) \qquad (5.9)$$

be the section taken to γ (see (5.7)) by the isomorphism in (5.5). On the other hand, the restriction of $(p_1^* \otimes \kappa) \otimes (p_2^* \otimes \kappa) \otimes \mathcal{O}_{C \times C}(\Delta)$ to the subscheme $2\Delta \subset C \times C$ has a canonical trivialization [BR, p. 688, Theorem 2.2]. Restricting the section $\widetilde{\Gamma}$ in (5.9) to 2Δ, and invoking the trivialization of the restriction of $(p_1^* \otimes \kappa) \otimes (p_2^* \otimes \kappa) \otimes \mathcal{O}_{C \times C}(\Delta)$ to $2\Delta \subset C \times C$, we obtain a section

$$\Gamma \in H^0\left(2\Delta, ((p_1^* \mathrm{ad}(E_G)) \otimes (p_2^* \mathrm{ad}(E_G)))\big|_{2\Delta}\right). \qquad (5.10)$$

The restriction of Γ to $\Delta \subset 2\Delta$ evidently coincides with γ (see (5.7)) using the identification of Δ with C. Since the composition of homomorphisms in (5.8) coincides with the identity map of $\mathrm{ad}(E_G)$, it follows that Γ defines an algebraic connection on the vector bundle $\mathrm{ad}(E_G)$.

Now using lemma 5.1, the above algebraic connection on $\mathrm{ad}(E_G)$ produces an algebraic connection on the principal G–bundle E_G. Therefore, we have the following:

Theorem 5.2. *Take a theta characteristic κ on C. We assume that if $\dim \mathfrak{g}$ is odd, then κ is an even theta characteristic. Take any principal G–bundle $E_G \in \mathcal{M}_G^0$ that lies in the nonempty Zariski open subset of $\mathcal{M}_G^0$ whose image under the map F in Corollary 4.2 lies in the complement of the divisor $\Xi_\kappa \subset \mathcal{M}_{\mathsf{SO}_{\dim \mathfrak{g}}}$. Then E_G has a natural algebraic connection.*

6. Isomorphism of torsors

Let Y be a smooth complex variety. A torsor on Y for the cotangent bundle $\psi : \Omega_Y^1 \longrightarrow Y$ is an algebraic fiber bundle $p : \mathcal{V} \longrightarrow Y$ together with an isomorphism

$$\Phi : \mathcal{V} \times_Y \Omega_Y^1 \longrightarrow \mathcal{V} \times_Y \mathcal{V}$$

such that

(1) $\psi \circ p_2 = p \circ \Phi$, where $p_2 : \mathcal{V} \times_Y \Omega_Y^1 \longrightarrow \Omega_Y^1$ is the natural projection, and
(2) Ψ defines an action of the fibers of Ω_Y^1 on the fibers of $\mathcal{V}$.

Let M_G^{rs} denote the moduli space of regularly stable topologically trivial principal G-bundles on C. Recall that the a G bundle is regularly stable if it is stable and it's automorphism group is the center $Z(G)$ of the group G. It is known that M_G^{rs} is the smooth locus [BiHo] of the moduli space of semistable topologically trivial principal G-bundles on C except in the only one case where $G = \mathrm{SL}(2, \mathbb{C})$ and $\mathrm{genus}(C) = 2$.

On M_G^{rs}, there are two natural torsors for the cotangent bundle $\Omega_{M_G^{rs}}^1$ which we will now describe. Note that any $E_G \in M_G^{rs}$ admits an algebraic connection [Ra], [AB].

The first torsor. Let $\mathcal{C}_G$ denotes the moduli space G-connections such that the underlying principal bundle is in M_G^{rs}. In other words, $\mathcal{C}_G$ parametrizes pairs of the (E_G, D), where $E_G \in M_G^{rs}$ and D is an algebraic connection on E_G. Let

$$\Phi : \mathcal{C}_G \longrightarrow M_G^{rs}$$

be the natural projection that sends any (E_G, D) to E_G. If a principal G-bundle F_G admits an algebraic connection, then the space of all algebraic connections on F_G is an affine space for $H^0(C, \mathrm{ad}(E_G) \otimes K_C)$. Therefore, for the projection Φ, the moduli space $\mathcal{C}_G$ is a torsor over M_G^{rs} for the cotangent bundle $\Omega_{M_G^{rs}}^1$.

