DIAGRAM AUTOMORPHISMS AND CONFORMAL BLOCKS DIVISORS

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Abstract. In this article, we compute how conformal blocks divisors over $\overline{\mathrm{M}}_{0,n}$ change under the action of the affine Dynkin diagram automorphisms of the associated Lie algebra. We show that the difference is supported only on those boundary divisors that separates the points 1 and n and give a formula for the multiplicity for classical Lie algebras.

1. Introduction

Let A be a partition of $\{1,\dots,n\}$ of cardinality $1 \leq i \leq n/2$. We denote the corresponding divisor by D_{A,A^c}. From Fakhruddin's formula, it is clear that the difference is only supported on D_{A,A^c} if A separates the points 1 and n.

2. Action of diagram automorphisms in type A

Let σ be the diagram automorphism of $\mathfrak{sl}(r)$ that sends ω_1 to ω_2 and so on. We will denote by $\sigma^m = \sigma \circ \cdots \circ \sigma$ composition m-times. Let $\lambda = (\lambda^1 \geq \cdots \geq \lambda^{r-1})$ be a reduced Young diagram of λ, that gives an irreducible representaion of $\mathfrak{sl}(r)$. Then we have the following Lemma

Lemma 2.1.

$$\Delta_{\sigma^m(\lambda)}(\mathfrak{sl}(\mathfrak{r}),k) = \Delta_\lambda(\mathfrak{sl}(r),k) + \frac{m(r-m)k}{2r} - \frac{m|\lambda|}{r} + \sum_{i=r-m+1}^{r} \lambda^i$$

The following is also easy to check

Lemma 2.2.

$$\sigma^m(\lambda)^\dagger = \sigma^{n-m}(\lambda^\dagger),$$

where $\lambda^\dagger = -w_0(\lambda)$.

2.1. **Action of diagram automorphisms on conformal blocks divisors.** Let $\vec{\lambda} = (\lambda_1,\dots,\lambda_n)$ and $\vec{\lambda}' = (\sigma(\lambda_1),\lambda_2,\dots,\lambda_{n-1},\sigma^{r-1}(\lambda_n))$. We would like to understand the difference of the Chern classes

$$c_1(\mathbb{V}_{\vec{\lambda}}(\mathfrak{sl}(r),k) - c_1(\mathbb{V}_{\vec{\lambda}'}(\mathfrak{sl}(r),k).$$

We will use Fakhruddin's version of the Chern class formula. We first observe that if $A \sqcup A^c = \{1,\dots,n\}$ is a partition of $\{1,\dots,n\}$, then the above difference is supported on the boundary divisor D_{A,A^c} if and only if $1 \in A$ and $n \notin A$. The next formula gives the multiplicity m_{A,A^c} of the divisor D_{A,A^C} that supports the divisor $c_1(\mathbb{V}_{\vec{\lambda}}(\mathfrak{sl}(r),k) - c_1(\mathbb{V}_{\vec{\lambda}'}(\mathfrak{sl}(r),k)$.

Proposition 2.3. *With the above notation, we have the following formula:*

$$m_{A,A^c} = \frac{\operatorname{rk}\mathbb{V}_{\vec{\lambda}}}{(n-1)(n-2)}\Big((n-i)(n-i-1)\frac{(r-1)k-2|\lambda_1|}{2r} + i(i-1)\big(\frac{(r-1)k}{2r} - \frac{(r-1)|\lambda_n|}{r} + \lambda_n^1\big)\Big) - \sum_{\lambda\in P_k(\mathfrak{sl}(r))}\frac{(r-1)k-2|\lambda|}{2r}\operatorname{rk}\mathbb{V}_{\lambda_A,\lambda}\operatorname{rk}\mathbb{V}_{\lambda_{A^c},\lambda^*}$$

The general formula for σ^m can be found by applying the above Proposition repeatedly.

