Graph potentials and the moduli space of vector bundles of rank two on a curve.

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(joint work with Pieter Belmans and Sergey Galkin)

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Moduli space

Consider the smooth intersection of two quadrics Q_1 and Q_2

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Reinterpretation as moduli space

- C will denote a smooth projective curve of genus $g \geq 2$.
- $\mathbb{L}$ be a fixed line bundle of odd degree on C.
- $M_C(\mathbb{L})$ will denote the moduli space of stable rank two bundles with determinant $\mathbb{L}$.

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$$M_C = \mathrm{OGr}_{q_1}(g-1, 2g+2) \cap \mathrm{OGr}_{q_2}(g-1, 2g+2).$$

- M_C is smooth, Fano of dimension $3(g-1)$. Moreover (Drezet-Narasimhan)

$$\mathrm{Pic}(M_C) = \mathbb{Z}\Theta.$$

The canonical class $K_{M_C} = -2[\Theta]$, i.e. M_C is of index two.

Properties... continued

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- As C varies in $\overline{\mathcal{M}}_g$, the spaces $H^0(M_C, \Theta^{\otimes \ell})$ form a vector bundle (Tsuchiya-Ueno-Yamada).

Pants

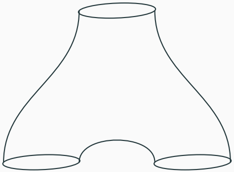


Figure 1: Pair of Pants

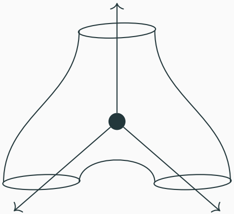
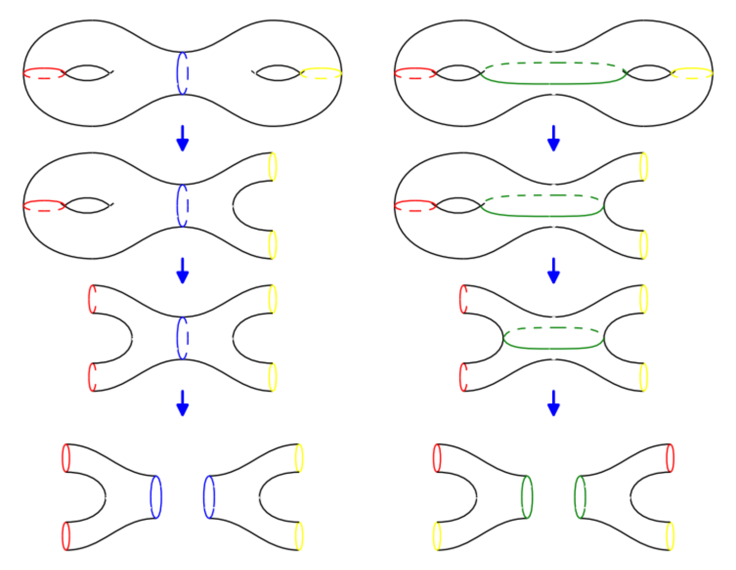


Figure 2: Trinion Graph

Pants decomposition



Decomposition of Hodge diamond

$$\begin{matrix} & 1 & \\ 3 & & 3 \\ & 1 & \end{matrix}$$

Hodge diamond of C

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Hodge diamond of $Sym^2 C$

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Theorem: Muñoz

The quantum multiplication $\star_0$ by $c_1(M_C)$ on quantum cohomology ring $QH^*(M_C)$ has the following eigen-space decomposition:

$$QH^*(M_C) = \bigoplus_{m=1-g}^{g-1} H_m,$$

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Remark: This decomposition is equivariant with respect to the natural $\mathrm{Sp}(2g)$ action on both sides.

