

DIAGRAM AUTOMORPHISMS AND RANK-LEVEL DUALITY

SWARNAVA MUKHOPADHYAY

ABSTRACT. We study the effect of diagram automorphisms on rank-level duality. We create new symplectic rank-level dualities from T. Abe's symplectic rank-level duality on genus zero smooth curves with marked points and chosen coordinates. We also show that rank-level dualities for the pair $\mathfrak{sl}(r), \mathfrak{sl}(s)$ in genus 0 arising from representation theory can also be obtained from the parabolic strange dualities of R. Oudompheng.

1. INTRODUCTION

Let \mathfrak{g} be a finite dimensional simple Lie algebra, ℓ a non-negative integer called the level and $\vec{\lambda} = (\lambda_1, \dots, \lambda_n)$ a n -tuple of dominant weights of \mathfrak{g} at level ℓ (cf Section 2). Consider n distinct points $\vec{z} = (z_1, \dots, z_n)$ on \mathbb{P}^1 with coordinates ξ_1, \dots, ξ_n and let \mathfrak{X} denote the corresponding data. One can associate a finite dimensional complex vector space $\mathcal{V}_{\vec{\lambda}}^{\dagger}(\mathfrak{X}, \mathfrak{g}, \ell)$ to this data. These spaces are known as conformal blocks. The dual space of $\mathcal{V}_{\vec{\lambda}}^{\dagger}(\mathfrak{X}, \mathfrak{g}, \ell)$ is denoted by $\mathcal{V}_{\vec{\lambda}}(\mathfrak{X}, \mathfrak{g}, \ell)$. More generally one can define conformal blocks on curves of arbitrary genus. We refer the reader to the paper of Tsuchiya-Ueno-Yamada (cf [15]) for more details on conformal blocks. Rank-level duality is a duality between conformal blocks associated to two different Lie algebras. Several rank-level duality isomorphisms are known due to the works of the author, T. Abe, Boysal-Pauly and Nakanishi-Tsuchiya (cf [1], [4], [11], [12]).

A diagram automorphism of a Dynkin diagram is a permutation of its nodes which leaves the diagram invariant. For every diagram automorphism, we can construct a finite order automorphism of the Lie algebra. These automorphisms are known as outer automorphisms. Let G be the simply connected group associated to \mathfrak{g} and $Z(G)$ denote the center of G . For each element of $Z(G)$, there exists a diagram automorphism of the Dynkin diagram of the affine Lie algebra $\widehat{\mathfrak{g}}$. In the following, we restrict ourselves to those outer automorphisms of $\widehat{\mathfrak{g}}$ that come from $Z(G)$. For $\omega \in Z(G)$, ω^* denotes the permutation of level ℓ weights of \mathfrak{g} induced from the diagram automorphism of $\widehat{\mathfrak{g}}$ that corresponds to ω . We refer the reader to Section 3 for more details.

Let $\vec{\omega} = (\omega_1, \dots, \omega_n) \in Z(G)^n$ such that $\prod_{s=1}^n \omega_s = \text{id}$. We denote by $\vec{\omega}\vec{\lambda}$ the n -tuple $(\omega_1^* \lambda_1, \dots, \omega_n^* \lambda_n)$ of level ℓ weights, where $\vec{\lambda} = (\lambda_1, \dots, \lambda_n)$ is an n -tuple of level ℓ weights of \mathfrak{g} . In [5], J. Fuchs and C. Schweigert identified the conformal blocks $\mathcal{V}_{\vec{\lambda}}^{\dagger}(\mathfrak{X}, \mathfrak{g}, \ell)$ and $\mathcal{V}_{\vec{\omega}\vec{\lambda}}^{\dagger}(\mathfrak{X}, \mathfrak{g}, \ell)$ via an isomorphism which is flat with respect to the KZ connection.

2010 *Mathematics Subject Classification*. Primary 17B67, Secondary 32G34, 81T40.
The author was partially supported by NSF grant DMS-0901249.

We now briefly recall the above isomorphism of conformal blocks following [5]. Let P^\vee (resp Q^\vee) denote the coweight (resp coroot) lattice of \mathfrak{g} and consider the following additive group:

$$\Gamma_n^{\mathfrak{g}} = \{(\mu_1, \dots, \mu_n) \in (P^\vee)^n \mid \sum_{i=1}^n \mu_i = 0\}.$$

The center $Z(G)$ is naturally identified with P^\vee/Q^\vee via the exponential map. For $1 \leq i \leq n$, we denote ω_i to be the image of μ_i in $Z(G)$. Let \vec{z} denote n distinct points on \mathbb{P}^1 . An explicit automorphism $\sigma_{\vec{\mu}, s}(\vec{z})$ of $\widehat{\mathfrak{g}}$ is constructed in [5], where $\vec{\mu} \in \Gamma_n(\mathfrak{g})$ and s is a positive integer. These automorphisms are referred to as multi-shift automorphisms. We refer the reader to Section 4 for more details on multi-shift automorphisms.

Multi-shift automorphisms for each component combine to give an automorphism $\sigma_{\vec{\mu}}(\vec{z})$ of the Lie algebra $\widehat{\mathfrak{g}}_n = \bigoplus_{i=1}^n \mathfrak{g} \otimes \mathbb{C}((\xi_i)) \oplus \mathbb{C}c$. Further the automorphism $\sigma_{\vec{\mu}}(\vec{z})$ preserves the subalgebra $\mathfrak{g} \otimes \Gamma(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(*\vec{z}))$ of $\widehat{\mathfrak{g}}_n$, where $\Gamma(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(*\vec{z}))$ denotes the algebra of meromorphic functions on \mathbb{P}^1 with poles along the points \vec{z} . To implement the action of $\sigma_{\vec{\mu}}(\vec{z})$ on tensor product of highest weight modules the following intertwining map is constructed:

$$\Theta_{\vec{\mu}}(\vec{z}) : \bigotimes_{i=1}^n \mathcal{H}_{\lambda_i} \rightarrow \bigotimes_{i=1}^n \mathcal{H}_{\omega_i^* \lambda_i},$$

where \mathcal{H}_{λ_i} is an integrable irreducible highest weight $\widehat{\mathfrak{g}}$ -module of highest weight λ_i , ω_i is the image of μ_i in $Z(G)$ and ω_i^* is a permutation of level ℓ weights associated to the diagram automorphism that corresponds to ω_i . The isomorphism of conformal blocks now follows by taking coinvariants with respect to $\mathfrak{g} \otimes \Gamma(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(*\vec{z}))$. Moreover the map $\Theta_{\vec{\mu}}(\vec{z})$ is chosen such that the induced map between the conformal blocks is flat with respect to the KZ connection. The main purpose of this article is to study functoriality of these isomorphisms constructed by J. Fuchs and C. Schweigert under embeddings of Lie algebras and its effect on rank-level duality isomorphisms.

We first want to have a conceptual understanding of multi-shift automorphisms. Since multi-shift automorphisms are generalizations of the single-shift automorphisms introduced by physicists (see [6], [7] and [10]), we focus our attention to single-shift automorphisms. We refer the reader to Section 4 for more details. First we realize single-shift automorphisms as ‘‘conjugations’’ of $\widehat{\mathfrak{g}}$. More precisely we have the following:

Let G be a simply connected simple Lie group with Lie algebra \mathfrak{g} . We assume that G is classical. Let $\mu \in P^\vee$, and consider $\tau_\mu = \exp(\ln \xi \cdot \mu)$ (well defined upon a choice of a branch of the complex logarithm). We consider conjugation by τ_μ which is independent of the branch of the chosen logarithm.