Let A be a subset of $\{1,\dots,n\}$ such that $n\in A$ and $1\notin A$. Then the difference $c_1(\mathbb{V}_{\vec{\lambda}'}(\mathfrak{sl}(r),k) - c_1(\mathbb{V}_{\vec{\lambda}})$ is also supported on D_{A,A^c}.

Proposition 2.4. *With the above notation, we have the following formula:*

$$m_{A,A^c} = \frac{\operatorname{rk}\mathbb{V}_{\vec{\lambda}}}{(n-1)(n-2)}\Big((n-i)(n-i-1)\big(\frac{(r-1)k}{2r} - \frac{(r-1)|\lambda_n|}{r} + \lambda_n^1\big) + i(i-1)\frac{(r-1)k-2|\lambda_1|}{2r}\Big) - \sum_{\lambda\in P_k(\mathfrak{sl}(r))}\frac{(r-1)k-2|\lambda|}{2r}\operatorname{rk}\mathbb{V}_{\lambda_A,\lambda}\operatorname{rk}\mathbb{V}_{\lambda_{A^c},\lambda^*}$$

3. Action of diagram automorphisms in type C

The affine Dynkin diagram symmetry for type C is $\mathbb{Z}/2$ and it takes ω_i to ω_{r-i}. As usual a dominant integral weights of $\mathfrak{sp}(2r)$ at level s is represented by a Young diagram Y with atmost r rows and s columns. Again we denote the generator of the affine Dynkin symmetry by $\mathbb{Z}/2$ and there for type C, we have $\lambda^\dagger=\lambda$. We have the following lemma.

Lemma 3.1.

$$\Delta_{\sigma(\lambda)}(\mathfrak{sp}(2r),s) = \Delta_\lambda(\mathfrak{sp}(2r),s) + \frac{1}{4}(rs-2|\lambda|)$$

3.1. **Action on the divisors.** Once again let $\vec{\lambda}$ and $\vec{\lambda}'$ be as before. Here the situation is easier, since σ is of order two. Again we are interested in the support of the difference of the conformal blocks $c_1(\mathbb{V}_{\vec{\lambda}})-c_1(\mathbb{V}_{\vec{\lambda}'})$. So we just need to compute the multiplicity of the divisors D_{A,A^c} where $1\in A$ and $n\notin A$. We denote the multiplicty for such a divisor by m_{A,A^c}. The following proposition gives a formula for that

Proposition 3.2.

$$m_{A,A^c} = \frac{\operatorname{rk}\mathbb{V}_{\vec{\lambda}}}{4(n-1)(n-2)}\Big((n-i)(n-i-1)(rs-2|\lambda_1|) + i(i-1)(rs-2|\lambda_n|)\Big) - \sum_{\lambda\in P_s(\mathfrak{sp}(2r))}\frac{1}{4}(rs-2|\lambda|)\operatorname{rk}\mathbb{V}_{\lambda_A,\lambda}\operatorname{rk}\mathbb{V}_{\lambda_{A^c},\lambda}$$

Let A be a subset of $\{1,\dots,n\}$ such that $n\in A$ and $1\notin A$. Then the difference $c_1(\mathbb{V}_{\vec{\lambda}'}(\mathfrak{sl}(r),k) - c_1(\mathbb{V}_{\vec{\lambda}})$ is also supported on D_{A,A^c}.

Proposition 3.3.

$$\begin{aligned} m_{A,A^c} &= \frac{\operatorname{rk}\mathbb{V}_{\vec{\lambda}}}{4(n-1)(n-2)}\Big((n-i)(n-i-1)(rs-2|\lambda_n|)+i(i-1)(rs-2|\lambda_1|)\Big) \\ &- \sum_{\lambda\in P_s(\mathfrak{sp}(2r))}\frac{1}{4}(rs-2|\lambda|)\operatorname{rk}\mathbb{V}_{\lambda_A,\lambda}\operatorname{rk}\mathbb{V}_{\lambda_{A^c},\lambda^*}\end{aligned}$$