Conjectural semi-orthogonal decomposition

Theorem: Bondal-Orlov

Let C be a smooth genus two curve, then

$$\mathbf{D}^b(M_C) = \langle \mathbf{D}^b(pt), \mathbf{D}^b(C), \mathbf{D}^b(pt) \rangle$$

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Conjecture I: Belmans-Galkin-M; Narasimhan

Let C be a smooth curve of genus g

$$\begin{aligned}\mathbf{D}^b(M_C) = &\langle \mathbf{D}^b(pt), \mathbf{D}^b(C), \cdots, \mathbf{D}^b(\mathrm{Sym}^{g-2} C), \\ &\mathbf{D}^b(\mathrm{Sym}^{g-1} C), \\ &\mathbf{D}^b(\mathrm{Sym}^{g-2} C), \cdots, \mathbf{D}^b(C), \mathbf{D}^b(pt) \rangle.\end{aligned}$$

Theorem: Belmans-Galkin-M (K-S. Lee for Chow motives)

The following identity holds in $K_0(Var)$.

$$[M_C] = \mathbb{L}^{g-1}[\mathrm{Sym}^{g-1} C] + \sum_{i=0}^{g-2}(\mathbb{L}^i + \mathbb{L}^{3g-3-2i})[\mathrm{Sym}^{\mathrm{i}} C] + T,$$

where $\mathbb{L} = [\mathbb{A}^1]$ and $(1+\mathbb{L})T = 0$.

Class in $\mathrm{K}_0(\mathrm{Var})$

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where $\mathbb{L} = [\mathbb{A}^1]$ and $(1+\mathbb{L})T = 0$.

Corollary

The following identity holds in $K_0(dgCat)$.

$$[\mathbf{D}^b(M_C)] = [\mathbf{D}^b(\mathrm{Sym}^{g-1} C)] + \sum_{i=0}^{g-2} 2[\mathbf{D}^b(\mathrm{Sym}^{\mathrm{i}} C)] + T',$$

where $2T' = 0$.

More motivation/evidence

Theorem: Narasimhan-2015; Fonarev-Kuznetsov-2017

$$\mathbf{D}^b(M_C) = \langle \mathbf{D}^b(pt), \mathbf{D}^b(pt), \mathbf{D}^b(C), \mathcal{A} \rangle$$

More motivation/evidence

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Theorem: Belmans-Mukhopadhyay-2019

$$\mathbf{D}^b(M_C(r)) = \langle \mathbf{D}^b(pt), \mathbf{D}^b(pt), \mathbf{D}^b(C), \mathbf{D}^b(C), \mathcal{B} \rangle,$$

where $M_C(r)$ is the moduli space of rank r bundles with fixed determinant of degree one.

Mirror Symmetry for Fano X and LG-models (Y, w)

B-side

- The bounded derived category $\mathbf{D}^b(X)$ and semi orthogonal decompositions.

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Decompositions: $\text{Eigen Values}\,(c_1(M)\star_0) = \text{Critical Values}\,(w)$.

Question on quantum periods for $X = M_C$

Let $X_{0,k,m}$ denote the Kontsevich moduli space of stable maps f from a rational curve with k marked points and $\deg f^*(-K_X) = m$.

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Definition

The $m \geq 2$-th descendent Gromov Witten number

$$p_m = \int_{X_{0,1,m}} \psi^{m-2} \operatorname{ev}_1^{-1}([pt]),$$

where ψ is the *Psi* class on $X_{0,1,m}$ and $\operatorname{ev}_1 : X_{0,1,m} \to X$.

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Compute

$$\sum_{m \geq 0} p_m t^m \quad \text{for } X = M_C; p_0 = 1, \; p_1 = 0.$$

Finding mirror potentials W

- (Hori-Vafa, Givental) If X is a smooth toric Fano or Fano complete intersection in a toric variety, then $W : (\mathbb{C}^\times)^{\dim X} \to \mathbb{C}$, such that

$$\frac{1}{m!}\text{Coefficient of Constant Term}(W^m) = p_m.$$

- (Coates-Corti-Galkin-Kasprzyk) If X is a smooth Fano three fold, then quantum periods are known.