The second torsor. To describe the second torsor over M_G^{rs} for the cotangent bundle $\Omega^1_{M_G^{rs}}$, first recall that

$$\mathrm{Pic}(M_G^{rs}) \;=\; \mathbb{Z} \oplus \mathrm{Tor}, \tag{6.1}$$

where Tor is a finite abelian group [BLS, p. 184, Theorem (a)]. Any line bundle ξ of finite order has a canonical integrable algebraic connection. In fact, if $\xi^{\otimes n}$ is the trivial line bundle, then there is a unique connection on ξ which induces the trivial connection on the trivial line bundle

Take a line bundle L on M_G^{rs}. Let $\mathrm{Conn}(L)$ denote the sheaf of algebraic connections on L, meaning the space of sections of $\mathrm{Conn}(L)$ over any open subset $U \subset M_G^{rs}$ is the space of all algebraic connections on $L\big|_U$. To describe $\mathrm{Conn}(L)$ explicitly, let

$$0 \longrightarrow \mathcal{O}_{M_G^{rs}} \longrightarrow \mathrm{At}(L) \longrightarrow TM_G^{rs} \longrightarrow 0$$

be the Atiyah exact sequence for L [At]. Tensoring it with $\Omega^1_{M_G^{rs}}$ we get the exact sequence

$$0 \longrightarrow \Omega^1_{M_G^{rs}} \longrightarrow \mathrm{At}(L) \otimes \Omega^1_{M_G^{rs}} \overset{\Psi}{\longrightarrow} (TM_G^{rs}) \otimes \Omega^1_{M_G^{rs}} = \mathrm{End}(TM_G^{rs}) \longrightarrow 0. \tag{6.2}$$

Let $\Psi^{-1}(\mathrm{Id}_{TM_G^{rs}}) \subset \mathrm{At}(L) \otimes \Omega^1_{M_G^{rs}}$ be the inverse image, under the map Ψ in (6.2), of the image of the section $M_G^{rs} \longrightarrow \mathrm{End}(TM_G^{rs})$ given by the identity map of TM_G^{rs}. From (6.2) it follows immediately that $\Psi^{-1}(\mathrm{Id}_{TM_G^{rs}})$ is a torsor over TM_G^{rs} for $\Omega^1_{M_G^{rs}}$.

This $\Omega^1_{M_G^{rs}}$-torsor $\Psi^{-1}(\mathrm{Id}_{TM_G^{rs}})$ is identified with the $\Omega^1_{M_G^{rs}}$-torsor $\mathrm{Conn}(L)$.

From the above observation that any line bundle ξ of finite order has a canonical integrable algebraic connection it follows immediately that

$$\mathrm{Conn}(L) \;=\; \mathrm{Conn}(L \otimes \xi).$$

Now let L be such that $c_1(L) \neq 0$ (equivalently, L is *not* of finite order because of (6.1)). We have the following theorem, that was originally shown in [BH] for the group $G = \mathrm{SL}(r, \mathbb{C})$.

Theorem 6.1. *The above two $\Omega^1_{M_G^{rs}}$-torsors $\mathcal{C}_G$ and $\mathrm{Conn}(L)$ over M_G^{rs} are isomorphic up to a constant rescaling of the action. This means that there is an algebraic isomorphic*

$$I : \mathrm{Conn}(L) \longrightarrow \mathcal{C}_G \tag{6.3}$$

of fiber bundles over M_G^{rs}, and a nonzero number $c \in \mathbb{R}$, such that

$$I(z + v) \;=\; I(z) + cv$$

for all $z \in \mathrm{Conn}(L)_y$, $y \in M_G^{rs}$, and $v \in (\Omega^1_{M_G^{rs}})_y$.

Proof. In view of Corollary 4.2 and Theorem 5.2, the proof of [BH] works for any semisimple G whose Lie algebra is simple. Therefore, the above result of [BH] holds for any such G. However, from the above result on $\mathrm{SL}(r, \mathbb{C})$ it is possible to deduce the same result for all semistable G whose Lie algebra $\mathfrak{g}$ is simple; this will be described below.

Take any semisimple G whose Lie algebra $\mathfrak{g}$ is simple. The dimension of $\mathfrak{g}$ is denoted by r. Let

$$P : M_G^{rs} \longrightarrow M_{\mathrm{SL}(r, \mathbb{C})} \tag{6.4}$$

be the finite morphism that sends any E_G to the vector bundle $\mathrm{ad}(E_G)$. Let

$$P^*I : P^*\mathrm{Conn}(L) \longrightarrow P^*\mathcal{C}_{\mathrm{SL}(r, \mathbb{C})} \tag{6.5}$$

be the pullback of the isomorphism in (6.3) by P in (6.4). Note that both $P^*\mathrm{Conn}(L)$ and $P^*\mathcal{C}_{\mathrm{SL}(r, \mathbb{C})}$ are torsors over M_G^{rs} for the pulled back vector bundle $P^*\Omega^1_{M_{\mathrm{SL}(r, \mathbb{C})}}$.