Theorem 1. *The map $x \rightarrow \tau_\mu x \tau_\mu^{-1}$ defines an automorphism φ_μ of the loop algebra $\mathfrak{g} \otimes \mathbb{C}((\xi))$. Further more the automorphism φ_μ coincide with the single-shift automorphism σ_μ restricted to $\mathfrak{g} \otimes \mathbb{C}((\xi))$.*

Since the extension $\widehat{\mathfrak{g}}$ of $\mathfrak{g} \otimes \mathbb{C}((\xi))$ is an universal central extension, an immediate corollary of Theorem 1 is the following:

Corollary 1. *The single-shift automorphism σ_μ is the unique extension of φ_μ to $\widehat{\mathfrak{g}}$.*

It follows directly from the definition multi-shift automorphisms and Theorem 1 that multi-shift automorphism can similarly be realized as “conjugations”. Since conjugations extend under embedding of Lie algebras, we consider the following situation:

Let $\phi : G_1 \times G_2 \rightarrow G$, where G_1, G_2 and G are simple, simply connected complex Lie groups with Lie algebras $\mathfrak{g}_1, \mathfrak{g}_2$ and \mathfrak{g} respectively. Let us also denote the map of the corresponding Lie algebras by ϕ . We assume that ϕ is an embedding of Lie algebras and extend ϕ to a map $\widehat{\phi} : \widehat{\mathfrak{g}}_1 \oplus \widehat{\mathfrak{g}}_2 \rightarrow \widehat{\mathfrak{g}}$. We prove the following theorem:

Theorem 2. *Let $\mathfrak{h}_1, \mathfrak{h}_2$ and \mathfrak{h} be Cartan subalgebras of $\mathfrak{g}_1, \mathfrak{g}_2$ and \mathfrak{g} such that $\phi(\mathfrak{h}_1 \oplus \mathfrak{h}_2) \subset \mathfrak{h}$. Consider $\vec{\mu} = (\mu_1, \dots, \mu_n) \in (P_1^\vee)^n$ such that $\vec{\mu} = (\phi(\mu_1), \dots, \phi(\mu_n)) \in (P^\vee)^n$, where P_1^\vee and P^\vee denote the coweight lattice of $(\mathfrak{g}_1, \mathfrak{h}_1)$ and $(\mathfrak{g}, \mathfrak{h})$ respectively. Then the following diagram commutes:*

$$\begin{array}{ccc} \widehat{\mathfrak{g}}_1 \oplus \widehat{\mathfrak{g}}_2 & \xrightarrow{\widehat{\phi}} & \widehat{\mathfrak{g}} \\ \downarrow \sigma_{\vec{\mu}, s} \oplus id & & \downarrow \sigma_{\vec{\mu}, s} \\ \widehat{\mathfrak{g}}_1 \oplus \widehat{\mathfrak{g}}_2 & \xrightarrow{\widehat{\phi}} & \widehat{\mathfrak{g}}. \end{array}$$

We now restrict ourselves to the situation where the embedding ϕ is conformal (cf Section 2.2) and apply Theorem 2 to rank-level duality isomorphisms. Let $\vec{\lambda} = (\lambda_1, \dots, \lambda_n), \vec{\gamma} = (\gamma_1, \dots, \gamma_n)$ and $\vec{\Lambda} = (\Lambda_1, \dots, \Lambda_n)$ be n -tuples of weights of $\mathfrak{g}_1, \mathfrak{g}_2$ and \mathfrak{g} of level ℓ_1, ℓ_2 and one respectively, where $\ell = (\ell_1, \ell_2)$ be the Dynkin multi-index (see Section 4) of the embedding ϕ . We assume that for $1 \leq i \leq n$, the module $\mathcal{H}_{\lambda_i} \otimes \mathcal{H}_{\gamma_i}$ appears in the branching of \mathcal{H}_{Λ_i} with multiplicity one. Many well known embeddings have this property. We include some examples below:

- (1) $\mathfrak{sl}(r) \oplus \mathfrak{sl}(s) \rightarrow \mathfrak{sl}(rs)$ and triples $(\lambda, \gamma, \Lambda)$ of the form $(\lambda, \lambda^T, \omega_{|\lambda|})$, where λ is considered as a Young diagram, λ^T is the transpose of λ and $|\lambda|$ denote the number of boxes in the Young diagram of λ
- (2) $\mathfrak{so}(p) \oplus \mathfrak{so}(q) \rightarrow \mathfrak{so}(pq)$ and triples $(\lambda, \lambda^T, \omega_1)$, where λ is consider as a Young diagram and above and $|\lambda|$ is odd. We refer the reader [11] for a completer description of such tuples.

Taking coinvariants we get the following map of conformal blocks. We refer the reader to a paper of the author (cf [11]) for more details.

$$\Psi : \mathcal{V}_{\vec{\lambda}}(\mathfrak{X}, \mathfrak{g}_1, \ell_1) \otimes \mathcal{V}_{\vec{\gamma}}(\mathfrak{X}, \mathfrak{g}_2, \ell_2) \rightarrow \mathcal{V}_{\vec{\Lambda}}(\mathfrak{X}, \mathfrak{g}, 1).$$

Let $\vec{\Omega} = (\Omega_1, \dots, \Omega_n)$, where for each $1 \leq i \leq n$, Ω_i is the image of $\phi(\mu_i)$ in $Z(G)$. Combining Theorem 2 with the isomorphism in [5], we have the following important corollary:

Corollary 2. *The following are equivalent:*

- (1) *The map $\mathcal{V}_{\vec{\lambda}}(\mathfrak{X}, \mathfrak{g}_1, \ell_1) \otimes \mathcal{V}_{\vec{\gamma}}(\mathfrak{X}, \mathfrak{g}_2, \ell_2) \rightarrow \mathcal{V}_{\vec{\Lambda}}(\mathfrak{X}, \mathfrak{g}, 1)$ is nondegenerate.*
- (2) *The map $\mathcal{V}_{\vec{\lambda}}(\mathfrak{X}, \mathfrak{g}_1, \ell_1) \otimes \mathcal{V}_{\vec{\omega}\vec{\gamma}}(\mathfrak{X}, \mathfrak{g}_2, \ell_2) \rightarrow \mathcal{V}_{\vec{\Omega}\vec{\Lambda}}(\mathfrak{X}, \mathfrak{g}, 1)$ is nondegenerate,*

where $\vec{\omega}\vec{\gamma} = (\omega_1^*\gamma_1, \dots, \omega_n^*\gamma_n)$ and $\vec{\Omega}\vec{\Lambda} = (\Omega_1^*\Lambda_1, \dots, \Omega_n^*\Lambda_n)$.

1.1. New symplectic rank-level dualities. Let $\mathcal{Y}_{r,s}$ denote the set of Young diagrams with at most r rows and s columns. For a Young diagram $Y = (a_1 \geq \dots \geq a_r) \in \mathcal{Y}_{r,s}$, we denote by Y^T the Young diagram obtained by exchanging the rows and columns. The map $Y \rightarrow Y^T$ defines a bijection of $\mathcal{Y}_{r,s}$ with $\mathcal{Y}_{s,r}$. There is a one to one correspondence between $P_s(\mathfrak{sp}(2r))$ and Young diagrams $\mathcal{Y}_{r,s}$. For $\lambda \in P_s(\mathfrak{sp}(2r))$, the corresponding Young diagram is denoted by $Y(\lambda)$.

Let us fix n distinct smooth points $\vec{p} = (P_1, \dots, P_n)$ on the projective line \mathbb{P}^1 . Let $\vec{z} = (z_1, \dots, z_n)$ be the local coordinates of \vec{p} . We denote the above data by \mathfrak{X} . Consider an n -tuple of level s weights $\vec{\lambda} = (\lambda_1, \dots, \lambda_n)$. Assume that both n and the number of boxes of the Young diagram $\sum_{i=1}^n |Y(\lambda_i)|$ are even. We apply Corollary 2 to T. Abe's symplectic rank-level duality (cf [1]) to get the following:

Corollary 3. *There is a linear isomorphism of the following spaces:*

$$\mathcal{V}_{\vec{\lambda}}(\mathfrak{X}, \mathfrak{sp}(2r), s) \rightarrow \mathcal{V}_{\vec{\lambda}^T}^\dagger(\mathfrak{X}, \mathfrak{sp}(2s), r),$$

where $\lambda^T = (\lambda_1^T, \dots, \lambda_n^T)$ and λ_i^T is the weight corresponding to the Young diagram $Y(\lambda_i)^T$ for all $1 \leq i \leq n$. Moreover these isomorphisms are flat with respect to the KZ connection.

1.2. Rank-level dualities for $\mathfrak{sl}(r)$. Next we consider the map of Lie algebras induced by the tensor product of vector spaces $\phi : \mathfrak{sl}(r) \oplus \mathfrak{sl}(s) \rightarrow \mathfrak{sl}(rs)$. It can be easily checked that this embedding is conformal and the Dynkin-multi index of the embedding is (s, r) . The branching rules for this conformal embeddings was completely described by D. Altschuer, M. Bauer and C. Itzykson (cf [2]). We recall the details of the branching rule in Section 7.