- Many other results due to works of Batryrev-Ciocan-Fontanine-Kim-van-Straten, Bondal-Galkin, Coates, Przyalkowski,...

Trinion Potential

$$W_+ = xyz + \frac{x}{yz} + \frac{y}{xz} + \frac{z}{xy}$$

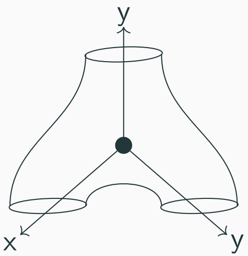


Figure 3: Trinion Graph

Colored Trinion Potential

$$W_- = \frac{1}{xyz} + \frac{xy}{z} + \frac{yz}{x} + \frac{zx}{y}$$

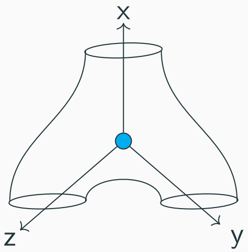


Figure 4: Trinion Graph

Remark

Think of the coloring scheme as taking a variable and replace it by its inverse. Changing it even number of times doesn't change color.

Graph potentials

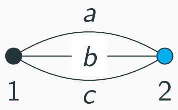
Let (Γ, c) be a colored trivalent graph (loops are allowed) and

$$c : V(\Gamma) \to \{\pm 1\}$$

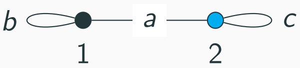
Definition of graph potential

$$W_{\Gamma,c} := \sum_{v \in V(\Gamma)} W_{v,c(v)}$$

Example $g = 2$



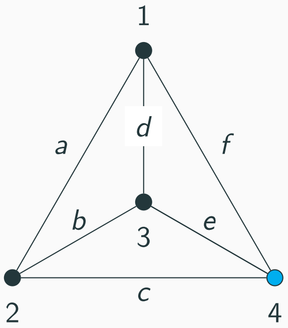
$$\left(abc + \frac{a}{bc} + \frac{b}{ac} + \frac{c}{ab}\right) + \left(\frac{1}{abc} + \frac{bc}{a} + \frac{ac}{b} + \frac{ab}{c}\right)$$



$$b^2 a + \frac{2}{a} + \frac{a}{b^2} + \left(\frac{1}{ac^2} + 2a + \frac{c^2}{a}\right)$$

Example $g = 3$

$$adf + \frac{f}{ad} + \frac{a}{df} + \frac{d}{af} +$$
$$bde + \frac{e}{bd} + \frac{b}{ed} + \frac{b}{de}$$
$$+abc + \frac{b}{ac} + \frac{c}{ab} + \frac{a}{bc}$$
$$+(\frac{1}{cef} + \frac{ef}{c} + \frac{cf}{e} + \frac{ce}{f})$$



Periods

Definition

Let $W : (\mathbb{C}^\times)^n \to \mathbb{C}$ be a Laurent polynomial. A classical period of W is the following Laurent series.

$$\pi_W(t) = \left(\frac{1}{2\pi\sqrt{-1}}\right)^n \int_{|x_1|=\cdots=|x_n|=1} \frac{1}{1 - tW(x_1, \ldots, x_n)} \operatorname{dlog} \vec{x}$$

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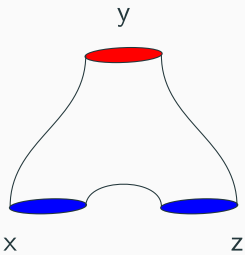
The Laplace transform

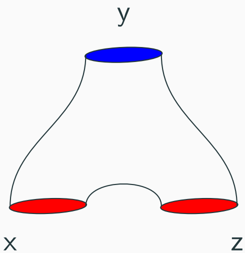
$$\widehat{\pi}_W(t) = \left(\frac{1}{2\pi\sqrt{-1}}\right)^n \int_{|x_1|=\cdots=|x_n|=1} e^{tW(x_1,\ldots,x_n)} \,\mathrm{dlog}\, \vec{x}$$