Next we will describe a subbundle of $P^*\Omega^1_{M_{\mathrm{SL}(r, \mathbb{C})}}$.

Consider the injective homomorphism

$$\mathfrak{g} \longrightarrow \mathfrak{sl}(\mathfrak{g}) = \mathfrak{sl}(r, \mathbb{C})$$

given by the adjoint action of G on $\mathfrak{g}$. The image of $\mathfrak{g}$ in $\mathfrak{sl}(\mathfrak{g})$ will also be denoted by $\mathfrak{g}$. The Killing form on $\mathfrak{sl}(\mathfrak{g})$ restricts to a nonzero constant multiple of the Killing form on $\mathfrak{g}$. Let

$$\mathfrak{sl}(\mathfrak{g}) = \mathfrak{g} \oplus \mathfrak{g}^{\perp} \tag{6.6}$$

be the orthogonal decomposition with respect to the Killing form on $\mathfrak{sl}(\mathfrak{g})$.

The decomposition of $\mathfrak{sl}(\mathfrak{g})$ in (6.6) is a decomposition of G-modules. Therefore, for any principal G-bundle E_G on C, the decomposition in (6.6) produces a decomposition

$$\mathrm{End}^0(\mathrm{ad}(E_G)) = \mathrm{ad}(E_G) \oplus \mathcal{V}(E_G), \tag{6.7}$$

where $\mathrm{End}^0(\mathrm{ad}(E_G)) \subset \mathrm{End}(\mathrm{ad}(E_G))$ is the subbundle of co-rank one defined by the endomorphisms of trace zero and

$$\mathcal{V}(E_G) := E_G(\mathfrak{g}^{\perp})$$

is the vector bundle on C associated to E_G for the G-module $\mathfrak{g}^{\perp}$ in (6.6). Let

$$H^0(C, \mathrm{End}^0(\mathrm{ad}(E_G)) \otimes K_C) = H^0(C, \mathrm{ad}(E_G) \otimes K_C) \oplus H^0(C, \mathcal{V}(E_G) \otimes K_C) \tag{6.8}$$

be the decomposition corresponding to the decomposition in (6.7). Note that for any $E_G \in M_G^{rs}$, the fiber of $P^*\Omega_{M_{\mathrm{SL}(r,\mathbb{C})}}$ (see (6.4)) over E_G is $H^0(C, \mathrm{End}^0(\mathrm{ad}(E_G)) \otimes K_C)$.

Let $\mathcal{V}$ denote the vector bundle over M_G^{rs} whose fiber over any $E_G \in M_G^{rs}$ is $H^0(C, \mathcal{V}(E_G) \otimes K_C)$. From (6.8) we conclude that $\mathcal{V}$ is a direct summand of $P^*\Omega_{M_{\mathrm{SL}(r,\mathbb{C})}}$.

Consider the isomorphism P^*I in (6.5) between the two $P^*\Omega^1_{M_{\mathrm{SL}(r,\mathbb{C})}}$-torsors $P^*\mathrm{Conn}(L)$ and $P^*\mathcal{C}_{\mathrm{SL}(r,\mathbb{C})}$. Quotienting both $P^*\Omega^1_{M_{\mathrm{SL}(r,\mathbb{C})}}$-torsors $P^*\mathrm{Conn}(L)$ and $P^*\mathcal{C}_{\mathrm{SL}(r,\mathbb{C})}$ by the subbundle $\mathcal{V} \subset P^*\Omega^1_{M_{\mathrm{SL}(r,\mathbb{C})}}$ we obtain a generalization of the isomorphism I in (6.3) for any semisimple G whose Lie algebra is simple. $\qquad\square$

7. A DECOMPOSITION OF THE TANGENT BUNDLE

Take a theta characteristic κ on C. We assume that if $\dim \mathfrak{g}$ is odd, then κ is an even theta characteristic. We further assume that κ has a section (there is always such a κ). Take any

$$\sum_{i=1}^{m} \mu_i.c_i \in |\kappa|; \tag{7.1}$$

so the line bundle $\mathcal{O}_C(\sum_{i=1}^{m} \mu_i.c_i)$, where $\{c_i\}_{i=1}^{m}$ are distinct points of C and μ_i are positive integers, is holomorphically isomorphic to κ. Note that

$$\sum_{i=1}^{m} \mu_i = g - 1,$$

where $g = \mathrm{genus}(C)$. Let

$$\mathcal{U} \subset M_G^{rs} \tag{7.2}$$

denote the nonempty Zariski open subset that parametrizes all $E_G \in M_G^{rs}$ such that

$$H^0(C, \mathrm{ad}(E_G) \otimes \kappa) = 0 = H^0(C, \mathrm{ad}(E_G) \otimes \kappa); \tag{7.3}$$

from Corollary 4.2 we know that $\mathcal{U}$ is nonempty. (Since $\chi(\mathrm{ad}(E_G) \otimes \kappa) = 0$, there is only one condition in (7.3).)