Let $\vec{\lambda} = (\lambda_1, \dots, \lambda_n)$, $\vec{\gamma} = (\gamma_1, \dots, \gamma_n)$ and $\vec{\Lambda} = (\Lambda_1, \dots, \Lambda_n)$ be n -tuples of weights of \mathfrak{g}_1 , \mathfrak{g}_2 and \mathfrak{g} and level s , r and one respectively. We assume that for $1 \leq i \leq n$ the $\mathcal{H}_{\lambda_i} \otimes \mathcal{H}_{\gamma_i}$ appear in the decomposition of \mathcal{H}_{Λ_i} as a $\widehat{\mathfrak{sl}(r)} \oplus \widehat{\mathfrak{sl}(s)}$ module. For this conformal embedding, the triples $(\lambda, \gamma, \Lambda)$ of the branching rule are of the form $(\lambda, \lambda^T, \omega_{|\lambda|})$ and their cyclic twists given by diagram automorphisms, where λ is considered as a Young diagram, λ^T -the transpose of λ and $|\lambda|$ denotes the number of boxes in the Young diagram of λ . This is described precisely in Section 7. We get the following rank-level duality map:

$$(1) \quad \Psi : \mathcal{V}_{\vec{\lambda}}(\mathfrak{X}, \mathfrak{sl}(r), s) \otimes \mathcal{V}_{\vec{\gamma}}(\mathfrak{X}, \mathfrak{sl}(s), r) \rightarrow \mathcal{V}_{\vec{\Lambda}}(\mathfrak{X}, \mathfrak{sl}(rs), 1).$$

We use R. Oudompheng's parabolic strange duality and Corollary 2 to give an alternate proof of the rank-level duality for $\mathfrak{sl}(r)$ proved by T. Nakanishi and A. Tsuchiya. More precisely we prove the following:

Corollary 4. *Assume that $\dim_{\mathbb{C}} \mathcal{V}_{\vec{\Lambda}}(\mathfrak{X}, \mathfrak{sl}(rs), 1) = 1$, then the map Ψ is nondegenerate. In particular one gets a rank-level duality isomorphism:*

$$\mathcal{V}_{\vec{\lambda}}(\mathfrak{X}, \mathfrak{sl}(r), s) \rightarrow \mathcal{V}_{\vec{\gamma}}^\dagger(\mathfrak{X}, \mathfrak{sl}(s), r).$$

We refer the reader to Section 7 for a more comprehensive discussion.

Remark 1. *A natural question is whether the rank-level dualities of Nakanishi-Tsuchiya (cf [12]) that arise from representation theory are same as those obtained from strange dualities of R. Oudompheng (cf [13]) for genus 0 curves using geometry of parabolic bundles. We apply Corollary 2 to (1) and “propagation of vacua”(cf [15]) to reduce it to the case where for all $1 \leq i \leq n$, the triples $(\lambda_i, \gamma_i, \Lambda_i)$ are of the form $(\lambda, \lambda^T, \omega_{|\lambda|})$ and $\sum_{i=1}^n |\lambda|$ is divisible by rs . The rank-level duality map for these cases are known to be non-degenerate by [13]. Hence the rank-level duality map Ψ is also non-degenerate. This also provides an affirmative answer that the rank-level duality arising from representation theory are same as rank-level duality arising in geometry. This was not known before. We refer the reader to Section 7 for more details. Similar questions about geometrizing the rank-level dualities for odd orthogonal groups (cf [11]) needs further investigation.*

An alternative proof of rank-level dualities of [12] without using the result of [13] can be obtained using the same strategy as the proof of the rank-level duality result in [11].

2. AFFINE LIE ALGEBRAS AND CONFORMAL BLOCKS

We recall some basic definitions from Tsuchiya-Ueno-Yamada (cf [15]) in the theory of conformal blocks. Let \mathfrak{g} be a simple Lie algebra over \mathbb{C} and \mathfrak{h} a Cartan subalgebra of \mathfrak{g} . We fix the decomposition of \mathfrak{g} into root spaces

$$\mathfrak{g} = \mathfrak{h} \oplus \sum_{\alpha \in \Delta} \mathfrak{g}_{\alpha},$$

where Δ is the set of roots decomposed into a union of $\Delta_+ \sqcup \Delta_-$ of positive and negative roots. Let $(,)$ denote the Cartan Killing form on \mathfrak{g} normalized such that $(\theta, \theta) = 2$, where θ is the longest root and we identify \mathfrak{h} with \mathfrak{h}^* using the form $(,)$.

2.1. Representation theory of affine Lie algebras. We define the affine Lie algebra $\widehat{\mathfrak{g}}$ to be

$$\widehat{\mathfrak{g}} := \mathfrak{g} \otimes \mathbb{C}((\xi)) \oplus \mathbb{C}c,$$

where c belongs to the center of $\widehat{\mathfrak{g}}$ and the Lie bracket is given as follows:

$$[X \otimes f(\xi), Y \otimes g(\xi)] = [X, Y] \otimes f(\xi)g(\xi) + (X, Y) \operatorname{Res}_{\xi=0}(gdf).c,$$

where $X, Y \in \mathfrak{g}$ and $f(\xi), g(\xi) \in \mathbb{C}((\xi))$.

Let $X(n) = X \otimes \xi^n$ for any $X \in \mathfrak{g}$ and $n \in \mathbb{Z}$. The finite dimensional Lie algebra \mathfrak{g} can be realized as a subalgebra of $\widehat{\mathfrak{g}}$ under the identification of X with $X(0)$.

The finite dimensional irreducible modules of \mathfrak{g} are parametrized by the set of dominant integral weights $P_+ \subset \mathfrak{h}^*$. Let V_{λ} denote the irreducible module of highest weight $\lambda \in P_+$ and v_{λ} denote the highest weight vector.

We fix a positive integer ℓ which we call the level. The set of dominant integral weights of level ℓ is defined as follows:

$$P_{\ell}(\mathfrak{g}) := \{\lambda \in P_+ | (\lambda, \theta) \leq \ell\}.$$

For each $\lambda \in P_{\ell}(\mathfrak{g})$ there is a unique irreducible integrable highest weight $\widehat{\mathfrak{g}}$ -module $\mathcal{H}_{\lambda}(\mathfrak{g})$.

2.2. Conformal Embeddings. Let $\mathfrak{s}, \mathfrak{g}$ be two simple Lie algebras and $\phi : \mathfrak{s} \rightarrow \mathfrak{g}$ an embedding of Lie algebras. Let $(\cdot, \cdot)_{\mathfrak{s}}$ and $(\cdot, \cdot)_{\mathfrak{g}}$ denote the normalized Cartan killing forms such that the length of the longest root is 2. We define the Dynkin index of ϕ to be the unique integer d_{ϕ} satisfying

$$(\phi(x), \phi(y))_{\mathfrak{g}} = d_{\phi}(x, y)_{\mathfrak{s}}$$

for all $x, y \in \mathfrak{s}$. When $\mathfrak{s} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ is semisimple, we define the Dynkin multi-index of $\phi = \phi_1 \oplus \phi_2 : \mathfrak{g}_1 \oplus \mathfrak{g}_2 \rightarrow \mathfrak{g}$ to be $d_{\phi} = (d_{\phi_1}, d_{\phi_2})$.

If \mathfrak{g} is simple, we define for any $\lambda \in P_{\ell}(\mathfrak{g})$ the conformal anomaly $c(\mathfrak{g}, \ell)$ and the trace anomaly $\Delta_{\lambda}(\mathfrak{g}, \ell)$ as

$$c(\mathfrak{g}, \ell) = \frac{\ell \dim \mathfrak{g}}{g^* + \ell} \quad \text{and} \quad \Delta_{\lambda}(\mathfrak{g}, \ell) = \frac{(\lambda, \lambda + 2\rho)}{2(g^* + \ell)},$$

where g^* is the dual Coxeter number of \mathfrak{g} and ρ denotes the half sum of positive roots. If \mathfrak{g} is semisimple, we define the conformal anomaly and trace anomaly by taking sum of the conformal anomalies over all simple components.

Definition 1. Let $\phi = (\phi_1, \phi_2) : \mathfrak{s} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \rightarrow \mathfrak{g}$ be an embedding of Lie algebras with Dynkin multi-index $\ell = (\ell_1, \ell_2)$. We define ϕ to be a conformal embedding \mathfrak{s} in \mathfrak{g} at level k if $c(\mathfrak{g}_1, \ell_1 k) + c(\mathfrak{g}_2, \ell_2 k) = c(\mathfrak{g}, k)$.

Many familiar embeddings of Lie algebras are conformal. A complete classification of conformal embedding can be found in [3].

2.3. Conformal blocks. We fix a n pointed curve C with formal neighborhood η_1, \dots, η_n around the n points $\vec{p} = (P_1, \dots, P_n)$, which satisfies the following properties :

- (1) The curve C has at most nodal singularities.
- (2) The curve C is smooth at the points P_1, \dots, P_n .
- (3) $C - \{P_1, \dots, P_n\}$ is an affine curve.
- (4) A stability condition (equivalent to the finiteness of the automorphisms of the pointed curve).
- (5) Isomorphisms $\eta_i : \widehat{\mathcal{O}}_{C, P_i} \simeq \mathbb{C}[[\xi_i]]$ for $i = 1, \dots, n$.