TFT via graph potentials

$$W_+ = xyz + \frac{x}{yz} + \frac{y}{xz} + \frac{z}{xy}$$
$$Z(Y_+) := e^{tW_+} \in L^2(S^1)^{\otimes 3}$$



$$W_- = \frac{1}{xyz} + \frac{xy}{z} + \frac{yz}{x} + \frac{xz}{y}$$
$$Z(Y_-) := e^{tW_-} \in L^2(S^1)^{\otimes 3}$$



Remark

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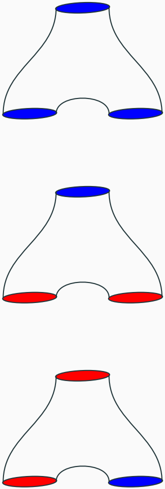

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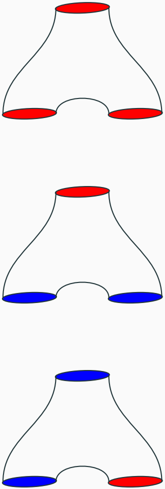


WDVV equations

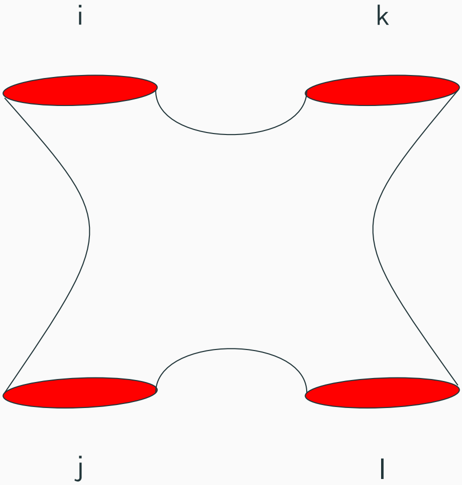
Theorem: Belmans-Galkin-M

Let $\mathcal{K}(i,j,k) = e^{tW_+(x_i,x_j,x_k)}$. Define

$$\mathcal{K}(i,j,k,l) := \langle \mathcal{K}(i,j,e) \otimes \mathcal{K}(k,l,e) \rangle \in L^2(S^1)^{\otimes 4}$$

The above assignment is symmetric in i,j,k,l i.e. it satisfies the WDVV equations.

WDVV and Hatcher-Thurston moves



$$(\mathbf{i},\mathbf{j});(\mathbf{k},\mathbf{l}) = (\mathbf{i},\mathbf{k});(\mathbf{j},\mathbf{l}) = (\mathbf{i},\mathbf{l});(\mathbf{j},\mathbf{k})$$

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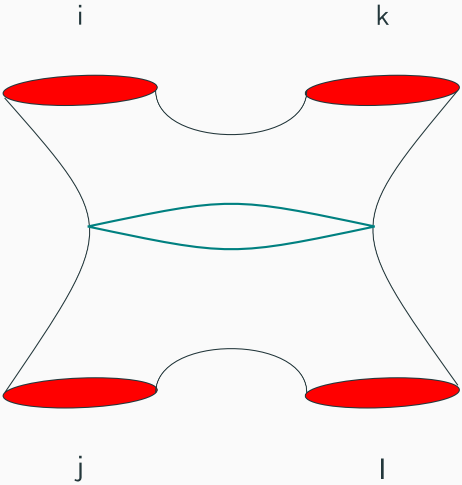