4. Action of diagram automorphisms in type B_r

Let $\omega_1,\dots,\omega_r$ denote the fundaments weights of $\mathfrak{so}(2r+1)$. Let λ be an dominant integral weights of $\mathfrak{so}(2r+1)$ and hence $\lambda=\sum_{i=1}^r a_i\omega_i$. Associate to λ, we attach integers $l_i(\lambda)$ for $1\le i\le r$, where $l_i(\lambda)=\frac{1}{2}a_r+\sum_{j=1}^{r-1}a_j$ for $1\le i\le r-1$ and $l_r(\lambda)=\frac{1}{2}a_r$.The level of λ is given by $l_1(\lambda)+l_2(\lambda)$. We say λ is of level k if $l_1(\lambda)+l_2(\lambda)\le k$.

The corresponding affine weight of $\lambda=\sum_{i=0}^r a_i\omega_i$, where $a_0=k-l_1(\lambda)-l_2(\lambda)$. The affine Dynkin diagram automorphism interchanges a_o with a_1. We denote $|\lambda|=\sum_{i=1}^n l_i(\lambda)$. Further for any dominant weight λ of $\mathfrak{so}(2r+1)$, we get $\lambda^\dagger=\lambda$. The following lemma tell us the difference between conformal weights under the action of the affine Dynkin automorphism σ. We have the following lemma

Lemma 4.1.

$$\Delta_{\sigma\lambda}(\mathfrak{so}(2r+1),k)=\Delta_\lambda(\mathfrak{so}(2r+1),k)+\frac{1}{2}(k-2l_1(\lambda))$$

4.1. **Action on the divisors.** We compute the multiplicity of the divisors D_{A,A^c} where $1\in A$ and $n\notin A$. We denote the multiplicty for such a divisor by m_{A,A^c}. The following proposition gives a formula for that

Proposition 4.2.

$$\begin{aligned} m_{A,A^c} &= \frac{\operatorname{rk}\mathbb{V}_{\vec{\lambda}}}{2(n-1)(n-2)}\Big((n-i)(n-i-1)(k-2l_1(\lambda_1))+i(i-1)(k-2l_1(\lambda_n))\Big) \\ &- \sum_{\lambda\in P_k(\mathfrak{so}(2r+1))}\frac{1}{2}(k-2l_1(\lambda))\operatorname{rk}\mathbb{V}_{\lambda_A,\lambda}\operatorname{rk}\mathbb{V}_{\lambda_{A^c},\lambda}\end{aligned}$$

Let A be a subset of $\{1,\dots,n\}$ such that $n\in A$ and $1\notin A$. Then the difference $c_1(\mathbb{V}_{\vec{\lambda}'}(\mathfrak{sl}(r),k)-c_1(\mathbb{V}_{\vec{\lambda}})$ is also supported on D_{A,A^c}.

Proposition 4.3.

$$\begin{aligned} m_{A,A^c} &= \frac{\operatorname{rk}\mathbb{V}_{\vec{\lambda}}}{2(n-1)(n-2)}\Big((n-i)(n-i-1)(k-2l_1(\lambda_n))+i(i-1)(k-2l_1(\lambda_1))\Big) \\ &- \sum_{\lambda\in P_k(\mathfrak{so}(2r+1))}\frac{1}{2}(k-2l_1(\lambda))\operatorname{rk}\mathbb{V}_{\lambda_A,\lambda}\operatorname{rk}\mathbb{V}_{\lambda_{A^c},\lambda}\end{aligned}$$

5. Action of diagram automorphisms in type D_r

Let $\omega_1,\dots,\omega_r$ denote the fundaments weights of $\mathfrak{so}(2r)$. The weights ω_{r-1} and ω_r are dual to each other. Let λ be an dominant integral weights of $\mathfrak{so}(2r)$ and hence $\lambda=\sum_{i=1}^r a_i\omega_i$. Associate to λ, we attach integers $l_i(\lambda)$ for $1\le i\le r$, where $l_i(\lambda)=\frac{1}{2}(a_r+a_{r-1})+\sum_{j=1}^{r-2}a_j$

for $1 \leq i \leq r-2$, $l_{r-1}(\lambda) = \frac{1}{2}(a_{r-1} + a_r)$ and $l_r(\lambda) = \frac{1}{2}(a_r - a_{r-1})$.The level of λ is given by $l_1(\lambda) + l_2(\lambda)$. We say λ is of level k if $l_1(\lambda) + l_2(\lambda) \leq k$ and we denote by $|\lambda| = \sum_{i=1}^{r} l_i(\lambda)$.