We denote by $\mathfrak{X} = (C; \vec{p}; \eta_1, \dots, \eta_n)$ the above data associated to the curve C . We define another Lie algebra

$$\widehat{\mathfrak{g}}_n := \bigoplus_{i=1}^n \mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}((\xi_i)) \oplus \mathbb{C}c,$$

where c belongs to the center of $\widehat{\mathfrak{g}}_n$ and the Lie bracket is given as follows:

$$\left[\sum_{i=1}^n X_i \otimes f_i, \sum_{i=1}^n Y_i \otimes g_i \right] := \sum_{i=1}^n [X_i, Y_i] \otimes f_i g_i + \sum_{i=1}^n (X_i, Y_i) \operatorname{Res}_{\xi_i=0} (g_i df_i) c.$$

We define the current algebra to be

$$\mathfrak{g}(\mathfrak{X}) := \mathfrak{g} \otimes \Gamma(C, \mathcal{O}_C(*\vec{p})),$$

where $\Gamma(C, \mathcal{O}_C(*\vec{p}))$ is the algebra of meromorphic functions on C with poles along the points \vec{p} . Consider an n -tuple of weights $\vec{\lambda} = (\lambda_1, \dots, \lambda_n) \in P_\ell^n(\mathfrak{g})$. We set $\mathcal{H}_{\vec{\lambda}} = \mathcal{H}_{\lambda_1}(\mathfrak{g}) \otimes \dots \otimes \mathcal{H}_{\lambda_n}(\mathfrak{g})$. The algebra $\widehat{\mathfrak{g}}_n$ acts on $\mathcal{H}_{\vec{\lambda}}$. For any $X \in \mathfrak{g}$ and $f \in \mathbb{C}((\xi_i))$, the action of $X \otimes f(\xi_i)$ on the i -th component is given by the following:

$$\rho_i(X \otimes f(\xi_i))|v_1 \otimes \dots \otimes v_n\rangle = |v_1 \otimes \dots \otimes (X \otimes f(\xi_i)v_i) \otimes \dots \otimes v_n\rangle,$$

where $|v_i\rangle \in \mathcal{H}_{\lambda_i}(\mathfrak{g})$ for each i .

Definition 2. We define the space of conformal blocks

$$\mathcal{V}_{\vec{\lambda}}^\dagger(\mathfrak{X}, \mathfrak{g}) := \text{Hom}_{\mathbb{C}}(\mathcal{H}_{\vec{\lambda}}/\mathfrak{g}(\mathfrak{X})\mathcal{H}_{\vec{\lambda}}, \mathbb{C}).$$

We define the space of dual conformal blocks, $\mathcal{V}_{\vec{\lambda}}(\mathfrak{X}, \mathfrak{g}) = \mathcal{H}_{\vec{\lambda}}/\mathfrak{g}(\mathfrak{X})\mathcal{H}_{\vec{\lambda}}$. These are both finite dimensional \mathbb{C} -vector spaces which can be defined in families. The dimensions of these vector spaces are given by the Verlinde formula.

3. DIAGRAM AUTOMORPHISMS OF SYMMETRIZABLE KAC-MOODY ALGEBRAS.

Consider a symmetrizable generalized Cartan Matrix $A = (a_{i,j})$ of size n and let $\mathfrak{g}(A)$ denote the associated Kac-Moody algebra. To a symmetrizable generalized Cartan Matrix, one can associate a Dynkin diagram which is a graph on n vertices, see [9] for details.

Definition 3. A diagram automorphism of a Dynkin diagram is a graph automorphism i.e. it is a bijection ω from the set of vertices of the graph to itself such that for $1 \leq i, j \leq n$ the following holds:

$$a_{\omega i, \omega j} = a_{i,j}.$$

3.1. Outer Automorphisms. We construct an automorphism of a symmetrizable Kac-Moody algebra $\mathfrak{g}(A)$ from a diagram automorphism ω of the Dynkin diagram of $\mathfrak{g}(A)$. The Kac-Moody Lie algebra $\mathfrak{g}(A)$ is the Lie algebra over \mathbb{C} generated by a Cartan subalgebra and the symbols $e_i, f_i, 1 \leq i \leq n$ with some relations. We refer the reader to [9] for a detailed definition.

We start by defining the action of ω on the generators e_i and f_i in the following way:

$$\omega(e_i) := e_{\omega i} \quad \text{and} \quad \omega(f_i) := f_{\omega i}.$$

For a simple coroot α_i^\vee , we know that $\alpha_i^\vee = [e_i, f_i]$. Since ω is a Lie algebra automorphism, it implies the following:

$$\omega(\alpha_i^\vee) = \omega[e_i, f_i] = [e_{\omega i}, f_{\omega i}] = \alpha_{\omega i}^\vee,$$

where α_i^\vee are the simple coroots.

In this way we have constructed an automorphism of the derived algebra $\mathfrak{g}(A)' = [\mathfrak{g}(A), \mathfrak{g}(A)]$. The extension of the action of ω to $\mathfrak{g}(A)$, follows from Lemma 1.3.1 in [8]. The order of the automorphism ω of the Lie algebra $\mathfrak{g}(A)$ is same as the corresponding diagram automorphism. We will refer to these automorphisms as outer automorphisms.

3.2. Action ω^* on affine fundamental weights. The map ω restricted to the Cartan subalgebra $\mathfrak{h}(A)$ defines an automorphism of $\omega : \mathfrak{h}(A) \rightarrow \mathfrak{h}(A)$. The adjoint action of ω^* on $\mathfrak{h}(A)^*$ is given by $\omega^*(\lambda)(x) = \lambda(\omega^{-1}x)$ for $\lambda \in \mathfrak{h}(A)^*$ and $x \in \mathfrak{h}(A)$. For $0 \leq i \leq n$, let Λ_i be the i -th affine fundamental weight. Then $\omega^*(\Lambda_i) = \Lambda_{\omega i}$.

4. SINGLE-SHIFT AUTOMORPHISMS OF $\widehat{\mathfrak{g}}$

Consider a finite dimensional complex simple Lie algebra \mathfrak{g} . For each $\alpha \in \Delta$, choose a non-zero element $X_\alpha \in \mathfrak{g}_\alpha$. Then, we have the following:

$$[X_\alpha, X_\beta] = \begin{cases} N_{\alpha,\beta}X_{\alpha+\beta} & \text{if } \alpha + \beta \in \Delta, \\ 0 & \text{if } \alpha + \beta \notin \Delta, \end{cases}$$

where $N_{\alpha,\beta}$ is a non-zero scalar. The coefficients $N_{\alpha,\beta}$ completely determine the multiplication table of \mathfrak{g} . However, they depend on the choice of the elements X_α . We refer the reader to [14] for a proof of the following proposition:

Proposition 1. *One can choose the elements X_α in such a way so that*

$$\begin{aligned} [X_\alpha, X_{-\alpha}] &= H_\alpha \quad \text{for all } \alpha \in \Delta, \\ N_{\alpha,\beta} &= -N_{-\alpha,-\beta} \quad \text{for all } \alpha, \beta, \alpha + \beta \in \Delta, \end{aligned}$$

where H_α is the coroot corresponding to α . The basis $\{X_\alpha, X_{-\alpha}, H_\alpha : \alpha \in \Delta_+\}$ is known as a Chevalley basis.

To every $\alpha_i \in \Delta_+$, the simple coroot $X_i \in \mathfrak{h}$ is defined by the property $X_i(\alpha_j) = \delta_{ij}$ for $\alpha_j \in \Delta_+$. The lattice generated by $\{X_i : 1 \leq i \leq \text{rank}(\mathfrak{g})\}$ is called the coweight lattice and is denoted by P^\vee . We identify \mathfrak{h} with \mathfrak{h}^* using the normalized Cartan killing form and let h_α denote the image of α under the identification.

For every $\mu \in P^\vee$, we define an map σ_μ of the Lie algebra $\widehat{\mathfrak{g}}$

$$\begin{aligned} \sigma_\mu(c) &:= c, \\ \sigma_\mu(h(n)) &:= h(n) + \delta_{n,0}\langle \mu, h \rangle c, \quad \text{where } h \in \mathfrak{h} \text{ and } n \in \mathbb{Z}, \\ \sigma_\mu(X_\alpha(n)) &:= X_\alpha(n + \langle \mu, \alpha \rangle). \end{aligned}$$

Proposition 2. *The map*

$$\sigma_\mu : \widehat{\mathfrak{g}} \rightarrow \widehat{\mathfrak{g}},$$

is a Lie algebra automorphism.