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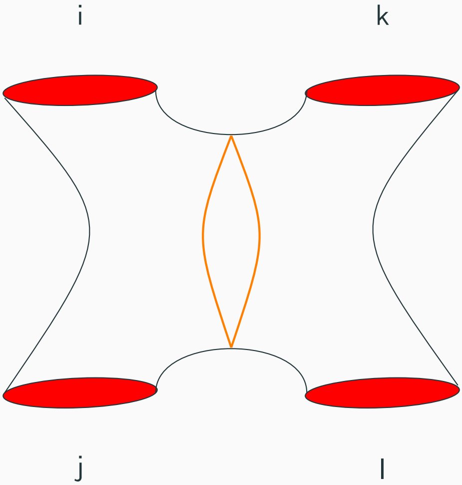

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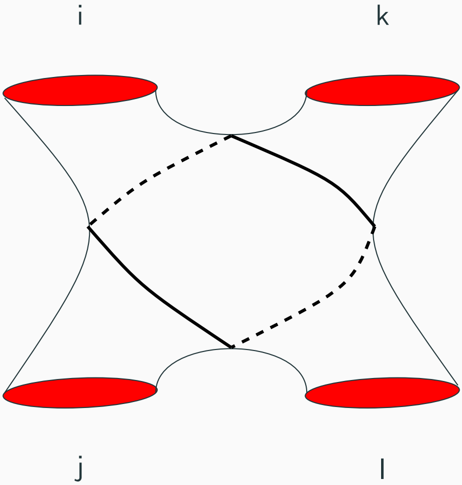

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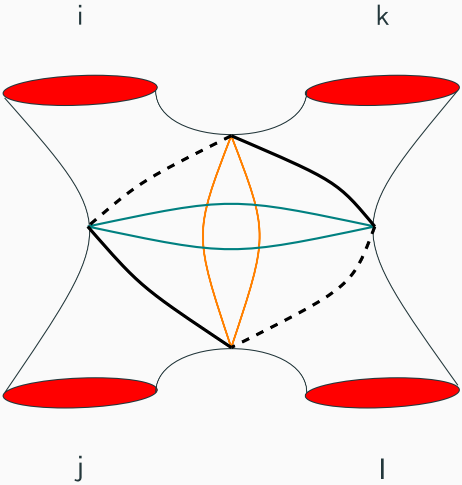

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Let $\Sigma_{g,n}$ be an oriented surface of genus g with n boundary components with the condition that $2g + n > 2$. To every pairs of pants decomposition of $\Sigma_{g,b}$, with dual graph (Γ, n) assign:

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General partition function

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Corollary

The above is well-defined, i.e. it does depends on the graph or equivalent on the pair of pants decomoposition.

Periods via TFT

$$B(x) := \sum_{m=0}^{\infty} \frac{x^{2m}}{(m!)^2}, \text{ Bessel function,}$$

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Periods via TFT

$$
\begin{aligned}
B(x) &:= \sum_{m=0}^{\infty} \frac{x^{2m}}{(m!)^2}, \text{ Bessel function,} \\
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Periods via TFT

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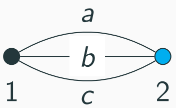
Theorem: Belmans-Galkin-M

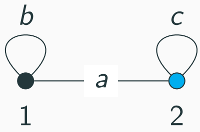
For any trivalent colored graph Γ, c with no half edges and genus $g \geq 2$, the Laplace transform

$$\widehat{\pi}_{W_{\Gamma,c}}(t) = [T_{g-1}(x, x^{(-1)^\epsilon})]_{x^0},$$

where ϵ is the parity of the number of colored vertices.

Example g=2





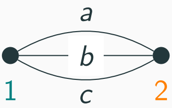
$$\widehat{\pi}_{W_{\Gamma,c}}(t) = \sum_{n\geq 0} \frac{(2n!)^2}{n!^2} t^{2n}.$$

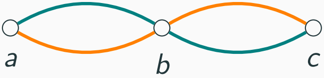
Remark

The series $\widehat{\pi}_{W_{\Gamma,c}}(t)$ computes the quantum period for $X_{2,2} \subset \mathbb{P}^5$.

