Exercise Sheet #1

1. Let $\mathrm{Gr}(d,n)$ be the set of d dimensional subspaces of k^n, where k is an algebraically closed field. The following steps show that the Grassmannian $\mathrm{Gr}(d,n)$ is an algebraic variety.

   (a) Let W be the set of $d \times n$ matrices of rank d and $\mathrm{GL}(d)$ acts on W by left multiplication. Show that there is a bijection between the orbits of the action and points of $\mathrm{Gr}(d,n)$.

   (b) For any $J = \{1 \leq j_1 < \cdots < j_d \leq n\}$, let U_J be the set of d dimensional spaces V of k^n which are complementary to the span of $n-d$ dimensional subspaces spanned by $\{e_j\}$ where $j \notin J$. Let W_J be the set of matrices in W whose J matrix is I_d. Show that $W_J \simeq U_J \simeq \mathbb{A}^{d(n-d)}$.

   (c) For $J \neq K$ of cardinality d, let $W_{J,K} \subset W_J$ be the set of matrices whose K-th minor is non zero. Show that there are isomorphism $\phi_{J,K} : W_{J,K} \to W_{K,J}$.

   (d) Conclude that $\mathrm{Gr}(d,n)$ is an abstract variety by gluing $\phi_{J,K}$.

2. Let $\mathrm{Gr}(d,n)$ be as defined before. Consider the map

$$\varphi : \mathrm{Gr}(d,n) \to \mathbb{P}(\bigwedge^d(k^n))$$

   defined by $W \to w_1 \wedge \cdots \wedge w_d$, where $w_1, \ldots, w_d$ is a basis of W. Show the following:

   (a) Check that ϕ is a well defined, injective morphism of varities.

   (b) Let $w \in \wedge k^r$ and consider the map $\phi_w(v) := v \wedge w$ from $k^r \to \wedge^{d+1}(k^r)$. Show that the map ϕ_w has rank $n-d$ if $w = w_1 \wedge \cdots \wedge w_d$, where w_i's are linearly independent. (This type of w's are called totally decomposable)

   (c) Show that $[w] \in \mathbb{P}(\wedge^d(k^n))$ is in the image of φ if and only if all $(n-d+1) \times (n-d+1)$ minors of the map ϕ_w vanish. Conclude that the Grassmannian is a projective variety.

3. Let (M, η) be a Coarse moduli space for a moduli problem $\mathcal{M}$. Show that (M, η) is a fine moduli space if and only if

   (a) There exists a family $\mathcal{U}$ over M that corresponds to the morphism id_M.

   (b) Two families $\mathcal{F}$ and $\mathcal{G}$ over S are similar over S if and only if $\eta_S(\mathcal{F}) = \eta_S(\mathcal{G})$.

4. Consider the naive moduli problem of classifying hypersurfaces of degree d in $\mathbb{P}^n$ upto equivalence by $PGL(n)$. Define a moduli functor associated to this naive moduli problem.