Proof. We only need to verify that σ_μ is a Lie algebra homomorphism. It is enough to check that σ_μ respects the following relations:

$$\begin{aligned} [H_{\alpha_i}(m), H_{\alpha_j}(n)] &= \langle H_{\alpha_i}, H_{\alpha_j} \rangle m \delta_{m+n,0} c, \\ [H_{\alpha_i}(m), X_\alpha(n)] &= \alpha(H_{\alpha_i}) X_\alpha(m+n), \\ [X_\alpha(m), X_\beta(n)] &= N_{\alpha,\beta} X_{\alpha+\beta}(m+n) \quad \text{if } \alpha + \beta \in \Delta, \\ [X_\alpha(m), X_{-\alpha}(n)] &= H_\alpha(m+n) + \langle X_\alpha, X_{-\alpha} \rangle m \delta_{m+n,0} c. \end{aligned}$$

It is trivial to see that σ_μ respects the first three relations. We only need to verify that σ_μ respects the last relation. Let us calculate the following:

$$\begin{aligned} [\sigma_\mu(X_\alpha(m)), \sigma_\mu(X_{-\alpha}(n))] &= [X_\alpha(m + \langle \mu, \alpha \rangle), X_{-\alpha}(n - \langle \mu, \alpha \rangle)], \\ &= H_\alpha(m + n) + \langle X_\alpha, X_{-\alpha} \rangle(m + \langle \mu, \alpha \rangle) \delta_{m+n, 0} c. \end{aligned}$$

If we apply σ_μ to the right hand side of the last relation, we get the following:

$$\begin{aligned} \sigma_\mu(H_\alpha(m + n) + \langle X_\alpha, X_{-\alpha} \rangle m \delta_{m+n, 0} c) &= H_\alpha(m + n) + (\langle \mu, H_\alpha \rangle + \langle X_\alpha, X_{-\alpha} \rangle m) \delta_{m+n, 0} c, \\ &= H_\alpha(m + n) + (\langle \mu, \langle X_\alpha, X_{-\alpha} \rangle h_\alpha \rangle + \langle X_\alpha, X_{-\alpha} \rangle m) \delta_{m+n, 0} c, \\ &= H_\alpha(m + n) + (\langle X_\alpha, X_{-\alpha} \rangle \langle \mu, h_\alpha \rangle + \langle X_\alpha, X_{-\alpha} \rangle m) \delta_{m+n, 0} c, \\ &= H_\alpha(m + n) + (\langle X_\alpha, X_{-\alpha} \rangle \langle \mu, \alpha \rangle + \langle X_\alpha, X_{-\alpha} \rangle m) \delta_{m+n, 0} c, \\ &= [\sigma_\mu(X_\alpha(m)), \sigma_\mu(X_{-\alpha}(n))]. \end{aligned}$$

This completes the proof. □

The automorphism σ_μ was studied in [6], [10] and [7] and is called a single-shift automorphism. It is easy to observe that single-shift automorphisms are additive, i.e. for $\mu_1, \mu_2 \in P^\vee$, we have $\sigma_{\mu_1 + \mu_2} = \sigma_{\mu_1} \circ \sigma_{\mu_2}$.

4.1. Proof of Theorem 1. Let G be a simply connected simple Lie group with Lie algebra \mathfrak{g} . We assume that G is classical. Let $\mu \in P^\vee$, consider $\tau_\mu = \exp(\ln \xi \cdot \mu)$. It is well defined upon a choice of a branch of the complex logarithm but conjugation by τ_μ is independent of the branch of the chosen logarithm. We prove the following:

Proposition 3. *For $\mu \in P^\vee$, the map $x \rightarrow \tau_\mu x \tau_\mu^{-1}$ defines an automorphism φ_μ of the loop algebra $\tilde{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}(\xi)$, where \mathfrak{g} is a Lie algebra of type A_r, B_r, C_r or D_r .*

Proof. We only need to show that is that $\tau_\mu x \tau_\mu^{-1}$ has no fractional powers. Let μ_1 and $\mu_2 \in P^\vee$, we first observe that $\tau_{\mu_1 + \mu_2} = \tau_{\mu_1} \circ \tau_{\mu_2}$. Thus it is enough to verify the claim for the fundamental coweights. For any classical Lie algebra, the roots spaces of \mathfrak{g} are integral linear combinations of the matrix $E_{i,j}$, where $E_{i,j}$ is a matrix with 1 at the (i, j) entry and zero every where else. Therefore it is enough to show that $\tau_\mu E_{ij}(a) \tau_\mu^{-1}$ is of the form $E_{ij}(b)$, where a and b are integers. The rest of the proof is by direct computation.

Consider the case when \mathfrak{g} is of type A_r . A basis for the coweight lattice is given by $X_i = \sum_{k=1}^i E_{k,k} - \frac{i}{r} \sum_{k=1}^r E_{k,k}$. In matrix notation, we have following:

$$\tau_{X_i} = \begin{bmatrix} \xi^{(1-\frac{i}{r})} & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & \xi^{(1-\frac{i}{r})} & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & \xi^{(1-\frac{i}{r})} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \xi^{-\frac{i}{r}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \xi^{-\frac{i}{r}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \xi^{-\frac{i}{r}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \xi^{-\frac{i}{r}} \end{bmatrix}$$

Thus for $a < b$ we get the following:

$$\tau_{X_i} E_{a,b}(n) \tau_{X_i}^{-1} = \begin{cases} E_{a,b}(n) & \text{if both } a \text{ and } b \text{ are less or greater than } i. \\ E_{a,b}(n+1) & \text{if } a \leq i < b. \end{cases}$$

The proof for B_r , C_r and D_r follows from a similar calculation. \square

Remark 2. For $\mu \notin Q^\vee$, then it is clear that φ_μ is not a conjugation by a element of the loop group $G(\mathbb{C}((\xi)))$.

The following lemma compares the single-shift automorphisms with the automorphism φ_μ defined above. The proof follows directly by a easy computation.

Lemma 1. For $\mu \in P^\vee$, the single-shift automorphism σ_μ on the loop algebra $\tilde{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}((\xi))$ is same as φ_μ .

Proof. Since both the automorphism σ_μ and φ_μ are additive, it is enough to prove the above lemma for the simple coweights. It is easy to see that σ_μ and φ_μ are coincide on $\mathfrak{h} \otimes (\mathbb{C}((\xi)))$. We just need to check for the equality on the non zero root spaces. This is an easy type dependent argument and follows from direct calculation. \square

5. MULTI-SHIFT AUTOMORPHISMS

We recall the definition of multi-shift automorphisms following Fuchs-Schweigert (cf [5]). Let us fix a sequence of pairwise distinct complex numbers z_s for $s \in \{1, \dots, n\}$, the coroot H_α corresponding to the roots α . Let P^\vee and Q^\vee denote the coweight and the coroot lattice of \mathfrak{g} respectively and $\Gamma_n^{\mathfrak{g}} = \{(\mu_1, \dots, \mu_n) \in (P^\vee)^n \mid \sum_{i=1}^n \mu_i = 0\}$. Consider a Chevalley basis given by $\{X_{-\alpha}, X_\alpha, H_\alpha : \alpha \in \Delta_+\}$. For $\vec{\mu} \in \Gamma_n^{\mathfrak{g}}$, we define a multi-shift automorphism $\sigma_{\vec{\mu}, t}(\vec{z})$

of $\widehat{\mathfrak{g}}$ as follows:

$$\begin{aligned}\sigma_{\vec{\mu},t}(\vec{z})(c) &:= c, \\ \sigma_{\vec{\mu},t}(\vec{z})(h) \otimes f &:= h \otimes f + \left(\sum_{s=1}^n \langle h, \mu_s \rangle \text{Res}(\varphi_{t,s} f) \right) c, \\ \sigma_{\vec{\mu},t}(\vec{z})(X_\alpha \otimes f) &:= X_\alpha \otimes f \prod_{s=1}^n \varphi_{t,s}^{-\alpha(\mu_s)},\end{aligned}$$

where $f \in \mathbb{C}((\xi))$, $\varphi_{t,s}(\xi) = (\xi + (z_t - z_s))^{-1}$, $h \in \mathfrak{h}$.

Let us now recall some important properties of the multi-shift automorphisms.

- (1) The multi-shift automorphism $\sigma_{\vec{\mu},t}(\vec{z})$ has the same outer automorphism class as the single shift automorphism σ_{μ_t} .
- (2) It is shown in [5] that $\sigma_{\vec{\mu},t}$ is a Lie algebra automorphism of $\widehat{\mathfrak{g}}$ and can be easily extended to an automorphism of $\widehat{\mathfrak{g}}_n$.
- (3) Multi-shift automorphisms of $\widehat{\mathfrak{g}}_n$ preserve the current algebra $\mathfrak{g} \otimes \Gamma(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(*\vec{p}))$, where $\vec{p} = (P_1, \dots, P_n)$ are n distinct points with coordinates z_1, \dots, z_n .