Let $\widetilde{C}$ be a maximally degenerate nodal curve of g:

- The dual graph (Γ, V, E) of $\widetilde{C}$ (V corresponds to components and E correspond to intersection of components) is trivalent.





Toric degenerations of M_C

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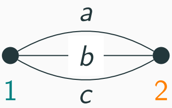

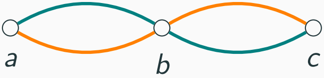

- Choose $3g-3$ disjoint circles in C and cut them such that C decomposes into a pair of pants.

Theorem: Manon, Galkin(g=2)

Let (Γ, c) be a trivalent graph with one (zero) colored vertex of genus g. The moduli spaces M_C (M_C^0-even degree determinant) degenerates to a toric variety $X_{\Gamma,c}$. whose moment polytope in $\mathbb{R}^{|E|}$ is given by:

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Remark

"Factorization of Conformal blocks" play a key role in Manon's theorem.

Theorem: Belmans-Galkin-M

- The variety $X_{\Gamma,c}$ has terminal singularities if the graph Γ has no separating edges.

Singularities of $X_{\Gamma,c}$

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- Let (Γ, c) and (Γ', c') be as above with no separating edges, then the toric varieties $X_{\Gamma,c} \cong X_{\Gamma',c'}$ if and only if Γ is isomorphic to Γ'. Moreover the class $[c] \in H^0(\Gamma, \mathbb{F}_2)$ to the class $[c']$. (Torelli type theorem)

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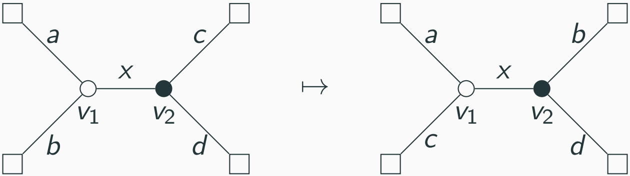
Conjecture II: (True for $g \leq 5$)

If Γ is three-connected, then $X_{\Gamma,c}$ admits a small resolution of singularities.

Hatcher-Thurston elementary mutations



What about mirror potentials

Theorem: Belmans-Galkin-M

- Let $T_{\Gamma,c}$ be the torus acting on $X_{\Gamma,c}$, then the graph potential defines a function.

$$W_{\Gamma,c} : \check{T}_{\Gamma,c} \to \mathbb{C},$$

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- Moreover (TFT construction), the periods are the same

$$\widehat{\pi}_{W_{\Gamma,c}}(t) = \widehat{\pi}_{W_{\Gamma',c'}}(t).$$

Quantum periods via potentials

Combining the works of Nishinou-Nohara-Ueda, Tomkonog, Bondal-Galkin and Conjecture II, we get

Corollary of Conjecture II

Let (Γ, c) be a any trivalent graph with odd number of colored vertices, then

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Remark

Our computations on the periods are very fast and we can compute up to high genus and degree. This can be used to compute examples of the quantum differential equation for M_C.

Properties

- The eigen-values of $c_1(M_C)\star_0$ of the quantum multiplication are also critical values i.e

$$\text{Eigen Values}(c_1(M_C)\star_0) \subseteq \text{Critical Values}(W_{\Gamma,c}).$$

- The critical set with critical values $8(1-g+k)(\sqrt{-1})^{\epsilon}$ for $0 \leq k \leq 2(g-1)$. Here ϵ is the parity of k.
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Remark

Usually the critical values of the mirror potential may miss some eigen-values of the quantum multiplication.

Landau-Ginzburg Models (Y, w)

Let Z be a smooth projective compactification of Y whose complement is s.n.c. and a commutative diagram

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The critical set of $|-K_Z|$ restricted to $\mathbb{A}^1$ is contained in Y.

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Conjecture III

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Question

Construct (Y, w) as a union of tori's $\check{T}_{\Gamma,c}$ for various graphs?