Exercise Sheet #9

The goal of this exercise sheet is to show the following proposition due to LePotier. The outline of the proof is given. This was generalized to higher dimensional varieties by C. Simpson.

1. Let n and d be fixed and d be such that $d > gn^2 + n(g-2)$. Let $\mathcal{F}$ be a locally free semistable sheaf on a smooth projective curve X of genus g with rank n and degree d. Then for all non-zero proper subsheaves $0 \neq \mathcal{F}' \subset \mathcal{F}$, we have

$$h^0(X, \mathcal{F}'). \operatorname{rk} \mathcal{F} \leq \chi(\mathcal{F}). \operatorname{rk} \mathcal{F}'$$

   and if equality holds, then $h^1(X, \mathcal{F}') = 0$ and $\mu(\mathcal{F}') = \mu(\mathcal{F})$

   (a) Show that for $\mu = d/n$, we can pick a constant C such that $2g - 2 < C < \mu - gn$.

   (b) Let $\mathcal{F}'$ be a proper subsheaf and $\mu_{min}(\mathcal{F}')$ be the minimal slope from the HN-filtration of $\mathcal{F}'$, then give a bound for $h^0(X, \mathcal{F}')/\operatorname{rk}(\mathcal{F}')$ using LePotier bounds.

   (c) If $\mu_{min}(\mathcal{F}') \leq C$, then show that $h^0(X, \mathcal{F}'). \operatorname{rk} \mathcal{F} < \chi(\mathcal{F}). \operatorname{rk} \mathcal{F}'$

   (d) If $\mu_{min}(\mathcal{F}') > C$, then $H^1(X, \mathcal{F}') = 0$. (Hint show that $H^1(X, \mathcal{F}'_i) = 0$)

   (e) Use Step (d) to show that $h^0(X, \mathcal{F}'). \operatorname{rk} \mathcal{F} \leq \chi(\mathcal{F}). \operatorname{rk} \mathcal{F}'$, and if equality holds $\mu(\mathcal{F}') = \mu(\mathcal{F})$.