Remark 3. *It is easy to observe that to prove Theorem 2 we need to prove the same identities in the proof of Proposition 5. Hence the proof of Theorem 2 follows from the proof of Proposition 5.*

5.1. Multi-shift automorphisms as conjugations. We restrict to the case \mathfrak{g} is of type A_r, B_r, C_r or D_r . It is easy to see that for $\vec{\mu} \in \Gamma_n^{\mathfrak{g}}$, the multi-shift automorphism $\sigma_{\vec{\mu},s}$ of $\widehat{\mathfrak{g}}$ descends to an automorphism $\sigma_{\vec{\mu},t}$ of the loop algebra $\tilde{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}((\xi))$. Let $\vec{\mu} \in \Gamma_n^{\mathfrak{g}}$ and consider n distinct points P_1, \dots, P_n on $\mathbb{P}^1 - \infty$ and denote their coordinates by $\xi_i(P_i) = z - z_i$, where z is a global coordinate of \mathbb{C} . We now consider $\tau_{\vec{\mu}} = \exp(\ln(\xi_1)\mu_1 + \ln(\xi_2)\mu_2 + \dots + \ln(\xi_n)\mu_n)$ which is only well defined up to a choice of a branch of the logarithms. We can rewrite $\xi_i = z - z_i$ as $\xi_t + z_t - z_i$ and expand $\exp(\ln(\xi_s)\mu_s)$ in terms of ξ_t . We rewrite $\tau_{\vec{\mu}}$ in the coordinate ξ_t and we rename it as $\tau_{\vec{\mu},t}$. We can conjugate by $\tau_{\vec{\mu},t}$ on the loop algebra $\tilde{\mathfrak{g}}$. Let us denote the conjugation by $c(\tau_{\vec{\mu},t})$. It is well defined and is independent of the branch of the logarithm chosen. We have the following proposition, the proof of which follows by a direct calculation:

Proposition 4. *The automorphisms $c(\tau_{\vec{\mu},t})$ and $\sigma_{\vec{\mu},t}$ coincide on the loop algebra $\tilde{\mathfrak{g}}$.*

Remark 4. *If one of the chosen points P_i is infinity, the formula of the multi-shift automorphism needs a minor modification to accommodate the new coordinate at infinity. This has been considered in [5].*

6. EXTENSION OF SINGLE-SHIFT AUTOMORPHISMS

Let $\mathfrak{g}_1, \mathfrak{g}_2$ and \mathfrak{g} be simple Lie algebras and $\mathfrak{h}_1, \mathfrak{h}_2$ and \mathfrak{h} be their Cartan subalgebras. For $i \in \{1, 2\}$, let $\phi_i : \mathfrak{g}_i \rightarrow \mathfrak{g}$ be an embedding of Lie algebras such that $\phi_i(\mathfrak{h}_i) \subset \mathfrak{h}$. Further let us denote the normalized Cartan Killing form on $\mathfrak{g}_1, \mathfrak{g}_2$ and \mathfrak{g} by $\langle \cdot, \cdot \rangle_{\mathfrak{g}_1}, \langle \cdot, \cdot \rangle_{\mathfrak{g}_2}$ and $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$

respectively, and let (ℓ_1, ℓ_2) be the Dynkin multi-index of the embedding $\phi = (\phi_1, \phi_2)$. We can extend the map $\phi = (\phi_1, \phi_2)$ to a map $\widehat{\phi}$ of $\widehat{\mathfrak{g}}_1 \oplus \widehat{\mathfrak{g}}_2 \rightarrow \widehat{\mathfrak{g}}$ as follows:

$$\begin{aligned}\widehat{\phi}_1 : \widehat{\mathfrak{g}}_1 &\rightarrow \widehat{\mathfrak{g}}, \\ X \otimes f + a \cdot c &\rightarrow \phi_1(X) \otimes f + a\ell_1 \cdot c,\end{aligned}$$

where $X \in \mathfrak{g}_1$, $f \in \mathbb{C}((\xi))$ and a is a constant. We similarly map

$$\begin{aligned}\widehat{\phi}_2 : \widehat{\mathfrak{g}}_2 &\rightarrow \widehat{\mathfrak{g}}, \\ Y \otimes g + b \cdot c &\rightarrow \phi_2(Y) \otimes g + b\ell_2 \cdot c,\end{aligned}$$

where $Y \in \mathfrak{g}_2$, $g \in \mathbb{C}((\xi))$ and b is a constant. We define $\widehat{\phi}$ to be $\widehat{\phi}_1 + \widehat{\phi}_2$. Consider an element $\mu \in P_1^\vee$ such that $\tilde{\mu} = \phi_1(\mu) \in P^\vee$, where P_1^\vee and P^\vee denote the coweight lattices of \mathfrak{g}_1 and \mathfrak{g} respectively.

Let α be a root in \mathfrak{g}_1 with respect to a Cartan subalgebra \mathfrak{h}_1 and X_α be a non-zero element of \mathfrak{g}_1 in the root space α . If $\phi_1(X_\alpha) = \sum_{i=1}^{\dim \mathfrak{h}} a_i h_i + \sum_{\gamma \in I_\alpha} a_\gamma X_\gamma$, where h_i 's be any basis of \mathfrak{h} . We have the following lemma:

Lemma 2. *For all i , we claim that $a_i = 0$.*

Proof. For any element $h \in \mathfrak{h}_1$, we consider the following Lie bracket.

$$\begin{aligned}[\phi(h), \phi(X_\alpha)] &= \phi([h, X_\alpha]), \\ &= \phi(\alpha(h)X_\alpha), \\ &= \sum_{i=1}^{\dim \mathfrak{h}} a_i \alpha(h) h_i + \sum_{\gamma \in I_\alpha} a_\gamma \alpha(h) X_\gamma.\end{aligned}$$

On the other hand $[\phi(h), \phi(X_\alpha)] = \sum_{\gamma \in I_\alpha} a_\gamma \gamma(\phi(h)) X_\gamma$. Comparing, we see that $a_i = 0$ for all i and $\gamma(\phi(h)) = \alpha(h)$ for all $h \in \mathfrak{h}_1$. \square

Next, we prove the following lemma:

Lemma 3. *If $\gamma \in I_\alpha$, then for all $h_2 \in \mathfrak{h}_2$*

$$\gamma(\phi(h_2)) = 0.$$

Proof. For $h_2 \in \mathfrak{h}_2$, we have the following:

$$\begin{aligned}\phi([h_2, X_\alpha]) &= [\phi(h_2), \phi(X_\alpha)], \\ &= \sum_{\gamma \in I_\alpha} a_\gamma \gamma(\phi(h_2)) X_\gamma, \\ &= 0.\end{aligned}$$

Thus comparing, we get $\gamma(\phi(h_2)) = 0$ for all $h_2 \in \mathfrak{h}_2$. \square

The following proposition is about the extension of single-shift automorphisms:

Proposition 5. *The single-shift automorphism $\sigma_{\tilde{\mu}}$ restricted to $\widehat{\phi}_1(\widehat{\mathfrak{g}}_1)$ is the automorphism σ_μ . Moreover $\sigma_{\tilde{\mu}}$ restricts to identity on $\widehat{\phi}_2(\widehat{\mathfrak{g}}_2)$.*

Proof. Let n be an integer and h_1, h_2 be elements of \mathfrak{h}_1 and \mathfrak{h}_2 . We need to show the following identities:

$$\begin{aligned}\sigma_{\tilde{\mu}}(\widehat{\phi}_1(h_1(n))) &= \widehat{\phi}_1(\sigma_{\mu}(h_1(n))), \\ \sigma_{\tilde{\mu}}(\widehat{\phi}_1(X_{\alpha}(n))) &= \widehat{\phi}_1(\sigma_{\mu}(X_{\alpha}(n))), \\ \sigma_{\tilde{\mu}}(\widehat{\phi}_2(h_2(n))) &= \widehat{\phi}_2(h_2(n)), \\ \sigma_{\tilde{\mu}}(\widehat{\phi}_2(X_{\beta}(n))) &= \widehat{\phi}_2(X_{\beta}(n)),\end{aligned}$$

where α, β are roots of $\mathfrak{g}_1, \mathfrak{g}_2$ respectively, and X_{α}, X_{β} are non-zero elements in the root space of α, β respectively. Let $h \in \mathfrak{h}_1$, we have the following:

$$\begin{aligned}\sigma_{\tilde{\mu}}(\widehat{\phi}_1(h(n))) &= \widehat{\phi}_1(h(n)) + \delta_{n,0} \langle \tilde{\mu}, \phi_1(h) \rangle_{\mathfrak{g}} c, \\ &= \widehat{\phi}_1(h(n)) + \delta_{n,0} \langle \phi_1(\mu), \phi_1(h) \rangle_{\mathfrak{g}} c, \\ &= \widehat{\phi}_1(h(n)) + \delta_{n,0} \ell_1 \langle \mu, h \rangle_{\mathfrak{g}_1} c, \\ &= \widehat{\phi}_1(\sigma_{\mu}(h(n))).\end{aligned}$$