As before the corresponding affine weight associated to λ is $\sum_{i=0}^{r} a_i\omega_i$, where $a_0 = k - l_1(\lambda) - l_2(\lambda)$. There affine Dynkin diagram automorphisms of D_r is $\mathbb{Z}/2 \times \mathbb{Z}/2$ and let σ and ϵ be the generator of the two components. The action of σ is as before in type B_r and ϵ exchanges a_i with a_{r-i}. The following lemma tell us the difference between the conformal weights

Lemma 5.1.

$$\Delta_{\sigma(\lambda)}(\mathfrak{so}(2r), k) = \Delta_{\lambda}(\mathfrak{so}(2r), k) + \frac{1}{2}(k - 2l_1(\lambda))$$
$$\Delta_{\epsilon(\lambda)}(\mathfrak{so}(2r), k) = \Delta_{\lambda}(\mathfrak{so}(2r), k) + \frac{1}{4}(|\epsilon(\lambda)| - |\lambda|)$$

5.1. **Action on the divisors.** We compute the multiplicity of the divisors D_{A,A^c} where $1 \in A$ and $n \notin A$. We denote the multiplicty for such a divisor by m_{A,A^c}. The following proposition gives a formula for that

Proposition 5.2. *Let* $\vec{\lambda} = (\lambda_1, \ldots, \lambda_n)$ *and* $\vec{\lambda}' = (\epsilon\lambda_1, \lambda_2, \ldots, \lambda_{n-1}, \epsilon\lambda_n)$. *Then coefficient* m_{A,A^c} *is gives the multplicity of the divisor* D_{A,A^c} *if* $1 \in A$ *and* $n \notin A$.

$$\begin{aligned} m_{A,A^c} &= \frac{\operatorname{rk} \mathbb{V}_{\vec{\lambda}}}{2(n-1)(n-2)}\Big((n-i)(n-i-1)(k-2l_1(\lambda_1)) + i(i-1)(k-2l_1(\lambda_n))\Big) \\ &\quad - \sum_{\lambda \in P_k(\mathfrak{so}(2r))} \frac{1}{2}(k-2l_1(\lambda)) \operatorname{rk} \mathbb{V}_{\lambda_A,\lambda} \operatorname{rk} \mathbb{V}_{\lambda_{A^c},\lambda^*} \end{aligned}$$

The formula for σ is same as in type B_r.

Let A be a subset of $\{1, \ldots, n\}$ such that $n \in A$ and $1 \notin A$. Then the difference $c_1(\mathbb{V}_{\vec{\lambda}'}(\mathfrak{sl}(r), k) - c_1(\mathbb{V}_{\vec{\lambda}})$ is also supported on D_{A,A^c}.

Proposition 5.3.

$$\begin{aligned} m_{A,A^c} &= \frac{\operatorname{rk} \mathbb{V}_{\vec{\lambda}}}{2(n-1)(n-2)}\Big((n-i)(n-i-1)(k-2l_1(\lambda_1)) + i(i-1)(k-2l_1(\lambda_n))\Big) \\ &\quad - \sum_{\lambda \in P_k(\mathfrak{so}(2r))} \frac{1}{2}(k-2l_1(\lambda)) \operatorname{rk} \mathbb{V}_{\lambda_A,\lambda} \operatorname{rk} \mathbb{V}_{\lambda_{A^c},\lambda^*} \end{aligned}$$