Exercise Sheet #4

1. Consider the diagonal action of $\mathbb{G}_m$ on $\mathbb{P}^3$ given by $t \to (t, t, t^{-1}, t^{-1})$. Find the invariants of the homogenous ring, semistable points, stable points and describe the quotient morphism

2. Let e_1, e_2 and e_3 be a basis of $\mathbb{C}^3$ and let $c \in \mathbb{G}_a$ acts as follows $c.e_1 = e_1$, $c.e_2 = 2ce_1 + e_2$ and $c.e_3 = c^2 e_1 + ce_2 + e_3$. Compute the ring $k[x_1, x_2, x_3]^{\mathbb{G}_a}$. Can you define a good quotient in this case.

3. Show that $\operatorname{Proj} R \simeq \operatorname{Proj} R^{(d)}$.

4. If R is finitely generated, then show that if d is large enough, then the Veronese subring $R^{(d)}$ is generated by R_d.

5. Assume that X is a smooth projective variety over $\mathbb{C}$ with an very ample line bundle L. Show that $X \simeq \operatorname{Proj}(\bigoplus_{k \geq 0} H^0(X, L^{\otimes k}))$ with the property that $\mathcal{O}_{\mathrm{Proj}}(1) \simeq L$. Feel free to assume finite generation of rings and also Serre Vanishing whenever required.

6. Let $a_0, \ldots, a_n$ be positive integers and $d = gcd(a_0, \ldots, a_n) > 1$. Then show that the following weighted projective spaces are isomorphic $\mathbb{P}(a_0, \ldots, a_n) \simeq \mathbb{P}(b_0, \ldots, b_n)$, where $b_i = a_i/d$.

7. Let S_4 act on $k[x_1, \ldots, x_4]$ by permuting the coordinates. Find $X//S_4$, where $X = \mathbb{P}^3$.

8. Let $(s, t) \in \mathbb{G}_m^2$ act by $\operatorname{dia}(st, s^{-1}t, s^{-1}t^{-1}, st^{-1})$ linearly on $\mathbb{P}^3$. Determine the stable and the semistable sets for this action and the G-invariant subring of $k[x_0, \ldots, x_4]$.