This completes the proof of the first identity. For the second identity, we use Lemma 2. For any non-zero element X_{α} in the root space of α , consider the following:

$$\begin{aligned}\widehat{\phi}_1(\sigma_{\mu}(X_{\alpha}(n))) &= \widehat{\phi}_1(X_{\alpha}(n + \alpha(\mu))), \\ &= \sum_{\gamma \in I_{\alpha}} a_{\gamma} X_{\gamma}(n + \alpha(\mu)), \\ &= \sum_{\gamma \in I_{\alpha}} a_{\gamma} X_{\gamma}(n + \gamma(\phi(\mu))), \\ &= \sigma_{\tilde{\mu}}(\widehat{\phi}_1(X_{\alpha}(n))).\end{aligned}$$

To prove the fourth identity we use Lemma 3. For a non-zero element X_{β} in the root space β , consider the following:

$$\begin{aligned}\sigma_{\tilde{\mu}}(\widehat{\phi}_2(X_{\beta}(n))) &= \sigma_{\phi_1(\mu)}\left(\sum_{\gamma \in I_{\beta}} X_{\gamma}(n)\right), \\ &= \sum_{\gamma \in I_{\beta}} \sigma_{\phi_1(\mu)}(X_{\gamma}(n)), \\ &= \sum_{\gamma \in I_{\beta}} X_{\gamma}(n + \gamma(\phi(\mu))), \\ &= \sum_{\gamma \in I_{\beta}} X_{\gamma}(n), \\ &= \widehat{\phi}_2(X_{\beta}(n)).\end{aligned}$$

We are only left to show that $\sigma_{\tilde{\mu}}(\widehat{\phi}_2(h_2(n))) = \widehat{\phi}_2(h_2(n))$ for $h_2 \in \mathfrak{h}_2$, which follows from the following lemma. \square

Lemma 4. $\langle \phi_1(h_1), \phi_2(h_2) \rangle_{\mathfrak{g}} = 0$ for any element h_1, h_2 of \mathfrak{h}_1 and \mathfrak{h}_2 respectively.

Proof. It is enough to prove the result for all H_β , where β is a root of \mathfrak{g}_2 . Let $\phi(X_\beta) = \sum_{\lambda \in I_\beta} a_\lambda X_\lambda$ and $\phi(X_{-\beta}) = \sum_{\gamma \in I_{-\beta}} a_\gamma X_\gamma$. Since $\phi_2(\mathfrak{h}_2) \subset \mathfrak{h}$, we get

$$\phi(H_\beta) = \sum_{\gamma \in I_\beta \cap (-I_{-\beta})} a_\gamma b_{-\gamma} H_\gamma.$$

Now $[\phi(h_1), \phi(X_\beta)] = \sum_{\gamma \in I_\beta} a_\gamma \gamma(\phi(h_1)) X_\gamma = 0$. Thus for $\gamma \in I_\beta$, we get $\gamma(\phi(h_1)) = \langle h_\gamma, \phi(h_1) \rangle_{\mathfrak{g}} = 0$ which implies $\langle \phi_1(h_1), H_\gamma \rangle_{\mathfrak{g}} = 0$. Thus, we have the following:

$$\begin{aligned} \langle \phi_1(h_1), \phi_2(H_\beta) \rangle_{\mathfrak{g}} &= \sum_{\gamma \in I_\beta \cap (-I_{-\beta})} a_\gamma b_{-\gamma} \langle \phi_1(h_1), H_\gamma \rangle_{\mathfrak{g}}, \\ &= 0. \end{aligned}$$

□

7. RANK-LEVEL DUALITY FOR $\mathfrak{sl}(r)$

We first describe the branching rules of the embedding $\mathfrak{sl}(r) \oplus \mathfrak{sl}(s) \subset \mathfrak{sl}(rs)$ following Altschuer-Bauer-Itzykson (cf [2]). Let $P_+(\mathfrak{sl}(r))$ denote the set of dominant integral weights of $\mathfrak{sl}(r)$ and $\Lambda_1, \dots, \Lambda_{r-1}$ denote the fundamental weights of $\mathfrak{sl}(r)$. If $\lambda = \sum_{i=1}^{r-1} \tilde{k}_i \Lambda_i$ for non-negative integers \tilde{k}_i , we rewrite λ as $\lambda = \sum_{i=0}^{r-1} \tilde{k}_i \Lambda_i$, where $\sum_{i=0}^{r-1} \tilde{k}_i = s$, and Λ_0 is the affine 0-th fundamental weight. The level one weights of $\mathfrak{sl}(rs)$ are given by $P_1(\mathfrak{sl}(rs)) = \{\Lambda_0, \dots, \Lambda_{rs-1}\}$. We identify it with the set $\{0, 1, \dots, rs - 1\}$.

7.1. Action of center and branching rule. Let $\hat{\rho} = r\Lambda_0 + \frac{1}{2} \sum_{\alpha \in \Delta_+} \alpha$ and Δ_+ is the set of positive roots with respect to a chosen Cartan subalgebra of $\mathfrak{sl}(r)$. Consider $\lambda + \hat{\rho} = \sum_{i=0}^{r-1} k_i \Lambda_i$, where $k_i = \tilde{k}_i + 1$ and $\sum_{i=0}^{r-1} k_i = r + s$. The center of $\text{SL}(r)$ is $\mathbb{Z}/r\mathbb{Z}$. The action of $\mathbb{Z}/r\mathbb{Z}$ induced from outer automorphisms on $P_s(\mathfrak{sl}(r))$ is described as follows:

$$\begin{aligned} \mathbb{Z}/r\mathbb{Z} \times P_s(\mathfrak{sl}(r)) &\longrightarrow P_s(\mathfrak{sl}(r)), \\ (\sigma, \Lambda_i) &\longrightarrow \Lambda_{((i+\sigma) \bmod r)}. \end{aligned}$$

Let $\Omega_{r,s} = P_s(\mathfrak{sl}(r))/(\mathbb{Z}/r\mathbb{Z})$ be the set of orbits under this action and similarly let $\Omega_{s,r}$ be the orbits of $P_r(\mathfrak{sl}(s))$ under the action of $\mathbb{Z}/s\mathbb{Z}$. The following map β parametrizes the bijection.

$$\beta : P_s(\mathfrak{sl}(r)) \rightarrow P_r(\mathfrak{sl}(s))$$

Set $a_j = \sum_{i=j}^r k_i$, for $1 \leq j \leq r$ and $k_r = k_0$. The sequence $\vec{a} = (a_1, \dots, a_r)$ is decreasing. Let (q_1, \dots, q_s) be the complement of \vec{a} in the set $\{1, 2, \dots, (r+s)\}$ in decreasing order. We define the following sequence:

$$b_j = r + s + q_s - q_{s-j+1} \text{ for } 1 \leq j \leq s.$$

The sequence b_j defined above is also decreasing. The map β is given by the following formula:

$$\beta(a_1, \dots, a_r) = (b_1, \dots, b_s).$$

Thus when λ runs over an orbit of $\Omega_{r,s}$, $\gamma = \sigma\beta(\lambda)$ runs over an orbit of $\Omega_{s,r}$ if σ runs over $\mathbb{Z}/s\mathbb{Z}$.

The elements λ of $P_s(\mathfrak{sl}(r))$ can be parametrized by Young diagrams $Y(\lambda)$ with at most $r-1$ rows and at most s columns. Let $Y(\lambda)^T$ be the modified transpose of $Y(\lambda)$. If $Y(\lambda)$ has rows of length s , then $Y(\lambda)^T$ is obtained by taking the usual transpose of $Y(\lambda)$ and deleting the columns of length s . We denote by λ^T the dominant integral weight of $\mathfrak{sl}(s)$ of level r that corresponds to $Y(\lambda)^T$. With this notation we recall the following proposition from [2]:

Proposition 6. *Let $\lambda \in P_s(\mathfrak{sl}(r))$ and $c(\lambda)$ be the number of columns of $Y(\lambda)$. Suppose $\sigma = c(\lambda) \bmod s$. Then $\sigma\beta(\lambda) = \lambda^T$.*

The following proposition describes the multiplicity $m_{\lambda,\gamma}^\Lambda$ of the component $\mathcal{H}_\lambda \otimes \mathcal{H}_\gamma$ in \mathcal{H}_Λ . We refer the reader to [2] for a proof.