For any $E_G \in \mathcal{U}$ (see (7.2)), the vector bundle $\text{ad}(E_G) \otimes \mathcal{O}_C(\sum_{i=1}^m \mu_i.c_i)$ will be denoted by $\widetilde{\text{ad}}(E_G)$ for notational convenience. Consider the natural short exact sequence of coherent sheaves on C

$$0 \longrightarrow \text{ad}(E_G) \longrightarrow \widetilde{\text{ad}}(E_G) \longrightarrow \sum_{i=1}^m \widetilde{\text{ad}}(E_G)\big|_{\mu_i c_i} \longrightarrow 0. \tag{7.4}$$

Let

$$H^0(C, \widetilde{\text{ad}}(E_G)) \longrightarrow \sum_{i=1}^m \widetilde{\text{ad}}(E_G)\big|_{\mu_i c_i} \xrightarrow{\Phi} H^1(C, \text{ad}(E_G)) \rceil$$

$$\lfloor\rightarrow H^1(C, \widetilde{\text{ad}}(E_G)) \tag{7.5}$$

be the long exact sequence of cohomologies associated to the short exact sequence of sheaves in (7.4). From (7.3) it follows immediately that the homomorphism Φ in (7.5) is an isomorphism. Note that $H^1(C, \text{ad}(E_G))$ is the fiber of the tangent bundle TM_G^{rs} at the point $E_G \in M_G^{rs}$.

For $1 \leq i \leq m$, let

$$\mathcal{W}_i \longrightarrow \mathcal{U} \tag{7.6}$$

be the vector bundle whose fiber over any $E_G \in M_G^{rs}$ is $\widetilde{\text{ad}}(E_G)\big|_{\mu_i c_i}$. Note that there is a unique universal adjoint bundle

$$\mathcal{A} \longrightarrow C \times M_G^{rs}. \tag{7.7}$$

It should be clarified that there may not be a universal principal G-bundle over $C \times M_G^{rs}$. The vector bundle $\mathcal{W}_i$ in (7.6) is the restriction of $\mathcal{A} \otimes p_1^* \mathcal{O}_C(\sum_{i=1}^m \mu_i.c_i)$ to $(\mu_i c_i) \times \mathcal{U} \subset C \times M_G^{rs}$, where $p_1 : C \times M_G^{rs} \longrightarrow C$ is the natural projection and $\mathcal{A}$ is the vector bundle in (7.7). Since Φ in (7.5) is an isomorphism, we have the following decomposition:

$$T\mathcal{U} = \bigoplus_{i=1}^m \mathcal{W}_i. \tag{7.8}$$

We have the following natural question:

Question 7.1. *Take any $1 \leq i \leq m$. Is the holomorphic distribution*

$$\mathcal{W}_i \subset T\mathcal{U}$$

on $\mathcal{U}$ (in (7.8)) integrable?

Next assume that all $m_i = 1$ for all $1 \leq i \leq m$ (see (7.1)). So we have $m = g - 1$.

Note that the Poincaré adjunction formula says that the fiber of $\mathcal{O}_C(c_1 + \cdots + c_{g-1})$ over any c_i, $1 \leq i \leq g-1$, is identified with the fiber $T_{c_i} C$ of the tangent bundle over the point c_i (see [GH, p. 146]). For $1 \leq i \leq g-1$, let

$$\mathcal{L}_i := \mathcal{U} \times (T_{c_i} C)^{\otimes 2} \longrightarrow \mathcal{U} \tag{7.9}$$

be the trivializable line bundle over $\mathcal{U}$ with fiber $(T_{c_i} C)^{\otimes 2}$. The Killing form on $\mathfrak{g}$ produces a fiberwise nondegenerate symmetric pairing

$$\mathcal{A} \otimes \mathcal{A} \longrightarrow \mathcal{O}_{C \times M_G^{rs}},$$

where $\mathcal{A}$ is the vector bundle in (7.7). Using it we have a homomorphism

$$\varphi_i : \mathcal{W}_i \otimes \mathcal{W}_i \longrightarrow \mathcal{L}_i, \tag{7.10}$$

where $\mathcal{W}_i$ and $\mathcal{L}_i$ are the vector bundles constructed in (7.6) and (7.9) respectively. The homomorphism φ_i in (7.10) is evidently symmetric and fiberwise nondegenerate. Now using (7.8) we get a holomorphic symmetric fiberwise nondegenerate bilinear form on $T\mathcal{U}$.

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