Proposition 7. *For $\lambda \in P_s(\mathfrak{sl}(r))$ and $\sigma \in \mathbb{Z}/s\mathbb{Z}$, let $\delta(\lambda, \sigma) = |Y(\lambda)| + rs + r(\sigma - c(Y(\lambda)))$. The multiplicity $m_{\lambda,\gamma}^\Lambda$ is given by the following formula:*

$$\begin{aligned} m_{\lambda,\gamma}^\Lambda &= 1 \text{ if } \gamma = \sigma\beta(\lambda), \quad \sigma \in \mathbb{Z}/s\mathbb{Z} \text{ and } \Lambda = \delta(\lambda, \sigma) \bmod(rs), \\ m_{\lambda,\gamma}^\Lambda &= 0 \text{ otherwise.} \end{aligned}$$

Moreover for fixed $\Lambda \in P_1(\mathfrak{sl}(rs))$ and $\lambda \in P_s(\mathfrak{sl}(r))$, there exists at most one $\gamma \in P_r(\mathfrak{sl}(s))$ such that $m_{\lambda,\gamma}^\Lambda$ is non zero.

7.2. Proof of rank-level duality of Nakanishi-Tsuchiya. Let us fix n distinct points $\vec{p} = (P_1, \dots, P_n)$ on the projective line \mathbb{P}^1 . Let $\vec{z} = (z_1, \dots, z_n)$ be the local coordinates of \vec{p} . We denote the above data by \mathfrak{X} . Consider an n -tuple $\vec{\Lambda} = (\Lambda_{i_1}, \dots, \Lambda_{i_n})$ of level one dominant integral weights of $\mathfrak{sl}(rs)$. Let $\vec{\lambda} = (\lambda_1, \dots, \lambda_n)$ and $\vec{\gamma} = (\gamma_1, \dots, \gamma_n)$ be such that for $1 \leq k \leq n$, the multiplicity $m_{\lambda_k, \gamma_k}^{\Lambda_k} = 1$. We get a map

$$\Psi : \mathcal{V}_{\vec{\lambda}}(\mathfrak{X}, \mathfrak{sl}(r), s) \otimes \mathcal{V}_{\vec{\gamma}}(\mathfrak{X}, \mathfrak{sl}(s), r) \rightarrow \mathcal{V}_{\vec{\Lambda}}(\mathfrak{X}, \mathfrak{sl}(rs), 1),$$

If rs divides $\sum_{k=1}^n i_k$, it is well known that $\dim_{\mathbb{C}} \mathcal{V}_{\vec{\Lambda}}(\mathfrak{X}, \mathfrak{sl}(rs), 1) = 1$. We get the following morphism well defined up to scalars:

$$\text{RL} : \mathcal{V}_{\vec{\lambda}}(\mathfrak{X}, \mathfrak{sl}(r), s) \rightarrow \mathcal{V}_{\vec{\gamma}}^\dagger(\mathfrak{X}, \mathfrak{sl}(s), r).$$

The rest of the section is devoted to the proof that RL is an isomorphism. It follows from Proposition 7 that for all $1 \leq i \leq n$, the weight γ_i is of the form $\sigma_i\beta(\lambda_i)$ for some $\sigma_i \in \mathbb{Z}/s\mathbb{Z}$. Without loss of generality we can assume that $\sigma_i - c(Y(\lambda_i))$ is non-negative for all i . Let Q_1 be a new point distinct from P_1, \dots, P_n on \mathbb{P}^1 and ξ be the new coordinate around Q_1 . Let $\tilde{\mathfrak{X}}$ be the data associated to the points P_1, \dots, P_n, Q_1 on \mathbb{P}^1 . We have the following proposition:

Lemma 5. *The following are equivalent:*

- (1) *The rank-level duality map*

$$\mathcal{V}_{\vec{\lambda}}(\mathfrak{X}, \mathfrak{sl}(r), s) \rightarrow \mathcal{V}_{\vec{\gamma}}^\dagger(\mathfrak{X}, \mathfrak{sl}(s), r),$$

is an isomorphism.

(2) *The rank-level duality map*

$$\mathcal{V}_{\tilde{\lambda},0}(\tilde{\mathfrak{X}}, \mathfrak{sl}(r), s) \rightarrow \mathcal{V}_{\tilde{\lambda}^T, \eta}^\dagger(\tilde{\mathfrak{X}}, \mathfrak{sl}(s), r),$$

is an isomorphism, where $\eta = r\omega_\sigma$, $\sigma = \sum_{i=1}^n (\sigma_i - c(Y(\lambda_i))) \bmod(s)$ and ω_σ is the σ -th fundamental weight.

Proof. The equivalence of (1) and (2) follows from “propagation of vacua” (see [15]) and Corollary 2. □

The proof that the rank-level duality map for $\mathfrak{sl}(r)$ is an isomorphism follows directly from Lemma 7.2 and Theorem 4.10 in [13].

Acknowledgements. I am grateful to P. Belkale for his suggestions, comments, clarifications and constant encouragement. I thank C. Pauly and S. Kumar for useful discussions. This paper is a part of my dissertation at University of North Carolina at Chapel Hill.

REFERENCES

- [1] T. Abe, *Strange duality for parabolic symplectic bundles on a pointed projective line*, Int. Math. Res. Not. (2008), Art. ID rnn121, 47 pp.
- [2] D. Altschuer, M. Bauer, C. Itzykson, *The branching rules of conformal embeddings*, Comm. Math. Phys. 132 (1990), no. 2, 349-364.
- [3] A. Bais, P. Bouwknegt, *A classification of subgroup truncations of the bosonic string*, Nuclear Phys. B 279 (1987), no. 3-4, 561-70.
- [4] A. Boysal, C. Paul, *Strange duality for Verlinde spaces of exceptional groups at level one*, Int. Math. Res. Not. (2009); doi:10.1093/imrn/rnp151.
- [5] J. Fuchs, C. Schweigert, *The Action of Outer Automorphisms on Bundles of Chiral Blocks*, Comm. Math. Phys. 206 (1999), no. 3, 691-736.
- [6] N. Gorman, W. McGlinn and O’Raifeartaigh, *Cartan preserving automorphisms of untwisted and twisted Kac-Moody algebras*, J. Math. Phys. no. 30, (1989) 1921-1932.
- [7] J. K. Freericks and M. B. Halpern, *Conformal deformation by the currents of affine g* , Ann. Phys. 188 (1988), no. 2, 258-306 [ibid. 190 (1989) 212, Erratum].
- [8] K. Hasegawa, *Spin module versions of Weyl’s reciprocity theorem for classical Kac-Moody Lie algebras – An application to branching rule duality*, Publ. Res. Inst. Math. Sci. 25 (1989), no. 5, 741-828.
- [9] V. Kac, *Infinite dimensional Lie algebras*, Third edition, Cambridge University Press, Cambridge(UK), (1990). xxii+400 pp.
- [10] W. Lerche, C. Vafa, and N. P. Warner, *Chiral rings in $N = 2$ superconformal theories*, Nuclear Phys. B 324 (1989), no. 2, 427-474.
- [11] S. Mukhopadhyay, *Rank-level duality of Conformal Blocks for odd orthogonal Lie algebras in genus 0*, arXiv:1211.2204.
- [12] T. Nakanishi, A. Tsuchiya, *Level-rank duality of WZW models in conformal field theory*, Comm. Math. Phys. 144 (1992), no. 2, 351-372.
- [13] R. Oudompheng, *Rank-level duality for conformal blocks of the linear group*, J. Algebraic Geom. 20 (2011), no. 3, 559-597.
- [14] J-P. Serre, *Complex semisimple Lie algebras*, Springer Monographs in Mathematics, Springer-Verlag, Berlin, (2001), x+74 pp.

- [15] A. Tsuchiya, K. Ueno, Y. Yamada, *Conformal field theory on universal family of stable curves with gauge symmetries*, Integrable systems in quantum field theory and statistical mechanics, 459-566, Adv. Stud. Pure Math. 19, Academic Press, Boston, MA, (1989).
- [16] D. Versteegen, *Conformal embeddings, rank-level duality and exceptional modular invariants*, Comm. Math. Phys. 137 (1991), no. 3, 567-586.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MARYLAND, DEPARTMENT OF MATHEMATICS, COLLEGE PARK, MD 20742-4015

E-mail address: `swarnava@umd.edu`